Forming societies and the Shapley NTU value[1]

Juan J. Vidal-Puga[2,3]

January 23, 2004

[1]An earlier version of this paper has benefited from helpful comments of Gustavo Bergantiños, David Pérez-Castrillo, Luis Corchón, María Gómez-Rúa, Lloyd S. Shapley, María Montero and the seminar participants at the Universidade de Santiago de Compostela (2003), Italian Meeting of Game Theory (Urbino, 2003), Summer Festival on Game Theory (Stony Brooks, 2003) and Summer Meeting of Economic Analysis (Vigo, 2003). Usual disclaimer applies. Financial support by the Spanish Ministerio de Ciencia y Tecnologia and FEDER through grant BEC2002-04102-C02-01 and by the Xunta de Galicia through grant PGIDIT03PXIC30002PN is gratefully acknowledged.

[2]Departamento de Estatistica e IO. Universidade de Vigo. 36200 Vigo (Pontevedra), SPAIN. Phone: +34 986 802014. Fax: +34 986 812401. Email: `vidalpuga@uvigo.es`

[3]The lastest version can be found at `http://webs.uvigo.es/vidalpuga`

Abstract

We design a simple protocol of coalition formation. A society grows up by sequentially incorporating new members. The negotiations are always bilateral. We study this protocol in the context of non-transferable utility (NTU) games in characteristic function form. When the corresponding NTU game (N, V) satisfies that $V(N)$ is flat, the only payoff which arises in equilibrium is the Shapley NTU value.

Keywords: Shapley NTU value, subgame perfect equilibrium, sequential formation of coalitions.

1 Introduction

Endogenous formation of coalitions has been widely studied in the literature. A common approach is to assume that many players can simultaneously join a coalition. For example, Hart and Kurz (1983), Chatterjee et al. (1993), Bloch (1996), Okada (1996), and Ray and Vohra (1999) consider situations where coalitions form and leave the game.

A different approach is to assume that only bilateral mergers occur, and the new created coalitions keep merging among themselves until a stable coalition structure is created. This is the approach followed by Gul (1989) and Macho-Stadler et al. (2002).

Following the latter approach, this paper studies situations in which a society is formed by the sequential incorporation of new members. In contrast with previous models, the collusion is not parallel. Instead, a size-increasing coalition arises swallowing other individuals, like a snowball. An individual agent may only join this coalition or remain single. International treaties such like the European Union or the NATO provide relevant examples of this coalition formation protocol. In the case of the European Union, since the custom convention between Belgium, Luxembourg and the Netherlands enters into force (1948), until the current negotiations with Eastern and Central Europe candidates, one refoundation (1951) and four additional enlargements (1973, 1981, 1986 and 1995) have taken place. Even thought more than a country officially joined the Union at the same time, the process of negotiation was individual for each candidate and independent from each other. Thus, in practice, it can be considered that the countries joined the union sequentially. Moreover, accession of new members may cause a change in the laws which rule the society. In the next scheduled enlargement (due to happen in 2004), the old voting system, used in a Union of 15, becomes obsolete and should be changed for a Union of 25. In any case, a change in the laws requires the unanimity of all members. Refusal from any current member may abort the enlargement. In 2001, people in Ireland voted in referendum against the Treaty of Niza, putting in jeopardy all the process of enlargement. In a second referendum (held in 2002), the Irish people voted in favour.

Another example of a society being formed by sequential entry of new members is given by situations in which a big company grows up by buying smaller companies, like Microsoft's policy in the second half of the nineties. Frequently, this enlargement is directed to new business fields and implies a change in the policy of the new enlarged company.

In this paper, we model this process in the set of NTU games by a simple mechanism of negotiation. The main idea of the mechanism is the creation and further enlargement of a union or society of players. The members of this society agree on a rule to share their resources. Players outside the society can apply to enter the society by agreeing on the established internal rule. However, in the admission negotiation, candidates may also propose to change the internal rule on entrance. Moreover, unanimity is required among every member of the society to change the rules.

We study the subgame perfect Nash equilibria (SPNE) payoffs of the above mechanism[1]. In a particular class of games, the Shapley NTU value (Shapley, 1969; see also Aumann, 1985) arises in equilibrium. This provides further support to this value.

Hart and Mas-Colell (1996) designed a non-cooperative mechanism such that the consistent value (Maschler and Owen, 1989 and 1992) arises in subgame perfect Nash equilibria. As far as we know, no similar result has been obtained for other extensions of the Shapley value (Shapley, 1953) and the Nash solution (Nash, 1950) to NTU games, such like the Harsanyi value (Harsanyi, 1963) or the Shapley NTU value.

The particular class of games we should restrict ourselves to are games (N, V) such that $V(N)$ is delimited by a hyperplane. This class includes the transfer utility (TU) games; but it also includes some proper NTU games, for example, prize games (Hart, 1994). Another example with pure exchange

[1] We use the term non-cooperative *mechanism* instead of non-cooperative *game* in order to avoid confusion with cooperative games.

economies is analyzed in Section 4.2 following a simple idea: when the grand coalition forms, the exchange economy is big enough for agents to have unlimited liability. However, when few agents are involved, they may have limited liability, so that a transfer of utility (money) may not be always possible.

In Section 2, we present the notation. In Section 3, we present the mechanism and the main results. In Section 4, we analyze several examples. In Section 5, we prove the results.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers. Similarly, $\mathbb{R}_{++}$ and $\mathbb{R}_{+}$ are the set of positive and nonnegative real numbers, respectively. Given any finite set S, we denote by $|S|$ the cardinality of S, and by $\mathbb{R}^S$ the set of all functions from S to $\mathbb{R}$. The sets $\mathbb{R}^S_{++}$ and $\mathbb{R}^S_{+}$ are defined accordingly. We also denote by 2^S the cardinal set of S, i.e. $2^S := \{T : T \subset S\}$. A member x of $\mathbb{R}^S$ is an $|S|$-dimensional vector whose coordinates are indexed by members of S; thus, when $i \in S$, we write x_i for $x(i)$. If $x \in \mathbb{R}^T$ (or $x \in \mathbb{R}^N$) and $T \subset S$ (or $T \subset N$), we write x_T for the restriction of x to T, i.e. the members of $\mathbb{R}^T$ whose ith coordinate is x_i. With some abuse of notation, given $x \in \mathbb{R}^S$ and $a \in \mathbb{R}$, we write $(x, a) \in \mathbb{R}^{S\cup\{i\}}$ for the member of $\mathbb{R}^{S\cup\{i\}}$ whose ith coordinate is a and whose restriction to S is x. Given $x, y \in \mathbb{R}^S$, we write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$.

Let $N = \{1, 2, ..., n\}$ be a finite set of *players*. Non-empty subsets of N are called *coalitions*. A *non-transferable utility (NTU) game* on N is a correspondence V that assigns to each coalition S a subset $V(S) \subset \mathbb{R}^S$ satisfying the following properties:

(A1) For each $S \subset N$, the set $V(S)$ is nonempty, closed, convex, *comprehensive* (i.e., if $x \in V(S)$ and $y \leq x$, then $y \in V(S)$), and *bounded above* (i.e., for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is compact).

(A2) *Normalization*: For each $i \in N$, the maximum of $\{x : x \in V(\{i\})\}$, which we denote by ω_i, is nonnegative.

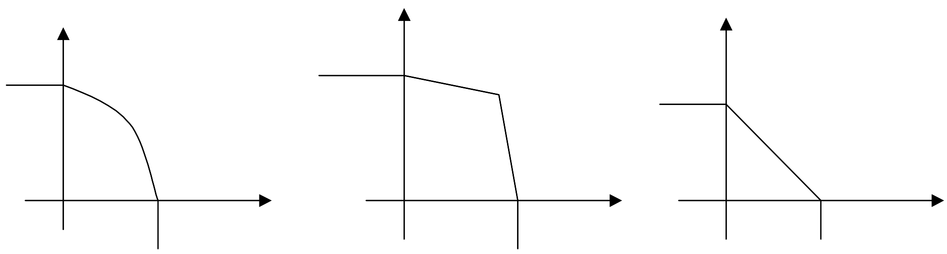

Figure 1: Valid examples.

(A3) *Zero-Monotonicity*: For each $S \subset N$, $x \in V(S)$ and $i \notin S$, we have $(x, \omega_i) \in V(S \cup \{i\})$. In particular, this implies that $(\omega_i)_{i \in S} \in \mathbb{R}^S$ belongs to $V(S)$.

(A4) The boundary of $V(N)$, which we denote by $\partial V(N)$, is nonlevel in the positive orthant (i.e., at any point of $\partial V(N) \cap \mathbb{R}_+^N$ there exists an outward vector with positive coordinates.)

(A5) For each $S \subset N$, if $x \in \partial V(S)$ with $x_i < 0$ for $i \in T$, then $\partial V(S)$ at x is parallel to the subspace $\mathbb{R}^T$.

Properties (A1), (A2), (A3), and (A4) are standard properties. The normalization given in (A2) does not affect our results. Property (A4) has been previously used by Hart and Mas-Colell (1996, in hypothesis (A2), page 359) and Serrano (1997, in assumption A4, page 61). The hypothesis in Hart and Mas-Colell (1996) is stronger, since it requires nonlevelness in every coalition $S \subset N$. Property (A5) is made so that all relevant action occurs in the positive orthant, and generalizes the property given in assumption A4 in Serrano (1997). See Figures 1 and 2 for some examples.

A *transfer utility (TU) game* on N is a function $v : 2^S \to \mathbb{R}$ that assigns to each coalition S a real number $v(S)$ and $v(\emptyset) = 0$. A TU game v on N may also be expressed as the following NTU game on N:

$$V'(S) = \left\{ x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S) \right\} \tag{1}$$

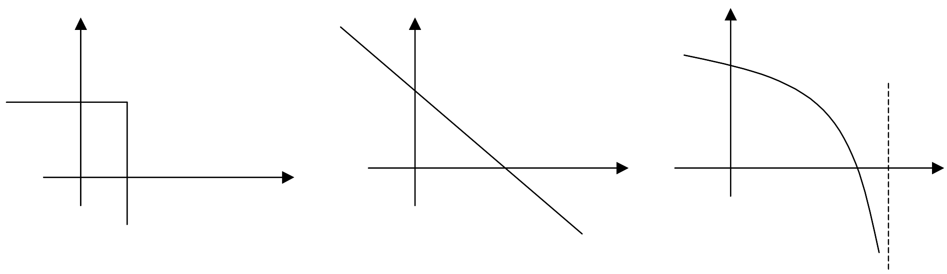

Figure 2: The first example does not satisfy (A4). The second and the third do not satisfy (A5).

for all $S \subset N$.

Let Π be the set of all orders of players in N. Given $\pi \in \Pi$ and $i \in N$, we define P_i^π as the set of players who come before i in the order π, namely

$$P_i^\pi := \{j \in N : \pi(j) < \pi(i)\}.$$

For notational convenience, we denote $P_{n+1}^\pi := N$.

