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Abstract

This short paper isolates a non-trivial class of games for which there exists a monotone relation between the size of pure strategy spaces and the number of pure Nash equilibria (Theorem). This class is that of two-player **nice** games, i.e., games with compact real intervals as strategy spaces and continuous and strictly quasi-concave payoff functions, assumptions met by many economic models. We then show that the sufficient conditions for Theorem to hold are tight.

Keywords: Strategic-form games, strategy spaces, Nash equilibrium, two players.

JEL Classification Numbers: C7.

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1 Introduction

An old and central problem in Game Theory is the enumeration of all Nash equilibria of a game.[1] This problem turns out to be a difficult and tedious one, and to date, there is even no complete answer for "small" games like $m \times m$ bi-matrix games.[2] However, a close look at this literature shows that there is, in general, a monotone relation between the number of pure strategies and the *maximal* number of (pure and mixed) Nash equilibria. Similarly, several papers tackle the issue of the *mean number* of equilibria (e.g., Berg and McLennan [1], McLennan [6]) and, again, show that there is a monotone relation between the number of strategies and the (mean) number of equilibria.[3]

In this paper, we are interested in a slightly different question: Can we isolate a class of games for which there exists a monotone relation between the size of pure strategy spaces and the *exact* number of pure Nash equilibria? An immediate answer to this question is the class of games with constant payoffs i.e., games such that for each strategy profile, the payoff to all players is the same, since for such games all strategy profiles are equilibria. However, this class is trivial and non-generic. The non-trivial answer we propose in this paper is the class of two-player *nice* games, i.e., games with non-empty compact real intervals as strategy spaces, and continuous and strictly quasi-concave payoff functions, assumptions met by many economic models. More precisely, we consider two two-player nice games G and G' such that both players have smaller strategy sets in G' compared to G, and show that each equilibrium strategy profile of G' can be paired with an equilibrium strategy profile of G. The core of our proof precisely consists then in showing that

[1] See McKelvey and McLennan [5] or von Stengel [10] for recent surveys.

[2] For instance, after proving that when $m = 3$ there is at most 7 equilibria (provided the game satisfies some regularity conditions), Quint and Shubik [8] conjectured that $2^m - 1$ would be an upper bound. This bound turns out to be correct when $m = 4$ but not when $m \geq 6$ (see McLennan and Park [7] and von Stengel [9]). The conjecture remains open for $m = 5$.

[3] For instance, using techniques from statistical mechanics, Berg and McLennan show that the mean number of Nash equilibria in bi-matrix games with m pure strategies for each player is $\exp(m[0.281644 + O(\log(m)/m)])$.

this pairing map is injective. It is also of interest to note that equilibrium sets are not necessarily nested.

Finally, we extensively discuss the conditions for our theorem to hold and show that they are tight. For instance, little reflexion suffices to realize that for strategic-form games with *finite* strategy spaces, our result does not hold. To see this, consider the game G' in Figure 1, that has two pure Nash equilibria. Adding to each player a strictly dominating strategy we obtain this way a game like the game G in Figure 1. Because G is dominance solvable, it has only one Nash equilibrium, the desired conclusion.[4]

	0	2
0	$1,1$	$-1,-1$
2	$0,0$	$3,3$

G'

	0	1	2
0	$1,1$	$2,4$	$-1,-1$
1	$4,0$	$4,4$	$4,1$
2	$0,0$	$1,4$	$3,3$

G

Figure 1: Games with finite strategy spaces

The paper is organized as follows. Section 2 presents the model and states our main result. Section 3 proves our main result while section 4 discusses the tightness of our sufficient conditions. Section 5 gives some final remarks.

