Comparative Statics with Never Increasing Correspondences[1]

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Comparative Statics with Never Increasing Correspondences

Abstract

This paper studies models where the correspondences (or functions) under consideration are never increasing (or weakly decreasing) in endogenous variables, and weakly increasing in exogenous parameters. Such models include games of strategic substitutes, and include cases where additionally, some variables may be strategic complements. It is shown that the equilibrium set in such models is a non-empty, complete lattice, if, and only if, there is a unique equilibrium. For a given parameter value, a pair of distinct equilibria are never comparable. Moreover, generalizing an existing result, it is shown that when a parameter increases, no new equilibrium is smaller than any old equilibrium. (In particular, in n-player games with real-valued action spaces, symmetric equilibria increase with the parameter.) Furthermore, when functions under consideration are weakly decreasing in endogenous variables, a sufficient condition is presented that guarantees existence of increasing equilibria (symmetric or asymmetric) at a new parameter value. This condition is applied to two classes of examples.

1 Introduction

Although comparative statics results for general games with strategic complements are well-developed,[2] results of similar generality are less commonly available for games with strategic substitutes, or in games in which functions under consideration are non-increasing (or weakly decreasing) in endogenous variables. As is well-known, games with strategic complements and strategic substitutes are found in many areas of economics. Such games are defined in Bulow, Geanakoplos, and Klemperer (1985), and as they show, models of strategic investment, entry deterrence, technological innovation, dumping in international trade, natural resource extraction, business portfolio selection, and others can be viewed in a more unifying framework according as the variables under consideration are strategic complements or strategic substitutes. Moreover, the important class of examples of Cournot oligopolies can be viewed as a model with strategic substitutes. Bargaining games can provide examples as well.[3] Additional classes of examples are described in Dubey, Haimanko, and Zapechelnyuk (2006), and include games of team projects with complementary or substitutable tasks, and tournaments.

For example, we are not aware of a general result for such games that can be applied to show increasing equilibria in a simple, parametrized, asymmetric, Cournot duopoly with

[2]Some of this work can be seen in Topkis (1979), Lippman, Mamer, and McCardle (1987), Sobel (1988), Vives (1990), Milgrom and Roberts (1990), Zhou (1994), Milgrom and Shannon (1994), Milgrom and Roberts (1994), Shannon (1995), Villas-Boas (1997), Edlin and Shannon (1998), Echenique (2002), and Echenique and Sabarwal (2003), among others. Extensive bibliographies are available in Topkis (1998) and in Vives (1999).

[3]For example, consider a game where each player bids on a share of a fixed prize, and if the sum of the bids is less than or equal to the prize, then each player gets her bid, else each player gets 0.

linear demand, constant marginal cost, and the standard product order on strategy spaces. Consider a linear inverse market demand curve given by $p = a - bQ$, with $Q = q_1 + q_2$, where q_1 is output of firm 1, and q_2 of firm 2. Suppose each firm has constant marginal cost c. Moreover, there is a subsidy of $t \leq c$ per unit, and this subsidy is split with an exogeously specified and fixed share $\frac{3}{5}$ for firm 1, and share $\frac{2}{5}$ for firm 2.[4] Thus, firm 1's marginal cost net of subsidy is $c - \frac{3}{5}t$, and that of firm 2 is $c - \frac{2}{5}t$. In this case, the unique equilibrium is given by $q^*(t) \equiv (q_1^*(t), q_2^*(t)) = (\frac{a-c+(\frac{9}{5}-1)t}{3b}, \frac{a-c+(2-\frac{9}{5})t}{3b})$, and it is increasing in t.

With the standard product order on strategy spaces, this example does not fit the framework of Milgrom and Shannon (1994), because the profit functions are not quasi-supermodular. (Denote profit of firm 1 at (q_1, q_2, t) by $\pi_1(q_1, q_2, t)$, and consider the values $a = 10, b = 1, c = 1, t = 0$, and consider $(q_1, q_2) = (3, 2)$, and $(q_1', q_2') = (4, 3)$. Then, $\pi_1(q_1', q_2, t) \geq \pi_1(q_1, q_2, t)$, but $\pi_1(q_1', q_2', t) < \pi_1(q_1, q_2', t)$.) Moreover, this implies that this game is not supermodular, and therefore, this example does not fit the framework of Topkis (1979), Sobel (1988), or Vives (1990). If the order on one of the strategy spaces is reversed, then it is known (see, for example, Milgrom and Shannon (1994), and a detailed application in Amir (1996)) that this example is a quasi-supermodular game with the single crossing property, and therefore, using Milgrom and Shannon (1994), equilibria are non-decreasing (in the new order) in t. Of course, this does not imply that equilibria are increasing or weakly increasing in the standard product order in t. Moreover, asymmetric Cournot conditions rule out an application of Amir and Lambson (2000), and of the intersection point theorem of

[4] Alternatively, the parameter t can be thought of as technological improvement, and $(\frac{3}{5}, \frac{2}{5})$ can be thought of as differential adaptation of technological improvement. A slightly more general example is presented later.

Tarski (1955).[5]

One general result is available for games where best responses of endogenous variables are weakly decreasing. As shown by Villas-Boas (1997), in such games, equilibria do not decrease when the exogenous parameter increases. Moreover, for Cournot oligopolies, if a new partial order can be chosen, then with some additional assumptions, there is a new partial order such that equilibria are increasing in this new order. Additionally, some aspects of non-monotone mappings that are increasing in some variables and decreasing in others are explored in Roy (2002).

For the models considered here, this paper sheds light on some reasons for the failure of the usual techniques to show increasing equilibria. Moreover, it generalizes an existing result and applies it to show that symmetric equilibria are increasing in the parameter. Furthermore, it provides a sufficient condition for existence of increasing equilibria that can be applied to asymmetric equilibria.

[5]Tarski's intersection point theorem applies to linearly ordered spaces. It is noteworthy that one trick that can work for the special duopoly case is to compose the reaction functions of the two firms. This yields an increasing function. In this case, an equilibrium can be shown to exist, and at least for one of the players, equilibrium can be shown to be increasing, but (in asymmetric Cournot) not necessarily for the other player. Indeed, as shown below, it is easy to formulate examples of simple Cournot duopolies where the equilibrium is increasing for one player, and decreasing for the other. The same point applies to techniques that apply when the best response of one player depends only on the aggregate best response of other players. Of course, such techniques have been formulated primarily to prove existence theorems for Cournot oligopolies, and not necessarily to show increasing equilibria. See, for example, Selten (1970), Roberts and Sonnenschien (1976), Bamon and Fraysee (1985), Novshek (1985), Kukushkin (1994), and Amir (1996), and additional discussion in Vives (1999).