Let v be a TU game on N and let $\pi \in \Pi$. Given $i \in N$, we define the *marginal contribution* of player i under the order π in the game v as

$$v(P_i^\pi \cup \{i\}) - v(P_i^\pi) \in \mathbb{R}.$$

The *Shapley value* (Shapley, 1953) of a TU game v on N is the vector $Sh(N, v) \in \mathbb{R}^N$ whose ith coordinate is given by

$$Sh_i(N, v) := \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \left[v(P_i^\pi \cup \{i\}) - v(P_i^\pi) \right] \in \mathbb{R}.$$

Let $\lambda \in \mathbb{R}_{++}^N$ and let $S \subset N$. We define

$$v^\lambda(S) := \sup \left\{ \sum_{i \in S} \lambda_i x_i : x \in V(S) \right\}.$$

Under our hypothesis, this supremum is a maximum:

Lemma 1 *For each $S \subset N$, there exists $x \in V(S)$ such that $\sum_{j \in S} \lambda_j x_j = v^\lambda(S)$.*

Proof. Let

$$A := \left\{ \sum_{i \in S} \lambda_i x_i : x \in V(S) \right\}$$

and

$$B := \left\{ \sum_{i \in S} \lambda_i x_i : x \in V(S) \cap \mathbb{R}^S_+ \right\}.$$

By (A1), B is bounded above and, by (A5), so is A. We will show that $\sup A = \sup B$. Clearly, $\sup A \geq \sup B$. Let $x \in V(S)$ and let $x^+ \in \mathbb{R}^S$ be defined by $x^+_i = \max\{0, x_i\}$ for all $i \in S$. By (A5), $x^+ \in V(S)$ and moreover $\sum_{i \in S} \lambda_i x_i \leq \sum_{i \in S} \lambda_i x^+_i$. Thus, $\sup A \leq \sup B$. Since B is compact, we conclude that there exists $\max B$. ■

A vector $x \in V(N)$ is a *Shapley NTU value* (Shapley, 1969) of V if there exists a vector $\lambda \in \mathbb{R}^N_{++}$ such that $\lambda_i x_i = Sh_i\left(N, v^\lambda\right)$ for all $i \in N$. Even though the Shapley NTU value may not be unique, Shapley (1969) points out that "it is sufficient [for uniqueness to hold] that the Pareto surface coincide with a hyperplane within the individually rational zone". We will refer to this property later (see Theorem 2 below) as $V(N)$ be delimited by a hyperplane. Of course, the vector $\lambda \in \mathbb{R}^N_{++}$ is outward to the hyperplane.

Players will negotiate to form a society, and their payoff will only depend on the identity of its members. Thus, we define a *rule* as a function γ which assigns to each coalition S a vector $\gamma(S) \in V(S)$. Formally, a rule is a "payoff configuration" (see Hart and Mas-Colell, 1996). However, a rule should not be interpreted as a payoff for every coalition, but as an index that indicates the payoff when a particular society is formed. We denote by Γ the set of all rules.

3 The non-cooperative mechanism

Players should form a society. First, an order of the players is randomly chosen. Assume the order is $\pi = (12...n)$. Player 1 should then present a rule $\gamma \in \Gamma$. No restrictions (apart from feasibility) are imposed on γ. Player 2 may either *agree* on γ and join the society, or *disagree* on γ and propose a new rule $\widetilde{\gamma}$ to player 1. If player 1 accepts (he *votes 'yes'*), the society $\{1,2\}$ forms with the new rule $\widetilde{\gamma}$, and turn passes to player 3. If player 2 rejects (he *votes 'no'*), he remains out of the society and turn passes to player 3.

In general, when the turn reaches player i, he faces a society $S \subset P_i^\pi$ with certain rule γ, and a set of players $E = P_i^\pi \backslash S$ who have chosen to stay out of the society. Players in S, E and $N \backslash P_i^\pi$ are called *active players*, *passive players* and *candidates*, respectively. Player i must then either agree to join the society (in that case, player i becomes an active player and turn passes to candidate $i+1$) or disagree and propose both a new rule $\widetilde{\gamma}$ and a new society $\widetilde{S} \subset P_i^\pi \cup \{i\}$ which includes himself and all the members of the old society (i.e. $S \cup \{i\} \subset \widetilde{S}$). The members of $\widetilde{S} \backslash \{i\}$ vote sequentially whether they accept or reject this proposal. If all of them vote 'yes', the new society $\widetilde{S}$ forms with the new rule (we say then that the proposal is accepted), and turn passes to candidate $i+1$. If at least one member of $\widetilde{S} \backslash \{i\}$ votes 'no', player i becomes a passive player and turn passes to candidate $i+1$.

Once there are no more candidates, we have a society $S \subset N$ of active players, a set $E = N \backslash S$ of passive players, and a rule γ for the society. Then, the final payoff for each player $i \in S$ is $\gamma_i(S)$ and every player $i \in E$ receives his individual payoff ω_i.

We now describe the mechanism M formally. We first describe the games $M(\pi, i, S, E, \gamma)$ and $\widetilde{M}(\pi, i, S, E, \gamma)$. $M(\pi, i, S, E, \gamma)$ is the subgame which begins when, given the order π, turn reaches player i and he faces a society of active players S with a proposed rule $\gamma \in \Gamma$, and a set of passive players E such that $S \cup E = P_i^\pi$ and $S \cap E = \emptyset$. $\widetilde{M}(\pi, i, S, E, \gamma)$ is the subgame which arises after player i disagrees in the subgame $M(\pi, i, S, E, \gamma)$.

> Let $\pi \in \Pi$ be an order of the players. We can assume without loss of generality that $\pi = (12...n)$. Given $i \in N \cup \{n+1\}$, $\gamma \in \Gamma$ and $S, E \subset P_i^\pi$ such that $S \cup E = P_i^\pi$ and $S \cap E = \emptyset$, we inductively define the mechanisms $M(\pi, i, S, E, \gamma)$ and $\widetilde{M}(\pi, i, S, E, \gamma)$ as follows.
>
> In both $M(\pi, n+1, S, E, \gamma)$ and $\widetilde{M}(\pi, n+1, S, E, \gamma)$, every player

$i \in S$ receives $\gamma_i(S)$ and every player $i \in E$ receives ω_i.

Assume both $M(\pi, j, S', E', \gamma')$ and $\widetilde{M}(\pi, j, S', E', \gamma')$ are defined for all $j > i$, $\gamma' \in \Gamma$ and S', E' such that $S' \cup E' = P_j^\pi$ and $S' \cap E' = \emptyset$.

In $\widetilde{M}(\pi, i, S, E, \gamma)$, player i proposes a rule $\widetilde{\gamma} \in \Gamma$ and sets $\widetilde{S} \supset S$ and $\widetilde{E} \subset E$ such that $i \in \widetilde{S}$, $\widetilde{S} \cup \widetilde{E} = P_i^\pi \cup \{i\}$ and $\widetilde{S} \cap \widetilde{E} = \emptyset$. If all the members of $\widetilde{S} \setminus \{i\}$ vote 'yes' (they are asked in some prespecified order) then the mechanism $M\left(\pi, i+1, \widetilde{S}, \widetilde{E}, \widetilde{\gamma}\right)$ is played. If at least one member of $\widetilde{S} \setminus \{i\}$ votes 'no', the mechanism $M(\pi, i+1, S, E \cup \{i\}, \gamma)$ is played.

In $M(\pi, i, S, E, \gamma)$, player i can either agree or disagree on (S, E, γ). If he disagrees, then $\widetilde{M}(\pi, i, S, E, \gamma)$ is played. If he agrees, then $M(\pi, i+1, S \cup \{i\}, E, \gamma)$ is played.

The mechanism M consists of choosing randomly an order $\pi' \in \Pi$, being each order equally likely to be chosen, and playing the game $M(\pi') := \widetilde{M}(\pi', i', \emptyset, \emptyset, \gamma^0)$, where $\pi'(i') = 1$.

Clearly, for any set of pure (mixed) strategies, this mechanism terminates in finite time. Thus, the (expected) payoffs at termination are well-defined. We will also assume that, if a player is indifferent to agreeing or rejecting an offer, he strictly prefers to agree. This assumption is made in order to avoid problems of coordination among players. In Section 4.3 we show that this tie-breaking rule is needed in our model. Note that we do not need to make any assumption when players sequentially vote 'yes' or 'no' to a proposal in the subgames $\widetilde{M}$.

Theorem 2 *If $V(N)$ is delimited by a hyperplane, then there exists a unique expected subgame perfect Nash equilibrium (SPNE) payoff in the negotiation mechanism M, and it is the Shapley NTU value. Furthermore, the strategy of a player in SPNE in the negotiation mechanism $M(\pi)$ for any π are robust to deviations by coalitions of his predecessors in π.*

The proof of Theorem 2 is located in the Appendix. The main idea of the proof is that, given any order π, each player has a strategy that ensures him his marginal contribution in the order π. Thus, in expected terms, the final payoff is the Shapley NTU value.

Theorem 3 *The negotiation mechanism M implements the Shapley value in zero-monotonic TU games.*

The proof of Theorem 3 is located in the Appendix.

4 Some examples

4.1 A classical example

In this section we apply the above procedure to an exchange economy which appeared in a series of papers in Econometrica in the 80's discussing the applicability of the Shapley NTU value. The reader is referred to Roth (1980), Shafer (1980), Harsanyi (1980), Aumann (1985b), Roth (1986), and Aumann (1986). This controversy has been recently revisited in Montero and Okada (2003).

Example 4 *(Shafer, 1980) Consider a pure exchange economy with three players $\{1,2,3\}$ and two commodities $\{x,y\}$. Initial endowments are given by:*

$$\begin{aligned} z_1^0 &= (1-\varepsilon, 0) \\ z_2^0 &= (0, 1-\varepsilon) \\ z_3^0 &= (\varepsilon, \varepsilon) \end{aligned}$$

and utility functions are given by

$$\begin{aligned} u_1(x,y) &= \min\{x,y\} \\ u_2(x,y) &= \min\{x,y\} \\ u_3(x,y) &= \frac{x+y}{2}. \end{aligned}$$

Commodities x and y may be considered as 'left gloves' and 'right gloves', respectively. Players 1 and 2 only get utility from pairs of gloves. However, player 3 only uses the leather of the gloves. Let i denote an element in $\{1,2\}$. Then, the non-transferable utility (NTU) game is given by

$$\begin{aligned}
V(\{i\}) &= \left\{t \in \mathbb{R}^{\{i\}} : t \leq 0\right\} \\
V(\{3\}) &= \left\{t \in \mathbb{R}^{\{3\}} : t \leq \varepsilon\right\} \\
V(\{1,2\}) &= \left\{(t_1,t_2) \in \mathbb{R}^{\{1,2\}} : t_1+t_2 \leq 1-\varepsilon, t_1 \leq 1-\varepsilon, t_2 \leq 1-\varepsilon\right\} \\
V(\{i,3\}) &= \left\{(t_i,t_3) \in \mathbb{R}^{\{i,3\}} : t_i+t_3 \leq \frac{1+\varepsilon}{2}, t_i \leq \varepsilon, t_3 \leq \frac{1+\varepsilon}{2}\right\} \\
V(\{1,2,3\}) &= \left\{(t_1,t_2,t_3) \in \mathbb{R}^{\{1,2,3\}} : t_1+t_2+t_3 \leq 1, t_1 \leq 1, t_2 \leq 1, t_3 \leq 1\right\}.
\end{aligned}$$

The Shapley NTU value proposes a payoff of $\left(\frac{5(1-\varepsilon)}{12}, \frac{5(1-\varepsilon)}{12}, \frac{5\varepsilon+1}{6}\right)$. For a discussion of this result, the reader is referred to Shafer (1980), Roth (1980) and Aumann (1980).

The TU game associated to this example is given by $\lambda = (1,1,1)$ and $v^\lambda(\{i\}) = 0$, $v^\lambda(\{3\}) = \varepsilon$, $v^\lambda(\{1,2\}) = 1-\varepsilon$, $v^\lambda(\{i,3\}) = (1+\varepsilon)/2$, and $v(N) = 1$ (see Figure 1.) In the order $\pi = (312)$, a possible equilibrium in the bargaining mechanism would proceed as follows: players 3 and 1 propose a rule γ that satisfies $\gamma(N) = ((1-\varepsilon)/2, (1-\varepsilon)/2, \varepsilon)$ – i.e. the vector of marginal contributions in the order π – and $\gamma(\{1,3\}) = ((1-\varepsilon)/2, -, \varepsilon)$. Player 2 cannot hope to suggest a more profitable outcome to himself. In fact, players 1 and 3 are indifferent to player 2 joining them or not. Player 2 accepts the offer and the final payoff is $\gamma(N)$.

Assume now we are in the NTU game of the example. Players 3 and 1 cannot propose a rule satisfying $\gamma(\{1,3\}) = ((1-\varepsilon)/2, -, \varepsilon)$, because this payoff is unfeasible for them. It is as if they wanted to make a non-credible threat to player 2 in case he does not join them.

