Distortion Risk Measures and Discrete Risks*

Antonella Campana[1] and Paola Ferretti[2]

[1] Department SEGeS - University of Molise
Via F. De Sanctis, I-86100 Campobasso, Italy
campana@unimol.it
[2] Department of Applied Mathematics - University Ca' Foscari Venice
Dorsoduro 3825/E, I-30123 Venice, Italy
ferretti@unive.it

Abstract

In this paper we consider the problem of determining approximations for distortion risk measures of sums of non-independent random variables. First, we give an overview of the recent actuarial literature on distortion risk measures and convex bounds for sums of random variables. Then, we examine the case of discrete risks with identical distribution. Upper and lower bounds for risk measures of sums of risks are presented in the case of concave distortion functions. The result is then extended to cover the case of non necessarily discrete risks.

KEYWORDS: Risk measures; dependency of risks; discrete risks with identical distribution; upper and lower bounds: concave risk measures.

J.E.L. CLASSIFICATION: D810
M.S.C. CLASSIFICATION: 62H20; 60E15

1 Introduction

Recently in actuarial literature, the study of the impact of dependence among risks has become a major and flourishing topic: even if in traditional risk theory, individual risks have been usually assumed to be independent, this assumption is very convenient for tractability but it is not generally realistic. Think for example to the aggregate claim amount in which any random variable represents the individual claim size of an insurer's risk portfolio. When the risk is represented by residential dwellings exposed to danger of an earthquake in a given location or by adjoining buildings in fire insurance, it is unrealistic to state that individual risks are not correlated, because they are subject to the same claim causing mechanism. Several notions of dependence were introduced in literature to model the fact that larger values of one of the component of a multivariate risk tend to be associated with larger values of the others. It is particularly interesting to study sums of random variables of which the marginal distribution is known

* This research was partially supported by MIUR

but the joint distribution is not specified or too complex to work with; in these cases it is possible to refer to upper and lower bounds in the sense of convex order, namely to the riskiest portfolio in which the multivariate distribution refers to mutually comonotonic risks, and to the safest portfolio in which the multivariate distribution refers to mutually exclusive risks, respectively. Some recent papers have investigated the compatibility of risk measures, for example insurance premium principles, with stochastic orders.

The present contribution is devoted to the analysis of a particular class of risk measures, namely of distortion risk measures given by Choquet Integrals, as well as to the definition of explicit formulas for distortion risk measures of upper and lower bounds of sums of risks, both in case of discrete identically distributed risks, both in case of identically distributed risks not necessarily discrete. The framework is that of multivariate risks with the same marginal distributions. The context is that of distortion risk measures, that is of measures of risks satisfying additivity for comonotonic risks, positive homogeneity, translation invariance, preservation of first order stochastic dominance; in the particular case of a concave distortion measure, the corresponding distortion risk measure is also sub-additive. Starting from the representation of risks as sums of layers it is possible to explicit the distorted risk measure of a risk as a particular sum. In this way we obtain upper and lower approximations for distortion risk measures of sums of discrete identically distributed risks, first, and of identically distributed risks not necessarily discrete, then.

The paper is organized as follows. In Section 2 we first review some basic settings for describing the problem of measuring a risk and then we remind some definitions and preliminary results in that field. Section 3 is devoted to the problem of detecting upper and lower bounds for sums of not mutually independent risks. Next Section 4 presents the study of the case of a discrete risk with finitely many mass points in such a way that it is possible to give an explicit formula for its distortion risk measure. In Section 5 the case of sums of discrete and identically distributed risks is investigated in order to obtain upper and lower bounds for concave distortion measures of aggregate claims of the portfolio. Finally Section 6 is devoted to the problem of setting bounds for distortion risk measures of sums of identically distributed risks by a limit result. Some concluding remarks end the paper.

2 Distortion risk measures: properties and preliminary results

An insurance risk is defined as a non-negative real-valued random variable X defined on some probability space. We will consider a set Γ of risks with bounded support on $[0, c]$. For each risk $X \in \Gamma$ we will denote by H_X its tail function, i.e. $H_X(x) = Pr[X > x]$, for all $x \geq 0$.

A risk measure is defined as a mapping from the set of random variables, namely losses or payments, to the set of real numbers. In actuarial science common risk measures are premium principles; other risk measures are used for determining provisions and capital requirements of an insurer in order to avoid insolvency (see e.g. [4]).