This paper considers models in which either (1) correspondences under consideration are never increasing in endogenous variables,[6] and weakly increasing in parameters, or (2) functions under consideration are weakly decreasing in endogenous variables, and weakly increasing in parameters.[7]

The first result shows that for models in which correspondences of endogenous variables are never increasing, the equilibrium set is a non-empty, complete lattice, if, and only if, there is a unique equilibrium. Indeed, for a given parameter value, a pair of distinct equilibria are never comparable. Therefore, with multiple equilibria, some of the established techniques for exhibiting increasing equilibria or computing equilibria that use the largest or smallest equilibrium, or the lattice structure of the equilibrium set do not apply to such models.[8]

The second result generalizes to the case of never increasing correspondences, the result by Villas-Boas (1997) for the case of weakly decreasing functions; that is, in such cases, when a parameter increases, no new equilibrium is smaller than any old equilibrium. In the particular case of n-player games with real-valued action spaces and symmetric equilibria, this implies that when a parameter increases, each symmetric equilibrium increases as well. Furthermore, it is shown by means of an example of a Cournot duopoly that in such models,

[6]Never increasing correspondences are a generalization of non-increasing functions in partially ordered spaces, and weakly decreasing functions in linearly ordered spaces. This class of models includes those in which best-response functions are increasing in some endogenous variables and decreasing in others. Thus, it includes models in which endogenous variables are strategic substitutes for each other, and includes cases where additionally, some endogenous variables may be strategic complements.

[7]This class of models includes those in which endogenous variables are strategic substitutes for each other.

[8]This results also shows that there are no ranked equilibria, and therefore, even with multiple equilibria, these models do not have inefficiencies that arise purely from existence of ranked equilbria.

in general, with asymmtric equilibria, (and with a fixed partial order,) increasing equilibrium selections may not exist, even when the product of best response functions is strictly decreasing in endogenous variables, and strictly increasing in exogenous parameters, and for every parameter value, there is a unique equilibrium.

The final result considers models in which the functions under consideration are weakly decreasing in endogenous variables, and weakly increasing in exogenous parameters, and presents a sufficient condition that guarantees existence of increasing equilibria at a new parameter value. This result applies to asymmetric equilibria. Intuitively, in games of strategic substitutes, there are two opposing effects of an increase in the parameter value. The direct effect increases each player's best response, but strategic substitutes imply that an increase in the best response of other players has an additional indirect and opposite effect on each player's best response. At a new parameter value, if this indirect effect does not dominate the direct effect, then a larger equilibrium exists.[9] The condition here applies to games with strategic substitutes, finite number of players, finite-dimensional strategy spaces, and continuous best response functions. The condition is tight in the sense that with a weakenened condition, the same result may not obtain. This result is applied to two classes of examples; the first includes team projects with substitutable tasks, and second includes tournaments; both classes are described in Dubey, Haimanko, and Zapechelnyuk (2006).

Notice that as shown by Villas-Boas (1997), in the case of a Cournot oligopoly, when a new partial order can be chosen as well, then there exists a new partial order in which equilibria are increasing. For a given partial order, it is not known under what conditions a similar result obtains. There may be cases when a given partial order is a natural one

[9]Notably, in games with strategic complements, both effects work in the same direction.

for the model under consideration. For example, for a Cournot oligopoly, the product order may be natural when considering the impact of taxes or subsidies on firm output. In games of strategic complements, the product order is used commonly for the same reason; that is, to investigate the impact of different parameters on each agent's choice. The results here apply to cases where a partial order is considered as fixed.

The paper proceeds as follows. Section 2 presents the model. Section 3 presents results for never-increasing correspondences. Section 4 presents a sufficient condition for the existence of increasing equilibria when the correspondences under consideration are weakly decreasing functions of endogenous variables, and applies this condition to two classes of examples.

2 Model

Suppose $(X, \preceq)$ is a partially ordered set, and A and B are subsets of X. Then A is ***weakly smaller*** than B, if for every $a \in A$, there is $b \in B$ such that $a \preceq b$, and for every $b \in B$, there is $a \in A$ such that $a \preceq b$. A correspondence $g : X \twoheadrightarrow X$ is ***weakly increasing***, if for every $x, y \in X$ with $x \preceq y$, it is the case that $g(x)$ is weakly smaller than $g(y)$.

A correspondence $g : X \twoheadrightarrow X$ is ***never increasing***, if for every $x, y \in X$ with $x \prec y$, for every $x' \in g(x)$, and for every $y' \in g(y)$, it is the case that $x' \npreceq y'$. In other words, g is never increasing, if regardless of which point (y') we choose in the image of a higher point

(y), this point is not higher than any point (x') in the image of a lower point (x).[10] [11]

The model space for endogenous variables is assumed to be a non-empty, compact, convex subset of Euclidean space, denoted X. The space for exogenous parameters is assumed to be a partially ordered set, denoted T. An ***admissible family of correspondences*** is a correspondence $g : X \times T \twoheadrightarrow X$ such that for every t, the correspondence $g(\cdot, t)$ is never increasing, non-empty valued, compact-valued, convex-valued, and upper hemi-continous, and for every x, the correspondence $g(x, \cdot)$ is weakly increasing.

Assumptions other than those regarding correspondences that are never increasing in

[10]Notice that when g is a function, this definition coincides with the standard definition of a non-increasing function; that is, $x \preceq y \Rightarrow g(x) \npreceq g(y)$, and moreover, for linearly ordered spaces, this definition coincides with that of a weakly decreasing function. For purposes of application, non-increasing functions are likely to be more useful than never-increasing correspondences. Indeed, we are not aware of particular economic applications in which correspondences are never-increasing, as defined here. The results here show that at least one extension to correspondences works, and our hope is that this version is available to researchers thinking about more general cases.

[11]An idea of the extent of strategic complements permissible in this model can be formed as follows. Consider a game with $N \geq 2$ players, each with a one-dimensional strategy space, and each with a payoff function, denoted $\pi_i(x)$, where $x \in R^N$ is a vector of endogenous variables, one component for each player. For player i, a pair of endogenous variables (x_j, x_k), $(j \neq k)$ are *strategic complements* if $\frac{\partial^2 \pi_i}{\partial x_j \partial x_k} \geq 0$, and *strategic substitutes* if $\frac{\partial^2 \pi_i}{\partial x_j \partial x_k} < 0$. Suppose that best responses are functions, rather than correspondences, each denoted g_n. For each player, there are $\binom{N}{2}$ pairs of variables that could be strategic complements or substitutes, for a total of $N\binom{N}{2}$ pairs of possible pairs that could be strategic complements or substitutes. Consider the following condition: there is n_0 such that for $m \neq n_o$, $\frac{\partial^2 \pi_{n_0}}{\partial x_m \partial x_{n_0}} < 0$, and there is $m_0 \neq n_0$ such that $\frac{\partial^2 \pi_{m_0}}{\partial x_{m_0} \partial x_{n_0}} < 0$. In this case, it is easy to see that the product of g_n is a non-increasing function, and therefore, a maximum of $N\binom{N}{2} - N$ pairs of variables can be strategic complements. Of course, other estimates would depend on the particular situation under consideration.

endogenous variables and weakly increasing in parameters are made to guarantee existence of equilibrium via Kakutani's theorem. Notably, both theorems of Tarski are not applicable to the general case considered here. Moreover, as mentioned in Vives (1999) (page 42), a general n-dimensional existence theorem for decreasing best responses does not appear to be available even for the case of functions.[12] Given existence of equilibrium, the results here apply to arbitrary, partially ordered X.

