Coalition Formation with Local Public Goods and Network Effect

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June 2005

Abstract

Many local public goods are provided by coalitions and some of them have network effects. Namely, people prefer to consume a public good in a coalition with more members. This paper adopts the Drèze and Greenberg (1980) type utility function where players have preferences over goods as well as coalition members. In a game with anonymous and separable network effect, the core is nonempty when coalition feasible sets are monotonic and players' preferences over public goods have connected support. All core allocations consist of connected coalitions and they are Tiebout equilibria as well. We also examine the no-exodus equilibrium for games whose feasible sets are not monotonic.

1

1 Introduction

Network effects appear in the consumption of many goods. The utility derived from consumption is higher as there are more people consuming the same good. Katz and Shapiro (1985) describe three major sources of this type of consumption externalities. First, the consumption of some goods constitutes physical networks like telephones, fax machines, and email. The users form a network that links all users together. Having more users makes the good more valuable to any single user. Second, the utility from a hardware platform depends on the availability of software, and the availability of software in turn depends on the number of people using the same platform. Third, a durable good can provide longer use if maintenance services are accessible, and a company will provide more service stations if it has more customers. The literature addresses the interactions among firms including issues such as price and quantity competitions, technology sharing, and standardization (see, for example, Church and Gandal 1992 and Economides 1996). Network effects, on the other hand, are less explored in the context of public goods. However, people form coalitions to provide local public goods and many public goods do have network effects. The following economic applications are examples of local public goods with network effects: (i) Buyers choose among different insurance policies in the market. Having more people purchasing the same contract ensures better risk-sharing. (ii) Political parties promote their policy platforms, and people prefer to join a larger party for a better chance of wining. (iii) Clubs form in order to provide entertainment for members. People may prefer to socialize with more members with common interest. (iv) Professors join academic departments to obtain research resources, and a larger faculty provides better interactions among scholars.

Cooperative coalition formation with nontransferable utility and no externalities among coalitions has been first studied in the characteristic function form. Aumann and Dréze (1975) adapt for coalition structures solution concepts originally designed for the grand coalition, such as the core, von Neumann-Morgenstern stable set, and the bargaining set. Kaneko and Wooders (1982) develop conditions, based on Scarf's (1967) balancedness condition, that guarantee the nonemptiness of the core independent of the payoff function. Le Breton, Owen and Weber (1992) study communication games on graphs where only connected coalitions are effective. The term "hedonic coalition" describes the situation that players

have preferences over coalition members. Dréze and Greenberg (1980) first consider the hedonic aspect where players derive utility from the consumption of private and public goods, as well as coalition members. Subsequent works focus on local public goods provided by coalitions. Guesnerie and Oddou (1981) and Greenberg and Weber (1986) show the nonemptiness of the core in local public economies where coalitions decide on levels of public expenditures that were financed by taxes. The notion of a consecutive game is developed in the latter. Greenberg and Weber (1993) and Demange (1994) study more abstract models. A stronger notion of stability, core allocations that are also Tiebout equilibria, is obtained. Their existence is shown in the former when preferences are single-peaked on a line and in the latter when preferences are intermediate on a tree graph. Coalition feasible sets are assumed to be monotonic in both. Pure hedonic coalitions where utility is solely derived from members are recently investigated by the following authors: Banerjee, Konishi and Sönmez (2001) show that the core is empty when several common assumptions are applied. However, the weak top-coalition property guarantees the nonemptiness of the core. Bogomolnaia and Jackson (2002) study the same model and obtain the existence of individually stable and Nash stable partitions. In a related work on strategic form games, Konishi, Le Breton and Weber (1997) study group formation with positive externalities from group size. A coalition is a set of players choosing the same pure strategy. A Nash equilibrium exists when preferences over strategies dominate group size effect. Demange and Wooders (2005) and Demange (2004) present comprehensive surveys of these issues.

This paper investigates the situation where players derive utility from public goods and coalition size. This is a special case of the Drèze and Greenberg (1980) type utility function.[1] We focus on a type of network effect that is anonymous and separable; preferences over members are represented by a separate increasing function. We impose a preference structure called "connected support." It requires that there is a tree graph linking players together such that the set of players whose utility difference from a pair of public goods is strictly bigger than any given constant is connected. When coalition feasible sets are monotonic and player's preferences over public goods satisfy connected support, the core is nonempty. With the above assumptions, we characterize the core as follows. An allocation is

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[1]Greenberg and Weber (1986) have another application where the utility from group size is derived indirectly through a tax sharing rule.

in the core if and only if it is not blocked by any connected coalition and it is a Tiebout equilibrium. Moreover, a core allocation consists of connected coalitions.

