# Formation of Collective Decision-Making Units: Stability and a Solution*

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**Abstract**

We study how individuals divide themselves into coalitions and choose a public alternative for each coalition. When preferences have consecutive support and coalition feasible sets are positively population-responsive, the proposed consecutive benevolence solution generates allocations belonging to the coalition structure core and that are also Tiebout equilibria. However, when each coalition follows a single-valued collective decision rule, the coalition structure core may be empty.

Our results show that if individual preferences are, in a sense, similar and if members can be as well off when a coalition enlarges, then a stable formation of collective decision-making units can be guaranteed. A predetermined decision rule makes coalitions less stable.

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## 1 Introduction

Individuals in a society face many collective decisions; for example, how much tax to levy, how much to spend on public schools, and whether to build a community center.

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Collective decisions are carried out in units. Depending on different classes of issues at stake, individuals belong to different decision-making units, and there are usually multiple units with respect to one class of issues. Countries, local governments, universities, unions, and political parties are such units. When national security and foreign affairs are concerned, countries are the units; the United States can have only one stance on these matters. When policies on abortion, gun control, and the legalization of marijuana are concerned, states are the corresponding units; each state has its own policies. When it comes to matters of local area construction, the decision units are zoning districts.

In most cases, individuals are free to switch to other units or form new units. Individuals want to switch to units that offer more preferable alternatives. For example, people move to other communities for better public schools or amenities. Recurrent migration also indicates that individuals move into other countries for better lives. Individuals can also form a new unit that offers a better alternative for all who join. For example, the former Soviet Union broke down into several republics; West and East Germany became a united country; East Timor separated itself from Indonesia. Moreover, countries, as units deciding on international trade issues, form international organizations, and thus emerged such institutions as the Word Trade Organization and the Europe Common Market.

In some situations, there is a pre-determined collective decision rule associated with these decision-making units. For instance, zoning policies are determined by the vote of residents; a newborn country is expected to adopt democratic rules. In other situations, a unit makes decisions in a more flexible manner, such as via the interaction of its members. There is no fixed formal rule to follow and the final decision may not be predicted from the preferences of members. Families, clubs, and business partnerships are this type of decision-making units.

An important question is, does there exist a stable formation of collective decision-making units? A formation of units is stable if everyone is satisfied with the alternative offered by the unit to which she belongs, so that no individuals want to form or switch to a new unit. A further question is, is there a solution to attaining stable formations of units?

The literature of local public finance addresses the issue of collective decision-making in multiple units. Tiebout (1956) initiated the approach that models local economies as competing entities which offer tax and expenditure packages to attract residents. Individuals are free to reside where they wish. Tiebout claims that there is an equilibrium and that it is efficient. This concept of "Tiebout equilibrium" has been criticized by many. For example, Bewley (1981) points out that without very restrictive conditions, an equilibrium may not exist or may not be efficient. Other works provide various conditions under which a Tiebout equilibrium exists in the general equilibrium framework. In these models, the allocation of private goods is determined by the market, while various mechanisms are proposed to determine the allocation of local public goods: via a public competitive equilibrium, which is a profile of tax-expenditure packages provided by governments (Greenberg 1977); by an assignment, which simply assigns individuals to jurisdictions (Ellickson 1979); by planners, who maximize local tax revenue or property value (Epple and Zelenitz 1981); or by vote, which decides an outcome according to aggregated preferences (Westhoff 1977, Denzau and Parks 1983). In many applied studies (Westhoff 1977, Rose-Ackerman 1979, Epple, Filimon and Romer 1984, 1993, and Epple and Romer 1990), freely mobile individuals vote on local tax rates, which affect the prices of private goods including housing (or land) in local communities. These general equilibrium models assume an exogenously fixed number of communities, which prohibits individuals from forming new units. In fact, it causes the "integer problem" in finite models, where an individual has to be divided into fractions in equilibria. Many examples of disequilibrium in Bewley (1981) can be resolved when individuals can form new communities (see Greenberg and Weber 1986). There is usually a fixed decision rule[1] in these models to make public decisions according to the composition of residents.

The issue of endogenous formation is discussed in the literature of coalition formation games. Many authors develop solution concepts supported by coalition structures (partitions of players). The $\psi$-stability in Luce and Raiffa (1985) requires that no new coalition, from a pre-determined list of coalitions, can block. Aumann and Dréze (1975) apply many solution concepts developed for the grand coalition to coalition structures. The structure equilibrium in Greenberg (1978, 1979) requires that no player can join another existing coalition and make herself and members of that coalition better off. The individual contractual stability in Dréze and Greenberg (1980) requires that no one can join another existing coalition and make members of the coalition she leaves better off. The C-stable solution in Guesnerie and Oddou (1981) requires that no new coalition can block. The $\mathcal{S}$-equilibrium in Greenberg and Weber

[1] A fixed rule is implied even when the decision is made by a community planner who maximizes residents' utility or tax revenue. Because the objective function of the planner is given, public decisions can be determined once the composition of residents is known.

(1986, 1993a) is C-stable and such that no individual wants to join another existing coalition. (See also the survey in Greenberg 1994.) Note that all these concepts apply to payoff profiles in coalitional form games, except for the $\mathcal{S}$-equilibrium, which combines the core and Tiebout equilibrium, and applies in a less abstract environment. Greenberg (1978), Guesnerie and Oddou (1981), and Greenberg and Weber (1986, 1993a) apply tools of coalition formation games to local public economies. They add a new dimension to the above literature on Tiebout equilibrium. This literature captures the situation where decision-making units form with more freedom: individuals can switch among and form units. However, these games focus on payoff profiles, and economic variables are reduced to utility levels. A drawback is that there may not be a one-to-one correspondence between payoff profiles and the "economic states" of individuals.

We employ the model introduced by Greenberg and Weber (1993a), where individuals form coalitions endogenously to decide on public alternatives. Each individual joins one and only one coalition. Each coalition has a set of feasible public alternatives, and members choose one from the feasible set. We study the combination of the following two notions of stability. The coalition structure core is the collection of allocations that partition the population such that no group wants to form a new coalition. A Tiebout equilibrium requires that no individual wants to join another coalition. (It is envy-free, which requires that no one wants to switch places with another.) We examine conditions that guarantee the existence of a stable formation of collective decision-making units (coalitions) and develop a solution to attaining such allocations.

The existence of a stable formation of coalitions is not unconditionally guaranteed. (We will see an example in Section 2.) To make sure there is a stable formation, we need to impose structures on individual preferences and coalition feasible sets: The preferences have "consecutive support" if, with respect to a linear order, the set of individuals who strictly prefer an alternative to the other in any pair of alternatives is consecutive. This assumption is weaker than three other preference restrictions commonly used in the literature: intermediate preferences, order restriction, and single-crossing (detailed in the Appendix). Coalition feasible sets are "positive population-responsive" if it is always possible to make existing members as well off when new members join the coalition. We propose a solution that generates Core-Tiebout equilibria when preferences have consecutive support and feasible sets are positive population-responsive. A Core-Tiebout equilibrium belongs to the coalition structure core and is also a Tiebout equilibria. Our result differ from Greenberg and Weber (1993a) and Demange (1994) in that we do not require alternatives to be one-dimensional; our preference restriction is weaker; we allow coalitions with empty feasible sets; and we do not require the feasible sets to be monotonic.

Furthermore, we study situations where each coalition follows a collective decision rule which chooses a set of public alternatives according to members' preferences. If coalitions follow single-valued decision rules, there may not exist a stable coalition structure even when preferences and feasible sets are restricted as mentioned above. We also show that a Core-Tiebout equilibrium can be guaranteed when the decision rule permits all of the Pareto optimal alternatives in a coalition.