However, they can still propose $\gamma(N) = ((1-\varepsilon)/2, (1-\varepsilon)/2, \varepsilon)$ and $\gamma(\{1,3\}) = (0, -, (1+\varepsilon)/2)$. This means that

- they propose a society N in which player 1 receives commodities (x_1, y_1)

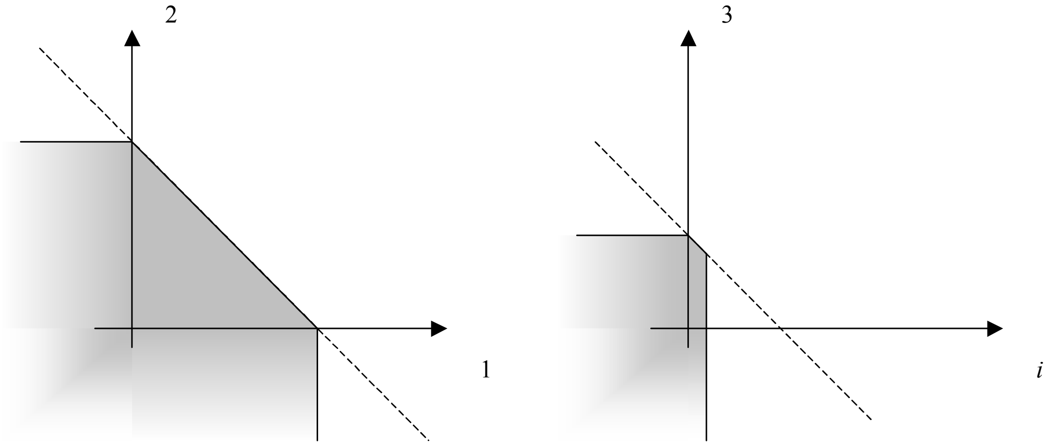

Figure 3: Feasible outcomes for $\{1,2\}$ and $\{i,3\}$.

with $x_1 = y_1 = (1-\varepsilon)/2$, player 2 receives commodities (x_2, y_2) with $x_2 = y_2 = (1-\varepsilon)/2$, and player 3 keeps his initial endowment; and

- in case player 2 does not join them, they threat to form a society in $\{1,3\}$ in which player 3 receives all their commodities (i.e. (x_3, y_3) with $x_3 = 1$ and $y_3 = \varepsilon$), and player 1 receives nothing.

In this case, the threat is credible, because this allocation is feasible for $\{1,3\}$. Again, player 2 cannot suggest a more profitable outcome to himself. Any feasible proposal giving him more than $(1-\varepsilon)/2$ would result in it being rejected by player 1 or 2. This means that, in equilibrium, player 2 would directly agree to join the society. Note that, in this case, players 1 and 3 are not indifferent to player 2 joining them or not.

4.2 An example with farmers and Factories

Consider a pure exchange economy where big *Factories* acquire products from *farmers*, who have limited liability. Suppose that a planner (e.g. the government) would like to favor the productivity of the farmers, avoiding the factories to take advantage of farmers' lack of liability. Our analysis shows that this handicap can be avoided by forcing the proposed mechanism. The Shapley NTU value, as opposed to other values, such like the Harsanyi value and the consistent value, provide all agents (both farmers and Factories) with the Shapley value of the game which arises from the economy when a common utility is freely transferable.

The next example is an adaptation of the game presented by Owen (1972). It has also been used by Hart and Kurz (1983) and Hart and Mas-Colell (1996):

Example 5 *Consider a pure exchange economy with three players* $\{1, 2, 3\}$ *and three commodities* $\{x, y_1, y_2\}$. *Initial endowments are given by:*

$$\begin{aligned} z_1^0 &= (0, 1, 0) \\ z_2^0 &= (0, 0, 1) \\ z_3^0 &= (1, 0, 0) \end{aligned}$$

and utility functions are given by

$$\begin{aligned} u_1(x, y_1, y_2) &= x + \min\{y_1, y_2\} \\ u_2(x, y_1, y_2) &= x + \frac{1}{4}\min\{y_1, y_2\} \\ u_3(x, y_1, y_2) &= x + \min\{y_1, y_2\} - 1. \end{aligned}$$

Thus, commodity x *(money) is additive and linear in every player's utility function. Commodities* y_1 *and* y_2 *may be considered as 'left gloves' and 'right gloves', respectively. Players only get utility from pairs of gloves. However, player* 2 *does not have as much production (or selling) ability as the rest of the players. If players had unlimited liability, players* 1 *and* 2 *could agree on the consumptions* $z_1 = \left(-\frac{1}{2}, 1, 1\right)$ *and* $z_2 = \left(\frac{1}{2}, 0, 0\right)$, *so that the final payoff would be* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

However, if we consider only nonnegative commodities, the above consumptions are not feasible. We are, thus, in the context of the NTU game *given by*

$$\begin{aligned}
V(\{i\}) &= \left\{t \in \mathbb{R}^{\{i\}} : t \leq 0\right\} \textit{ for all } i \in \{1,2,3\} \\
V(\{1,2\}) &= \left\{(t_1,t_2) \in \mathbb{R}^{\{1,2\}} : t_1 + 4t_2 \leq 1, t_1 \leq 1, t_2 \leq \frac{1}{4}\right\} \\
V(\{i,3\}) &= \left\{(t_i,t_3) \in \mathbb{R}^{\{i,3\}} : t_i + t_3 \leq 0, t_i \leq 1, t_3 \leq 0\right\} \textit{ for all } i \in \{1,2\} \\
V(\{1,2,3\}) &= \left\{(t_1,t_2,t_3) \in \mathbb{R}^{\{1,2,3\}} : t_1 + t_2 + t_3 \leq 1, t_1 \leq 2, t_2 \leq \frac{5}{4}, t_3 \leq 1\right\}.
\end{aligned}$$

Thus, player 3 *(the banker) is needed as a catalyst. Players* 1 *and* 2 *may then agree to share part of their resources (pair of gloves) with player* 3 *in exchange of his services.*

In particular, the Harsanyi value proposes a payoff of $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$*. For example, players* 1 *and* 2 *sell their gloves to player* 3 *at a exchange rate of 5 pairs for 4 currency units.*

The consistent value, however, proposes a payoff of $\left(\frac{1}{2}, \frac{3}{8}, \frac{1}{8}\right)$*, i.e. since player* 2 *has the low production ability, he is the one who has to pay to player* 3*.*

Finally, the Shapley NTU value proposes a payoff of $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$*. For example, players* 1 *and* 2 *sell their gloves to player* 3 *at a exchange rate of* 1 *pair for* 1 *currency unit. Notice that this payoff is the same players would have agreed upon player* 1 *should initially have enough money.*

It may be argued that, since player 3 *is not a dummy in the game* V *(the final payoff of* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ *is not attainable without him), he must receive more than* 0*. However, player* 3 *does not contribute with any additional production capability. He just provides the other players with money so that trade may freely happen. We may want to incentive the production of goods and not the lending of money. Thus, player* 3 *should not get profit from the simple fact to have money when others do not have it. In this context, the Shapley NTU value seems a much fairer allocation.*

In our pure exchange economy, two conditions must hold:

1. The farmers have limited liability. The Factories have unlimited liability.

2. Production in Factories at least as efficient as in farms. Thus, it is optimal (in the sense of maximizing aggregate utility) for the Factories to hold *all* the non-monetary commodities.

The first condition implies that the farmers may be in an inferior position with respect to the Factories. If every player had unlimited liability, we would be in a transfer utility (TU) context, and our mechanism would implement the Shapley value. In Example 4, players 1 and 2 play the role of farmers, and player 3 is the Factory.

The second condition implies that the farms produce not for domestic consumption, but for selling to the Factories. Efficiency may be achieved by assigning all the commodities, but money, to the Factories. In example 4, the consumptions $z_1 = \left(\frac{1}{2}, 0, 0\right), z_2 = \left(\frac{1}{2}, 0, 0\right), z_3 = (0, 1, 1)$, which held the Shapley NTU value, maximize the aggregate utility and give player 3 (the Factory) all the gloves.

We present the model more formally. Let $\{N_f, N_F\}$ be a partition of N. We say that the members of N_f are *farmers* and the members of N_F are *Factories*. We consider l commodities.

Properties 1 and 2 are formally stated as follows:

1. A *consumption* z_i for player $i \in N_f$ (resp. N_F) is a pair (x_i, y_i) such that $x_i \in \mathbb{R}_+$ (resp. $\mathbb{R}$) and $y_i \in \mathbb{R}_+^{l-1}$. Player $i \in N$ is characterized by an *initial endowment* $z_i^0 = (x_i^0, y_i^0) \in \mathbb{R}_+^l$ and a utility function $u_i : \mathbb{R}_+^l \to \mathbb{R}$ (if $i \in N_F$, $u_i : \mathbb{R} \times \mathbb{R}_+^{l-1} \to \mathbb{R}$) such that $u_i(z_i) = x_i + \hat{u}_i(y_i)$ for some continuous, nondecreasing function $\hat{u}_i : \mathbb{R}_+^{l-1} \to \mathbb{R}$ satisfying $\hat{u}_i(y_i^0) \geq 0$ (this is a normalization condition made to fit (A2) but it has no more consequences).

Notice that the additivity separability and linearity in x_i of u_i permits utility transfers among players. However, the nonnegativeness of x_i when $i \in N_f$ restricts these transfers when farmers are involved (they have limited liability).

Given a coalition $S \subset N$, $V(S)$ is given by

$$V(S) = \left\{t \in \mathbb{R}^S : t_i \leq u_i(z_i) \, \forall i \in S \text{ for some } z \in \Omega^S\right\}$$

where $\Omega^S :=$

$$\left\{(z_i)_{i\in S} : z_i \in \mathbb{R}^l_+ \forall i \in N_f \cap S, z_i \in \mathbb{R} \times \mathbb{R}^{l-1}_+ \forall i \in N_F \cap S, \sum_{i\in S} z_i \leq \sum_{i\in S} z_i^0\right\}.$$

2. There exists a $y^M \in \mathbb{R}^{l-1}_+$ with $y_i^M = 0$ for all $i \in N_f$ such that

$$\sum_{i\in N} \hat{u}_i\left(y_i^M\right) = \max\left\{\sum_{i\in N} \hat{u}_i(y_i) : (x,y) \in \Omega^N \text{ for some } x \in \mathbb{R}^N\right\}.$$

Under these conditions, we have the following result:

Theorem 6 *The negotiation mechanism M implements the NTU Shapley value in pure exchange economies which satisfy conditions 1 and 2.*

The proof of Theorem 6 is similar to those of Theorem 2 and we omit it.

4.3 The tie-breaking rule

Assume the tie-breaking rule does not hold. Then, the Shapley NTU value is still an equilibrium outcome. However, another equilibrium payoffs may arise, as the next example shows.

Example 7 *Let $N = \{1,2,3,4\}$ and let v be the defined by $v(S) = 1$ if $\{1,2,3\} \subset S$ and $v(S) = 0$ otherwise, i.e. the game v is the unanimity game with carrier $\{1,2,3\}$ and 4 as null player. Let $\pi = (1234)$. In this example, the vector of marginal contributions in the order π is $d^\pi = (0,0,1,0)$. We consider the following strategies for players in the order π: Players $\{1,2\}$ propose a rule γ satisfying $\gamma(\{1,2\}) = (1,-1)$, $\gamma(\{1,2,4\}) = (-1,1,0)$ and $\gamma(N) = (0,1,0,0)$. Player 4, when facing a society $S = \{1,2\}$ and a set of passive players $E = \{3\}$ with a rule γ such that $\gamma_4(\{1,2,4\}) = 0$, would use the following tie-breaking rule: If player 3 was excluded after proposing*

$\widetilde{\gamma}$ with $\widetilde{\gamma}_1(N) \geq \widetilde{\gamma}_2(N)$, then player 4 agrees to join the society[2]*. If player 3 was excluded after proposing $\widetilde{\gamma}$ with $\widetilde{\gamma}_1(N) < \widetilde{\gamma}_2(N)$, then player 4 disagrees and proposes an unacceptable offer, for example $(0,0,0,1)$.*

These strategies can be supported as part of an equilibrium. Player 3 agrees because he cannot hope to propose a positive payoff to himself. If he disagrees and proposes $\widetilde{\gamma}$ with $\widetilde{\gamma}_1(N) \geq \widetilde{\gamma}_2(N)$, then player 2 would get 1 by voting 'no'. This means that $\widetilde{\gamma}$ is not accepted unless $\widetilde{\gamma}_1(N) \geq \widetilde{\gamma}_2(N) \geq 1$, which leaves player 3 with a negative payoff. If player 3 disagrees and proposes $\widetilde{\gamma}$ with $\widetilde{\gamma}_1(N) < \widetilde{\gamma}_2(N)$, then player 1 would get 1 by voting 'no'. Again, this means that $\widetilde{\gamma}$ is not accepted unless $\widetilde{\gamma}_2(N) > \widetilde{\gamma}_1(N) \geq 1$.