Monotonicity appears in all previous results of public good models even though different preference structures are used (such as in Guesnerie and Oddou 1981, Greenberg and Weber 1986, 1993, and Demange 1994). It looks benign but there are no results without assuming it. When monotonicity is not assumed, individually stable allocations may not exist even if players have intermediate preferences that are also single-peaked. We examine another stability notion that is relaxed from the core, the "no-exodus equilibrium." When feasible sets are not monotonic, it strictly contains the core. It captures the idea of free migration into a country that welcomes all newcomers. Still, there may not exist a no-exodus equilibrium without monotonicity.

Section 2 introduces the model and presents the results. Section 3 introduces the no-exodus equilibrium. Section 4 concludes.

2   The model

The set of players is denoted by N and the set of public goods is denoted by X. Each player $i \in N$ has preferences over X that are represented by a continuous function $u_i : X \to \mathbb{R}$. A *coalition* is a subset $S \subset N$. The set of feasible public goods of coalition S is $\phi(S)$ where $\phi : 2^N \to 2^X$ is called a *feasibility correspondence*. A coalition may have an empty feasible set. We assume that there exists an $S \subset N$ such that $\phi(S) \neq \varnothing$ to eliminate triviality. The network effect is anonymous and separable. Utility function $v_i : 2^N \times X$ represents player i's preferences over pairs of coalition and public good. If player i consumes public good $x \in X$ in coalition S, her utility is

$$v_i(S, x) = u_i(x) + f(|S|)$$

where u_i is continuous, and f is nonnegative and strictly increasing. The pair (S, x) is called a coalition public good pair. Using an identical function f for every player only has a normalizing effect since preferences are ordinal. The profile of utility functions of N is denoted by $(v_i)_{i \in N}$. A *public good game with network effect* $\left(N, X, \phi, (v_i)_{i \in N}\right)$ consists of a set of players, a set of public goods, a feasibility correspondence, and utility functions where N is finite, X is closed, and ϕ is compact-valued.

A *coalition structure* $\Pi \subset 2^N$ is a partition of N such that $\phi(S) \neq \varnothing$ for all $S \in \Pi$. An *allocation* $a : N \rightarrow 2^N \times X$ assigns a coalition public good pair $a(i)$ to individual i. Allocation a is *feasible* if there is a coalition structure and a list of public goods $\left(\Pi_a, (x_S)_{S \in \Pi_a}\right)$ with $x_S \in \phi(S)$ for all $S \in \Pi_a$ such that $a(i) = (S, x_S)$ for all $i \in S$ and all $S \in \Pi_a$. To simplify notation, we denote $v_i(a(i)) = v_i(S, x_S)$. In many cases, a coalition can do more if it has more members. The feasibility correspondence ϕ is assumed to be *monotonic*: $\phi(S) \subset \phi(S')$ for all $S, S' \in 2^N$, $S \subset S'$.

Previous works impose preference structures over public goods such as single-peaked preferences and intermediate preferences. These structures link players on a line or a tree graph and regulate players' preferences to change gradually over the line or the tree. However, the network effect in our model invalidates these preference structures: a player's desire for a bigger coalition size can dominate preferences over public goods. For example, suppose player i prefers x to y in any coalition and the network effect has $f(3) - f(2) > u_i(x) - u_i(y)$. Then, i prefers to consume y in a three-person coalition than x in a smaller coalition. Hence, previous results are not valid when the network effect is not negligible and tips over preferences over public goods in an arbitrary way. We resort to a stronger restriction that regulates gradual changes in the relative strength of preferences over public goods and sizes.

- Players' preferences over public goods satisfy *connected support* $(G)^2$ if there is a tree graph G on N such that for any pair $x, y \in X$ and any $t \in \mathbb{R}$ the set $\{i \in N \mid u_i(x) - u_i(y) > t\}$ is connected on G.