Section 2 introduces the model and discusses the notions of stability. Section 3 defines the consecutive benevolence solution. Section 4 shows that the solution generates stable coalition structures. Section 5 discusses the situation where coalitions use decision rules. Section 6 concludes.

## 2 The Model

A *society*, $\mathcal{S} = (N, X, R, \varphi)$, is composed of a set of individuals $N$, a nonempty set of alternatives $X$, a family of preference relations $R$, and a feasible correspondence $\varphi$. Each individual $i \in N$ has a preference relation $R_i$ over $X$, which is a weak order. Let $P$ denote strict preference and $I$ indifference. The family of preference relations $R = \{R_i\}_{i \in N}$ on set $N$ is the collection of preference relations of all $i \in N$. Let $\mathcal{P}(.)$ denote the set of all nonempty subsets of its argument. Each subset of $N$, $S \in \mathcal{P}(N)$, is a coalition of individuals. The feasible correspondence $\varphi : \mathcal{P}(N) \rightarrow \mathcal{P}(X) \cup \{\emptyset\}$ denotes the feasible set of public alternatives of a coalition. (Coalitions with empty feasible sets are permitted.) We require that there exists a coalition $S \in \mathcal{P}(N)$ such that $\varphi(S) \neq \emptyset$. Each coalition chooses an alternative from its feasible set. Each individual belongs to one and only one coalition. The formation of coalitions generates a partition of $N$.

A *coalition structure* $C$ in society $\mathcal{S}$ is a partition of $N$, where $C \subset \mathcal{P}(N)$ and (i) $S \cap S' = \emptyset$ for all $S, S' \in C$, $S \neq S'$; (ii) $\cup_{S \in C} S = N$; (iii) $\varphi(S) \neq \emptyset$ for all $S \in C$. An *allocation* in society $\mathcal{S}$ is a map $a : N \rightarrow X$ which assigns alternative $a(i)$ to individual $i$. Allocation $a$ is *feasible* if there is a pair $(C, x)$ of coalition structure $C$ and alternative profile $x = \{x_S\}_{S \in C}$ with $x_S \in \varphi(S)$ for all $S \in C$ such that $a(i) = x_S$

for all $i \in S$ for all $S \in C$.

Given a coalition structure, individuals may want to switch to other coalitions or form a new coalition. The following two stability concepts are applicable to our model.

**Definition 2.1.** A feasible allocation $a$ in society $\mathcal{S}$ is a *Tiebout equilibrium* if (i) there is no $S \in C$ with $\omega \in \varphi(S)$ such that $\omega P_i x_S$ for all $i \in S$, and (ii) $a(i) R_i a(j)$ for all $j \neq i$ for all $i, j \in N$.

In a Tiebout equilibrium, coalitions cannot find a better alternative, and there is no individual who strictly prefers an alternative offered in another coalition. Tiebout (1956) wrote the following (p. 418):

> "The consumer-voter may be viewed as picking that community which best satisfies his preference pattern for public goods. ... Given these revenue and expenditure patterns, the consumer-voter moves to that community whose local government best satisfies his set of preferences."

With "community" replaced by "coalition" and "local government" by "collective decision", the above passage describes a situation where a freely mobile individual chooses an existing coalition that offers one of her most preferred alternatives of all that are offered. In a Tiebout equilibrium, coalitions cannot exclude members, and an individual can move to another coalition without considering the effect of her arrival. That is, a coalition can not reject a new member even if she reduces the welfare of the existing members. Actually, a newcomer assumes that the that same alternative will be offered after she joins. However, the arrival of a new member may reduce a coalition's feasible set, and consequently make the existing members worse off.

**Definition 2.2.** A feasible allocation $a$ in society $\mathcal{S}$ belongs to the *coalition structure core*, if there is no coalition $S \in \mathcal{P}(N)$ that blocks it. A coalition $S \in \mathcal{P}(N)$ *blocks* a feasible allocation $a$ if there is an alternative $\omega \in \varphi(S)$ such that $\omega P_i a(i)$ for all $i \in S$.

It requires that there are no new coalition that can offer an alternative strictly preferred by all of its members to the status quo. Coalition structure core describes a situation where new coalitions can exclude members. A coalition forms if it can afford an alternative preferred by all of its members. There may be more individuals who want to join the new coalition. But it does not need to accommodate all who want to join.

Different concepts of stability apply to different categories of decision-making units in the real world. Tiebout equilibrium applies to situations where individual entry and exit are not restricted but new coalitions are not permitted, such as public school districts. Residents can move to anywhere they prefer. On the other hand, coalition structure core applies to situations where new coalitions are permitted and can exclude other people, such as business partnerships. Everyone is supposed to benefit in a partnership. A combination of the above two has attracted some research attentions (see Greenberg and Weber 1993a and Demange 1994).

**Definition 2.3.** A *Core–Tiebout equilibrium* is a Tiebout equilibrium which is also in the coalition structure core.

The mixing of Tiebout equilibrium and coalition structure core requires a stronger notion of stability. Not only does no group of individuals want to form a new coalition, but also no one wants to switch to another coalition. The existence of a stable coalition structure is not unconditionally guaranteed. Example 2.1 shows a society with an empty coalition structure core. It is adapted from the famous Condorcet's Voting Paradox.

**Example 2.1.** Consider the society $\mathcal{S} = (N, X, R, \varphi)$ where $N = \{1,2,3\}$, $X = \{x, y, z\}$, $\varphi(\{1\}) = \{z\}$, $\varphi(\{2\}) = \{x\}$, $\varphi(\{3\}) = \{y\}$, and $\varphi(S) = X$ if $|S| \geq 2$. The preferences are the following.

$$xP_1yP_1z$$
$$yP_2zP_2x$$
$$zP_3xP_3y.$$

First, we check the coalition structure composed of three one-person coalitions. Allocation $(\{\{1\},\{2\},\{3\}\},\{z,x,y\})$ is blocked by $\{1,2\}$ with $y$. Second, if the grand coalition $\{1,2,3\}$ choose $x$, $y$, or $z$, it will be blocked by $\{2,3\}$ with $z$, $\{1,3\}$ with $x$, or $\{1,2\}$ with $y$, respectively. Third, allocations $(\{\{1,2\},\{3\}\},\{x,y\})$, $(\{\{1,2\},\{3\}\},\{y,y\})$, and $(\{\{1,2\},\{3\}\},\{z,y\})$ are blocked by $\{2,3\}$ with $z$, $\{1,3\}$

with $x$, $\{1,3\}$ with $x$, respectively. Thus, by the symmetry, all allocations with coalition structures $\{\{1\},\{2,3\}\}$ or $\{\{1,3\},\{2\}\}$ are blocked also. Hence, the coalition structure core is empty in this society. ∎

In the above example, a stable coalition structure does not exist because the preferences contain a cycle. Individuals, with diverse preferences, tend to form multiple coalitions, since the grand coalition may not be able to offer an alternative appealing to all. But preferences cannot be too diverse: unrestricted preferences undermine the stability of coalitions. In later sections, we impose structures on preferences and feasible sets. The domain of the problem, hence, is reduced to a class of "reasonable" societies.

# 3 The Consecutive Benevolence Solution

In this section, we present the consecutive benevolence solution. It is an algorithm which selects allocations for a society. It is defined with respect to a linear order[2] on $N$. This solution is defined for societies satisfying the following assumptions:

C. *(Compact, continuous, closed, and finite)* $X$ is compact in a metric space, $R_i$ is continuous in $X$, $\varphi$ is closed-valued, and $N$ is a finite set[3].
PPR. *(Positive Population-Responsive)* For all $S, S' \in \mathcal{P}(N)$ with $S \subset S'$, for all $y \in \varphi(S)$, there is an $x \in \varphi(S')$ such that $xR_iy$ for all $i \in S'$.