Consider an admissible family of correspondences g, and define the following sets. Let $\underline{S}(t) = \{x \in X \mid \exists x' \in g(x,t), x' \preceq x\}$, let $\bar{S}(t) = \{x \in X \mid \exists x' \in g(x,t), x \preceq x'\}$, let $\min \underline{S}(t)$ be the minimal elements of $\underline{S}(t)$, let $\max \bar{S}(t)$ be the maximal elements of $\bar{S}(t)$, and let $FP(t) = \{x \in X \mid x \in g(x,t)\}$ be the fixed points of g at t. Kakutani's theorem implies that for every t, $FP(t)$ is non-empty.

3 Non-Lattice Equilibrium Sets and Nowhere Decreasing Equilibria

It is useful to consider one particular reason for the failure of a standard proof of Tarski's theorem when correspondences are never-increasing.[13] This particular failure is notable, because it is related to a modification that does apply in the models considered here, and this modification helps understand comparability of equilibria when correspondences are never-increasing.

[12] Recent developments showing existence of equilibrium for aggregative games are given in Kukushkin (1994), and in Dubey, Haimanko, and Zapechelnyuk (2006).

[13] For a version of the standard proof, see Topkis (1998), page 39.

In a standard proof, the set $\underline{S}(t)$ has an infimum, $\inf \underline{S}(t) \in \underline{S}(t)$, and $\inf \underline{S}(t)$ is the smallest fixed point. Similarly, $\bar{S}(t)$ has a supremum, $\sup \bar{S}(t) \in \bar{S}(t)$, and $\sup \bar{S}(t)$ is the largest fixed point. Monotone increasing selections can then be exhibited by considering these extremal fixed points. With never increasing correspondences, it is easily possible that the set $\underline{S}(t)$ does not contain an infimum, and the set $\bar{S}(t)$ does not contain a supremum. In such cases, the supremum and infimum cannot be fixed points. For example, consider figure 1, which gives best response functions of two agents. These functions can be viewed as best responses in a Cournot duopoly where firm 1 has a lower marginal cost at a higher level of output, and firm 2 has constant marginal cost. As shown, the product of best responses is a weakly decreasing function. Moreover, $\underline{S}(t)$ is the area with lower boundary given by ABDE, and it does not contain a smallest point, $\bar{S}(t)$ is the area with upper boundary given by FBCDG, and it does not contain a largest point, $\inf \underline{S}(t) \notin \underline{S}(t)$, $\inf \underline{S}(t) \notin FP(t)$, $\sup \bar{S}(t) \notin \bar{S}(t)$, and $\sup \bar{S}(t) \notin FP(t)$.

Nevertheless, as shown in the following lemma, equilibrium points are minimal elements of $\underline{S}(t)$, and maximal elements of $\bar{S}(t)$. These properties are useful in trying to understand when are equilibria comparable, and when is the equilibrium set a lattice.

Lemma 1. *Let $g : X \times T \twoheadrightarrow X$ be an admissible family of correspondences. If $x^* \in FP(t)$, then $x^* \in \min \underline{S}(t) \cap \max \bar{S}(t)$.*

Proof. Let $x^* \in FP(t)$. Then $x^* \in g(x^*, t)$, and $x^* \preceq x^*$, so $x^* \in \underline{S}(t)$. Suppose, by way of contradiction, there is $\hat{x} \in \underline{S}(t)$ with $\hat{x} \neq x^*$, and $\hat{x} \preceq x^*$; that is, $\hat{x} \prec x^*$. As $\hat{x} \in \underline{S}(t)$, there is $x' \in g(\hat{x}, t)$ such that $x' \preceq \hat{x}$. In other words, $\hat{x} \prec x^*$, and there exist $x' \in g(\hat{x}, t)$ and $x^* \in g(x^*, t)$ such that $x' \preceq x^*$, contradicting the fact that g is never increasing.

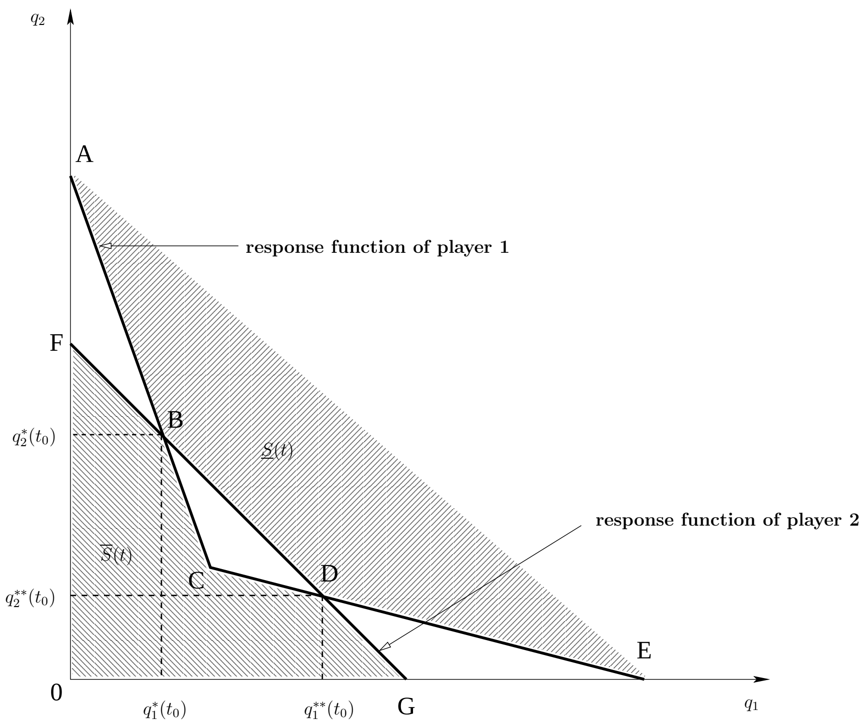

Figure 1: $\max \overline{S}(t)$, $\min \underline{S}(t)$ and Non-lattice Equlibrium Sets

Similarly, $x^* \in FP(t) \Rightarrow x^* \in \bar{S}(t)$. If $x^* \notin \max \bar{S}(t)$, then there is $\hat{x} \in \bar{S}(t)$ such that $x^* \prec \hat{x}$. Consequently, there is $x' \in g(\hat{x}, t)$, and $x^* \in g(x^*, t)$ such that $x^* \preceq x'$, contradicting the fact that g is never increasing. ∎

In the example provided in figure 1, the minimal elements of $\underline{S}(t)$ are given by the boundary depicted by ABDE, and maximal elements of $\bar{S}(t)$ are given by the boundary depicted by FBCDG, and each of the two equilibria satisfies the conclusion of the lemma. This lemma is useful in proving the following sets of results; presented in theorems 1 and 2.