Connected support requires that there is a tree graph linking players together such that the set of players whose utility difference from any pair of public goods is strictly bigger than any given constant is connected. It says that there is a way to link players on a tree according to their preferences, and players with similar preferences are connected on the tree. A related preference structure is intermediate preferences (Grandmont 1978, Demange 1994) which requires that, with respect to a tree and for any pair of public goods, the following two sets of players are connected: players with the same strict preferences and players with the same weak preferences. So, in other words, intermediate preferences requires that strict and weak preferences change gradually over a tree, while connected

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[2]The parenthesis, (G), denotes "with respect to G."

support requires that the strength of strict preferences change gradually over a tree. We illustrate this type of preferences with a simple example: the Euclidean utility function is $u_i(x) = \|x - a_i\|$ where $x, a_i \in \mathbb{R}$ and constant a_i is player i's idea point. For $x, y \in X$ and a real number t, the set $\{i \in N \mid u_i(x) - u_i(y) > t\}$ is equivalent to $\{i \in N \mid a_i > (x - y + t)/2\}$ and it is connected.[3]

The local public good literature focuses on the following two stability notions. The core is the set of allocations where no group of players can break away while improve every member's payoff. A Tiebout equilibrium is an allocation such that no one wants to move to another coalition.

A feasible allocation a is in the *core* if there is no coalition public good pair (S, x) such that $x \in \phi(S)$ and $v_i(S, x) > v_i(a(i))$ for all $i \in S$.

A feasible allocation a is a *Tiebout equilibrium* if $v_i(a(i)) \geq v_i(a(j))$ for all $i, j \in N$.[4]

We have the following characterization of the core which is not obtained in models without network effect.

Proposition 1. *In a public good game with network effect where preferences satisfy connected support (G) and feasible sets are monotonic, a is in the core if and only if (i) a is not blocked by connected (G) coalitions, and (ii) a is a Tiebout equilibrium. Moreover, if a is in the core, a consists of connected (G) coalitions.*

Proof. (1) In the first part of the proof, we show that conditions (i) and (ii) characterize the core. First, suppose a satisfies (i) and (ii). We will show that no coalition can block. Suppose S blocks with x and S is not connected. Let T be the minimal connected set containing S. That is, $S \subset T \in 2^N$ and there is no connected $T' \in 2^N$, $T' \neq T$ such that $S \subset T' \subset T$. For all $h \in T \backslash S$, we can find $i, j \in S$ such that h is on the path linking i and j. Denote $a(h) = (S', y)$. We have $v_i(S, x) > v_i(a(i)) \geq v_i(S', y)$ and $v_j(S, x) > v_j(a(j)) \geq v_j(S', y)$ because a is a Tiebout equilibrium. Since the set $\{k \in N \mid u_k(x) - u_k(y) > f(|S'|) - f(|S|)\}$

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[3]Examples of utility functions satisfying intermediate preferences used in Demange (1994, p. 50) also satisfy connected support.

[4]This version of Tiebout equilibrium coincides with *envy-freeness*, which is a notion of fairness (Foley 1967).

is connected, $v_h(S, x) > v_h(S', y)$ as well. $v_i(T, x) > v_i(S, x)$ for all $i \in T$ since T is larger than S. Moreover, $x \in \phi(T)$ by monotonicity. Thus, T is a connected coalition that blocks a; this is a contradiction.

Since condition (i) is implied by the core, what remains is to show that a core allocation is also a Tiebout equilibrium. This is done via the following property which is satisfied by all core allocations: any edge linking two adjacent coalitions divides players into two groups with the opposite weak preferences.

A feasible allocation a has the *separation property (G)* if for any linking edge ij of two adjacent coalitions on a tree G,

$$v_h(a(i)) \geq v_h(a(j)) \text{ for all } h \in M(ij),$$
$$v_h(a(j)) \geq v_h(a(i)) \text{ for all } h \in M(ji),$$

where $M(ij) = \{h \in N \mid ij \notin p(i, h)\}$.

Lemma 1. *If a feasible allocation a satisfies the separation property (G), it is a Tiebout Equilibrium.*

Proof. Each pair $i, j \in N$ are linked on G by a unique path that passes through adjacent coalitions. Let $i_0 = i$, $i_k = j$ and $p(i, j) = \{i_0 i_1, i_1 i_2, ..., i_{k-1} i_k\}$. For all $m = 1, ..., k$, either i_{m-1} and i_m belong to the same coalition and $a(i_{m-1}) = a(i_m)$, or $i_{m-1} i_m$ links two adjacent coalitions and $i \in M(i_{m-1} i_m)$, which means $v_i(a(i_{m-1})) \geq v_i(a(i_m))$. So, $v_i(a(i_0)) \geq v_i(a(i_k))$. ∎

Lemma 2. *When preferences satisfy connected support (G) and feasible sets are monotonic, any core allocation a has the separation property (G).*