2 Definitions and Theorem

Let $G(Y) :=< N, (Y_i, u_i)_{i \in N} >$ be a strategic-form game with $Y := \prod_{i \in N} Y_i$. $N = \{1, \dots, n\}$ is the set of players, Y_i is the set of pure strategies available to player i. Denote $Y_{-i} = \prod_{j \neq i}(Y_j)$ and y_{-i} an element of Y_{-i}. Player i's payoff function is $u_i : Y_i \times Y_{-i} \to \mathbb{R}$. A strategic-form game $G(Y)$ is a ***nice*** game if for each player $i \in N$, Y_i is a non-empty real intervals (i.e., a compact, convex subset of the real line), and the payoff function u_i is continuous in all its arguments, and strictly quasi-concave in y_i, that is, for all $y_{-i} \in Y_{-i}$, for

[4]Observe that the "trick" of using mixed strategies to bypass the finiteness of pure strategies does not work either.

all $y_i \in Y_i$, $y'_i \in Y_i$ and $\alpha \in (0,1)$,

$$u_i(\alpha y_i + (1-\alpha)y'_i, y_{-i}) > \min(u_i(y_i, y_{-i}), u_i(y'_i, y_{-i})).$$

We denote $N(G(Y))$ the set of *pure* Nash equilibria of $G(Y)$.

For two strategic-form games $G(X) :=< N, (X_i, v_i)_{i\in N} >$ and $G(Y) :=< N, (Y_i, u_i)_{i\in N} >$, we say that $G(X)$ is a **restriction** of $G(Y)$ if for each player $i \in N$, $X_i \subseteq Y_i$ and $v_i(x) = u_i(x)$ for all $x \in X$. In words, a game $G(X)$ is a restriction of a game $G(Y)$ if $G(X)$ is obtained from $G(Y)$ by restricting the set of pure strategies of some players.

The restriction of strategy sets can be interpreted in different ways. For instance, we could assume that players have committed not to play some of their strategies, as it is the case in competition models in which firms must choose capacity constrains, or that players have signed contracts that limit their course of actions as in Bernheim and Winston [2]. It is also interesting to note that there exists a formal relationship between "more strategies" and "more information" in games. For instance, Gossner [4] shows that for any restriction G' of a strategic-form game G, there exists two equivalent games of incomplete information, with strategic-form games G' and G, such that players are more informed in G than in G'. Thus, "more strategies" is merely a synonym for more information.

We can also note that if $G(X)$ is a restriction of $G(Y)$ and $G(X')$ is a restriction of $G(X)$, then $G(X')$ is a restriction of $G(Y)$. However, if both $G(X)$ and $G(X')$ are restrictions of $G(Y)$, we do not necessarily have that $G(X)$ is a restriction of $G(X')$ or $G(X')$ is a restriction of $G(X)$. $G(Y)$ is obviously a restriction of itself.

Theorem *Let $G(X)$ and $G(Y)$ be two two-player nice games such that $G(X)$ is a restriction of $G(Y)$. We have*

$$\sharp N(G(X)) \leq \sharp N(G(Y)).$$

Theorem states that for any two two-player nice games G and G' such both players have smaller strategy sets in G' compared to G, G has more Nash equilibria in pure strategies than G'. In particular, this implies that if G has a unique Nash equilibrium, then G' has also a unique equilibrium.

3 Proof of Theorem

Let $G(X)$ and $G(Y)$ be two two-player nice games with $G(X)$ a restriction of $G(Y)$. To prove Theorem, we show that there exists an *injective* mapping from the set of equilibria $N(G(X))$ of the restricted game $G(X)$ to the set of equilibria $N(G(Y))$. First, observe that $N(G(Y))$ as well as $N(G(X))$ are non-empty sets and generically finite (Harsanyi [3]). Second, note that we can obviously map each equilibrium of $G(X)$, that is also an equilibrium of $G(Y)$, to itself. Therefore, the crucial part of the proof consists in showing that there exists an injective mapping that associates an equilibrium in $N(G(Y)) \setminus N(G(X))$ to each equilibrium of the restricted game in $N(G(X)) \setminus N(G(Y))$. Lastly, observe that if $N(G(X)) \setminus N(G(Y)) = \emptyset$, there is nothing to prove. From now on, suppose that $N(G(X)) \setminus N(G(Y)) \neq \emptyset$.