Positive population-responsiveness requires that it is always possible to make existing members as well off when new members join a coalition. This condition is satisfied when the feasible correspondence $\varphi$ is monotonic (that is, $\varphi(S) \subseteq \varphi(S')$ for all $S, S' \in \mathcal{P}(N)$ with $S \subset S'$). This is common in many economic situations. For example, when there is no congestion, the feasible sets of local economies are

[2] A binary relation $>$ is a *linear order on a set* $T$ if for every two distinct elements $t, t' \in T$, either $t > t'$ or $t < t'$. Since it will not cause confusion between the linear order $>$ with the binary relation "larger than" of the real numbers, the same symbol is adopted for both, and $t$ is said to be larger than $t'$ if $t > t'$. Note that for $t \in T$ and $A \in \mathcal{P}(T)$, $t > A$ means that $t > t'$ for all $t' \in A$; and for two sets $A, B \in \mathcal{P}(T)$, $A > B$ means that $t > t'$ for all $t \in A$ and $t' \in B$. Moreover, a coalition $S \in \mathcal{P}(N)$ is *consecutive* with respect to linear order $>$ on $N$ if for all $i \in N \backslash S$, either $i > S$ or $i < S$.

[3] Assumption C can be replaced with that $\Omega$ and $N$ are both finite. All results still hold.

monotonic; the more residents, the less tax one must pay for a unit of public good.

We introduce an algorithm, called the consecutive benevolence solution. Suppose individuals are arranged according to a linear order $>$. They are labeled with integers 1 to $n$ and $1 < 2 < ... < n$. A consecutive set $N_k$, which is composed of individuals 1 to $k$ is called a subsociety. We construct, recursively, an allocation for each subsociety with a nonempty feasible set, starting at the smallest one. In each $N_k$ the largest individual, $k$, is the "benevolent dictator" who gets one of her most preferred alternatives from the alternatives from those that (i) are feasible to a consecutive coalition containing herself, and (ii) make all other members of that coalition at least as well off as when they are dictators themselves. Let $S$ be the largest consecutive coalition containing the dictator that affords one of her most preferred alternatives $\omega$ such that (i) and (ii) are satisfied. Next, form coalition $S$ if some of the remaining individuals (that is $N_k \backslash S$) can form a coalition with a nonempty feasible set; otherwise, form the coalition containing the whole subsociety (that is $N_k$). Alternative $\omega$ is assigned to this coalition containing the dictator. If there are individuals left, let the largest member among them be a benevolent dictator for the remaining individuals. Then, form a coalition containing the dictator and assign an alternative in the same way described above. Keep going if there are individuals left. When there are no individuals left, an allocation of subsociety $N_k$ is determined. The constructed allocation is a "benevolent dictator allocation" for the subsociety. Then, we proceed to construct an allocation for $N_{k+1}$. Construct a benevolent dictator allocation for each subsociety in an increasing order up to the $N$. This determines a final allocation for the whole society. The consecutive benevolence solution is the collection of all potential benevolent dictator allocations for the whole society. This is defined formally in the following.

For a society $\mathcal{S}$ satisfying C and PPR, the *consecutive benevolence solution* with respect to $>$, $\mathcal{D}_{>}(\mathcal{S})$, is the collection of allocations constructed from the following algorithm:

**Algorithm.** Label individuals with integers 1 to $n$ according to linear order $>$. Namely, $N = \{1, 2, ..., n\}$ and $1 < 2 < ... < n$. Let $N_k = \{1, ..., k\}$, where $k \in N$, denote the consecutive coalition containing individual 1 with $k$ as the largest member. Take $\underline{k} = \min k$ s.t. $\varphi(N_k) \neq \emptyset$. $N_{\underline{k}}$ is the minimum consecutive coalition containing individual 1 that has a nonempty feasible set. Note that there exists $S \in \mathcal{P}(N)$ such that $\varphi(S) \neq \emptyset$. Thus, $\varphi(N) \neq \emptyset$ by PPR and $\underline{k}$ exists. Let $\mathcal{S}|_k = \left(N_k, X, R_{N_k}, \varphi_{N_k}\right)$ where $R_{N_k} = \{R_i\}_{i \in N_k}$ and $\varphi : \mathcal{P}(N_k) \to \mathcal{P}(X)$ such that $\varphi_{N_k}(S) = \varphi(S)$ for all

$S \in \mathcal{P}(N_k)$. $\mathcal{S}|_k$ is the subsociety which contains individuals 1 to $k$. We will construct an allocation $a|_k$ for each $\mathcal{S}|_k$.

First, let $\underline{\omega}_i$ denote the least preferred alternative of $i$ in $X$; i.e., $\omega' R_i \underline{\omega}_i$ for all $\omega' \in X$. Since $X$ is compact and $R_i$ continuous, $\underline{\omega}_i$ exists[4]. Let $a|_j(i) = \underline{\omega}_i$ for all $i \leq j < \underline{k}$. We assign their worst alternative to all individual $i$ in $\mathcal{S}|_j$ when the whole subsociety cannot afford anything (i.e., $N_j$ has an empty feasible set).

**Step 1.** We start at $\mathcal{S}|_{\underline{k}}$. Note that $\varphi(N_{\underline{k}}) \neq \emptyset$. Moreover, when C is satisfied, $\varphi(N_{\underline{k}}) \subset X$ is compact. Let $\omega(\underline{k})$ be one of individual $\underline{k}$'s most preferred alternatives in $\varphi(N_{\underline{k}})$, i.e.,

$$\omega(\underline{k}) \in \{\omega \mid \omega R_{\underline{k}} \omega', \forall \omega' \in \varphi(N_{\underline{k}})\}.$$

The existence of a maximizer $\omega(\underline{k})$ is guaranteed by that $R_{\underline{k}}$ is continuous and $\varphi(N_{\underline{k}})$ is compact.

Form coalition $C(\underline{k}) = N_{\underline{k}}$ and assign it alternative $\omega(\underline{k})$. Thus, $a|_{\underline{k}}(i) = \omega(\underline{k})$ for all $i \in N_{\underline{k}}$. Note that $\omega(\underline{k}) \in \varphi(N_{\underline{k}})$ and $\omega(\underline{k}) R_i a|_i(i)$ for all $i \in N_{\underline{k}} \setminus \{\underline{k}\}$. For the remaining subsocieties, allocations are constructed recursively.

**Step 2.** Suppose $a|_j$ is constructed for $\mathcal{S}|_j$ for all $j \leq k-1$. We construct allocation $a|_k$ for $\mathcal{S}|_k$. Let $C(k-1)$ denote the coalition containing $k-1$ in allocation $a|_{k-1}$ and $\omega(k-1)$ the alternative assigned to $C(k-1)$. (Thus $a|_{k-1}(i) = \omega(k-1)$ for all $i \in C(k-1)$.) Also, suppose that $\omega(k-1) \in \varphi(C(k-1))$ and $\omega(k-1) R_i a|_i(i)$ for all $i \in C(k-1) \setminus \{k-1\}$. Let $\tilde{S}(k)$ denote the set of all consecutive subsets of $N_k$ containing $k$, i.e., $\tilde{S}(k) = \{S \in \mathcal{P}(N_k) \mid k \in S \text{ and } S \text{ is consecutive}\}$. Let

$$B(k) = \left\{(S, \omega) \in \mathcal{P}(N) \times X \mid S \in \tilde{S}(k) \text{ and } \omega \in \varphi(S) \text{ s.t. } \omega R_i a|_i(i)\ \forall i \in S \setminus \{k\}\right\}.$$

$B(k)$ is the set of all coalition-alternative pairs such that (i) the coalition is consecutive and contains $k$, (ii) the alternative is feasible to the coalition, and (iii) the alternative makes all other members $i$ as well off as in allocations $a|_i$ (i.e., when $i$ is the dictator).