Theorem 1. *Let $g : X \times T \twoheadrightarrow X$ be an admissible family of correspondences.*

1. *If $x^*, \hat{x} \in FP(t)$, and $x^* \neq \hat{x}$, then x^* and $\hat{x}$ are non-comparable.*

2. *The following are equivalent:*

(a) FP(t) is a non-empty lattice,

(b) FP(t) is a singleton, and

(c) FP(t) is a non-empty, complete lattice.

Proof. The first statement follows from the lemma above, as follows. If x^* and $\hat{x}$ are distinct fixed points of g at t, then these are maximal elements of $\bar{S}(t)$, and hence, these are non-comparable. The only part of the second statement that needs to be checked is that the first sub-statement implies the second. (The other implications are trivial.) Suppose $FP(t)$ is a non-empty, complete lattice, and suppose it contains at least two distinct points, say x^* and $\hat{x}$, with $x^* \neq \hat{x}$. Then it contains the join and meet of these points, the join and meet are distinct points, and the join and meet are comparable, contradicting part (1) above. ∎

This theorem shows that for a given parameter value, a pair of distinct equilibria are always non-comparable. In particular, in contrast to equilibria in games with complementarities, this theorem implies that models considered here do not have ranked equilibria. A graphical example with two equilibria is presented in figure 1.

Moreover, as compared to the complete lattice structure of the equilibrium set when functions of endogenous variables are increasing, (see Zhou (1994),) the equilibrium set here is a non-empty, complete lattice exactly in the trivial case of a unique equilibrium. Otherwise, the equilibrium set is totally unordered. (A graphical example with two equilibria, and in which the equilibrium set is not a lattice is provided in figure 1.) Consequently, with multiple equilibria, techniques using the lattice structure of the equilibrium set, or the existence of a smallest and largest equilibrium do not apply to models considered here.

Furthermore, this result implies that in the special case when X is linearly ordered, there is a unique equilibrium for every parameter value.

Theorem 2. *Let $g : X \times T \twoheadrightarrow X$ be an admissible family of correspondences. For every $t_1, t_2 \in T$, if $t_1 \preceq t_2$, $x^* \in FP(t_1)$, $x^{**} \in FP(t_2)$, and $x^* \neq x^{**}$, then $x^{**} \npreceq x^*$.*

Proof. When $t_1 = t_2$, the result follows from part (1) of Theorem 1. Suppose that $t_1 \prec t_2$, and consider distinct fixed points $x^* \in FP(t_1)$, $x^{**} \in FP(t_2)$, and suppose $x^{**} \preceq x^*$. Recall that $x^* \in FP(t_1) \subset \min \underline{S}(t_1) \subset \underline{S}(t_1)$. Moreover, $g(x^{**}, \cdot)$ is weakly increasing in t implies that $g(x^{**}, t_1)$ is weakly smaller than $g(x^{**}, t_2)$. As $x^{**} \in g(x^{**}, t_2)$, let $x' \in g(x^{**}, t_1)$ be such that $x' \preceq x^{**}$. Then $x^{**} \in \underline{S}(t_1)$, contradicting the fact that x^* is a minimal element of $\underline{S}(t_1)$.

∎

This theorem generalizes to the case of never increasing correspondences, the result by Villas-Boas for the case of decreasing functions; that is, in such cases, when a parameter increases, no new equilibrium is smaller than any old equilibrium. Thus, equilibria are nowhere decreasing in t.

In particular, this result implies that in the models considered here, there are no decreasing selections of equilibria.

Moreover, combined with the previous theorem, it follows that if X is linearly ordered, then there is a unique equilibrium for every t, and this equilibrium selection is increasing in t. In particular, for games with real-valued strategies, symmetric equilibria are increasing, as formalized in the following corollaries.[14]

[14] As discussed in the introduction, with decreasing best responses, symmetric equilibria can be shown to be increasing using Tarski's intersection point theorem (see, for example, Milgrom and Roberts (1994)). The corollaries here present another proof as an application of the previous theorem, and this proof does not require Tarski's theorem.

Corollary 1. *Consider a game of n-players, each with a non-empty, compact, convex strategy space $X_i \subset \Re$, $X = \times_{i=1}^{n} X_i$, a parameter space T, and $g : X \times T \twoheadrightarrow X$ a product of best response correspondences, and g is never-increasing. Say that an equilibrium $x^* \in FP(t)$ is a symmetric equilibrium if for all i, j, $x_i^* = x_j^*$. Let $SE(t)$ be the (possibly empty) set of symmetric equilibria. In this case, the following is true.*

For every $t_0, \hat{t} \in T$, if $t_0 \preceq \hat{t}$, $x^* \in SE(t_0)$, and $x^{**} \in SE(\hat{t})$, then $x^* \preceq x^{**}$.

Proof. We know that $x^{**} \not\preceq x^*$. Therefore, there is i such that $x_i^* \leq x_i^{**}$. As x^*, x^{**} are symmetric equilibria, this implies that $x^* \preceq x^{**}$. ∎

Thus, in the class of games with non-increasing best response functions and symmetric equilibria, equilibria are increasing. Indeed,

Corollary 2. *Suppose the same class of games as in the previous corollary. Then the following is true.*

If for every t, $SE(t) \neq \emptyset$, then every selection from $SE(t)$ is a (weakly) increasing selection.

More generally, with asymmetric players, the conclusion of the theorem and corollaries above cannot be strengthened to conclude the existence of increasing equilibria, even when there is always a unique equilibrium, as shown in the following example.

Example 1. Consider a standard Cournot duopoly with a linear inverse market demand curve given by $p = a - bQ$, with $Q = q_1 + q_2$, where q_1 is output of firm 1, and q_2 of firm 2. Suppose each firm has constant marginal cost c, but firm 1 gets a subsidy of $t \leq c$ per unit, so that firm 1's marginal cost net of subsidy is $c - t$. Then best response function of firm 1 is $g_1(q_2, t) = \frac{a-c+t}{2b} - \frac{q_2}{2}$, and that of firm 2 is $g_2(q_1, t) = \frac{a-c}{2b} - \frac{q_1}{2}$. It is easy to check that $g(q_1, q_2, t) \equiv (g_1(q_2, t), g_2(q_1, t))$ is a strictly decreasing correspondence in (q_1, q_2), it is strictly increasing in t, and the unique equilibrium at t is $q^*(t) \equiv (q_1^*(t), q_2^*(t)) = (\frac{a-c+2t}{3b}, \frac{a-c-t}{3b})$. Consequently, and as shown figure 2, there are no increasing equilibria, regardless of the parameter value $t \leq c$.