3.1 Characterization of $N(G(X)) \setminus N(G(Y))$

Define $br_i^X : X_{-i} \to X_i$ the best-reply map of player i in the game $G(X)$ with for all $x_{-i} \in X_{-i}$,

$$br_i^X(x_{-i}) := \{x_i \in X_i : u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i}) \text{ for all } x_i' \in X_i\}.$$

It is worth noting that br_i^X is a continuous, non-empty, single-valued map in any nice game $G(X)$. For simplicity, we denote BR_i the best-reply of player i in the game $G(Y)$, that is $BR_i := br_i^Y$. For any non-empty compact real interval Z, we denote $\overline{z}$ its least upper bound and $\underline{z}$ its greatest lower bound. Our first lemma characterizes br_i^X as a function of BR_i and X_i.

Lemma 1 *Player i's best-reply function $br_i^X : X_{-i} \to X_i$ in $G(X)$ is given by*

$$br_i^X(x_{-i}) = \begin{cases} \underline{x}_i & \text{if } \; BR_i(x_{-i}) < \underline{x}_i \\ BR_i(x_{-i}) & \text{if } \; \underline{x}_i \leq BR_i(x_{-i}) \leq \overline{x}_i \\ \overline{x}_i & \text{if } \; \overline{x}_i > BR_i(x_{-i}) \end{cases}.$$

Proof First, it is clear that for any x_{-i} in the set $\{x_{-i} \in X_{-i} : BR_i(x_{-i}) \in X_i\}$, $br_i^X(x_{-i}) = BR_i(x_{-i})$. Second, choose a $x_{-i} \in X_{-i}$ such that $BR_i(x_{-i}) <$

$\underline{x}_i$, and suppose that $br_i^X(x_{-i}) > \underline{x}_i$. The single-valuedness of the best-reply maps implies that $u(BR_i(x_{-i}), x_{-i}) > u(\underline{x}_i, x_{-i})$ and $u(br_i^X(x_{-i}), x_{-i}) > u(\underline{x}_i, x_{-i})$. It follows that $(BR_i(x_{-i}), x_{-i})$ and $(br_i^X(x_{-i}), x_{-i})$ both belong to the strict upper contour set of $(\underline{x}_i, x_{-i})$. Since $BR_i(x_{-i}) < \underline{x}_i < br_i^X(x_{-i})$, we have a contradiction with the strict quasi-concavity of u_i. An analogous reasoning holds if we choose a $x_{-i} \in X_{-i}$ such that $BR_i(x_{-i}) > \overline{x}_i$, and suppose $br_i^X(x_{-i}) < \overline{x}_i$. □

In words, the best-reply map br_i^X of the restricted game $G(X)$ agrees with the best-reply map BR_i of the game $G(Y)$ on the set $\{x_{-i} \in X_{-i} : BR_i(x_{-i}) \in X_i\}$, and is on the boundary ∂X_i of X_i, otherwise. The next lemma states a quite obvious property of a restricted game, that is, if $G(Y)$ has a Nash equilibrium y^*, which is also a feasible action profile of the restricted game $G(X)$, then y^* is also a Nash equilibrium of $G(X)$.

Lemma 2 *If there exists a Nash equilibrium $y^* = (y_i^*)_{i \in N}$ of $G(Y)$ such that $y^* \in X$, then $y^* \in N(G(X))$.*

Proof Since y^* is a Nash equilibrium of $G(Y)$ and $y_i^* \in X_i \subseteq Y_i$ for all $i \in N$, we have that $u_i(y_i^*, y_{-i}^*) \geq u_i(x_i, y_{-i}^*)$, for all $x_i \in X_i \subseteq Y_i$ and all $i \in N$, hence $y^* \in N(G(X))$. □

In the previous lemma, we have seen that any equilibrium of $G(Y)$, which belongs to the restricted set of strategies X, is also an equilibrium of $G(X)$. The converse is obviously not true. However, we can prove that any *interior* equilibrium of $G(X)$ is also an equilibrium of $G(Y)$.

Lemma 3 *If there exists a Nash equilibrium $x^* = (x_i^*)_{i \in N}$ of $G(X)$ such that $x^* \in \operatorname{int} X$, then $x^* \in N(G(Y))$.*

Proof Since $x^* \in \operatorname{int} X$, we have that $\underline{x}_i < x_i^* < \overline{x}_i$ for all $i \in N$. Furthermore, since $x^* \in N(G(X))$, we have that $x_i^* = br_i^X(x_{-i}^*)$ for all $i \in N$. From Lemma 1, it follows that $br_i^X(x_{-i}^*) = BR_i(x_{-i}^*)$ for all $i \in N$, hence x^* is a Nash equilibrium of $G(Y)$. □

It immediately follows that the equilibria of $G(X)$, which are not equilibria of $G(Y)$, are on the *boundary* of X. The next proposition formally states this result.