**Lemma 3.1.** *$B(k) \neq \emptyset$.*

**Proof.** Apparently, $\omega(k-1) \in \varphi(C(k-1))$. By PPR, there exists $x \in \varphi(C(k-1) \cup \{k\})$ such that $x R_i \omega(k-1)$ for all $i \in C(k-1)$. Since $\omega(k-1) R_i a|_i(i)$

[4] Each $R_i$ can be represented by a continuous utility function. The existence of a maximizer and a minimizer of a continuous function in a compact domain is guaranteed by the Weierstrauss Theorem.

for all $i \in C(k-1) \setminus \{k-1\}$ and $\omega(k-1) = a|_{k-1}(k-1)$, $xR_i\omega(k-1)\, R_i a|_i(i)$ for all $i \in C(k-1)$. Therefore, $(C(k-1) \cup \{k\}, x) \in B(k)$. ∎

Let $W(k) \subseteq B(k)$ be the set of coalition-alternative pairs consisting of individual $k$'s most preferred alternatives in $B(k)$, i.e.,

$$W(k) \in \{(S,\omega) \in B(k) \mid \omega R_k \omega',\ \forall (S',\omega') \in B(k)\}.$$

**Lemma 3.2.** $W(k) \neq \emptyset$.

**Proof.** Let $G(k) = \{\omega \in X \mid \exists S \text{ s.t. } (S,\omega) \in B(k)\}$. $W(k) \neq \emptyset$ if there is a maximum element in $G(k)$ according to $R_k$. We show that $G(k)$ is compact. $G(k) \subseteq X$ is bounded since $X$ is bounded. Let $R_i(\omega)$ denote the upper contour set of $R_i$ at $\omega$; i.e., $R_i(\omega) = \{x \in X \mid xR_i\omega\}$. Note that

$$G(k) = \cup_{S \in \tilde{S}(k)} \left( \cap_{i \in S \setminus \{k\}} R_i(a|_i(i)) \cap \varphi(S) \right).$$

C assures that $R_i(.)$ and $\varphi(.)$ are closed, and $S$ and $\tilde{S}(.)$ are finite sets. Thus $G(k)$ is closed.

Since $R_k$ is continuous, it can be represented by a continuous utility function. It has a maximizer in compact set $G(k)$. ∎

Let $|S|$ denote the cardinality of coalition $S$. Let $(C(k), \omega(k))$ denote one of the coalition-alternative pairs in $W(k)$ with the largest coalition (note that $C(k)$ is unique); i.e.,

$$(C(k), \omega(k)) \in \{(S,\omega) \in W(k) \mid |S| > |S'| \text{ for all } (S',\omega') \in W(k)\}.$$

Let $l(k)$ denote the largest member outside $C(k)$ in $N_k$; i.e., $l(k) \geq j$ for all $j \in N_k \backslash C(k)$. (Note that the function $l(.)$ is strictly decreasing in $N$ with respect to $>$.) Form coalition $C(k)$ and assign it $\omega(k)$. Note that either $C(k) \supseteq N_{\underline{k}}$ or $l(k) \geq \underline{k}$. To see this, suppose $l(k) < \underline{k}$. Since $\omega(k) \in \varphi(C(k))$, by PPR, there is $x \in \varphi(C(k) \cup N_{\underline{k}})$ such that $xR_i\omega(k)$ for all $i \in C(k)$. Moreover, $\omega(k)\, R_i a|_i(i)$ for all $i < l(k) < \underline{k}$ (note that $a|_i(i)$ are their worst alternatives). So, $(C(k) \cup N_{\underline{k}}, x) \in W(k)$ and $|C(k) \cup N_{\underline{k}}| > |C(k)|$; a contradiction.

Allocation $a|_k$ is constructed as the following:

$$a|_k(i) = \omega(k) \text{ for all } i \in C(k),$$
$$a|_k(j) = a|_{l(k)}(j) \text{ for all } j \in N_k \backslash C(k).$$

$a|_k$ is a benevolent dictator allocation for $\mathcal{S}|_k$.

**Step 3.** Construct allocation $a|_k$ for each $\mathcal{S}|_k$ for all $\underline{k} \leq k \leq n$. Hence, the sequence of coalitions $C(\underline{k}), C(\underline{k}+1), ..., C(n-1), C(n)$ are defined. Let

$$l^r(.) = \underbrace{l(l(...(l(.))))}_{r \text{ times}}$$

denote the composition of $l(.)$ for $r$ repetitions. (Note that $l^0(k) = k$.)

Since $l(k)$ is decreasing, $N$ is finite, and either $C(l(k)) \supseteq N_{\underline{k}}$ or $l(k) \geq \underline{k}$ for all $k$, there exists an integer $p \leq n - \underline{k}$ such that $1 \in C(l^p(n))$. A collection of coalitions is constructed as the following:

$$C = \left\{C(n), C(l(n)), C\left(l^2(n)\right), ..., C(l^p(n))\right\}.$$

$C$ is a partition of $N$. Furthermore, $\omega(l^r(n)) \in \varphi(C(l^r(n)))$; thus, $\varphi(C(l^r(n))) \neq \emptyset$ for all $0 \leq r \leq p-1$. So, $C$ is a coalition structure in society $\mathcal{S}$.

Let $a = a|_n$, where $a|_{l^r(n)}(i) = \omega(l^r(n))$ for all $i \in C(l^r(n))$ for all integer $r$, $0 \leq r \leq p$. Alternative $\omega(l^r(n))$ is assigned to coalition $C(l^r(n))$ That is,

$$a(i) = a|_n(i) \text{ for all } i \in C(n),$$
$$a(i) = a|_{l(n)}(i) \text{ for all } i \in C(l(n)),$$
$$........$$
$$a(i) = a|_{l^p(n)}(i) \text{ for all } i \in C(l^p(n)).$$

All alternatives are feasible to the assigned coalition since $\omega(l^r(n)) \in \varphi(C(l^r(n)))$ for all $0 \leq r \leq p$. Therefore, $a$ is a feasible allocation in $\mathcal{S}$. Allocation $a$ is a benevolent dictator allocation for society $\mathcal{S}$. ∎

We can see in the above construction that $\mathcal{D}_>(\mathcal{S})$ is well-defined.

**Proposition 3.1.** *For any society $\mathcal{S}$ satisfying C and PPR, $\mathcal{D}_>(\mathcal{S}) \neq \emptyset$.*

The following example illustrates how this solution works.