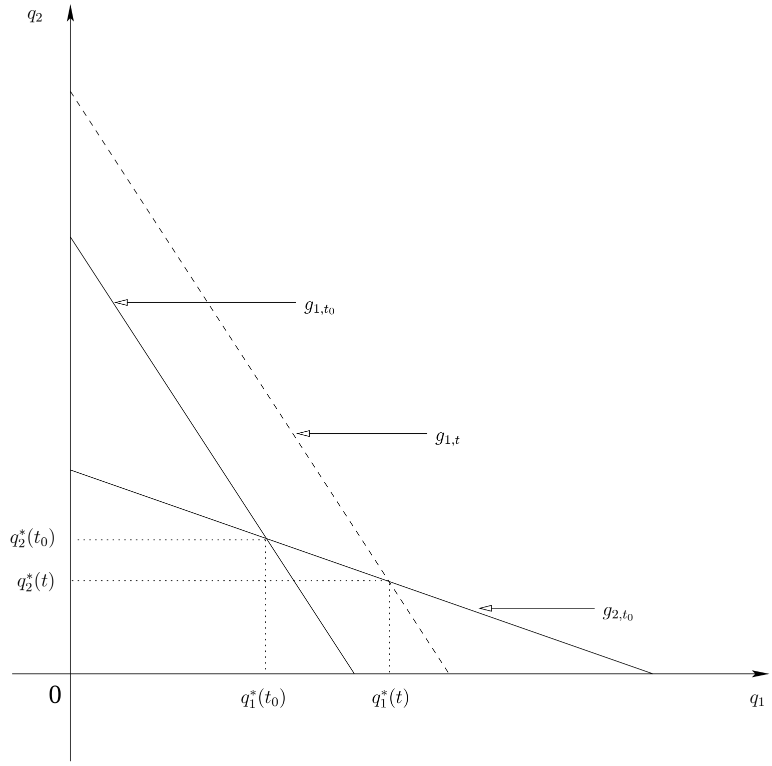

Figure 2: Non-existence of Increasing Equilibria

Notice that the lemma and two theorem in this section apply as stated when X is an arbitrary, partially ordered set.

4 Existence of Increasing Equilibria

This section considers functions that are weakly decreasing in endogenous variables, and weakly increasing in parameters, and provides a sufficient condition, which when satisfied at a new parameter value, guarantees existence of increasing equilibria for the new parameter value. This condition is applied to two classes of examples to show existence of increasing equilibria.

To develop a better understanding of the general result, it is helpful to view it explicitly in the special case of a game with two agents, each with a decreasing best response function, each with a one-dimensional action space, and with the partial order determined by the product order. This case is considered below, and for additional insight, in this case, a direct proof is provided as well.

Consider a game with two agents, indexed $i = 1, 2$. Agent i's action space is a non-empty, compact, convex interval I_i of the real numbers, and there is a partially ordered parameter space T. Agent i's response function is $g_i : I_j \times T \to I_i$, with $i \neq j$. For each i and t, suppose that $g_i(\cdot, t)$ is strictly decreasing, and for each i, and for each $x_j \in I_j$, suppose that $g_i(x_j, \cdot)$ is strictly increasing. Let $X = I_1 \times I_2$, and with the product order (denoted $\leq$). Suppose $g(x_1, x_2, t) \equiv (g_1(x_2, t), g_2(x_1, t))$ is a continuous function in (x_1, x_2), and let $FP(t)$ be the set of fixed points of g at t.[15] For notational convenience, let $g_{i,t}(\cdot) \equiv g_i(\cdot, t)$, and $g_t(\cdot) \equiv g(\cdot, t)$.

Theorem 3. *Fix $t_0 \in T$, let $x^* = (x_1^*, x_2^*) \in FP(t_0)$, and consider $\hat{t} \in T$ with $t_0 \preceq \hat{t}$ such that (1) $x_2^* \leq g_{2,\hat{t}}(g_{1,\hat{t}}(x_2^*))$, and (2) $x_1^* \leq g_{1,\hat{t}}(g_{2,\hat{t}}(x_1^*))$.*
Then there is $\hat{x} = (\hat{x}_1, \hat{x}_2) \in FP(\hat{t})$ such that $x^* \leq \hat{x}$.

Proof. Notice that by (2), at x_1^*, $g_{2,\hat{t}}(x_1^*) \leq g_{1,\hat{t}}^{-1}(x_1^*)$. Moreover, $g_{2,\hat{t}}(g_{2,\hat{t}}^{-1}(x_2^*)) \geq g_{1,\hat{t}}^{-1}(g_{2,\hat{t}}^{-1}(x_2^*))$, because

$$g_{1,\hat{t}}^{-1}(g_{2,\hat{t}}^{-1}(x_2^*)) \quad \leq \quad g_{1,\hat{t}}^{-1}(g_{1,\hat{t}}(x_2^*)) \quad = \quad x_2^* \quad = \quad g_{2,\hat{t}}(g_{2,\hat{t}}^{-1}(x_2^*)),$$

[15]For reference, notice that this game allows for multiple equilibria. (An example to show this can be constructed from figure 1.) Moreover, this games allows for agent conditions to be asymmetric.

where the weak inequality follows from (1) and the fact that $g^{-1}_{1,\hat{t}}(\cdot)$ is decreasing. Furthermore, $x_1^* \leq g^{-1}_{2,\hat{t}}(x_2^*)$, because $g_{2,t_0}(x_1^*) = x_2^* \Rightarrow x_1^* = g^{-1}_{2,t_0}(x_2^*) \leq g^{-1}_{2,\hat{t}}(x_2^*)$, where the last inequality follows from the fact that $g^{-1}_{2,\hat{t}}(\cdot)$ is weakly increasing in t.

By continuity, there is $\hat{x}_1 \in [x_1^*, g^{-1}_{2,\hat{t}}(x_2^*)]$ such that $g_{2,\hat{t}}(\hat{x}_1) = g^{-1}_{1,\hat{t}}(\hat{x}_1)$. Let $\hat{x}_2 = g_{2,\hat{t}}(\hat{x}_1)$, and notice that $g_{1,\hat{t}}(\hat{x}_2) = g_{1,\hat{t}}(g^{-1}_{1,\hat{t}}(\hat{x}_1)) = \hat{x}_1$, whence $\hat{x} = (\hat{x}_1, \hat{x}_2) \in FP(\hat{t})$.

Finally, by the fact that $\hat{x}_1 \in [x_1^*, g^{-1}_{2,\hat{t}}(x_2^*)]$, we conclude that $x_1^* \leq \hat{x}_1$, and moroever, $\hat{x}_1 \leq g^{-1}_{2,\hat{t}}(x_2^*)$ implies that $\hat{x}_2 = g_{2,\hat{t}}(\hat{x}_1) \geq g_{2,\hat{t}}(g^{-1}_{2,\hat{t}}(x_2^*)) = x_2^*$, whence $x^* \leq \hat{x}$. ∎

The conditions in this theorem can be viewed explicitly, as follows. Starting from an existing equilibrium, $x^* = (x_1^*, x_2^*)$ at $t = t_0$, an increase in t has two effects on $g_{2,t}(\cdot)$. One effect is an increase in $g_{2,t}$, because response functions are increasing in t. (This is a direct effect of an increase in t.) The other effect is a decrease in $g_{2,t}(\cdot)$, because an increase in t increases $g_{1,t}(x_2^*)$, and x_1 and x_2 are strategic substitutes. (This is an indirect effect arising from the response of player 1 to an increase in t.) Similar statements are valid for player 1 as well. Taken together, conditions (1) and (2) say that for each player, as long as the indirect strategic substitute effect does not dominate the direct parameter effect, there is a new equilibrium that is larger than $x^* = (x_1^*, x_2^*)$. A graphical illustration of these conditions is presented in figure 3.