Proposition 1 *If $x^* \in N(G(X)) \setminus N(G(Y))$, then $x^* \in \partial X$.*

Proof It is a direct consequence of Lemmata 2 and 3. $\square$

We also have the following lemma.

Lemma 4 *Let $G(X)$ and $G(Y)$ be two two-player nice games such that $G(X)$ is a restriction of $G(Y)$. Then,*

$$\sharp(N(G(X)) \setminus N(G(Y))) \leq 4.$$

Proof From Proposition 1, any $x \in N(G(X)) \setminus N(G)$ is such that $x \in \partial X$. Moreover, since X is the product of two compact real intervals, ∂X is composed of 4 edges. By strict quasi-concavity of the payoff functions, we have at most one Nash equilibrium on each edge of X, hence the desired result. $\square$

Next, consider the sequence of strategy sets $X^0 = Y_1 \times Y_2,\ X^1 = X_1 \times Y_2$ and $X^2 = X_1 \times X_2$, and the sequence of games $G(X^0)$, $G(X^1)\,, G(X^2)$. Observe that $G(X^2)$ is a **restriction** of $G(X^1)$ with the property that $G(X^2)$ differs from $G(X^1)$ only in the set of strategies of player 2. Similarly, $G(X^1)$ is a restriction of $G(X^0)$ with the property that $G(X^1)$ differs from $G(X^0)$ only in the set of strategies of player 1. By transitivity, $G(X^2)$ is a restriction of $G(X^0)$. As a direct corollary of Proposition 1, we have that for any $x^* \in N(G(X^1)) \setminus N(G(X^0))$, $x_1^* \in \partial X_1$.

Thus, if $G(X^1)$ is a restriction of $G(X^0)$, which differs only in the set of strategies of a single player, say player 1, then any equilibrium of $G(X^1)$, which is not an equilibrium of $G(X^0)$ is such that player 1's equilibrium strategy is on the boundary of his strategy space. This result will prove extremely useful in proving Theorem as our strategy of proof will consist in showing that $\sharp N(G(X^1)) \leq \sharp N(G(X^0))$, and thus, by an induction argument, that $\sharp N(G(X)) \leq \sharp N(G(Y))$.

3.2 Injective mapping

With a slight abuse of notation, we denote br_i the best-reply function of player i in $G(X^1)$, i.e., $br_i := br_i^{X^1}$. Let us show that $\sharp N(G(X^1)) \leq \sharp N(G(X^0))$.

Choose a $x^* = (x_1^*, x_2^*) \in N(G(X^1)) \setminus N(G(X^0))$. From a previous argument, we have that $x_1^* \in \partial X_1$. Suppose that $x_1^* = \overline{x}_1$. (We can reason in analogy if $x_1^* = \underline{x}_1$ holds.) Since x^* is a Nash equilibrium of $G(X^1)$, x_1^* satisfies the equation $f(x_1^*) = 0$ with $f : X_1 \to \mathbb{R}$, $f(x_1) = br_1(br_2(x_1)) - x_1$.

Remember that BR_i is the best-reply function of player i in the game $G(X^0)$, and define $F : Y_1 \to \mathbb{R}$ with $F(y_1) = BR_1(BR_2(y_1)) - y_1$. Observe that $F(\underline{y}_1) \geq 0$ and $F(\overline{y}_1) \leq 0$. Moreover, since $x^* \notin N(G(X^0))$, we have $F(x_1^*) \neq 0$.

From Lemma 1, br_2 is the restriction of BR_2 to the domain $X_1 \subseteq Y_1$, hence $br_2(x_1^*) = BR_2(x_1^*)$. It follows that $F(x_1^*) > 0$. For otherwise, we would have $BR_1(BR_2(x_1^*)) < x_1^* = br_1(BR_2(x_1^*))$, a contradiction with Lemma 1.