In this paper, we will concentrate on those risk measures associated with the risks $X \in \Gamma$ which belong to the class of distortion risk measures introduced by Wang [8]. They can be written as

$$W_g(X) = \int_0^\infty g(H_X(x))dx \tag{1}$$

where the distortion function g is defined as a non-decreasing function $g : [0,1] \rightarrow [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

The *quantile risk measure* and the *Tail Value-at-Risk* belong to this class.

In fact, the *quantile risk measure* $Q_p[X]$ at level $p \in (0,1)$, often called VaR (Value-at-Risk), corresponds to the distortion function

$$g(x) = I_{(x>1-p)}, \quad 0 \le x \le 1, \tag{2}$$

where $I_{(x>1-p)}$ is the indicator function which equals 1 if $x > 1-p$ and 0 otherwise.

The *Tail Value-at-Risk* $TVaR_p[X]$ at level $p \in (0,1)$ corresponds to another distortion function, namely to the function

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), \quad 0 \le x \le 1. \tag{3}$$

If g is a power function, i.e. $g(x) = x^{1/\rho}$, $\rho \ge 1$, the corresponding risk measure is the PH-transform risk measure proposed by Wang [**?**].

Let $\mathbf{X}$ be a random vector, i.e. $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, with $X_i \in \Gamma$, $i = 1, 2, \ldots, n$.

For any $\mathbf{X}$, not necessarily comonotonic, we will denote by $\mathbf{X}^c = (X_1^c, X_2^c, \ldots, X_n^c)$ its comonotonic counterpart, i.e. a random vector with the same marginal distributions and with the comonotonic dependence structure.

Let S^c be the sum of the components of $\mathbf{X^c}$.

The concept of comonotonicity, its characterizations and applications in actuarial science and finance are examined in [2] and [3].

It can be proved that a random vector is comonotonic if and only if all its marginal distribution function are non-decreasing (or non-increasing) transformations of the same random variable (see e.g. [2]).

Any distortion risk measure obeys the following properties (see [8] and [4]).

P1. Additivity for comonotonic risks

For any distortion function g and all random variables X_i,

$$W_g(S^c) = \sum_{i=1}^{n} W_g(X_i). \tag{4}$$

P2. Positive homogeneity

For any distortion function g, any random variable X and any non-negative constant a, we have

$$W_g(a\,X) = a\,W_g(X). \tag{5}$$

P3. Translation invariance

For any distortion function g, any random variable X and any constant b, we have

$$W_g(X + b) = W_g(X) + b. \tag{6}$$

P4. Monotonicity

For any distortion function g and any two random variables X and Y where $X \leq Y$ with probability 1, we have

$$W_g(X) \leq W_g(Y). \tag{7}$$

The following theorem says that stochastic dominance (of first order) can be characterized in terms of distortion risk measures.

Theorem 1 *For any random pair (X, Y) we have that X is smaller than Y in stochastic dominance sense if and only if their respective distortion risk measures are ordered:*

$$X \leq_{st} Y \iff W_g(X) \leq W_g(Y), \quad \textit{for all distortion functions } g.$$

Proof See [4].

An important subclass of distortion functions is represented by the class of concave distortion functions. A risk measure with a concave distortion function is called a concave distortion risk measure. Examples of these risk measures are the TVaR and the PH-transform risk measure, whereas the quantile risk measure is not a concave risk measure.

As shown in [4], the stop-loss order can be characterized in terms of ordered concave distortion risk measures.

Theorem 2 *For any random pair (X, Y) we have that X precedes Y in the stop-loss order sense if and only if their respective concave distortion risk measures are ordered:*

$$X \leq_{sl} Y \iff W_g(X) \leq W_g(Y), \quad \textit{for all concave distortion functions } g.$$

Proof See [4].

3 Convex bounds for sums of risks

In financial or actuarial situations one often encounters random variables of the type

$$S = \sum_{i=1}^{n} X_i$$

where the terms X_i are not mutually independent and the multivariate distribution function of the random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is not completely specified because one only knows the marginal distribution functions of the risks X_i. In such cases, to be able to make decisions, it may be helpful to determine approximations for the distribution of S.

Recently, in actuarial literature several authors have derived stochastic lower and upper bounds for sums S. These bounds are in the sense of convex order. As shown in the following theorem, the least attractive random vector with given marginal distribution functions has the comonotonic joint distribution. It is also shown how to obtain a lower bound for S, in the sense of convex order, by using a conditioning random variable.