It is useful to note that if either condition is not satisfied, this theorem may not necessarily apply. This can be seen in the following generalized version of example 1, and graphically in figure 4, where condition (1) is violated but (2) is satisfied, and in figure 5, where the reverse is true.

Example 2. Consider a standard Cournot duopoly with a linear inverse market demand curve given by $p = a - bQ$, with $Q = q_1 + q_2$, where q_1 is output of firm 1, and q_2 of firm

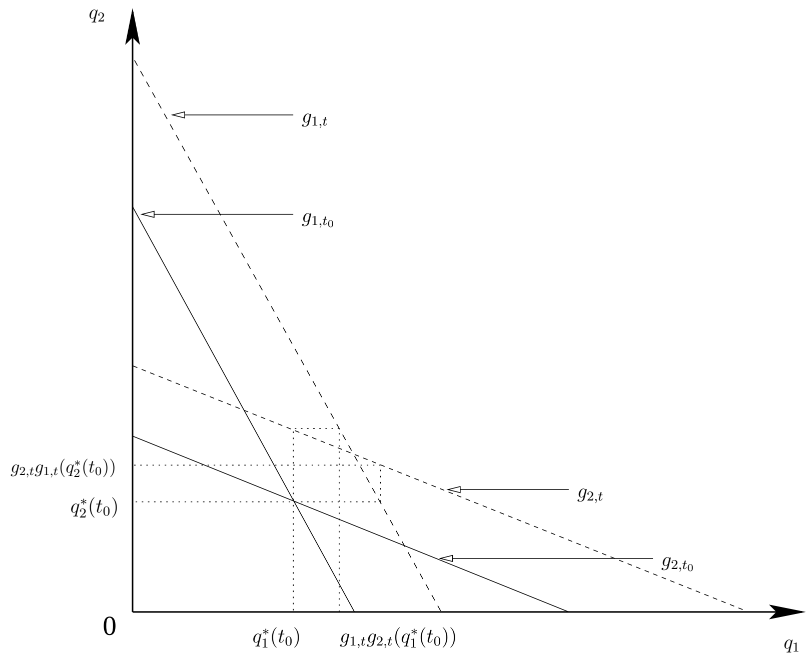

Figure 3: Existence of Increasing Equilibria

2. Suppose each firm has constant marginal cost c. Moreover, there is a subsidy of $t \leq c$ per unit, and this subsidy is split with share $\xi \in [0, 1]$ for firm 1, and share $1 - \xi$ for firm 2. (Example 1 is the case where $\xi = 1$, and the example in the introduction is the case where $\xi = \frac{3}{5}$.) Thus, firm 1's marginal cost net of subsidy is $c - \xi t$, and that of firm 2 is $c - (1 - \xi)t$. Then best response function of firm 1 is $g_1(q_2, t) = \frac{a-c+\xi t-bq_2}{2b}$, and that of firm 2 is $g_2(q_1, t) = \frac{a-c+(1-\xi)t-bq_1}{2b}$. It is easy to check that $g(q_1, q_2, t) \equiv (g_1(q_2, t), g_2(q_1, t))$ is a strictly decreasing correspondence in (q_1, q_2), it is strictly increasing in t, and the unique equilibrium at t is $q^*(t) \equiv (q_1^*(t), q_2^*(t)) = (\frac{a-c+(3\xi-1)t}{3b}, \frac{a-c+(2-3\xi)t}{3b})$. Consequently,

$$\xi < \tfrac{1}{3} \quad \Leftrightarrow \quad q_1^*(t) \text{ is decreasing in } t, \text{ and } q_2^*(t) \text{ is increasing in } t,$$

$$\tfrac{1}{3} \leq \xi \leq \tfrac{2}{3} \quad \Leftrightarrow \quad q_1^*(t) \text{ is increasing in } t, \text{ and } q_2^*(t) \text{ is increasing in } t, \text{ and}$$

$$\tfrac{2}{3} < \xi \quad \Leftrightarrow \quad q_1^*(t) \text{ is increasing in } t, \text{ and } q_2^*(t) \text{ is decreasing in } t.$$

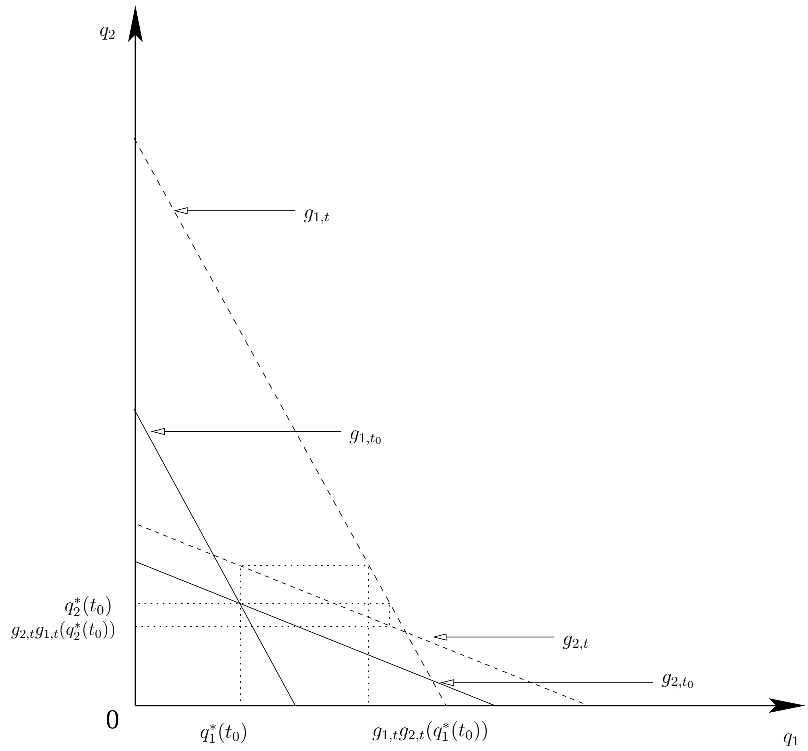

Figure 4: Violation of condition (1)

Moreover, for $t_0 \leq \hat{t}$, $g_{2,\hat{t}}(g_{1,\hat{t}}(q_2^*(t_0))) = \frac{1}{12b}[4(a-c)+(6-9\xi)(\hat{t}-t_0)+(8-12\xi)t_0]$, whence for $t_0 \leq \hat{t}$, $q_2^*(t_0) \leq g_{2,\hat{t}}(g_{1,\hat{t}}(q_2^*(t_0))) \Leftrightarrow \xi \leq \frac{2}{3}$. Similarly, for $t_0 \leq \hat{t}$, $g_{1,\hat{t}}(g_{2,\hat{t}}(q_1^*(t_0))) = \frac{1}{12b}[4(a-c) + (9\xi - 3)(\hat{t} - t_0) + (12\xi - 4)t_0]$, whence for $t_0 \leq \hat{t}$, $q_1^*(t_0) \leq g_{1,\hat{t}}(g_{1,\hat{t}}(q_1^*(t_0))) \Leftrightarrow \xi \geq \frac{1}{3}$. Thus, if $\xi > \frac{2}{3}$, then condition (1) is violated, but (2) is satisfied, and if $\xi < \frac{1}{3}$, then condition (1) is satisfied, but (2) is violated.