**Example 3.1.** Consider society $\mathcal{S} = (N, X, R, \varphi)$ where $N = \{1, 2, 3\}$, $X = \{x, y, z\}$, $\varphi(\{1\}) = \{x\}$, $\varphi(\{2\}) = \emptyset$, $\varphi(\{3\}) = \{y\}$, $\varphi(\{1, 2\}) = \{x, z\}$, $\varphi(\{2, 3\}) = \{x, y\}$, $\varphi(\{1, 3\}) = \{x, y\}$, and $\varphi(N) = X$. The preferences are the following:

$$xP_1yP_1z$$
$$yP_2xP_2z$$
$$zP_3yP_3x.$$

Order individuals according to their labels; that is $1 < 2 < 3$. We start at the subsociety composed of $\{1\}$ and assign allocation $(\{\{1\}\}, \{x\})$, since $x$ is individual 1's most preferred feasible alternative. This is the dictator allocation for individual 1. Next, for the subsociety composed of $\{1, 2\}$, individual 2 can form coalition $\{1, 2\}$ (she cannot stay alone) and we assign allocation $(\{\{1, 2\}\}, \{x\})$. This is the dictator allocation for individual 2. Finally, for the whole society, individual 3 can stay alone with $y$, form $\{2, 3\}$ with $x$ or $y$, or form $\{1, 2, 3\}$ with $x$. Note that she cannot choose $y, z$ in $\{1, 2, 3\}$ since individual 1 should not enjoy an alternative worse than $x$. Since alternative $y$ is the most preferred by 3 among $x, y$ and $\{2, 3\}$ is larger than $\{3\}$, we assign $y$ to $\{2, 3\}$ and individual 1 forms a one-person coalition. The resulting allocation $(\{\{1\}, \{2, 3\}\}, \{x, y\})$ is the consecutive benevolence solution for this society. ∎

The dictator is benevolent because she guarantees members of her coalition to be as well off as when they are dictators themselves in the subsocieties. By the recursive nature of the solution, this argument extends to the whole society. This solution has an interesting equity property: each individual enjoys a "minimum welfare level". Any allocation guarantees each individual a minimum welfare level, which is what she enjoys in a benevolent dictator allocation when she is a dictator.

**Proposition 3.2.** *For all society $\mathcal{S}$, for all $a \in \mathcal{D}_{>}(\mathcal{S})$, $a(i)\, R_i a|_i(i)$ for all $i \in N$.*

**Proof.** Every individual in $N$ belongs to a coalition in $C$ since $C$ is a partition of $N$. Suppose individual $i \in C(l^r(n))$ for some $r \leq p$ where $p = |C|$. By the construction of $a$, $a|_k(j) = a|_{l(k)}(j)$ for all $j \in N_k \backslash C(k)$ for all $k$. Hence, $a|_{l^t(n)}(j) = a|_{l^{t+1}(n)}(j)$ for all $j \in N_{l^t(n)} \backslash C(l^t(n))$ for all $t$. Moreover, $i \in C(l^r(n))$ means $i \in N_{l^t(n)} \backslash C(l^t(n))$ for all $t < r$. Therefore,

$$a|_n(i) = a|_{l(n)}(i) = a|_{l^2(n)}(i) = ... = a|_{l^r(n)}(i).$$

By the construction of $a|_{l^r(n)}$, $a|_{l^r(n)}(i) = \omega(l^r(n)) R_i a|_i(i)$ because $i \in C(l^r(n))$. Therefore, $a(i) = a|_{l^r(n)}(i) R_i a|_i(i)$. ∎

# 4 Stable Coalition Formation

We define consecutive support and show that the proposed solution generates Core-Tiebout equilibria when preferences have consecutive support and feasible sets are positive population-responsive.

CS. (*consecutive support*) The family of preference relations $R$ has *consecutive support* on $N$ if there is a linear order $>$ on $N$ such that for any pair $x, y \in X$, the set $\{i \in N \mid xP_iy\}$ is consecutive with respect to $>$.

Consecutive support requires that the set of individuals who strictly prefer an alternative to the other in any pair of alternatives is consecutive with respect to a linear order. It is strictly weaker than the following three preference restrictions: *intermediate preferences*, *order restriction*, and *single-crossing* (see the Appendix). It is satisfied by many economic models: for example, those in Westhoff (1977), Roberts (1977), Epple, Filimon and Romer (1984), and Epple and Romer (1990). In models with public goods where the individual's marginal rate of substitution between the private and the public good is monotonic with respect to a characteristic (such as income), preferences have consecutive support.

One of the reasons for individuals to form a coalition is that a coalition can offer better alternatives. For example, a homeowner may not be able to afford a tennis court nearby the house, but a community can provide a gym; a local community cannot afford to build a baseball stadium while a city can. With the freedom to exit, an individual who remains in the coalition actually prefers the offered alternative to what she can afford on her own. Therefore, "better alternatives with more members" makes coalitions more stable. Consecutive support and positive population-responsiveness guarantee a Core-Tiebout equilibrium.

**Proposition 4.1.** *For every society $\mathcal{S} = (N, X, R, \varphi)$ satisfying* C, PPR *and* CS*, if $a \in \mathcal{D}_>(\mathcal{S})$ where $>$ is the linear order that suffices* CS*, $a$ is a Core-Tiebout equilibrium.*

**Proof.** Suppose $a \in \mathcal{D}_{>}(\mathcal{S})$ is constructed with corresponding $C(.)$, $\omega(.)$, $B(i)$, $W(i)$, and $a|_i$ for all $i \in N$. The proof consists of the following three lemmas.

**Lemma 4.1.** *For any two adjacent coalitions $C(m)$ and $C(l)$ in allocation a with $m < l$,*

$$\omega(m)\, R_i \omega(l) \text{ for all } i \leq m,$$
$$\omega(l)\, R_i \omega(m) \text{ for all } i > m.$$

**Proof.** Let $\alpha = \omega(m)$ and $\beta = \omega(l)$. First, suppose $\beta R_m \alpha = a|_m(m)$. There exists $x \in \varphi(C(l) \cup \{m\})$ such that $x R_i \beta$ for all $i \in C(l) \cup \{m\}$ by PPR. Thus, $x R_m \beta R_m a|_m(m)$ and $x R_i \omega(l)\, R_i a|_i(i)$ for all $i \in C(l)$. Therefore, $(C(l) \cup \{m\}, x) \in W(l)$ and $|C(l) \cup \{m\}| > |C(l)|$; a contradiction. It must be that $\alpha P_m \beta$. Second, $\beta R_{m+1} a|_{m+1}(m+1)$ by construction. Since $\alpha \in \varphi(C(m))$ and $\alpha R_i a|_i(i)$ for all $i \in C(m)$, by PPR, there is $x' \in \varphi(C(m) \cup \{m+1\})$ such that $x' R_i \alpha R_i a|_i(i)$ for all $i \in C(m)$. So, $(C(m) \cup \{m+1\}, x') \in B(m+1)$. Hence, $a|_{m+1}(m+1)\, R_{m+1} x' R_{m+1} \alpha$ by construction. So, $\beta R_{m+1} \alpha$. Third, suppose $\beta I_i \alpha$ for all $i \in C(l)$, then $\alpha I_i \beta R_i a|_i(i)$ for all $i \in C(l) \setminus \{l\}$. Also, $\alpha R_j a|_j(j)$ for all $j \in C(m)$ by construction. Moreover, there exists $x'' \in \varphi(C(m) \cup C(l))$ such that $x'' R_i \alpha$ for all $i \in C(l)$. So, $(C(m) \cup C(l), x'') \in W(l)$ and $|C(m) \cup C(l)| > |C(l)|$; a contradiction. Therefore, there exists $i \in C(l)$ such that $\beta P_i \alpha$.