Vidal-Puga and Bergantiños (2003) model this tie-breaking rule by "punishing" with a small penalty $\varepsilon > 0$ the players involved in an exclusion. In our model, this means that any excluded player i would get an utility of almost (but strictly less than) ω_i. The mechanism would also work if we restrict ourselves to strict zero-monotonic games, i.e. for each $S \subset N$, $x \in V(S)$ and $i \notin S$, the payoff (x, ω_i) belongs to the interior of $V(S \cup \{i\})$.

5 Appendix

5.1 Proof of Theorem 2

The proof is structured as follows: First, we introduce some additional notation. Second, we construct a SPNE that yields the Shapley NTU value. Third, we prove that any SPNE yields the Shapley NTU value as expected outcome.

[2] For simplicity, we assume that player 3 makes an acceptable offer to player 4 (i.e. $\widetilde{\gamma}_4(N) = 0$). A more precise description of player 4's strategy is given in Section 5.3.

5.1.1 Additional notation

Let $(\lambda_i)_{i\in N} \in \mathbb{R}^N_{++}$ and $k \in \mathbb{R}_+$ be such that

$$V(N) = \left\{ x \in \mathbb{R}^N_+ : \sum_{i\in N} \lambda_i x_i = k \right\} - \mathbb{R}^N_+.$$

Clearly, $v^\lambda(N) = k$.

In order to prove Theorem 2, we need some additional notation. Given $x \in \mathbb{R}^S$, we define $x^+ \in \mathbb{R}^S_+$ as the vector whose coordinates are given by $x_i^+ := \max\{0, x_i\}$ for all $i \in S$. By (A5), $x \in V(S)$ implies $x^+ \in V(S)$ for all $S \subset N$.

Let $\pi \in \Pi$. From now on, we assume without loss of generality that $\pi = (12...n)$. In particular, this implies that $P^\pi_{i+1} = P^\pi_i \cup \{i\}$ for all $i \in N$.

Let $\lambda_i d_i^\pi$ be the marginal contribution of player i to the game v^λ in the order π, namely

$$d_i^\pi := \frac{1}{\lambda_i} \left[v^\lambda\left(P^\pi_{i+1}\right) - v^\lambda\left(P^\pi_i\right) \right].$$

Given $x \in \mathbb{R}^{P^\pi_i}$, we define

$$f_i^\pi(x) := \max\left\{ y_i : \left(x, y_i, d^\pi_{N\setminus P^\pi_{i+1}}\right) \in V(N) \right\}$$

when this maximum exists. In particular, if $x \in V(P^\pi_i)$, this value is well-defined and nonnegative.

Note that $f_i^\pi(x)$ represents the maximum payoff that player i can obtain when the players before him get x and the players after him get d^π.

Lemma 8 *Let $x \in \mathbb{R}^{P^\pi_i}$ be such that $f_i^\pi(x)$ is well-defined. Then,*

$$f_i^\pi(x) = \frac{1}{\lambda_i} \left[v^\lambda\left(P^\pi_{i+1}\right) - \sum_{j<i} \lambda_j x_j^+ \right]. \tag{2}$$

Proof. Clearly, $\left(x, f_i^\pi(x), d^\pi_{N\setminus P^\pi_{i+1}}\right) \in \partial V(N)$. By (A5), $f_i^\pi(x) \geq 0$ and $\left(x^+, f_i^\pi(x), d^\pi_{N\setminus P^\pi_{i+1}}\right) \in \partial V(N)$. Thus,

$$\sum_{j<i} \lambda_j x_j^+ + \lambda_i f_i^\pi(x) + \sum_{j>i} \lambda_j d_j^\pi = v^\lambda(N).$$

Since $\sum_{j>i} \lambda_j d_j^\pi = v^\lambda(N) - v^\lambda\left(P_{i+1}^\pi\right)$, (2) holds. ■

Given $S \neq N$, we define

$$\kappa(S) := \min\left\{j \in N : P_j^\pi \subset S\right\}.$$

Thus, players in $P_{\kappa(S)}^\pi$ are the first players out of S who come together in the order π. This minimum always exists, because $P_1^\pi = \emptyset$. We also define

$$\Gamma^\pi := \left\{\gamma \in \Gamma : \gamma(S) = \left(\gamma^+\left(P_{\kappa(S)}^\pi\right), \omega_{S \setminus P_{\kappa(S)}^\pi}\right) \text{ for all } S \subsetneq N\right\}$$

where $\gamma^+(S)$ is such that $\gamma_i^+(S) = \max\left\{0, \gamma_i^+(S)\right\}$ for all $i \in S$.

Thus, Γ^π is the set of (positive) rules which do not share the resources of the players after the first 'gap' in the coalition (with respect to π). Note that, given $\gamma \in \Gamma^\pi$, we can change $\gamma(N)$ and the resulting rule will still be in Γ^π.

We also define

$$K^\pi := \left\{\gamma \in \Gamma^\pi : \gamma(P_i^\pi) \in \arg\min_{x \in V\left(P_i^\pi\right)} \{f_i^\pi(x)\} \text{ for all } i \in N\right\}.$$

This K^π is the set of rules out of Γ^π which give each coalition P_i^π the payoff x that minimizes $f_i^\pi(x)$. The next lemma provides an alternative definition for K^π.

Lemma 9 $K^\pi = \left\{\gamma \in \Gamma^\pi : \sum_{j<i} \lambda_j \gamma_j(P_i^\pi) = v^\lambda(P_i^\pi) \textit{ for all } i \in N\right\}.$

Proof. Note that

$$\begin{aligned} \min_{x \in V\left(P_i^\pi\right)} \{f_i^\pi(x)\} &= \min_{x \in V\left(P_i^\pi\right)} \left\{\frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j<i} \lambda_j x_j^+\right]\right\} \\ &= \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \max_{x \in V\left(P_i^\pi\right)} \sum_{j<i} \lambda_j x_j^+\right] = d_i^\pi. \end{aligned}$$

Thus, $\gamma(P_i^\pi) \in \arg\min_{x \in V\left(P_i^\pi\right)} \{f_i^\pi(x)\}$ iff $f_i^\pi(\gamma(P_i^\pi)) = d_i^\pi$, i.e.

$$\frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j<i} \lambda_j \gamma_j(P_i^\pi)\right] = \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - v^\lambda(P_i^\pi)\right].$$

■

We denote by Θ_i^π the set of feasible (S, E, γ)'s in the subgame $M(i, S, E, \gamma)$, namely

$$\Theta_i^\pi := \{(S, E, \gamma) : S \cup E = P_i^\pi, S \cap E = \emptyset \text{ and } \gamma \in \Gamma\}.$$

5.1.2 Existence of equilibria

Given any $(S, E, \gamma) \in \Theta_{n+1}^\pi$, we define

$$b(n+1, S, E, \gamma) := (\gamma(S), \omega_E) \in V(N).$$

Thus, $b(n+1, S, E, \gamma)$ is the final payoff in the subgame $M(\pi, n+1, S, E, \gamma)$.

Consider the following strategies in the subgames $M(\pi, n, S, E, \gamma)$ and $\widetilde{M}(\pi, n, S, E, \gamma)$:

In the subgame $M(\pi, n, S, E, \gamma)$, player n agrees on (S, E, γ) if and only if

$$\gamma_n(S \cup \{n\}) \geq f_n^\pi(\gamma(S), \omega_E),$$

which can be restated as

$$b_n(n+1, S \cup \{n\}, E, \gamma) \geq f_n^\pi\left(b_{P_n^\pi}(n+1, S, E \cup \{n\}, \gamma)\right). \tag{3}$$

In the subgame $\widetilde{M}(\pi, n, S, E, \gamma)$, player n proposes $\left(\widetilde{S}, \widetilde{E}, \widetilde{\gamma}\right)$ such that $\widetilde{S} = N$, $\widetilde{E} = \emptyset$ and

$$\widetilde{\gamma}(N) = (t, f_n^\pi(t)) \tag{4}$$

with $t := (\gamma(S), \omega_E)$ and $\widetilde{\gamma}(T)$ for all $T \neq N$ given by

$$\widetilde{\gamma}(T) = \left(y, \omega_{T \setminus P_{\kappa(T)}^\pi}\right) \tag{5}$$

with $y \in V\left(P_{\kappa(T)}^\pi\right)$ such that

$$\sum_{j<\kappa(T)} \lambda_j y_j = v^\lambda\left(P_{\kappa(T)}^\pi\right). \tag{6}$$

Clearly, $\widetilde{\gamma} \in K^\pi$. In the subgame $\widetilde{M}(\pi, n, S, E, \gamma)$, assume player n proposes $\left(\tilde{S}, \tilde{E}, \tilde{\gamma}\right)$ and $j \in \tilde{S} \backslash \{n\}$. Then, player j votes 'yes' if and only if

$$b_j\left(n+1, \tilde{S}, \tilde{E}, \tilde{\gamma}\right) \geq b_j\left(n+1, S, E \cup \{n\}, \gamma\right). \tag{7}$$

Thus, we have defined the strategies of the players in $M(\pi, n, S, E, \gamma)$ for any $(S, E, \gamma) \in \Theta_n^\pi$. Let $b(n, S, E, \gamma) \in V(N)$ be the final payoff in the subgame $M(\pi, n, S, E, \gamma)$ when players follow these strategies. This payoff is well-defined. Moreover, next claims apply. Claim I will imply that every player agrees on the offers. Claim II specifies the final payoff when there is a disagreement. Claim III and Claim VI are technical ones. Claim IV says that every player receives at least d^π. Claim V says that passive players receive nothing.

Claim I(n): Assume we are in $M(\pi, n, P_n^\pi, \emptyset, \gamma)$ such that $\gamma_n(N) = d_n^\pi$ and $\gamma \in K^\pi$. Then, under these strategies, player n agrees.

Proof. We need to prove that (3) is satisfied.

$$b_n\left(n+1, P_n^\pi \cup \{n\}, \emptyset, \gamma\right) = \gamma_n(N) = d_n^\pi$$

and

$$\begin{aligned} f_n^\pi\left(b_{P_n^\pi}\left(n+1, P_n^\pi, \{n\}, \gamma\right)\right) &= f_n^\pi\left(\gamma\left(P_n^\pi\right)\right) \\ &= \frac{1}{\lambda_n}\left[v^\lambda(N) - \sum_{j<n} \lambda_j \gamma_j^+\left(P_n^\pi\right)\right] \\ &= \frac{1}{\lambda_n}\left[v^\lambda(N) - v^\lambda\left(P_n^\pi\right)\right] = d_n^\pi. \end{aligned}$$

■

Claim II(n): Assume we are in $\widetilde{M}(\pi, n, S, E, \gamma)$. Then, under these strategies, the final payoff is given by $(t, f_n^\pi(t))$ with $t = b_{P_n^\pi}\left(n+1, S, E \cup \{n\}, \gamma\right)$.