A similar tradeoff between direct and indirect effects is useful in proving a more general theorem. Consider an admissible family of correspondences, $g : X \times T \twoheadrightarrow X$, where for each $t \in T$, $g(\cdot, t)$ is a weakly decreasing function, and for each $x \in X$, $g(x, \cdot)$ is a weakly increasing function. To distinguish this from the general case of correspondences, denote this family by $g : X \times T \to X$. For notational convenience, let $g_t(\cdot) \equiv g(\cdot, t)$.

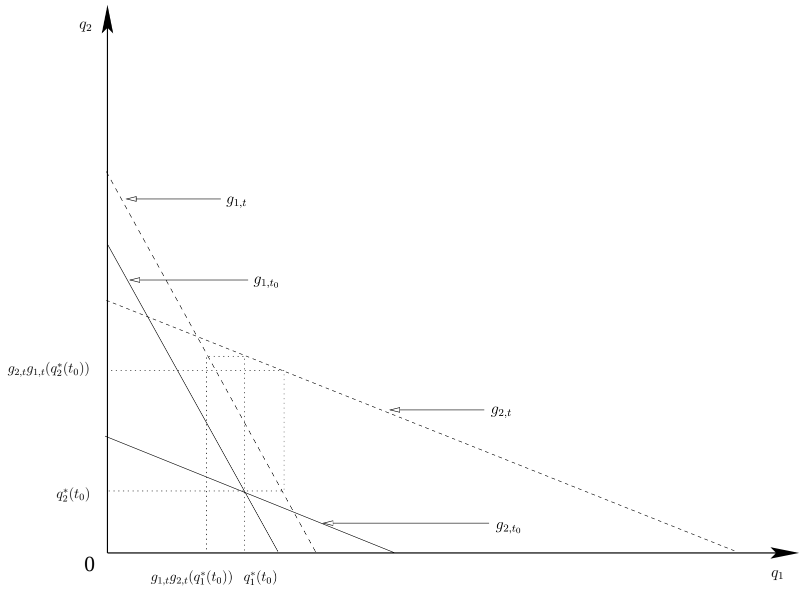

Figure 5: Violation of condition (2)

Theorem 4. *Consider $g : X \times T \to X$ as above, fix $t_0 \in T$, and let $x^* \in FP(t_0)$. Consider $\hat{t} \in T$ such that $t_0 \preceq \hat{t}$, and let $\hat{y} = g_{\hat{t}}(x^*)$.*

If $x^* \preceq g_{\hat{t}}(\hat{y})$, then there is $\hat{x} \in FP(\hat{t})$ such that $x^* \preceq \hat{x}$.

Proof. Notice that as g is weakly increasing in t, $x^* \preceq \hat{y}$. Moreover, for every x in $[x^*, \hat{y}]$, $g_{\hat{t}}(x) \in [x^*, \hat{y}]$, and this can be seen as follows. Suppose $x^* \preceq x \preceq \hat{y}$. Then $x \preceq \hat{y}$ implies that $g_{\hat{t}}(x) \succeq g_{\hat{t}}(\hat{y}) \succeq x^*$, where the first inequality follows from the fact that $g_{\hat{t}}(\cdot)$ is weakly decreasing, and the second follows from the condition in the theorem. Moreover, $x^* \preceq x$ implies that $g_{\hat{t}}(x) \preceq g_{\hat{t}}(x^*) = \hat{y}$, where the inequality follows from weakly decreasing $g_{\hat{t}}(\cdot)$, and the equality follows from definition of $\hat{y}$. Therefore, the restriction of $g_{\hat{t}}$ to $[x^*, \hat{y}]$ is a map from $[x^*, \hat{y}]$ to $[x^*, \hat{y}]$. By Kakutani's theorem, there is $\hat{x} \in [x^*, \hat{y}]$ such that $g_{\hat{t}}(\hat{x}) = \hat{x}$, and consequently, there is $\hat{x} \in FP(\hat{t})$ such that $x^* \preceq \hat{x}$. ∎

Notice that for the special case considered in theorem 3, the conditions here specialize to those in theorem 3. The intuition for the general case is the same as for the special case. Suppose g_t is a product of best response functions of finitely many players, and consider an equilibrium x^* at t_0. Then for each given player, a rise in t has two opposing effects. The direct effect leads to an increase in the best reponse of the given player. The indirect effect leads to a decrease in the best response of a given player, because responses of each player are strategic substitutes for every other player. At a new parameter value, if the indirect effect does not dominate the direct effect, then there is a new equilibrium larger than x^*.

Notice that existence of increasing equilibria is shown here starting from an arbitrary equilibrium point. Therefore, this result applies to any equilibrium point that is selected by some theory of equilibrium selection. If a different equilibrium point is selected by a different theory of equilibrium selection, then the condition in this theorem applies to the different equilibrium point. Moreover, with finitely many equilibria at a parameter value t_0, there are finitely many conditions, one for each equilibrium, such that if all conditions are satisfied, then regardless of which equilibrium obtains at t_0, there is a larger equilibrium. Furthermore, a condition that applies with potentially infinite number of equilibria, and that is independent of a theory of equilibrium selection is given in the corollary below.

Corollary 3. *Consider $g : X \times T \to X$ as in the theorem above, fix $t_0 \in T$, and let $x^* \in FP(t_0)$. Consider $\hat{t} \in T$ such that $t_0 \preceq \hat{t}$, and suppose $\tilde{x} = \bigvee_{x^* \in FP(t_0)} x^*$ and $\hat{y} = \bigvee_{x^* \in FP(t_0)} g_{\hat{t}}(x^*)$ are both in X.*

If $\tilde{x} \preceq g_{\hat{t}}(\hat{y})$, then there is $\hat{x} \in FP(\hat{t})$ such that $x^* \preceq \hat{x}$.

This corollary can be proved by following the proof of the previous theorem, and noticing that for every $x^* \in FP(t_0)$, $g_{\hat{t}}(\tilde{x}) \preceq g_{\hat{t}}(x^*) \preceq \hat{y}$.

The idea of competing direct and indirect effects helps relate the conditions here to those that arise in models with strategic complements. In those models, the direct and indirect effects work in the same direction, and therefore, once a parameter increases, both effects serve to move the new equilibrium set higher. Moreover, in those models, once increasing equilibria have been demonstrated, additionally higher parameter values serve to further increase equilibria, and do not reverse any increases. When direct and indirect effects work in opposite directions, increasing equilibria are no longer guaranteed. Moreover, even when the tradeoff between indirect and direct effects implies a larger equilibrium at a higher parameter value, that tradeoff might not necessarily hold at additionally higher parameter values, and therefore, a demonstration of a favorable tradeoff at a parameter value does not necessarily imply increasing equilibria at additionally higher parameter values.