From the above results, we can see that if there is $i > m$ such that $\alpha P_i \beta$, or $i \leq m$ such that $\beta P_i \alpha$, then CS is violated. Therefore, $\beta R_i \alpha$ for all $i > m$ and $\alpha R_i \beta$ for all $i \leq m$. ∎

**Lemma 4.2.** *For all $i, j \in N$, $a(i)\, R_i a(j)$.*

**Proof.** Take any $i, j \in N$. Without loss of generality, suppose $i < j$ and $i \in C(m)$, $j \in C(l)$. If $C(m) = C(l)$, then $a(i) = a(j)$. If not, suppose the following are adjacent coalitions $C(m), C(n_1), ..., C(n_h), C(l)$. By Lemma 4.1, $i \leq m$ implies $\omega(m)\, R_i \omega(n_1)$; $i < n_1$ implies $\omega(n_1)\, R_i \omega(n_2)$;...; $i < n_h$ implies $\omega(n_h)\, R_i \omega(l)$. So, $a(i) = \omega(m)\, R_i \omega(l) = a(j)$. Similarly, by Lemma 4.1, $j > n_h$ implies $\omega(l)\, R_j \omega(n_h)$; $j > n_{h-1}$ implies $\omega(n_h)\, R_j \omega(n_{h-1})$;...; $j > m$ implies $\omega(n_1)\, R_j \omega(m)$. So, $a(j) = \omega(l)\, R_j \omega(m) = a(i)$. ∎

**Lemma 4.3.** *There exists no $S \in \mathcal{P}(N)$ that blocks a.*

**Proof.** (i) First, we show that there is no consecutive coalition that blocks $a$. Suppose coalition $T$ is consecutive and blocks $a$ with $\omega$. Note that, by Lemma 3.3, $a(i)\,R_i a|_i(i)$, so $\omega P_i a(i)\,R_i a|_i(i)$ for all $i \in T$. Suppose $l$ is the largest member in $T$, then $(T, \omega) \in B(l)$. $\omega P_l a|_l(l)$ implies that $(C(l), a|_l(l)) \notin W(l)$; a contradiction.

(ii) Second, we show that if there is a blocking coalition, then there is a consecutive blocking coalition. Suppose $S$ blocks $a$ with $\omega \in \varphi(S)$. Suppose the smallest member of $S$ is $\underline{m}$ and the largest is $\overline{m}$. Take $T = \{i \in N \mid \underline{m} \leq i \leq \overline{m}\}$; $T \supseteq S$ and $T$ is consecutive. If $T = S$, then $S$ is a consecutive coalition that blocks $a$. If $T \neq S$, then $\underline{m} < i < \overline{m}$ for all $i \in T \backslash S$. Since $S$ blocks $a$, $\omega P_j a(j)$ for all $j \in S$. Moreover, by Lemma 4.2, $a(\underline{m})\,R_{\underline{m}} a(i)$ and $a(\overline{m})\,R_{\overline{m}} a(i)$ for all $i \in T \backslash S$. So, $\omega P_{\underline{m}} a(i)$ and $\omega P_{\overline{m}} a(i)$. This implies, by CS, $\omega P_i a(i)$ for all $i \in T \backslash S$. Hence, $\omega P_i a(i)$ for all $i \in T$. By PPR, there exists $x \in \varphi(T)$ such that $x R_i \omega$ for all $i \in T$ since $\omega \in \varphi(S)$. This means that $T$ blocks $a$ with $x$. ∎

We have shown that, by Lemma 4.3, $a$ is in the coalition structure core. Moreover, by Lemma 4.2 and the fact that "no coalition can block" implies "each coalition chooses a weakly Pareto optimal alternative" (condition ii in Definition 2.1), $a$ is a Tiebout equilibrium. Thus, $a \in \mathcal{D}_>(\mathcal{S})$ is a Core-Tiebout equilibrium. ∎

**Theorem 4.1.** *Every society $\mathcal{S} = (N, X, R, \varphi)$ that satisfies* C, PPR *and* CS *has a Core-Tiebout equilibrium.*

**Proof.** Take linear order $>$ that suffices CS. Since $\mathcal{S}$ satisfies C and PPR, $\mathcal{D}_>(\mathcal{S}) \neq \emptyset$ by Proposition 3.1. Thus any allocation $a \in \mathcal{D}_>(\mathcal{S})$ is a Core-Tiebout equilibrium by Proposition 4.1. ∎

The following examples illustrate that the conditions in our result are necessary. First, Example 2.1 shows a society with an empty coalition structure core. It satisfies PPR but not CS. Second, Example 4.1 shows that without PPR, the coalition structure core may be empty. Third, Example 4.2 shows that a stronger notion of stability, "strong coalition structure core[5]" cannot be guaranteed even with CS and

[5] A feasible allocation $a$ in society $\mathcal{S}$ belongs to the *strong coalition structure core* if there is no coalition $S \in \mathcal{P}(N)$ and alternative $\omega \in \phi(S)$ such that $\omega R_i a(i)$ for all $i \in S$ and $\omega P_j a(j)$ for some $j \in S$. There is no inclusion relationship between the Core-Tiebout equilibrium and the strong coalition structure core in general. However, when feasible sets are monotonic, the set of Core-Tiebout equilibria contains the strong coalition structure core.

monotonic feasible sets.

**Example 4.1.** Consider the society $\mathcal{S} = (N, X, R, \varphi)$ where $N = \{1, 2, 3\}$, $X = \{x, y, z\}$, $\varphi(\{1\}) = \{z\}$, $\varphi(\{2\}) = \{y\}$, $\varphi(\{3\}) = \{x\}$, $\varphi(\{1, 2\}) = \{x\}$, $\varphi(\{2, 3\}) = \{a\}$, $\varphi(\{1, 3\}) = \{y\}$, and $\varphi(N) = \{y\}$. The preferences are the following:

$$xP_1aP_1yP_1z$$
$$aP_2xI_2zP_2y$$
$$zP_3yP_3aP_3x.$$

Note that CS is satisfied with respect to the order of individuals' labels. However, PPR is violated since $x \in \varphi(\{1, 2\})$, $xP_1y$, and $y$ is the only alternative in $\varphi(\{1, 2, 3\})$. Allocations $(\{\{1\}, \{2\}, \{3\}\}, \{z, y, x\})$ and $(\{\{1, 2, 3\}\}, \{y\})$ are blocked by $\{1, 2\}$ with $x$. Allocation $(\{\{1, 2\}, \{3\}\}, \{x, x\})$ is blocked by $\{2, 3\}$ with $a$. Allocation $(\{\{1\}, \{2, 3\}\}, \{z, a\})$ is blocked by $\{1, 3\}$ with $y$. Allocation $(\{\{1, 3\}, \{2\}\}, \{y, y\})$ is blocked by $\{1, 2\}$ with $x$. Therefore, the coalition structure core is empty. ∎

**Example 4.2.** Consider the society $\mathcal{S} = (N, X, R, \varphi)$ where $N = \{1, 2, 3\}$, $X = \{x, y, z\}$, $\varphi(S) = X$ if $|S| \geq 2$, $\varphi(\{1\}) = \{z\}$, $\varphi(\{2\}) = \{y\}$, and $\varphi(\{3\}) = \{x\}$. The preferences are the following:

$$xP_1yP_1z$$
$$xI_2zP_2y$$
$$zP_3yP_3x.$$

Apparently, CS is satisfied with respect to the order of their labels and feasible sets are monotonic. Note that the coalition structure core contains allocations $a(\{1\}) = a(\{2\}) = a(\{3\}) = x$ with coalition structure $\{\{1, 2, 3\}\}$ or $\{\{1, 2\}, \{3\}\}$, and $a'(\{1\}) = a'(\{2\}) = a'(\{3\}) = z$ with coalition structure $\{\{1, 2, 3\}\}$ or $\{\{1\}, \{2, 3\}\}$. However, allocation $a$ is not in the strong coalition structure core since $zR_2x$, $zP_3x$, and $z \in \varphi(\{2, 3\})$; and allocation $a'$ is not in the strong coalition structure core since $xP_1z$, $xR_2z$, and $x \in \varphi(\{1, 2\})$. Therefore, the strong coalition structure core is empty. ∎

Two related existence theorems are in Greenberg and Weber (1993a) and Demange (1994). Demange (1994) provides the existence of a Core-Tiebout equilibrium with intermediate preferences on a tree[6]. Greenberg and Weber (1993a) provide the

[6] A tree is a graph with no loop and a graph is a collection of paths linking elements of a set.

existence of a Core-Tiebout equilibrium with single-peaked preferences in the same setting as ours. Single-peakedness requires that each individual has an "ideal point", which is strictly preferred to any other alternatives, and the closer an alternative is to the ideal point, the more preferred it is. It is used in a wide range of studies. However, single-peakedness is not satisfied in many economic models (see Stiglitz 1974 and Roberts 1977). On the other hand, consecutive support can be assumed in a broad range of models, especially in public good economies.