Theorem 3 *For any random vector* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *and any random variable* Λ*, we have that*

$$S^l \leq_{cx} S \leq_{cx} S^c, \tag{8}$$

with S^l *given by*

$$S^l = \sum_{i=1}^{n} E[X_i \mid \Lambda].$$

Proof See [2] and [6].

Obviously the marginal distributions of the random vector $(E[X_1 \mid \Lambda], E[X_2 \mid \Lambda], \ldots, E[X_n \mid \Lambda])$ will not be, in general, the same of the random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. In order to obtain a lower bound which is the sum of n comonotonic random variables, one can find a conditioning random variable Λ with the property that all the random variables $E[X_i \mid \Lambda]$ are non-decreasing (or non-increasing) functions of Λ.

If we restrict our study to a Fréchet space consisting of all n-dimensional random vectors $\mathbf{X}$ possessing $(H_{X_1}, H_{X_2}, \ldots, H_{X_n})$ as marginal tail functions, for which the condition $\sum_{i=1}^{n} H_{X_i}(0) \leq 1$ is fulfilled, we find that the safest dependence structure is obtained with the Fréchet lower bound and precisely it corresponds to the mutually exclusive risks of the Fréchet space. We recall that the risks $X_1^e, X_2^e, \ldots, X_n^e$ are said to be mutually exclusive when $Pr[X_i^e > 0, X_j^e > 0] = 0$ for all $i \neq j$.

Let S^e denote the sum of mutually exclusive risks $X_1^e, X_2^e, \ldots, X_n^e$. Its tail function is given by

$$H_{S^e}(x) = \sum_{i=1}^{n} H_{X_i}(x), \quad \text{for all } x \geq 0. \tag{9}$$

It is proved in [1] that

$$S^e \leq_{sl} \sum_{i=1}^{n} S_i \tag{10}$$

holds true for any random vector **X** with the given marginal distributions.

It is well known that the concept of stop-loss order is closely related to the notion of convex order: in fact it can be defined as follows (see e.g. [2]).

Definition 1 *The random variable X is said to precede the random variable Y in the convex order sense, notation $X \leq_{cx} Y$, if and only if $X \leq_{sl} Y$ and in addition $E[X] = E[Y]$.*

It follows that the statements of Theorem 3 are also true in the stop-loss order.

We recall that a risk measure W is said to be sub-additive if for any pair of random variables X and Y, one has

$$W(X+Y) \leq W(X) + W(Y). \tag{11}$$

As we have seen, a concave distortion risk measure preserves stop-loss order (Theorem 2) and satisfies the property *P1* of additivity for comonotonic risks. From Theorem 3 we have:

$$W_g(S) \leq \sum_{i=1}^{n} W_g(X_i) \tag{12}$$

i.e. a concave risk measure is sub-additive.

4 Discrete risks

As it is well-known, each risk $X \in \Gamma$ can be written as sum of layers (see [5]), where a layer at (a,b) of X is defined as the loss from an excess-of-loss cover, namely

$$L(a,b) = \begin{cases} 0 & 0 \leq X \leq a \\ X - a & a < X < b \\ b - a & X \geq b \end{cases} \tag{13}$$

The tail function of the layer $L(a,b)$ is given by

$$H_{L(a,b)}(x) = \begin{cases} H_X(a,b) & 0 \leq x < b-a \\ 0 & x \geq b-a. \end{cases} \tag{14}$$

For any positive integer k and any sequence $0 \equiv y_0 < y_1 < \ldots < y_k \equiv c$, we can write

$$X = \sum_{j=0}^{k-1} L(y_j, y_{j+1}). \tag{15}$$

The interest in the proposed decomposition of X relies on the fact that the layers $L(y_j, y_{j+1})$, $j = 0, 1, 2, \ldots, k-1$, are pairwise mutually comonotonic risks.

In the particular case of a discrete risk $X \in \Gamma$ with finitely many mass points it is possible to deduce an explicit formula of the distortion risk measure $W_g(X)$ of X. In fact it is possible to set the following result.