Since F is continuous, it follows from the Intermediate Value Theorem that there exists a y_1^* in $(x_1^*, \overline{y}_1]$ such that $F(y_1^*) = 0$. Hence, we can associate to $x^* \in N(G(X^1)) \setminus N(G(X^0))$ an equilibrium $y^* = (y_1^*, BR_2(y_1^*))$ in $N(G(X^0)) \setminus N(G(X^1))$. Note that if x^* is such that $x_1^* = \underline{x}_1$, we associate to x^* an equilibrium y^* such that $y_1^* \in [\underline{y}_1, \underline{x}_1)$.

For completeness, suppose that $\partial X_1 \cap \partial Y_1$, and that $x^* \in N(G(X^1)) \setminus N(G(X^0))$ is such that $x_1^* = \overline{y}_1$. By Lemma 1 and compactness, we then have

$$\overline{y}_1 \geq BR_1(x_2^*) > br_1(x_2^*) = x_1^* = \overline{y}_1,$$

a contradiction. A similar reasoning holds if $x_1^* = \underline{y}_1$. Hence, for any Nash equilibrium $x^* \in N(G(X^1)) \setminus N(G(X^0))$, we can associate a Nash equilibrium y^* in $N(G(X^0)) \setminus N(G(X^1))$. The last part of the proof consists in showing that for any two Nash equilibria $x^* \neq x^{**}$ of $G(X^1)$, we can associate two different Nash equilibria $y^* \neq y^{**}$ of $G(X^0)$.

Clearly, if $x_1^* \neq x_1^{**}$, then $y_1^* \neq y_1^{**}$ since $x_1^* \neq x_1^{**}$ implies that x_1^* is $\overline{x}_1$ and x_1^{**} is $\underline{x}_i$ or the inverse, hence y_1^* is a zero of F in $(\overline{x}_1, \overline{y}_1]$ and y_1^{**} is a zero of F in $[\underline{y}_1, \underline{x}_1)$. Suppose that $x_1^* = x_1^{**}$. But, then since best-reply map are single-valued, we have that $br_2(x_1^*) = br_2(x_1^{**})$, a contradiction with $x^* \neq x^{**}$. Therefore, we have that $\sharp N(G(X^1)) \leq \sharp N(G(X^0))$.

To complete the proof, apply the preceding arguments to prove that $\sharp N(G(X^2)) \leq \sharp N(G(X^1))$, hence that $\sharp N(G(X)) \leq \sharp N(G(Y))$.

4 Tightness of Theorem

To recapitulate, the sufficient conditions for our theorem to hold are that both $G(X)$ and $G(Y)$ are two-player nice games i.e., games with non-empty real intervals as strategy spaces and continuous and strictly quasi-concave payoff functions. We first show that Theorem does neither extend to n-player games, nor to games with multidimensional compact-convex strategy spaces, nor to games with payoff functions that are not strictly quasi-concave. Interestingly enough, in all three cases, our proof breaks down at the same point: the mapping from $N(G(X))$ to $N(G(Y))$ needs not be an injection.

Example 1 (Two-player games) Consider the following three-player nice game[5] $G(Y)$. Assume that the strategy spaces are $Y_1 = Y_2 = Y_3 = [0, 1]$. Player 1's payoff when he plays y_1, player 2 plays y_2 and player 3 plays y_3 is

$$u_1(y_1, y_2, y_3) = y_1.$$

Similarly, player 2's payoff is

$$u_2(y_1, y_2, y_3) = -0.1(y_2)^2 + (1 - y_1)y_3y_2,$$

and player 3's payoff is

$$u_3(y_1, y_2, y_3) = -(y_3 - y_2)^2.$$

Since player 1's payoff is strictly increasing in its own strategy, it follows that 1 is a strictly dominant strategy. The Nash equilibria of $G(Y)$ are then the points at which the graph of the restriction of BR_2 to $\{1\} \times Y_3$ intersects the graph of the restriction of BR_3 to $\{1\} \times Y_2$. It follows that there exists a unique Nash equilibrium of $G(Y)$, namely $(1, 0, 0)$. Assume now that player 1 has a restricted set of strategy $X_1 = [0, 0.5]$. Player 1's new strictly dominant action is 0.5. $G(X)$ has a multiplicity of Nash equilibria: $(0.5, 0, 0)$ and $(0.5, 1, 1)$.