Our result shows that if individual preferences are, in a sense, similar and if members can be kept as well off when a coalition enlarges, then there is a stable formation of collective decision-making units. Our results differ from others in the following ways. First, Greenberg and Weber (1993a) assumes a one-dimensional set of alternatives. We impose no restriction on the dimension of alternatives. Second, our preference restriction is weaker than that used in Demange (1994). Third, both studies assume that every potential coalition has a nonempty feasible set. Our results apply to more general situations where coalitions can have empty feasible sets. Fourth, we relax monotonicity, assumed in both, to positive population-responsiveness.

# 5 Coalition Formation with a Collective Decision Rule

In this section, we study the case where a coalition decides its public alternative according to a pre-determined collective decision rule which selects an alternative from the feasible set according to the preferences of members.

For coalition $S$, a *collective decision rule* $f : \mathcal{P}(R) \times \mathcal{P}(X) \to \mathcal{P}(X)$ selects a subset of alternatives $f(R_S, \varphi(S))$ from its feasible set $\varphi(S)$ according to preference relations $R_S$.

The coalition formation problem with a collective decision rule is a society $\mathcal{S}^f = (N, X, R, \varphi, f)$. The rule selects a set $f(R_S, \varphi(S)) \subseteq \varphi(S)$ for coalition $S$. Let $\hat{f}(S) = f(R_S, \varphi(S))$ be the induced feasible set of coalition $S$, then definitions for $\mathcal{S}$ can be modified for $\mathcal{S}^f$ with $\hat{f}(.)$ replacing $\varphi(.)$; for example, $a$ is a *feasible allocation* in society $\mathcal{S}^f$ if there is a pair $(C, x)$ of coalition structure $C$ and alternative profile $x = \{x_S\}_{S \in C}$ with $x_S \in \hat{f}(S)$ for all $S \in C$ such that $a(i) = x_S$ for all $i \in S$

for all $S \in C$.

To compare the cases with a collective decision rule with those without, suppose each coalition has a "coalition planner" (as a city planner to a city) in charge of selecting the public alternative to attract members and sustain stability. First, when a decision rule is not present, a planner has the freedom to select any alternatives from the feasible set of a coalition. She can offer members the most attractive feasible alternative against alternatives offered elsewhere. On the other hand, when the decision rule is present, the power of the planner is reduced to a subset of feasible alternatives. The freedom of a planner helps to stabilize a coalition. Second, without a decision rule, other coalitions have more freedom to attract members away as well. The planner of a new coalition can choose the most damaging alternative that undermines the stability of existing coalitions. Therefore, coalitions without a collective decision rule are more competitive than those with one. This translates into a problem of stability: a decision rule undermines the stability of coalitions. Example 5.1 shows that when the decision rule is single-valued, the coalition structure core may be empty in a society satisfying C and CS and with monotonic feasible sets. The reason is that, in contrast to Theorem 4.1, PPR is not satisfied by the induced feasible sets.

**Example 5.1.** Consider the society $\mathcal{S} = (N, X, R, \varphi)$ where $N = \{1,2,3\}$, $X = \{x, y, z\}$, $\varphi(S) = X$ for all $S \in \mathcal{P}(N)$, $\hat{f}(\{1\}) = \{z\}$, $\hat{f}(\{2\}) = \{y\}$, $\hat{f}(\{3\}) = \{x\}$, $\hat{f}(\{1,2\}) = \{x\}$, $\hat{f}(\{2,3\}) = \{a\}$, $\hat{f}(\{1,3\}) = \{y\}$, and $\hat{f}(N) = \{y\}$. The preferences are the following:

$$xP_1aP_1yP_1z$$
$$aP_2xI_2zP_2y$$
$$zP_3yP_3aP_3x.$$

Allocations $(\{\{1\},\{2\},\{3\}\},\{z,y,x\})$ and $(\{1,2,3\},\{y,y,y\})$ are blocked by $\{1,2\}$ with $x$. Allocation $(\{\{1,2\},\{3\}\},\{x,x,x\})$ is blocked by $\{2,3\}$ with $a$. Allocation $(\{\{1\},\{2,3\}\},\{z,a,a\})$ is blocked by $\{1,3\}$ with $y$. Allocation $(\{\{1,3\},\{2\}\},\{y,y,y\})$ is blocked by $\{1,2\}$ with $x$. Therefore, the coalition structure core is empty. ∎

Hence we have reached the following result.

**Proposition 5.1.** *If the decision rule is single-valued, a society may have an empty coalition structure core, even when C, CS and monotonicity are satisfied.*

If the induced feasible correspondence $\hat{f}$ satisfies PPR, Theorem 4.1 applies here and $\mathcal{S}^f$ has a Core-Tiebout equilibrium given it satisfies C and CS also. Let the Pareto set of coalition $S$ be

$$P(S) = \{\omega \in \varphi(S) \mid \nexists \omega' \in \varphi(S) \text{ s.t. } \omega' R_i \omega \ \forall i \in S, \omega' P_j \omega \ \exists j \in S\},$$

When the feasible sets are monotonic, if $f$ permits the Pareto set, then $\hat{f}$ is PPR. Consequently, there is a Core-Tiebout equilibrium under $f$.

**Theorem 5.1.** *If society $\mathcal{S}^f = (N, X, R, \varphi, f)$ satisfies* C *and* CS*, $\varphi$ is monotonic, and $f(R_S, \varphi(S)) \supseteq P(S)$, then there is a Core-Tiebout equilibrium.*

**Proof.** Let $\hat{f}(S) = f(R_S, \varphi(S))$; define $\widehat{\mathcal{S}} = \left(N, X, R, \hat{f}\right)$. If $\widehat{\mathcal{S}}$ satisfies C, CS and PPR, it has a Core-Tiebout equilibrium (Theorem 4.1) which is also a Core-Tiebout equilibrium in $\mathcal{S}^f$. Evidently, $\widehat{\mathcal{S}}$ satisfies C and CS. We show that PPR is satisfied. Suppose $S, S' \in \mathcal{P}(N)$ with $S \subset S'$, and $y \in \hat{f}(S) \subseteq \varphi(S)$. By monotonicity, $y \in \varphi(S')$. We show that there is an alternative $\omega \in \hat{f}(S')$ such that $\omega R_i y$ for all $i \in S'$.

Note that since $R_i$ is continuous and $X$ is compact, the upper contour set $R_i(\omega)$ of $R_i$ at $\omega$ is compact for all $\omega \in X$ for all $i \in N$. Also, there is a continuous utility function $u_i$ that represents $R_i$ for all $i \in N$. Moreover, take function $h(\omega) = \sum_{i \in N} u_i(\omega) : X \to R^1$; it is continuous. Define $I(y) = \cap_{i \in S'} R_i(y) \cap \varphi(S')$; it is the intersection of the feasible set of $S'$ and all members' upper contour sets at $y$. This set is compact. So, $h$ has a maximum in $I(y)$. Let $\bar{\omega} = \arg\max_{\omega \in I(y)} h(\omega)$. There is no other alternative $x \in \varphi(S')$ that Pareto dominates $\bar{\omega}$. Suppose there is; $x R_i \bar{\omega}$ for all $i \in S'$ and $x P_i \bar{\omega}$ for some $j \in S'$. Thus, $h(x) > h(\bar{\omega})$. Also, $x R_i \bar{\omega} R_i y$ means $x \in I(y)$; a contradiction. So, $\bar{\omega} \in P(S') \subseteq \hat{f}(S')$. Since $\bar{\omega} R_i y$ for all $i \in S'$; PPR is satisfied. ∎

The function of a decision rule is to narrow down the alternatives of a coalition to a permissible subset. Unfortunately, if it is too selective (the selected subset is small compared with the feasible set), our approach to the stability of coalitions cannot apply. There may not exist a stable coalition structure if the rule is single-valued. Theorem 5.1 shows that if the rule permits the Pareto set, stability exists. However, since many decision rules are actually selections from the Pareto set, our result cannot apply. Whether there are conditions guaranteeing stable coalition structures for other decision rules in general is still an open question.