Proposition 1 *Let $X \in \Gamma$ be a discrete risk. Then for any distortion function g*

$$W_g(X) = \sum_{i=0}^{m-1} (x_{j+1} - x_j)\, g(p_j) \tag{16}$$

where the finite sequence $\{p_i\}$, $(i = 0, \ldots, m-1)$, is defined in such a way that the tail function of X is

$$H_X(x) = \sum_{j=0}^{m-1} p_j\, I_{(x_j \leq x < x_{j+1})}, \quad x \geq 0. \tag{17}$$

$I_{(x_j \leq x < x_{j+1})}$ is the indicator function of the set $\{x : x_j \leq x < x_{j+1}\}$.

Proof

By hypothesis, $X \in \Gamma$ is a discrete risk with finitely many mass points: then, there exist a positive integer m, a finite sequence $\{x_j\}$, $(j = 0, \cdots, m)$, $0 \equiv x_0 < x_1 < \ldots < x_m \equiv c$ and a finite sequence $\{p_j\}$, $(j = 0, \cdots, m-1)$, $1 \geq p_0 > p_1 > p_2 > \ldots p_{m-1} > 0$ such that the tail function H_X of X is so defined

$$H_X(x) = \sum_{j=0}^{m-1} p_j\, I_{(x_j \leq x < x_{j+1})}, \quad x \geq 0, \tag{18}$$

where $I_{(x_j \leq x < x_{j+1})}$ is the indicator function of the set $\{x : x_j \leq x < x_{j+1}\}$.

We can use representation (15) for X, i.e.

$$X = \sum_{j=0}^{m-1} L(x_j, x_{j+1}). \tag{19}$$

In this case, the tail function of the layer $L(x_j, x_{j+1})$ is given by

$$H_{L(x_j, x_{j+1})}(x) = \begin{cases} p_j & 0 \leq x < x_{j+1} - x_j \\ 0 & x \geq x_{j+1} - x_j \end{cases} \tag{20}$$

and so $L(x_j, x_{j+1})$ is a two-points distributed random variable which satisfies the following equality in distribution

$$L(x_j, x_{j+1}) \stackrel{\mathrm{d}}{=} (x_j - x_{j+1})\, B_{p_j}, \tag{21}$$

where B_{p_j} denotes a Bernoulli random variable such that

$$Pr[B_{p_j} = 1] = p_j = 1 - Pr[B_{p_j} = 0].$$

Consider a distortion risk measure W_g. From the property of additivity for comonotonic risks, we have

$$W_g(X) = \sum_{i=0}^{m-1} W_g(L(x_j, x_{j+1})). \tag{22}$$

Further, the property of positive homogeneity leads to

$$W_g(X) = \sum_{i=0}^{m-1} (x_{j+1} - x_j)\, W_g(B_{p_j}) \tag{23}$$

with

$$W_g(B_{p_j}) = g(p_j). \tag{24}$$

and this completes the proof.

■

5 Distortion risk measures for sums of discrete risks

In this paragraph the components of the random vector $\mathbf{X}$ are supposed to be discrete and identically distributed risks with common tail function given by (18). In this framework it is possible to refer to Theorem 3 in order to have upper and lower approximations of distortion risk measures of sums S. More precisely, it is possible to obtain upper and lower bounds in the case of one particularly interesting class of distortion risk measures: namely, the class of concave distortion risk measures. It is possible to state, in fact, that

Theorem 4 *Let $\mathbf{X}$ be a random vector with discrete and identically distributed risks $X_i \in \Gamma$. Let the common tail function of X_i be written as*

$$H_{X_i}(x) = \sum_{j=0}^{m-1} p_j\, I_{(x_j \leq x < x_{j+1})}, \quad x \geq 0 \tag{25}$$

where m is a positive integer and $1 \geq p_0 > p_1 > p_2 > \dots p_{m-1} > 0$, $0 \equiv x_0 < x_1 < \dots < x_m \equiv c$. Let $n\, p_0 \leq 1$.