Proof. Suppose $a(i) = (S, x)$ and $a(j) = (T, y)$. Suppose $v_j(S, x) \geq v_j(T, y)$. Then $v_j(S \cup \{j\}, x) > v_j(T, y)$ and $v_i(S \cup \{j\}, x) > v_i(T, y)$ by network effect. Moreover, $x \in \phi(S \cup \{j\})$ by monotonicity, and this means that a is not in the core. So, $v_j(T, y) > v_j(S, x)$; by the same argument, $v_i(S, x) > v_i(T, y)$. Suppose there is $h \in M(ij)$ such that $v_h(T, y) > v_h(S, x)$, then $j, h \in \{k \in N \mid u_k(y) - u_k(x) > f(|S|) - f(|T|)\}$. Since i is on the path linking j, h, the above set is not connected, and this violates connected support. So, there is no $h \in M(ij)$ such that $v_h(\hat{a}(j)) > v_h(\hat{a}(i))$. By the same argument, there is no $h \in M(ji)$ such that $v_h(a(i)) > v_h(a(j))$. ∎

(2) In this part of the proof, we show that a consists of connected coalitions if a is in the core. Suppose there is a coalition $S \in \Pi_a$ which is not connected. Let T be the minimal connected coalition containing S. Take any $i \in T \backslash S$. Note that all $i \in T \backslash S$ is on a path linking two players in S. Then, $v_j(T, x_S) > v_j(S, x_S) \geq v_j(a(i))$ for all $j \in S$ and all $i \in T \backslash S$ by network effect and Tiebout equilibrium respectively. Denote $a(i) = (T', y)$. Since the set $\{k \in N \mid u_k(x_S) - u_k(y) > f(|T'|) - f(|S|)\}$ is connected, $v_i(S, x_S) > v_i(a(i))$ for all $i \in T \backslash S$. Moreover, $v_i(T, x_S) > v_i(a(i))$ for all $i \in T \backslash S$ by network effect. Finally, $x_S \in \phi(T)$ by monotonicity. So, T blocks with x_S, and this contradicts with a being in the core. ∎

In a model without network effect, Demange (1994) show that conditions (i) and (ii) together with that a consist of connected coalitions imply the core. The network effect brings a stronger result: the core also implies the above three conditions. In a core allocation, every player prefers own coalition to another, and each coalition is composed of players with similar preferences. A core allocation also has a *diversity property*: no two coalitions choose the same public good. If there are two such coalitions, they would join together because of the network effect. We present the nonemptiness result in the following.

Theorem 1. *When preferences satisfy connected support (G) and feasible sets are monotonic, a public good game with network effect has a nonempty core.*

Proof. An algorithm that constructs a core allocation is defined in the following.[5] Take $r \in N$ to be the root. Rooted tree G^r assigns priorities to players. The distance between player i and r is $\delta(r, i) = k$ if r and i are linked by a path of length k. We say that i has priority-$(k+1)$. Let $\bar{k} = \max_{i \in N} \delta(r, i)$ be the maximal length on G^r. Let N^i denote the subtree originated from i that contains i and players with lower priorities. Note that $N^i \cap N^j = \varnothing$ if i, j have the same priority.

For the convenience of the construction, we temporarily assign a null public

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[5] This algorithm is based on a chapter of my dissertation, Kung (2002). Several hierarchical algorithms can be found in Demange (2004) for games without preference structures where only coalitions that are connected on a hierarchy can form.

good μ to coalitions that have empty feasible sets and $u_i(x) > u_i(\mu) + f(|N|)$ for all $x \in X$ and all $i \in N$. μ is the least preferred for all players. Let $X' = X \cup \{\mu\}$. We will show later that the final construction does not involve μ.

Next, we define the *top choice set* C^i for player i.

$$C^i = \arg\max \left\{ v_i(S, x) \mid (S, x) \in \tilde{N}^i \text{ and } v_j(S, x) \geq \bar{v}_j, \forall j \in S \backslash i \right\}$$

where $\bar{v}_j = v_j(S^*, x^*)$ for some $(S^*, x^*) \in C^j$ and

$$\tilde{N}^i = \left\{ (S, x) \in 2^{N^i} \times X \mid x \in \phi(S), \ i \in S, \ S \text{ is connected} \right\}.$$

Set C^i consists of player i's most preferred coalition public good pairs among all feasible coalition public good pairs that consist of connected coalitions on i's subtree N^i containing i and give other coalition members utility levels no worse than their top choice sets. The next lemma shows that the top choice set is well-defined.