# 6 Conclusion

We propose the consecutive benevolence solution for coalition formation with public alternatives. It selects Core-Tiebout equilibria when preferences have consecutive support and feasible sets are positive population-responsive. We also study the situation where coalitions follow a collective decision rule to make decisions. We show that if the decision rule is single-valued, there may not exist a stable coalition structure even when preferences and feasible sets are restricted. Yet, a Core-Tiebout equilibrium exists when the decision rule permits the Pareto set, given consecutive supporte and monotonic feasible sets.

Our study shows that two things are instrumental to the existence a stable formation of collective decision-making units: (i) individual preferences are similar, and (ii) members can be kept as well off when a coalition enlarges. One surprising finding is that a pre-imposed decision rule can destabilize the society.

# Appendix: Preference Restrictions

This section discusses the relationships among four preference restrictions: intermediate preferences, order restriction, single-crossing, and consecutive support. They all concern a linear order on the set of individuals. Intermediate preferences[7] is proposed by Grandmont (1978), and order restriction by Rothstein (1990, 1991), for the transitivity of the majority rule (see also ch. 4 in Austen-Smith and Banks 1999). Single-crossing appears many times in the literature (a general version is formulated by Milgrom and Shannon 1994). It also implies the transitivity of the majority rule (see Gans and Smart 1996).

The family of preference relations $R$ are *intermediate preferences* on $N$ if there is a linear order $>$ on $N$ such that for any pair $x, y \in X$ for all $i, j, k \in N$ with $i < j < k$, (i) $xR_iy$ and $xR_ky$ implies $xR_jy$, and (ii) $xP_iy$ and $xP_ky$ implies $xP_jy$.

It requires one to agree with two agreeing individuals if they are on the opposite sides.

The family of preference relations $R$ satisfies *order restriction* on $N$ if there is a linear order $>$ on $N$ such that for any pair $x, y \in X$, either

[7] This notion, weaker than the one used in Grandmont (1978), coicides with the "betweeness" of perferences in Kemeny and Snell (1972), p. 10; and is adopted in Demange (1994).

(i) $\{i \in N \mid xP_iy\} > \{i \in N \mid xI_iy\} > \{i \in N \mid yP_ix\}$, or
(ii) $\{i \in N \mid xP_iy\} < \{i \in N \mid xI_iy\} < \{i \in N \mid yP_ix\}$.

It requires that individuals can be divided into three consecutive sets by any pair of alternatives according to their preferences, and those who are indifferent are ordered in between individuals with opposite strict preferences.

The family of preference relations $R$ satisfies *single-crossing* on $N$ if there is a linear order $>$ on $N$ such that there is a linear order $>'$on $X$ such that for any pair $x, y \in X$ with $x >' y$ and for all $i, j \in N$ with $i > j$, $xR_iy$ implies $xR_jy$ and $xP_iy$ implies $xP_jy$.

It requires that there is also an order on the alternatives such that one has to agree with a larger individual if she strictly prefers a larger alternative to a smaller one.

Intermediate preferences, order restriction, and single-crossing are equivalent, and consecutive support is weaker than all of them.

**Proposition A.1.** *If $R$ satisfies order restriction on $N$ with respect to $>$, then it has consecutive support on $N$ with respect to $>$.*

**Proof.** (This is obvious.)

**Proposition A.2.** *For a family of preference relations $R$ on $N$ with a linear order $>$, the following are equivalent:*
*(i) $R$ is intermediate on $N$ with respect to $>$,*
*(ii) $R$ satisfies order restriction on $N$ with respect to $>$,*
*(iii) $R$ satisfies single-crossing on $N$ with respect to $>$.*

**Proof.**

(i) *(Intermediate preferences implies order restriction.)*

If $R$ is intermediate on $N$ with respect to $>$, then for any pair $x, y \in X$, the sets $A_1 = \{i \in N \mid xP_iy\}$, $A_2 \{i \in N \mid xI_iy\}$, and $A_3 = \{i \in N \mid yP_ix\}$ are all consecutive. Suppose $A_1$ is not; there exists $i < j < k$ such that $i, k \in A_1$ and $yP_jx$ or $yI_jx$; a contradiction. Suppose $A_3$ is not; there exists $i < j < k$ such that $i, k \in A_3$ and $xP_jy$ or $xI_jy$; a contradiction. Suppose $A_2$ is not; there exists $i < j < k$ such that $i, k \in A_2$ and $xP_jy$ or $yP_jx$; a contradiction.

Suppose $R$ is not order restricted, suppose first that either $A_1 < A_3 < A_2$ or $A_2 < A_3 < A_1$. Then for any $i \in A_1$, $j \in A_2$, $k \in A_3$, either $i < k < j$ or $j < k < j$. Thus $xR_iy$ and $xR_jy$ implies $xR_ky$ by intermediateness; a contradiction to $k \in A_3$. The same contradiction occurs when $A_2 < A_1 < A_3$ or $A_3 < A_1 < A_2$. Hence it can only be the case that either $A_1 < A_2 < A_3$ or $A_3 < A_2 < A_1$.

(ii) *(Order restriction implies intermediate preferences.)*

Suppose $R$ is not intermediate; either there exist $i < j < k$ and $x, y \in X$ such that $xR_iy$ and $xR_ky$ and $yP_jx$ or $xP_iy$ and $xP_ky$ and $yR_jx$. Take the pair $x, y$. In the first case, $j \in A_3$ and $i, k \in A_1 \cup A_2$ ($A_1$, $A_2$, $A_3$ are defined in i). Thus, neither $j < A_1 \cup A_2$ nor $j > A_1 \cup A_2$. In the second case, $i, k \in A_1$ and $j \in A_2 \cup A_3$. So, neither $j < A_2 \cup A_3$ nor $j > A_2 \cup A_3$. Both case violate $A_1 < A_2 < A_3$ and $A_3 < A_2 < A_1$. So, $R$ is not order restricted.

(iii) *(Single-crossing is equivalent to order restriction.)*

See Theorem 3 in Gans and Smart (1996). ∎

The following example illustrates the difference between consecutive support and order restriction.

**Example A.1.** There are four individuals.

$$R = \begin{array}{l} xI_1yP_1z \ * \\ xP_2yP_2z \\ yP_3xP_3z \\ zP_4yP_4x \\ zP_5xI_5y \ * \end{array} \qquad R' = \begin{array}{l} xP_1yP_1z \\ xI_2yP_2z \ * \\ yP_3xP_3z \\ zP_4yP_4x \end{array}$$

$R$ has consecutive support but does not satisfy order restriction; $R'$ satisfy order restriction and also consecutive support. The difference is about the starred individuals, who are indifferent between $x, y$. In order restriction, indifferent individuals have to be in-between individuals with strict preferences, while this is relaxed in consecutive support; they can be any place as long as the set of individuals with strict preferences are consecutive. ∎

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