[5] We thank Andrew McLennan for having suggested us this example.

Example 2 (Actions are subset of the real line) Consider the following two-player game $G(Y)$ that is nice except for the assumption that player 1's strategy space is multidimensional. Player 1's action is the unit square $Y_1 = [0,1] \times [0,1]$, and his payoff when he plays $y_1 = (y_1^1, y_1^2) \in Y_1$ and player 2 plays $y_2 \in Y_2 := [0,1]$, is

$$u_1(y_1, y_2) = y_1^1 - (y_2 - y_1^2)^2,$$

while player 2's payoff is

$$u_2(y_1, y_2) = -0.1(y_2)^2 + (1 - y_1^1)y_1^2 y_2.$$

Clearly, this example is a very close relative of the previous example, and has similar equilibria. The strategy profile $((1,0),0)$ is the unique Nash equilibrium of the game $G(Y)$. However, the game $G([0,0.5]\times[0,1],[0,1])$ has more than one equilibrium: $((0.5,0),0)$ and $((0.5,1),1)$ are both equilibrium profiles.

Example 3 (Strict quasi-concavity) Consider a two-player game $G(Y)$ that is nice except for the assumption that player 2's payoff function is strictly quasi-concave. We have $Y_1 = Y_2 = [0,1]$ and the payoff to player 1 when he plays y_1 and player 2 plays y_2 is $u_1(y_1, y_2) = y_1$ while player 2's payoff is $u_2(y_1, y_2) = (0.5 - y_1)y_2$. $G(Y)$ has a unique Nash equilibrium $(1,0)$. Now, let us restrict player 1's strategy space to $[0,0.5]$. The game $G([0,0.5]\times[0,1])$ has infinitely many equilibria: any $(0.5, \lambda)$ with $\lambda \in [0,1]$ is a Nash equilibrium.

If we drop any one of the other sufficient conditions i.e., convexity or compactness of strategy spaces or continuity of payoff functions, our proof might fail for an additional reason. Any two-player game with either discontinuous payoffs or non-compact strategy spaces that does not have an equilibrium can serve as an example to illustrate this point. To see this, take any such game $G(Y)$ and restrict both players' strategy spaces to singletons $\{x_1\}$ and $\{x_2\}$. The strategy profile (x_1, x_2) is an equilibrium of $G(\{x_1\} \times \{x_2\})$, which clearly cannot be mapped to an equilibrium in $N(G(Y))$ since this set is empty. However, even if $G(Y)$ has a Nash equilibrium, our proof still fails as the following examples show.

Example 4 (Convexity) Consider the example in the introduction. G is a nice game except for the assumption that strategies space are convex. To see that strict quasi-concavity is satisfied, observe that $u_i(1, y_{-i}) > \min(u_i(0, y_{-i}), u_i(2, y_{-i}))$ for all $y_{-i} \in \{0, 1, 2\}$, for all $i \in \{1, 2\}$. Continuity refers to order-continuity (we endow $\{0, 1, 2\}$ with the usual order $\geq$).

Example 5 (Compactness) Consider a two-player symmetric game $G(Y)$ that is nice except for the assumption that strategy spaces are compact. For all $i \in \{1, 2\}$, $Y_i = \mathbb{R}$, $u_i(y_i, y_{-i}) = (1 - 2y_{-i})y_i - \frac{1}{2}y_i^2$. $G(Y)$ has unique Nash equilibrium $(\frac{1}{3}, \frac{1}{3})$. Now, consider the restriction $G(X)$ of $G(Y)$, with $X = [0, \frac{1}{2}] \times \mathbb{R}$. It is easy to see that $G(X)$ has three Nash equilibria: $(\frac{1}{3}, \frac{1}{3}), (0, 1)$ and $(\frac{1}{2}, 0)$.