The following examples apply the condition in the previous theorem to exhibit increasing equilibria in cases where equilibria may be asymmetric, and might not necessarily be computable analytically.

Example 3. Consider games of team projects with substitutable tasks, as follows.[16] Suppose a project is to be accomplished by a team of $n \geq 2$ players, each choosing task (or effort) $x_i \in [0, 1]$, with probability of success x_i. The quadratic cost of effort x_i is $\frac{c_i}{2}x_i^2$, and is allowed to be asymmetric across players. Tasks are substitutable in the sense that each player by herself can make the project successful. The probability of success is $1 - \prod_{j=1}^{n}(1 - x_j)$. If the project is successful, player i receives a parameterized reward (or utility) $f(t) > 0$ (with $0 \leq t \leq T$, and $f'(t) > 0$.)[17] Otherwise, the player receives zero. Therefore, the payoff to

[16] The version used here is the one presented in Dubey, Haimanko, and Zapechelnyuk (2006).

[17] The parameter t can be viewed as technological improvement or subsidy provided or reward provided

player i is

$$f(t)(1-\prod_{j=1}^{n}(1-x_j))-\frac{c_i}{2}x_i^2.$$

It is easy to calculate that the best response of player i is

$$x_i=\frac{f(t)}{c_i}\prod_{j=1,\dots,n;j\neq i}(1-x_j),$$

and this best response is decreasing in other player actions, and increasing in t. Denote by $g_{i,t}(x_{-i})$ the best response function of player i when parameter is t and other player actions are x_{-i}. Let the equilibrium at $t=t^*$ be given by $(x_1^*,\dots,x_n^*)$. Then it is easy to check that $g_{i,t}(x_{-i}^*)=\frac{f(t)}{f(t^*)}x_i^*$. Consequently, for player 1,

$$g_{1,t}(g_{2,t}(x_{-2}^*),g_{3,t}(x_{-3}^*),\dots,g_{n,t}(x_{-n}^*))=\frac{f(t)}{c_1}\prod_{j=2}^{n}(1-\frac{f(t)}{f(t^*)}x_j^*).$$

For ease of computation, let $\phi_j(t)=(1-\frac{f(t)}{f(t^*)}x_j^*)$, and let $\Phi(t)=\prod_{j=2}^{n}\phi_j(t)$. Then the above can be re-written as

$$g_{1,t}(g_{2,t}(x_{-2}^*),g_{3,t}(x_{-3}^*),\dots,g_{n,t}(x_{-n}^*))=\frac{f(t)}{c_1}\Phi(t).$$

When viewed as a function of t, the derivative of this function[18] evaluated at t^* is

$$\Phi(t^*)\frac{f'(t^*)}{c_1}(1-\sum_{j=2}^{n}\frac{x_j^*}{1-x_j^*}).$$

This expression is positive when $\sum_{j=2}^{n}\frac{x_j^*}{1-x_j^*}<1$. One sufficient condition for this to hold is that for all j, $x_j^*\leq\frac{1}{n}$, with strict inequality for one player. A similar condition holds for the other players as well. Thus, an increasing equilibrium obtains, if for every j, $x_j^*<\frac{1}{n}$.

that can induce an increase in effort (or probability) of task completion. As shown below, the best response function depends on $\frac{f(t)}{c_i}$, where c_i measures player i's costs, and therefore, $f(t)$ can be viewed as a relative reward enhancement parameter, relative to a player's costs.

[18]Note that $\Phi'(t)=\Phi(t)(\sum_{j=2}^{n}\frac{\phi_j'(t)}{\phi_j(t)})$.

Example 4. Consider games of tournaments.[19] Suppose a tournament has 3 players, where a parametrized reward $f(t)$ (with $0 \leq t \leq T$, and $f'(t) > 0)$[20] is shared by the players who succeed in the tournament. If one player succeeds, he gets $f(t)$ for sure, if two players succeed, each gets $f(t)$ with probability one-half, and if all players succeed, each gets $f(t)$ with probability one-third. Expected reward for player i is

$$f(t)x_i(1-x_j)(1-x_k) + \frac{f(t)}{2}x_ix_j(1-x_k) + \frac{f(t)}{2}x_ix_k(1-x_j) + \frac{f(t)}{3}x_ix_jx_k.$$

The quadratic cost of effort x_i is $\frac{c_i}{2}x_i^2$, and is allowed to be asymmetric across players. The payoff to player i is expected reward minus cost of effort. It is easy to calculate that the best response of player i is

$$x_i = \frac{f(t)}{c_i}(1 - \frac{1}{2}(x_j + x_k) + \frac{1}{3}x_jx_k),$$

and this best response is decreasing in other player actions, and increasing in t. Denote by $g_{i,t}(x_{-i})$ the best response function of player i when parameter is t and other player actions are x_{-i}. Let the equilibrium at $t = t^*$ be given by (x_1^*, x_2^*, x_3^*). Then it is easy to check that $g_{i,t}(x_{-i}^*) = \frac{f(t)}{f(t^*)}x_i^*$. Consequently, for player i,

$$g_{i,t}(g_{j,t}(x_{-j}^*), g_{k,t}(x_{-k}^*)) = \frac{f(t)}{c_i}(1 - \frac{1}{2}\frac{f(t)}{f(t^*)}(x_j^* + x_k^*) + \frac{1}{3}(\frac{f(t)}{f(t^*)})^2 x_j^* x_k^*).$$

When viewed as a function of t, the derivative of this function evaluated at t^* is

$$\frac{f'(t^*)}{c_i}(1 - (x_j^* + x_k^*) + x_j^*x_k^*) \;=\; \frac{f'(t^*)}{c_i}(1-x_j^*)(1-x_k^*).$$

[19] The version used here is the one presented in Dubey, Haimanko, and Zapechelnyuk (2006).

[20] As shown below, the best response function depends on $\frac{f(t)}{c_i}$, where c_i measures player i's costs, and therefore, $f(t)$ can be viewed as a relative reward enhancement parameter, relative to a player's costs.

Given the domain restriction of strategies to the unit interval, this expression is positive exactly when either $x_j^* < 1$ or $x_k^* < 1$. Similar results hold for players j and k. Consequently, if the equilibrium is not degenerate, (that is, no player wins the tournament for sure,) then equilibrium increases with the parameter. The analogous result holds for the n-player case. Its notationally intensive details are available from the authors, if desired.

The idea of both examples is that if an estimate of an equilibrium is available (perhaps because we observe a particular equilibrium under given economic conditions), then it can be concluded whether an increase in economic conditions will increase the equilibrium. Similar applications of the theorem can be made when an estimate of an equilibrium is available, and best response functions are computable. In particular, an application of this theorem does not require that best response functions have analytically closed forms. Therefore, from a practical point of view, this theorem can have broader applications.

One limitation of this work is the current absence of conditions on the payoff functions that guarantee the sufficient conditions of the last theorem. This remains a subject of continuing work.

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