Proof. Notice first that $f_n^\pi(t)$ is well-defined because

$$b_{P_n^\pi}\left(n+1, S, E \cup \{n\}, \gamma\right) = \left(\gamma(S), \omega_E\right) \in V\left(P_n^\pi\right).$$

Let $\left(\widetilde{S}, \widetilde{E}, \widetilde{\gamma}\right)$ be player n's proposal. This means that $\widetilde{S} = N$, $\widetilde{E} = \emptyset$ and $\widetilde{\gamma}$ is given as in (4), (5) and (6).

Given $j \in \widetilde{S} \setminus \{n\}$, we need to prove that player j votes 'yes', i.e. (7) holds:

$$b_j\left(n+1, \widetilde{S}, \widetilde{E}, \widetilde{\gamma}\right) = b_j\left(n+1, N, \emptyset, \widetilde{\gamma}\right) = \widetilde{\gamma}_j\left(N\right) = b_j\left(n+1, S, E \cup \{n\}, \gamma\right).$$

■

Claim III(n): Assume we are in $M(\pi, n, S, E, \gamma)$. Then, there exists a $T \supset S$, $E \cap T = \emptyset$ such that $b_S(n, S, E, \gamma) = \gamma_S(T)$.

Proof. If player n agrees, then

$$b_S(n, S, E, \gamma) = b_S(n+1, S \cup \{n\}, E, \gamma) = \gamma_S(S \cup \{n\})$$

and thus $T = S \cup \{n\}$.

If player n disagrees, by Claim II(n),

$$b_S(n, S, E, \gamma) = b_S(n+1, S, E \cup \{n\}, \gamma) = \gamma(S)$$

and thus $T = S$. ■

Claim IV(n): $b_n(n, S, E, \gamma) \geq d_n^\pi$ for all $(S, E, \gamma) \in \Theta_n^\pi$.

Proof. Assume we are in the subgame $M(\pi, n, S, E, \gamma)$. If player n disagrees, then by Claim II(n),

$$\begin{aligned} b_n(n, S, E, \gamma) &= f_n^\pi\left(b_{P_n^\pi}(n+1, S, E \cup \{n\}, \gamma)\right) \\ &= f_n^\pi(\gamma(S), \omega_E) \\ &= \frac{1}{\lambda_n}\left[v^\lambda(N) - \sum_{j \in S} \lambda_j \gamma_j^+(S) - \sum_{j \in E} \lambda_j \omega_j\right] \\ &\geq \frac{1}{\lambda_n}\left[v^\lambda(N) - v^\lambda(P_n^\pi)\right] = d_n^\pi. \end{aligned}$$

If player n agrees, then by (3)

$$\begin{aligned} b_n(n, S, E, \gamma) &= b_n(n+1, S \cup \{n\}, E, \gamma) \\ &\geq f_n^\pi\left(b_{P_n^\pi}(n+1, S, E \cup \{n\}, \gamma)\right) \end{aligned}$$

and we proceed as before. ■

Claim V(n): $b_E(n, S, E, \gamma) = \omega_E$ for each $(S, E, \gamma) \in \Theta_n^\pi$.

Proof. Assume we are in the subgame $M(\pi, n, S, E, \gamma)$. If player n agrees, $b(n, S, E, \gamma) = (\gamma(S \cup \{n\}), \omega_E)$ and so the result holds. If player n disagrees, by Claim II(n), $b_E(n, S, E, \gamma) = \omega_E$. ■

Claim VI(n): $\sum_{j \in S} \lambda_j b_j^+(n, S, E, \gamma) \leq v^\lambda(S)$ for all $(S, E, \gamma) \in \Theta_n^\pi$.

Proof. Assume we are in $M(\pi, n, S, E, \gamma)$. If player n disagrees, by Claim II(n),

$$b_S(n, S, E, \gamma) = b_S(n+1, S, E \cup \{n\}, \gamma) = \gamma(S).$$

By (A5), $\gamma^+(S) \in V(S)$ and thus

$$\sum_{j \in S} \lambda_j b_j^+(n, S, E, \gamma) = \sum_{j \in S} \lambda_j \gamma_j^+(S) \leq v^\lambda(S).$$

If player n agrees,

$$b(n, S, E, \gamma) = b(n+1, S \cup \{n\}, E, \gamma) = (\gamma(S \cup \{n\}), \omega_E)$$

and (3) holds, i.e.

$$\begin{aligned}
\gamma_n(S \cup \{n\}) &\geq f_n^\pi(\gamma(S), \omega_E) \\
&= \frac{1}{\lambda_n}\left[v^\lambda(N) - \sum_{j \in S} \lambda_j \gamma_j^+(S) - \sum_{j \in E} \lambda_j \omega_j\right] \\
&\geq \frac{1}{\lambda_n}\left[v^\lambda(N) - v^\lambda(S) - \sum_{j \in E} \lambda_j \omega_j\right].
\end{aligned}$$

Then, by Claim V(n), $\sum_{j \in S} \lambda_j b_j^+(n, S, E, \gamma) =$

$$\begin{aligned}
&\sum_{j \in N} \lambda_j b_j^+(n, S, E, \gamma) - \sum_{j \in E} \lambda_j b_j^+(n, S, E, \gamma) - \lambda_n b_n^+(n, S, E, \gamma) \\
= \; &\sum_{j \in N} \lambda_j b_j^+(n, S, E, \gamma) - \sum_{j \in E} \lambda_j \omega_j - \lambda_n \gamma_n^+(S \cup \{n\}) \\
\leq \; &v^\lambda(N) - \sum_{j \in E} \lambda_j \omega_j - \left[v^\lambda(N) - v^\lambda(S) - \sum_{j \in E} \lambda_j \omega_j\right] = v^\lambda(S).
\end{aligned}$$

■

Assume that for each $j > i$ and each $(S, E, \gamma) \in \Theta_j^\pi$, we have defined the strategy profiles in $M(\pi, j, S, E, \gamma)$ and $\widetilde{M}(\pi, j, S, E, \gamma)$. Let $b(j, S, E, \gamma) \in V(N)$ be the final payoff when players follow these strategies in $M(\pi, j, S, E, \gamma)$. Furthermore, assume next claims hold:

Claim I$(i{+}1)$: Assume we are in $M\left(\pi, i+1, P_{i+1}^\pi, \emptyset, \gamma\right)$ such that $\gamma_j(N) = d_j^\pi$ for all $j \geq i+1$ and $\gamma \in K^\pi$. Then, under these strategies, player $i+1$ agrees.

Claim II$(i+1)$: Assume we are in $\widetilde{M}(\pi, i+1, S, E, \gamma)$. Then, under these strategies, the final payoff is given by $\left(t, f_{i+1}^\pi(t), d_{N \setminus P_{i+2}^\pi}^\pi\right)$ with $t = b_{P_{i+1}^\pi}(i+2, S, E \cup \{i+1\}, \gamma)$.

Claim III$(i+1)$: Assume we are in $M(\pi, i+1, S, E, \gamma)$. Then, there exists a $T \supset S$, $E \cap T = \emptyset$ such that $b_S(i+1, S, E, \gamma) = \gamma_S(T)$.

Claim IV$(i{+}1)$: $b_j(i+1, S, E, \gamma) \geq d_j^\pi$ for all $j \geq i{+}1$ and all $(S, E, \gamma) \in \Theta_{i+1}^\pi$.

Claim V$(i+1)$: $b_E(i+1, S, E, \gamma) = \omega_E$ for all $(S, E, \gamma) \in \Theta_{i+1}^\pi$.

Claim VI$(i+1)$: $\sum_{j \in S} \lambda_j b_j^+(i+1, S, E, \gamma) \leq v^\lambda(S)$ for all $(S, E, \gamma) \in \Theta_{i+1}^\pi$.

We now describe the strategies in $M(\pi, i, S, E, \gamma)$ and $\widetilde{M}(\pi, i, S, E, \gamma)$. In $M(\pi, i, S, E, \gamma)$, player i agrees on (S, E, γ) if and only if

$$b_i(i+1, S \cup \{i\}, E, \gamma) \geq f_i^\pi\left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma)\right). \tag{8}$$

This value is well-defined because, by Claim IV$(i+1)$, Claim V$(i+1)$ and comprehensiveness

$$\left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma), \omega_i, d_{N \setminus P_{i+1}^\pi}^\pi\right) \leq b(i+1, S, E \cup \{i\}, \gamma) \in V(N)$$

and thus there exists

$$\max\left\{y_i : \left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma), y_i, d_{N \setminus P_{i+1}^\pi}^\pi\right) \in V(N)\right\} \geq \omega_i.$$

The strategies in $\widetilde{M}(\pi, i, S, E, \gamma)$ are as follows: Assume we are in the subgame $\widetilde{M}(\pi, i, S, E, \gamma)$. Then, player i proposes $\left(P_{i+1}^\pi, \emptyset, \widetilde{\gamma}\right)$ with $\widetilde{\gamma}$ given by

$$\widetilde{\gamma}(N) = \left(t, f_i^\pi(t), d_{N \setminus P_{i+1}^\pi}^\pi\right) \tag{9}$$

with $t = b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma)$. It is not difficult to check that $\widetilde{\gamma}(N)$ is well-defined (i.e. $\widetilde{\gamma}(N) \in V(N)$). For $T \neq N$, $\widetilde{\gamma}(T)$ is given as in (5) and (6). Hence, $\widetilde{\gamma} \in K^\pi$.

In the subgame $\widetilde{M}(\pi, i, S, E, \gamma)$, assume player i proposes $\left(\tilde{S}, \tilde{E}, \tilde{\gamma}\right) \in \Theta_{i+1}^\pi$ and $j \in \tilde{S} \setminus \{i\}$. Then, player j votes 'yes' if and only if

$$b_j\left(i+1, \tilde{S}, \tilde{E}, \tilde{\gamma}\right) \geq b_j(i+1, S, E \cup \{i\}, \gamma). \tag{10}$$

We need to prove the claims:

Claim I(i): Assume we are in $M(\pi, i, P_i^\pi, \emptyset, \gamma)$ such that $\gamma_j(N) = d_j^\pi$ for all $j \geq i$, and $\gamma \in K^\pi$. Then, under these strategies, player i agrees.

Proof. We need to prove that (8) holds. Given $j > i$, by Claim I(j), player j is bound to agree on $\left(P_j^\pi, \emptyset, \gamma\right)$. Thus,

$$b_i(i+1, P_i^\pi \cup \{i\}, \emptyset, \gamma) = \gamma_i(N) = d_i^\pi.$$

Moreover, by Claim III($i+1$), there exists a $T \supset P_i^\pi$, $i \notin T$, such that

$$b_{P_i^\pi}(i+1, P_i^\pi, \{i\}, \gamma) = \gamma_{P_i^\pi}(T)$$

which is $\gamma(P_i^\pi)$ because $\gamma \in \Gamma^\pi$. Thus,

$$\begin{aligned} f_i^\pi\left(b_{P_i^\pi}(i+1, P_i^\pi, \{i\}, \gamma)\right) &= f_i^\pi\left(\gamma\left(P_i^\pi\right)\right) \\ &= \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j<i} \lambda_j \gamma_j^+\left(P_i^\pi\right)\right] \\ &= d_i^\pi. \end{aligned}$$

■

Claim II(i): Assume we are in $\widetilde{M}(\pi, i, S, E, \gamma)$. Then, under these strategies, the final payoff is given by

$$\left(t, f_i^\pi(t), d_{N \setminus P_{i+1}^\pi}^\pi\right)$$

with $t = b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma)$.

Proof. Let $(P_i^\pi, \emptyset, \tilde{\gamma})$ be given as in (9), (5) and (6). We need to prove that player i's proposal is accepted, i.e. that (10) holds for all $j \in P_i^\pi$. By

Claim I(k) for all $k \geq i+1$, we know that each player $k \geq i+1$ is bound to agree on $(P_k^\pi, \emptyset, \widetilde{\gamma})$. Thus,

$$b_j\left(i+1, P_{i+1}^\pi, \emptyset, \widetilde{\gamma}\right) = \widetilde{\gamma}_j(N) = b_j(i+1, S, E \cup \{i\}, \gamma)$$

for each $j \in P_i^\pi$. ■

Claim III(i): Assume we are in $M(\pi, i, S, E, \gamma)$. Then, there exists a $T \supset S$, $E \cap T = \emptyset$ such that $b_S(i, S, E, \gamma) = \gamma_S(T)$.