Then for any concave distortion function g the following inequality is verified:

$$\sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(n\, p_j) \leq W_g(S) \leq n \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(p_j). \tag{26}$$

Proof

Under these assumptions, the tail function of the sum S^e of mutually exclusive risks given by (9) becomes

$$H_{S^e}(x) = n \sum_{j=0}^{m-1} p_j \, I_{(x_j \leq x < x_{j+1})}, \quad \text{for all } x \geq 0, \tag{27}$$

provided that the condition $n\,p_0 \leq 1$ is satisfied. Note that S^e can be written as a sum of layers

$$S^e = \sum_{j=0}^{m-1} \tilde{L}(x_j, x_{j+1}) \tag{28}$$

where $\tilde{L}(x_j, x_{j+1})$ is a two-points distribution with

$$Pr[\tilde{L}(x_j, x_{j+1}) = x_{j+1} - x_j] = n\,p_j = 1 - Pr[\tilde{L}(x_j, x_{j+1}) = 0]$$

i.e. (see (21)):

$$\tilde{L}(x_j, x_{j+1}) \stackrel{\mathrm{d}}{=} B_{n\,p_j}. \tag{29}$$

By considering a concave distortion risk measure, we find:

$$W_g(S^e) = \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(n\,p_j) \tag{30}$$

¿From (10) we obtain

$$W_g(S^e) \leq W_g(S). \tag{31}$$

In order to apply Theorem 3, we observe that, under these assumptions, the following equality distribution holds:

$$S^c \stackrel{\mathrm{d}}{=} n\,X_1, \tag{32}$$

and then

$$W_g(S) \leq W_g(nX_1). \tag{33}$$

Hence, for any concave distortion function g, we have:

$$\sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(n\,p_j) \leq W_g(S) \leq n \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(p_j). \tag{34}$$

■

Note that in the case in which all, except $n\,p_0 \leq 1$, the conditions in Theorem 4 are verified, for any random variable Λ the following inequality may be stated

$$n \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, W_g(E[B_{p_j} \mid \Lambda]) \leq W_g(S) \leq n \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(p_j). \quad (35)$$

In fact owing to the fact that the following equality in distribution is verified

$$S^l \overset{\mathrm{d}}{=} n\, E[X_1 \mid \Lambda] \quad (36)$$

according to (19) and (21) we can also write

$$n\, E[X_1 \mid \Lambda] \overset{\mathrm{d}}{=} n \sum_{j=0}^{m-1} E[L(x_j, x_{j+1}) \mid \Lambda] \overset{\mathrm{d}}{=} n \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, E[B_{p_j} \mid \Lambda]. \quad (37)$$

From Theorem 3 it follows

$$n \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, E[B_{p_j} \mid \Lambda] \leq_{cx} S \leq_{cx} n \sum_{j=0}^{m-1} L(x_j, x_{j+1}) \quad (38)$$

and so it follows (35).

Lower and upper bounds in (26) represent approximations for distortion risk measures of sums of discrete risks X_i, $i = 1, 2, \ldots, n$, identically distributed. The reference to them is particularly interesting when dependency structure between the X_i is unknown or too cumbersome to work with.

For example, if we consider the PH-transform risk measure introduced by Wang (1995), we obtain:

$$n^{1/\rho} \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, p_j^{1/\rho} \leq W_g(S) \leq n \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, p_j^{1/\rho}. \quad (39)$$

In this case we have:

$$W_g(S^c) = n^{1-1/\rho}\, W_g(S^e) \quad (40)$$

This means that, for a portfolio of a given size n, the performance of the obtained approximations depends on ρ which is called the risk-averse index: the smaller ρ, the better the performance.

6 Distortion risk measures for sums of i.d. risks: an extension result

In the case in which the risk $X \in \Gamma$ is not discrete, it is possible to approximate its tail function H_X by the following piecewise constant tail function:

$$H_{\tilde{X}_m}(x) = \sum_{j=0}^{2^m-1} H_X(\frac{j+1}{2^m}\,c)\, I_{(\frac{j}{2^m}\,c \le x < \frac{j+1}{2^m}\,c)}, \quad x \ge 0. \tag{41}$$

Let W_g be a concave distortion function. In [5] it is proved that:

$$\lim_{m\to\infty} W_g(\tilde{X}_m) = W_g(X) \tag{42}$$

where $W_g(\tilde{X}_m)$ is given by:

$$W_g(\tilde{X}_m) = \sum_{j=0}^{2^m-1} \frac{c}{2^m}\, g(H_X(\frac{j+1}{2^m}\,c)). \tag{43}$$

The previous observations have an appealing interpretations in the case in which it is assumed that the components of the random vector $\mathbf{X}$ are identically distributed risks with tail function H_X.