Example 6 (Continuity) Consider a two-player game $G(Y)$ that is nice except for the assumption that payoff function u_2 of player 2 is continuous. Player 1's payoff $u_1(y_1, y_2)$ when he plays $x_1 \in [\frac{3}{5}, \frac{4}{5}]$ and player 2 plays $y_2 \in [\frac{1}{5}, \frac{4}{5}]$ is

$$(\frac{1}{2}x_2 + \frac{1}{2})x_1 - \frac{1}{2}(y_1)^2.$$

Player 2's payoff $u_2(y_1, y_2)$ when she plays y_2 and player 1 plays y_1 is

$$\begin{cases} (y_1 - 1)y_2 & \text{if } y_2 > \frac{1}{2} \\ y_1(y_2 - 1) & \text{if } y_2 \leq \frac{1}{2} \end{cases}.$$

$G(Y)$ has no Nash equilibrium in pure strategies. (In fact, it is because u_2 fails to be upper semi-continuous). Now, consider the restricted game $G(X)$ with $X = [\frac{3}{5}, \frac{4}{5}] \times [\frac{1}{5}, \frac{1}{3}]$. Then, $G(X)$ has a Nash equilibrium $(\frac{2}{3}, \frac{1}{3})$.

5 Final remarks

Theorem states that any two-player nice game $G(X)$, which is a restriction of the two-player nice game $G(Y)$, has fewer Nash equilibria in pure strategies than the game $G(Y)$. The theorem is silent on whether a restriction $G(X)$

admits *strictly* less Nash equilibria than the game $G(Y)$. However, it is easy to construct examples for which some restrictions have strictly less Nash equilibria than the game $G(Y)$ while some others have exactly the same number of Nash equilibria. For instance, consider the two-player symmetric game $G(Y)$, and for all $i \in N$, $Y_i = [0,1]$, $(y_i, y_{-i}) \mapsto u_i(y_i, y_{-i}) = (1 - 2y_{-i})x_i - \frac{1}{2}(y_i)^2$. It is straightforward to check that $G(Y)$ is indeed a nice game. $G(Y)$ has three Nash equilibria in pure strategies $(1,0),(0,1)$ and $(\frac{1}{3}, \frac{1}{3})$. To see this, observe that i's best-reply to y_{-i} is $1 - 2y_{-i}$ if $y_{-i} \in [0, \frac{1}{2}]$ and 0, otherwise. Consider the restriction $G(X)$ of $G(Y)$, with $X = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$. $G(X)$ has a unique Nash equilibrium $(\frac{1}{3}, \frac{1}{3})$. Similarly, consider the restriction $G(X')$ of $G(Y)$ with $X' = [\frac{1}{5}, \frac{2}{5}] \times [\frac{1}{5}, \frac{2}{5}]$. It is easy to see that $G(X')$ has three Nash equilibria: $(\frac{1}{3}, \frac{1}{3}), (\frac{1}{5}, \frac{2}{5})$ and $(\frac{2}{5}, \frac{1}{5})$.

We can also mention a discrete counterpart[6] of Theorem. Suppose that strategy spaces are discrete $Y_i := \{1, 2, \cdots, m_i\}$ for all $i \in \{1,2\}$, and payoff functions are order-continuous, strictly quasi-concave and have increasing differences in (y_i, y_{-i}). Consider two such games $G(X)$ and $G(Y)$ such that $G(X)$ is a restriction of $G(Y)$. We then have that $\sharp N(G(X)) \leq \sharp N(G(Y))$. The proof is almost identical to our proof. First, observe that since $G(Y)$ as well as $G(X)$ are supermodular games, hence $N(G(Y)$ and $N(G(X))$ are non-empty. Second, the characterization of $N(G(X)) \setminus N(G(Y))$ is the same as in our proof. Finally, it suffices to apply Tarsky fixed-point Theorem (instead of the Intermediate Value Theorem) to show that the map from $N(G(X))$ to $N(G(Y))$ is injective.

Finally, we remark that Theorem holds for regular as well as irregular nice games $G(Y)$. Moreover, it might well be that $G(Y)$ is a regular game, hence has a finite and odd number of Nash equilibria (Harsanyi ([3]), but $G(X)$ has an even number of Nash equilibria, hence is not regular.

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