Proof. If player i agrees, by Claim III(i+1), $b_{S\cup\{i\}}(i+1, S \cup \{i\}, E, \gamma) = \gamma_{S\cup\{i\}}(T)$ with $T \supset S \cup \{i\}$ (thus $T \supset S$) and $T \cap E = \emptyset$. Then,

$$b_S(i, S, E, \gamma) = b_S(i+1, S \cup \{i\}, E, \gamma) = \gamma_S(T).$$

If player i disagrees, by Claim III($i+1$), $b_S(i+1, S, E \cup \{i\}, \gamma) = \gamma_S(T)$ with $T \supset S$ and $T \cap (E \cup \{i\}) = \emptyset$ (thus $T \cap E = \emptyset$). Then, by Claim II(i),

$$b_S(i, S, E, \gamma) = b_S(i+1, S, E \cup \{i\}, \gamma) = \gamma_S(T).$$

■

Claim IV(i): $b_j(i, S, E, \gamma) \geq d_j^\pi$ for all $j \geq i$ and all $(S, E, \gamma) \in \Theta_i^\pi$.

Proof. Assume we are in the subgame $M(\pi, i, S, E, \gamma)$. If player i disagrees, then by Claim II(i),

$$b_i(i, S, E, \gamma) = f_i^\pi\left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \omega_E)\right).$$

If player i agrees, by (8),

$$\begin{aligned} b_i(i, S, E, \gamma) &= b_i(i+1, S \cup \{i\}, E, \gamma) \\ &\geq f_i^\pi\left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma)\right). \end{aligned}$$

Thus, it is enough to prove that $f_i^\pi\left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma)\right) \geq d_i^\pi$. By Lemma 8 and Claim V($i+1$),

$$\begin{aligned} & f_i^\pi\left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma)\right) \\ = \; & \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j<i} \lambda_j b_j^+(i+1, S, E \cup \{i\}, \gamma)\right] \\ = \; & \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j\in S} \lambda_j b_j^+(i+1, S, E \cup \{i\}, \gamma) - \sum_{j \in E} \lambda_j \omega_j\right] \end{aligned}$$

by Claim VI$(i+1)$,

$$\geq \frac{1}{\lambda_i}\left[v^\lambda\left(P^\pi_{i+1}\right) - v^\lambda\left(S\right) - \sum_{j\in E}\lambda_j\omega_j\right]$$

by zero-monotonicity, $v^\lambda\left(S\right) + \sum_{j\in E}\lambda_j\omega_j \leq v^\lambda\left(P^\pi_i\right)$ and thus

$$\geq \frac{1}{\lambda_i}\left[v^\lambda\left(P^\pi_{i+1}\right) - v^\lambda\left(P^\pi_i\right)\right] = d^\pi_i.$$

■

Claim V(i): $b_E\left(i, S, E, \gamma\right) = \omega_E$ for each $(S, E, \gamma) \in \Theta^\pi_i$.

Proof. Assume we are in the subgame $M\left(\pi, i, S, E, \gamma\right)$ and let $j \in E$. If player i agrees, $b_j\left(i, S, E, \gamma\right) = b_j\left(i+1, S\cup\{i\}, E, \gamma\right)$, which is ω_j by Claim V$(i+1)$. If player i disagrees, by Claim II(i),

$$b_j\left(i, S, E, \gamma\right) = b_j\left(i+1, S, E\cup\{i\}, \gamma\right)$$

which is ω_j by Claim V$(i+1)$. ■

Claim VI(i): $\sum_{j\in S}\lambda_j b^+_j\left(i, S, E, \gamma\right) \leq v^\lambda\left(S\right)$ for all $(S, E, \gamma) \in \Theta^\pi_i$.

Proof. Assume we are in $M\left(\pi, i, S, E, \gamma\right)$. If player i disagrees, by Claim II(i),

$$b^+_{P^\pi_i}\left(i, S, E, \gamma\right) = b^+_{P^\pi_i}\left(i+1, S, E\cup\{i\}, \gamma\right).$$

Then, by Claim VI$(i+1)$,

$$\sum_{j\in S}\lambda_j b^+_j\left(i, S, E, \gamma\right) = \sum_{j\in S}\lambda_j b^+_j\left(i+1, S, E\cup\{i\}, \gamma\right) \leq v^\lambda\left(S\right).$$

If player i agrees,

$$b^+\left(i, S, E, \gamma\right) = b^+\left(i+1, S\cup\{i\}, E, \gamma\right)$$

and (8) holds, i.e.

$$\begin{aligned} b_i\left(i+1, S\cup\{i\}, E, \gamma\right) &\geq f^\pi_i\left(b_{P^\pi_i}\left(i+1, S, E\cup\{i\}, \gamma\right)\right) \\ &= \frac{1}{\lambda_i}\left[v^\lambda\left(P^\pi_{i+1}\right) - \sum_{j<i}\lambda_j b^+_j\left(i+1, S, E\cup\{i\}, \gamma\right)\right] \end{aligned}$$

by Claim V(i),

$$= \frac{1}{\lambda_i}\left[v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in S}\lambda_j b_j^{+}\left(i+1, S, E\cup\{i\}, \gamma\right) - \sum_{j\in E}\lambda_j\omega_j\right]. \qquad (11)$$

Then, by Claim V(i) and Claim IV($i+1$),

$$\begin{aligned}
\sum_{j\in S}\lambda_j b_j^{+}(i,S,E,\gamma) &= \sum_{j\in N}\lambda_j b_j^{+}(i,S,E,\gamma) - \sum_{j\in E}\lambda_j b_j^{+}(i,S,E,\gamma) \\
&\quad -\lambda_i b_i^{+}(i,S,E,\gamma) - \sum_{j>i}\lambda_j b_j^{+}(i,S,E,\gamma) \\
&\leq v^{\lambda}(N) - \sum_{j\in E}\lambda_j\omega_j - \lambda_i b_i^{+}(i,S,E,\gamma) - \sum_{j>i}\lambda_j d_j^{\pi} \\
&= v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in E}\lambda_j\omega_j - \lambda_i b_i^{+}(i,S,E,\gamma) \\
&= v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in E}\lambda_j\omega_j - \lambda_i b_i^{+}\left(i+1, S\cup\{i\}, E, \gamma\right)
\end{aligned}$$

by (11),

$$\begin{aligned}
&\leq v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in E}\lambda_j\omega_j \\
&\quad -\left[v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in S}\lambda_j b_j^{+}\left(i+1, S, E\cup\{i\}, \gamma\right) - \sum_{j\in E}\lambda_j\omega_j\right] \\
&= \sum_{j\in S}\lambda_j b_j^{+}\left(i+1, S, E\cup\{i\}, \gamma\right)
\end{aligned}$$

by Claim VI($i+1$),

$$\leq v^{\lambda}(S).$$

■

Thus, under these strategies, player 1 proposes $(\{1\}, \emptyset, \gamma^{\pi})$ with $\gamma^{\pi}(N) = d^{\pi}$ and the rest of players agree on it. Society is then formed with all the players and the final outcome is d^{π}.

We now prove that these strategies form a SPNE.

Assume we are in the subgame $\widetilde{M}(\pi, i, S, E, \gamma)$ and player i proposes $\left(\tilde{S}, \tilde{E}, \tilde{\gamma}\right)$ with $j \in \tilde{S}\setminus\{i\}$.

If some player after player j is bound to vote 'no' should turn reach him, player j is indifferent to voting 'yes' or 'no'. Assume then the offer is bound to be accepted should player j vote 'yes'. By doing so, and given the strategies of the rest of the players, player j gets $b_j\left(i+1,\tilde{S},\tilde{E},\tilde{\gamma}\right)$. By rejecting, however, player j gets $b_j\left(i+1,S,E\cup\{i\},\gamma\right)$. Thus, it is optimal for player i to vote 'yes' if and only if (10) holds, as prescribed by the strategy profiles.

Assume now we are in the subgame $\widetilde{M}\left(\pi,i,S,E,\gamma\right)$. By Claim II, given this set of strategies, player i's final payoff is

$$f_i^{\pi}\left(b_{P_i^{\pi}}\left(i+1,S,E\cup\{i\},\gamma\right)\right).$$

Assume player i changes his strategy and proposes a different $\left(\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)\in\Theta_{i+1}^{\pi}$. If (10) does not hold for some $j\in\widetilde{S}\backslash\{i\}$, this player j will vote 'no' and, by Claim V, the final payoff for player i is ω_i, which is not more than with the original strategy.

Assume then (10) holds for all $j\in\widetilde{S}\backslash\{i\}$. The proposal is then accepted and the final payoff for player i is at most $b_i\left(i+1,\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)$.

We must prove that $b_i\left(i+1,\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)\leq f_i^{\pi}\left(b_{P_i^{\pi}}\left(i+1,S,E\cup\{i\},\gamma\right)\right)$.

$$\begin{aligned}
& \lambda_i b_i\left(i+1,\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)\leq\lambda_i b_i^{+}\left(i+1,\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)\\
\leq\; & v^{\lambda}(N)-\sum_{j\in\widetilde{S}\backslash\{i\}}\lambda_j b_j^{+}\left(i+1,\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)\\
& -\sum_{j\in\widetilde{E}}\lambda_j b_j^{+}\left(i+1,\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)-\sum_{j>i}\lambda_j b_j^{+}\left(i+1,\widetilde{S},\widetilde{E},\widetilde{\gamma}\right)
\end{aligned}$$

by (10), Claim IV and Claim V,

$$\begin{aligned}
\leq\; & v^{\lambda}(N)-\sum_{j\in\widetilde{S}\backslash\{i\}}\lambda_j b_j^{+}\left(i+1,S,E\cup\{i\},\gamma\right)-\sum_{j\in\widetilde{E}}\lambda_j\omega_j-\sum_{j>i}\lambda_j d_j^{\pi}\\
=\; & v^{\lambda}\left(P_{i+1}^{\pi}\right)-\sum_{j\in P_i^{\pi}}\lambda_j b_j^{+}\left(i+1,S,E\cup\{i\},\gamma\right)\\
=\; & \lambda_i f_i^{\pi}\left(b_{P_i^{\pi}}\left(i+1,S,E\cup\{i\},\gamma\right)\right)
\end{aligned}$$

and thus player i does not improve his final payoff.

Finally, assume we are in the subgame $M(\pi, i, S, E, \gamma)$.

If (8) holds and player i disagrees on (S, E, γ), he will get at most

$$f_i^\pi \left(b_{P_i^\pi}(i+1, S, E \cup \{i\}, \gamma)\right) \tag{12}$$

which is not more than what he would get by agreeing. Thus, he will not improve his final payoff by deviating.

If (8) does not hold and player i agrees on (S, E, γ), he will get less than (12), which is the payoff he obtains by not deviating. Thus, it is optimal for him to disagree.

5.1.3 Unicity of equilibrium payoffs

We now prove that every SPNE in $M(\pi)$ has d^π as final outcome. Assume we are in an SPNE of $M(\pi)$. Let $B(\pi) \subset V(N)$ be the set of final payoffs in $M(\pi)$. We will prove that $B(\pi) = \{d^\pi\}$.

Given $i \in N$ and $(S, E, \gamma) \in \Theta_i^\pi$, let $B(i, S, E, \gamma) \subset V(N)$ be the set of expected SPNE payoffs in the subgame $M(\pi, i, S, E, \gamma)$.

We proceed by a series of claims. Claim A says that, in equilibrium, every passive player receives nothing. Claim B gives a sufficient condition for candidates to agree. Claim C specifies the final payoff when a candidate disagrees and makes a new proposal. Claim D says that there is a unique payoff in equilibrium for any subgame. Claim F says that every candidate receives at least d^π. Claims E and G are technical ones.