Theorem 5 *Let $\mathbf{X}$ be a random vector with identically distributed risks $X_i \in \Gamma$ with common tail function $H_X(x)$. Let $n\,H_X(0) \le 1$.*

Then for any concave distortion function g the following inequality is verified:

$$\lim_{m\to\infty} W_g(\tilde{S}^e_m) \le W_g(S) \le n\,(\lim_{m\to\infty} W_g(\tilde{X}_m)) \tag{44}$$

where

$$W_g(\tilde{S}^e_m) = \sum_{j=0}^{2^m-1} \frac{c}{2^m}\, g(n\,H_X(\frac{j+1}{2^m}\,c)) \tag{45}$$

and

$$W_g(\tilde{X}_m) = \sum_{j=0}^{2^m-1} \frac{c}{2^m}\, g(H_X(\frac{j+1}{2^m}\,c)). \tag{46}$$

Proof

Assume that the condition $n\,H_X(0) \le 1$ is satisfied. Now we can approximate the tail function of the sum S^e of mutually exclusive risks by the following piecewise constant tail function:

$$H_{\tilde{S}^e_m}(x) = n \sum_{j=0}^{2^m-1} H_X(\frac{j+1}{2^m}\,c)\, I_{(\frac{j}{2^m}\,c \le x < \frac{j+1}{2^m}\,c)}, \quad x \ge 0. \tag{47}$$

By considering a concave distortion risk measure, as in (30), we obtain:

$$W_g(\tilde{S}^e_m) = \sum_{j=0}^{2^m-1} \frac{c}{2^m}\, g(n\, H_X(\frac{j+1}{2^m}\, c)). \tag{48}$$

It can be proven that

$$\lim_{m\to\infty} W_g(\tilde{S}^e_m) = W_g(S^e) \tag{49}$$

holds true like (42).

In our assumption we also have:

$$W_g(\sum_{i=1}^{n} X^c_i) = n\, W_g(X_1) = n\, (\lim_{m\to\infty} W_g(\tilde{X}_m)). \tag{50}$$

Then, we can write:

$$\lim_{m\to\infty} W_g(\tilde{S}^e_m) \le W_g(S) \le n\, (\lim_{m\to\infty} W_g(\tilde{X}_m)). \tag{51}$$

■

Note that the previous results can be extended to the case of a risk with an unbounded support. In fact, if the risk X has an unbounded support, for any $c \ge 0$ the risk $\min(X, c)$ is bounded and the concave distortion risk measure satisfies (see [5]):

$$\lim_{c\to\infty} W_g(\min(X, c)) = W_g(X) \tag{52}$$

7 Concluding remarks

In this contribution we considered the problem of deriving distorted risk measures of upper and lower bounds, in the sense of convex order, for sums of possibly dependent random variables with known marginal distribution. First, we assumed that the risks are discrete and identically distributed, then, that they are identically distributed but not necessarily discrete. Starting from the representation of risks as sums of layers, we derived explicit formulas for risk measures of upper and lower bounds of sums of risks, in the particular case of concave distortion risk measures. In the case of the PH-transform risk measure introduced by Wang [7], we obtained that the performance of the upper and lower approximations depends on the risk-averse index.

References

1. DHAENE, J., DENUIT, M. (1999). The safest dependence structure among risks. *Insurance: Mathematics and Economics*, 25: 11-21.
2. DHAENE, J., DENUIT, M., GOOVAERTS, M.J., KAAS, R., VYNCKCE, D. (2002). The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics and Economics*, 31: 3-33.
3. DHAENE, J., DENUIT, M., GOOVAERTS, M.J., KAAS, R., VYNCKCE, D. (2002). The concept of comonotonicity in actuarial science and finance: applications. *Insurance: Mathematics and Economics*, 31: 133-161.
4. DHAENE, J., VANDUFFEL, S., TANG, Q.H., GOOVAERTS, M.J., KAAS, R., VYNCKE, D. (2004). Solvency capital, risk measures and comonotonicity: a review. *Research Report OR 0416*, Department of Applied Economics, K.U. Leuven.
5. GOOVAERTS, M.J., DHAENE, J. (1998). On the characterization of Wang's class of premium principles. *Transactions of the 26th International Congress of Actuaries*, 4:121-134.
6. KAAS, R., DHAENE, J., GOOVAERTS, M.J. (2000). Upper and lower bounds for sums of random variables. *Insurance: Mathematics and Economics*, 27: 151-168.
7. WANG, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazard transforms. *Insurance: Mathematics and Economics*, 17: 43-54.
8. WANG, S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin*, 26: 71-92.