Claim A(n): $b_E = \omega_E$ for all $b \in B(n, S, E, \gamma)$ with $(S, E, \gamma) \in \Theta_n^\pi$.

Proof. Assume we are in the subgame $M(\pi, n, S, E, \gamma)$. If player n agrees (or disagrees and his new proposal is rejected), then the final payoff for players in E is clearly ω. Assume player n disagrees and makes an acceptable proposal. It is well-known that, in equilibrium, player n would make a proposal that leaves the responders indifferent to voting 'yes' or 'no'. Since any responder $i \in E$ receives ω_i if he votes 'no', we conclude the result. ■

Claim B(n): Assume we are in $M(\pi, n, P_n^\pi, \emptyset, \gamma)$ such that $\gamma_n(N) \geq d_n^\pi$ and $\gamma \in K^\pi$. Then, player n agrees in equilibrium.

Proof. By agreeing, player n assure himself a payoff of $\gamma_n(N)$. Assume player n disagrees and proposes a different $\widetilde{\gamma}$. If this proposal is rejected, player n receives $\omega_n \leq d_n^\pi$ and thus he is strictly worse off (note the tie-breaking rule). If the proposal is accepted by players in P_n^π, this means that each of them receives at least what they get by rejecting, i.e. $\widetilde{\gamma}_i(N) \geq \gamma_i(P_n^\pi)$ for all $i < n$. Hence, the final payoff for player n is not more than $f_n^\pi(\gamma(P_n^\pi))$. But $\gamma \in K^\pi$, which means that $\sum_{i<n} \lambda_i \gamma_i(P_n^\pi) = v^\lambda(P_n^\pi)$, and then

$$\begin{aligned} f_n^\pi(\gamma(P_n^\pi)) &= \frac{1}{\lambda_n}\left[v^\lambda(N) - \sum_{i<n} \lambda_i \gamma_i^+(P_n^\pi)\right] \\ &= \frac{1}{\lambda_n}\left[v^\lambda(N) - v^\lambda(P_n^\pi)\right] = d_n^\pi \end{aligned}$$

and again player n is strictly worse off. ■

Claim C(n): Assume we are in $\widetilde{M}(\pi, n, S, E, \gamma)$. Then, the only final payoff in equilibrium is given by $(t, f_n^\pi(t))$ with $t = (\gamma(S), \omega_E)$.

Proof. Since the final payoff for any $i < n$, in case of rejection is t_i, for any $\varepsilon > 0$ player n can propose $(N, \emptyset, \widetilde{\gamma})$ with $\widetilde{\gamma}(N) = (t, f_n^\pi(t)) + x^\varepsilon$, being $\lambda_i x_i^\varepsilon = \varepsilon$ for all $i < n$ and $\lambda_n x_n^\varepsilon = -(n-1)\varepsilon$, so that his proposal is always accepted. Thus, player n can get at least $f_n^\pi(t)$ in equilibrium. Moreover, player n cannot get more by doing an acceptable offer, because any player $i \in P_n^\pi$ can assure himself a payoff of t_i by rejecting any new offer. ■

Claim D(n): $B(n, S, E, \gamma)$ is a singleton for any $(S, E, \gamma) \in \Theta_n^\pi$.

Proof. Assume we are in $M(\pi, n, S, E, \gamma)$. By Claim C(n), by rejecting player n gets a final payoff of $f_n^\pi(\gamma(S), \omega_E)$. By the tie-breaking rule, player n will agree iff $f_n^\pi(\gamma(S), \omega_E) \geq \gamma_n(S \cup \{n\})$. In any case, the final payoff is unique. ■

Claim E(n): Assume we are in $M(\pi, n, S, E, \gamma)$. Then, there exists a $T \supset S$, $E \cap T = \emptyset$ such that $b_S = \gamma_S(T)$ for all $b \in B(n, S, E, \gamma)$.

Proof. If player n accepts, then $b_S = \gamma_S(S \cup \{n\})$. If player n disagrees, by Claim C(n) we know that $b_S = \gamma(S)$. ■

Claim F(n): If $b \in B(n, S, E, \gamma)$ for some $(S, E, \gamma) \in \Theta_n^\pi$, then $b_n \geq d_n^\pi$.

Proof. By Claim C(n), player n can, by rejecting, assure himself a payoff of

$$\begin{aligned} f_n^\pi(\gamma(S), \omega_E) &= \frac{1}{\lambda_n}\left[v^\lambda(N) - \sum_{i \in S} \lambda_i \gamma_i^+(S) - \sum_{i \in E} \lambda_i \omega_i\right] \\ &\geq \frac{1}{\lambda_n}\left[v^\lambda(N) - \max_{x \in V(P_n^\pi)} \sum_{i \in P_n^\pi} \lambda_i x_i\right] = d_n^\pi. \end{aligned}$$

■

Claim G(n): $\sum_{j \in S} \lambda_j b_j^+ \leq v^\lambda(S)$ for each $b \in B(n, S, E, \gamma)$ with $(S, E, \gamma) \in \Theta_n^\pi$.

Proof. Assume we are in $M(\pi, n, S, E, \gamma)$. If player n disagrees, by Claim C(n) the final payoff $b \in B(n, S, E, \gamma)$ satisfies

$$b_{P_n^\pi} = (\gamma(S), \omega_E).$$

Then,

$$\sum_{j \in S} \lambda_j b_j^+ = \sum_{j \in S} \lambda_j \gamma_j^+(S) \leq v^\lambda(S).$$

Moreover, player n gets $f_n^\pi(\gamma(S), \omega_E)$. Thus, if player n agrees, we deduce that

$$\begin{aligned} \gamma_n(S \cup \{n\}) &\geq f_n^\pi(\gamma(S), \omega_E) \\ &= \frac{1}{\lambda_n}\left[v^\lambda(N) - \sum_{j \in S} \lambda_j \gamma_j^+(S) - \sum_{j \in E} \lambda_j \omega_j\right] \\ &\geq \frac{1}{\lambda_n}\left[v^\lambda(N) - v^\lambda(S) - \sum_{j \in E} \lambda_j \omega_j\right]. \end{aligned}$$

Moreover,

$$b = (\gamma(S \cup \{n\}), \omega_E).$$

Then, by Claim A(n),

$$\begin{aligned}
\sum_{j\in S}\lambda_j b_j^+ &= \sum_{j\in N}\lambda_j b_j^+ - \sum_{j\in E}\lambda_j b_j^+ - \lambda_n b_n^+ \\
&= \sum_{j\in N}\lambda_j b_j^+ - \sum_{j\in E}\lambda_j \omega_j - \lambda_n \gamma_n^+ \left(S\cup\{n\}\right) \\
&\leq v^\lambda (N) - \sum_{j\in E}\lambda_j\omega_j - \left[v^\lambda (N) - v^\lambda (S) - \sum_{j\in E}\lambda_j\omega_j\right] = v^\lambda (S).
\end{aligned}$$

■

Assume now Claims A-G hold for $i+1$. Namely, we have

Claim A($i+1$): $b_E = \omega_E$ for all $b \in B(i+1, S, E, \gamma)$ with $(S, E, \gamma) \in \Theta_{i+1}^\pi$.

Claim B($i+1$): Assume we are in $M\left(\pi, i+1, P_{i+1}^\pi, \emptyset, \gamma\right)$ such that $\gamma_j(N) \geq d_j^\pi$ for all $j \geq i+1$ and $\gamma \in K^\pi$. Then, player $i+1$ agrees in equilibrium.

Claim C($i+1$): Assume we are in $\widetilde{M}(\pi, i+1, S, E, \gamma)$. Then, the final payoff in equilibrium is given by $\left(t, f_{i+1}^\pi(t), d_{N\setminus P_{i+2}^\pi}^\pi\right)$ with $t = b_{P_{i+1}^\pi}$ for some $b \in B(i+2, S, E\cup\{i+1\}, \gamma)$.

Claim D($i+1$): $B(i+1, S, E, \gamma)$ is a singleton for any $(S, E, \gamma) \in \Theta_{i+1}^\pi$.

Claim E($i+1$): Assume we are in $M(\pi, i+1, S, E, \gamma)$. Then, there exists a $T \supset S$, $E\cap T = \emptyset$ such that $b_S = \gamma_S(T)$ for all $b \in B(i+1, S, E, \gamma)$.

Claim F($i+1$): If $b \in B(i+1, S, E, \gamma)$ for some $(S, E, \gamma) \in \Theta_{i+1}^\pi$, then $b_j \geq d_j^\pi$ for all $j \geq i+1$.

Claim G($i+1$): $\sum_{j\in S}\lambda_j b_j^+ \leq v^\lambda(S)$ for all $b \in B(i+1, S, E, \gamma)$ with $(S, E, \gamma) \in \Theta_{i+1}^\pi$.

We prove the claims for $i < n$,

Claim A(i): $b_E = \omega_E$ for all $b \in B(i, S, E, \gamma)$ with $(S, E, \gamma) \in \Theta_i^\pi$.

Proof. Assume we are in the subgame $M(\pi, i, S, E, \gamma)$. If player i agrees (or he disagrees and his new proposal is rejected), by Claim A($i+1$), players in E get ω. Assume then player i disagrees and makes an acceptable proposal. It is well-known that, in equilibrium, player i would make a proposal that leaves the responders indifferent to voting 'yes' or 'no'. By Claim A$(i+1)$, any responder $j \in E$ receives ω_j if he votes 'no'. Thus, we conclude the result. $\blacksquare$

Claim B(i): Assume we are in $M\left(\pi, i, P_i^\pi, \emptyset, \gamma\right)$ such that $\gamma_j\left(N\right) \geq d_j^\pi$ for all $j \geq i$ and $\gamma \in K^\pi$. Then, player i agrees in equilibrium.

Proof. The hypothesis of Claim B(i) hold for $i+1, ..., n$ if player i agrees. Thus, by induction hypothesis applied to Claim B, we know that player i gets a payoff of $\gamma_i\left(N\right)$ by agreeing. Assume player i disagrees and proposes a different $\widetilde{\gamma}$. If this proposal is rejected, by Claim A$(i+1)$ player i receives $\omega_i \leq d_i^\pi$ and thus he is strictly worse off (note the tie-breaking rule). If the proposal is accepted by players in P_i^π, this means that each of them receives at least what they get by rejecting. By Claim E$(i+1)$, this is $\gamma_{P_i^\pi}\left(T\right)$ for some $T \supset P_i^\pi$ with $i \notin T$. Since $\gamma \in \Gamma^\pi$ and $i \notin T$, we know $\gamma_{P_i^\pi}\left(T\right) = \gamma\left(P_i^\pi\right)$. By Claim F$(i+1)$, the final payoff for player i is not more than $f_i^\pi\left(\gamma\left(P_i^\pi\right)\right)$. But $\gamma \in K^\pi$, which means that $\sum_{j<i} \lambda_j \gamma_j\left(P_i^\pi\right) = v^\lambda\left(P_i^\pi\right)$, and then

$$
\begin{aligned}
f_i^\pi\left(\gamma\left(P_i^\pi\right)\right) &= \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j<i} \lambda_j \gamma_j^+\left(P_i^\pi\right)\right] \\
&= \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - v^\lambda\left(P_i^\pi\right)\right] = d_i^\pi
\end{aligned}
$$

and again player i is strictly worse off. $\blacksquare$

Claim C(i): Assume we are in $\widetilde{M}\left(\pi, i, S, E, \gamma\right)$. Then, the final payoff in equilibrium is given by $\left(t, f_i^\pi\left(t\right), d_{N \backslash P_{i+1}^\pi}^\pi\right)$ with $t = b_{P_i^\pi}$ for $b \in B\left(i+1, S, E \cup \{i\}, \gamma\right)$.

Proof. By Claim D$(i{+}1)$, there exists a unique b in $B\left(i+1, S, E \cup \{i\}, \gamma\right)$. Let $t = b_{P_i^\pi}$, and consider the following strategy for player i: Given $\varepsilon > 0$, he proposes $\left(P_{i+1}^\pi, \emptyset, \widetilde{\gamma}\right)$ such that $\widetilde{\gamma} \in K^\pi$ and $\widetilde{\gamma}\left(N\right) = \left(t, f_i^\pi\left(t\right), d_{N \backslash P_{i+1}^\pi}^\pi\right) + x^\varepsilon$, where $\lambda_j x_j^\varepsilon = \varepsilon$ for all $j < i$, $x_j^\varepsilon = 0$ for all $j > i$, and $\lambda_i x_i^\varepsilon = -\left(i-1\right)\varepsilon$. By induction hypothesis applied to Claim B, we know that this proposal is bound to be accepted should players in P_i^π vote 'yes'. Then, by voting 'yes', players in P_i^π get something more than what they get by voting 'no'. Thus, they would vote 'yes' and player i gets a final payoff of almost $f_i^\pi(t)$. Since this is true for any $\varepsilon > 0$, we conclude that player i can get at least $f_i^\pi(t)$ in equilibrium. Moreover, player i cannot get more by doing an acceptable offer, because any player $j \in P_i^\pi$ can assure himself a payoff of t_j by rejecting any new offer. ■

Claim D(i): $B(i, S, E, \gamma)$ is a singleton for any $(S, E, \gamma) \in \Theta_i^\pi$.

Proof. Assume we are in $M(\pi, i, S, E, \gamma)$. By Claim C(i), by rejecting player i gets a final payoff of $f_i^\pi\left(b_{P_i^\pi}\right)$ where b is the only payoff in $B(i+1, S, E \cup \{i\}, \gamma)$ (by Claim D($i+1$)). By the tie-breaking rule, player i will agree iff $f_i^\pi\left(b_{P_i^\pi}\right) \geq b'_i$, where b' is the only payoff in $B(i+1, S \cup \{i\}, E, \gamma)$ (by Claim D($i+1$)). In any case, the final payoff is unique. ■

Claim E(i): Assume we are in $M(\pi, i, S, E, \gamma)$. Then, there exists a $T \supset S$, $T \cap E = \emptyset$ such that $b_S = \gamma_S(T)$ for all $b \in B(i, S, E, \gamma)$.

Proof. If player i agrees, by Claim E($i+1$), $b_{S\cup\{i\}} = \gamma_{S\cup\{i\}}(T)$ with $T \supset S \cup \{i\}$ (thus $T \supset S$) and $T \cap E = \emptyset$. Then, $b_S = \gamma_S(T)$.

If player i disagrees, by Claim C(i), $b_S = b'_S$ where $b' \in B(i+1, S, E \cup \{i\}, \gamma)$. By Claim E($i+1$), $b'_S = \gamma_S(T)$ with $T \supset S$ and $T \cap (E \cup \{i\}) = \emptyset$ (thus $T \cap E = \emptyset$). Then, $b_S = \gamma_S(T)$. ■

Claim F(i): If $b \in B(i, S, E, \gamma)$ for some $(S, E, \gamma) \in \Theta_i^\pi$, then $b_j \geq d_j^\pi$ for all $j \geq i$.

Proof. By induction hypothesis applied to Claim F, the result is true for $j > i$. Let b be the only element in $B(i+1, S, E \cup \{i\}, \gamma)$ (Claim D($i+1$)). By Claim C(i), player i can, by rejecting, assure himself a payoff of

$$f_i^\pi\left(b_{P_i^\pi}\right) = \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j<i} \lambda_j b_j^+\right]$$

by Claim A($i+1$),

$$= \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - \sum_{j\in S} \lambda_j b_j^+ - \sum_{j\in E} \lambda_j \omega_j\right]$$

by Claim G($i+1$),

$$\geq \frac{1}{\lambda_i}\left[v^\lambda\left(P_{i+1}^\pi\right) - v^\lambda(S) - \sum_{j\in E} \lambda_j \omega_j\right]$$

by zero-monotonicity, $v^{\lambda}(S) + \sum_{j\in E} \lambda_j \omega_j \le v^{\lambda}(P_i^{\pi})$ and thus

$$\ge \frac{1}{\lambda_i}\left[v^{\lambda}\left(P_{i+1}^{\pi}\right) - v^{\lambda}\left(P_i^{\pi}\right)\right] = d_i^{\pi}.$$

■

Claim G(i): $\sum_{j\in S} \lambda_j b_j^{+} \le v^{\lambda}(S)$ for all $b \in B(i, S, E, \gamma)$ with $(S, E, \gamma) \in \Theta_i^{\pi}$.

Proof. Assume we are in $M(\pi, i, S, E, \gamma)$. By Claim D, there exists a single imputation b in $B(i, S, E, \gamma)$ and a single imputation $\dot{b}$ in $B(i+1, S, E \cup \{i\}, \gamma)$. If player i disagrees, by Claim C(i), players in P_i^{π} get $b_{P_i^{\pi}} = \dot{b}_{P_i^{\pi}}$. By Claim G$(i+1)$,

$$\sum_{j\in S} \lambda_j b_j^{+} = \sum_{j\in S} \lambda_j \dot{b}_j^{+} \le v^{\lambda}(S).$$

If player i agrees, then $b = \ddot{b}$ with $\ddot{b}$ the only imputation in $B(i+1, S \cup \{i\}, E, \gamma)$. Moreover, by Claim C$(i)$, player i would not agree if $\ddot{b}_i < f_i^{\pi}\left(\dot{b}_{P_i^{\pi}}\right)$. Thus,

$$b_i = \ddot{b}_i \ge f_i^{\pi}\left(\dot{b}_{P_i^{\pi}}\right) = \frac{1}{\lambda_i}\left[v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j<i} \lambda_j \dot{b}_j^{+}\right]$$

by Claim A(i),

$$= \frac{1}{\lambda_i}\left[v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in S} \lambda_j \dot{b}_j^{+} - \sum_{j\in E} \lambda_j \omega_j\right]. \tag{13}$$

Then, by Claim A(i) and Claim F(i),

$$\begin{aligned}
\sum_{j\in S} \lambda_j b_j^{+} &= \sum_{j\in N} \lambda_j b_j^{+} - \sum_{j\in E} \lambda_j b_j^{+} - \lambda_i b_i^{+} - \sum_{j>i} \lambda_j b_j^{+} \\
&\le v^{\lambda}(N) - \sum_{j\in E} \lambda_j \omega_j - \lambda_i b_i^{+} - \sum_{j>i} \lambda_j d_j^{\pi} \\
&= v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in E} \lambda_j \omega_j - \lambda_i b_i^{+}
\end{aligned}$$

by (13),

$$\le v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in E} \lambda_j \omega_j - \left[v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j\in S} \lambda_j \dot{b}_j^{+} - \sum_{j\in E} \lambda_j \omega_j\right] = \sum_{j\in S} \lambda_j \dot{b}_j^{+}$$

by Claim G$(i+1)$,

$$\leq v^{\lambda}(S).$$

■

Since, by Claim F, every player $i \in N$ can assure himself a final payoff of at least d_i^{π}, and d^{π} is an efficient payoff, we conclude that the only possible final payoff in SPNE for the subgame $M(\pi)$ is d^{π} and, moreover, the strategy of player i is robust to deviations by coalitions of P_i^{π}.

5.2 Proof of Theorem 3

Note that a zero-monotonic TU game V' as given in (1) satisfies (A1), (A2), (A3), and (A4). Furthermore, it is clear that $V'(N)$ is delimited by a hyperplane whose outward normal vector is given by $\lambda_i = 1$ for all $i \in N$.

Moreover, property (A5) is only used in the proof of Lemma 1 and in Lemma 8. However, it is straightforward to prove that Lemma 1 still holds for a TU game. Moreover, (2) can be replaced by

$$f_i^{\pi}(x) = \frac{1}{\lambda_i}\left[v^{\lambda}\left(P_{i+1}^{\pi}\right) - \sum_{j<i} \lambda_j x_j\right]$$

for TU games. The rest of the proof is analogous to those of Theorem 2.

5.3 The tie-breaking rule

A more precise description for the strategy of player 4 in Example 7 is the following: If player 3 was excluded after proposing $\widetilde{\gamma}$ with $\widetilde{\gamma}_1(N) \geq \widetilde{\gamma}_2(N)$ and $\widetilde{\gamma}_4(N) \geq 0$, or $\widetilde{\gamma}_1(\{1,2,3\}) \geq \widetilde{\gamma}_2(\{1,2,3\})$ and $\widetilde{\gamma}_4(N) < 0$, then player 4 agrees to join the society. If player 3 was excluded after proposing $\widetilde{\gamma}$ with $\widetilde{\gamma}_1(N) < \widetilde{\gamma}_2(N)$ and $\widetilde{\gamma}_4(N) \geq 0$, or $\widetilde{\gamma}_1(\{1,2,3\}) < \widetilde{\gamma}_2(\{1,2,3\})$ and $\widetilde{\gamma}_4(N) < 0$, then player 4 disagrees and proposes an unacceptable offer, such like $(0,0,0,1)$.

6 References

1. Aumann, R., 1985. An axiomatization of the non-transferable utility value. Econometrica 53, 599-612.

2. Aumann, R., 1985b. On the non-transferable utility value: A comment on the Roth-Shafer examples. Econometrica 53, 667-677.

3. Aumann, R., 1986. Rejoinder. Econometrica 54, 985-989.

4. Bloch, F., 1996. Sequential formation of coalitions in games with externalities and fixed payoff division. Games Econ. Behav. 14, 90-123.

5. Chatterjee, K., Dutta, B., Ray, D., Sengupta, K., 1993. A noncooperative theory of coalitional bargaining. Rev. Econ. Stud. 60, 463-477.

6. Harsanyi, J.C., 1963. A simplified bargaining model for the n-person cooperative game. Int. Econ. Rev. 4, 194-220.

7. Harsanyi, J.C., 1980. Comments on Roth's paper, 'Values for games without side payments'. Econometrica 48, 477.

8. Hart, S., Kurz, M., 1983. Endogenous formation of coalitions. Econometrica 51, 1047-1064.

9. Hart, S., Mas-Colell, A., 1996. Bargaining and value. Econometrica 64, 357-380.

10. Macho-Stadler, I., Pérez-Castrillo, D., Porteiro N., 2002. Sequential formation of coalitions through bilateral agreements. UFAE and IAE Working Papers 515.02.

11. Maschler, M., Owen, G., 1989. The consistent Shapley value for hyperplane games. Int. J. Game Theory 18, 389-407.

12. Maschler, M., Owen, G., 1992. The consistent Shapley value for games without side payments. In: Selten, R. (Ed.) Rational Interaction. New York. Springer-Verlag, pp. 5-12.

13. Montero, M., Okada, A., 2003. Riskly versus riskless bargaining procedures: the Aumnann-Roth controversy revisited. Mimeo.

14. Nash, J.F., 1950. The bargaining problem. Econometrica 18, 155-162.

15. Okada, A., 1996. A noncooperative coalitional bargaining game with random proposers. Games Econ. Behav. 16, 97-108.

16. Roth, A.E., 1980. Values of games without side payments: some difficulties with current concepts. Econometrica 48, 457-465.

17. Roth, A.E., 1986. The non-transferable value: a reply to Aumann. Econometrica 54, 981-984.

18. Serrano, R., 1997. Reinterpreting the kernel. Journal of Economic Theory 77, 58-80.

19. Shafer, W., 1980. On the existence and interpretation of value allocations. Econometrica 48, 467-477.

20. Shapley, L.S., 1953. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to the Theory of Games II. Princeton University Press, Princeton NJ, pp. 307-317.

21. Shapley, L.S., 1969. Utility comparison and the theory of games. In: La Decision, Aggregation et Dynamique des Ordres de Preference, Editions du Centre de la Recherche Scientifique. Paris, pp. 251-263.

22. Vidal-Puga, J.J., Bergantiños, G., 2003. An implementation of the Owen value. Games Econ. Behav. 44, 412-427.