Lemma 3. $C^i \neq \varnothing$ *for all* $i \in N$.

Proof. First, let $R_i(S, \bar{v}_i) = \{(T, x) \in \{S\} \times X \mid x \in \phi(T), \ v_i(T, x) \geq \bar{v}_i\}$ denote i's upper contour set with utility no less than $\bar{v}_i$ when in coalition S. Let

$$D^i = \cup_{\left\{ S \mid (S,x) \in \tilde{N}^i \right\}} \left(\cap_{j \in S \backslash i} R_j(S, \bar{v}_i) \cap \{(S, x) \mid x \in \phi(S)\} \right).$$

Then, D^i is the set of all feasible coalition public good pairs that consist of connected coalitions on i's subtree N^i containing i and give other coalition members utility levels no worse than their top choice sets. C^i is the set of i's most preferred pairs in D^i. Since player i can always form a one-person coalition, the set $D^i \neq \emptyset$. All $R_i(S, .)$ and $\phi(S)$ are compact, and the set $\left\{ S \mid (S, x) \in \tilde{N}^i \right\}$ is finite. Thus, D^i is the union of finitely many compact sets. Since v_i is continuous, $C^i \neq \varnothing$. ∎

In the following, we construct an allocation $\hat{a}$ using top choice sets. Given a collection of pairs $\{(S^i, x^i)\}_{i \in N}$ such that $(S^i, x^i) \in C^i$ for all $i \in N$, we assign coalition public good pairs sequentially starting from r. Let $L^0 = \{r\}$.

$\hat{a}(i) = (S^r, x^r)$ for all $i \in S^r$. Let $L^1 = \{j \in N \backslash S^r \mid \nexists h \in N \backslash S^r \text{ s.t. } \delta(r, h) < \delta(r, j)\}$.

$\hat{a}(i) = (S^j, x^j)$ for all $i \in S^j$ and all $j \in L^1$.

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Suppose $\hat{a}(i)$ is assigned for all $i \in S^j$ and all $j \in L^{m-1}$.

Let $\hat{S}(m-1) = \cup_{i \in \cup_{k=0}^{m-1} L^k} S^i$ and

$$L^m = \left\{ j \in N \backslash \hat{S}(m-1) \mid \nexists h \in N \backslash \hat{S}(m-1) \ \ s.t. \delta(r,h) < \delta(r,j) \right\}.$$

$\hat{a}(i) = (S^j, x^j)$ for all $i \in S^j$ and all $j \in N \backslash L^m$.

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Since N is finite, there is an integer $\bar{l} \leq \bar{k} - 1$ such that $L^{\bar{l}+1}(r) = \varnothing$.

Note that all coalitions that have been assigned are connected. Let $L = \cup_{k=0,\ldots,\bar{l}} L^k(r)$. Thus, $\{S^i\}_{i \in L}$ is a partition of N. The collection of pairs $\{(S^i, x^i)\}_{i \in L}$ constitute allocation $\hat{a}$.

Lemma 4. $v_i(\hat{a}(i)) \geq \bar{v}_i$ for all $i \in N$.

Proof. For all $i \in N$, either $i \in L$ and $\hat{a}(i) = (S^i, x^i) \in C^i$, or $i \notin L$, $i \in S^j$ for some $j \in L$, then, $\hat{a}(i) = (S^j, x^j)$ and $v_i(S^j, x^j) \geq \bar{v}_i$. ∎

Next, we show that $\hat{a}$ does not involve the null public good μ.

Lemma 5. For all $i \in N$, $\hat{a}(i) \neq (S, \mu)$ for any $S \subset N$.

Proof. Suppose there is a coalition S that consumes μ. Suppose there is a coalition T that is adjacent to S and T consumes a public good $x \neq \mu$. We will show that every T adjacent to S must also consume μ.

Suppose ij is the linking edge of S and T, and $i \in S$, $j \in T$. First, suppose i has a higher priority than j. Thus, $T \subset N^i$ and for all $h \in T$, $v_h(T \cup \{i\}, x) > v_h(\hat{a}(h)) \geq \bar{v}_h$. By monotonicity, $x \in \phi(T \cup \{i\})$. Therefore, $(T \cup \{i\}, x) \in D^i$. Since $(S, \mu) = \hat{a}(i)$, lemma 3 implies $v_i(S, \mu) > \bar{v}_i > v_i(T \cup \{i\}, x)$; this is a contradiction.

Second, suppose j has a higher priority than i. Suppose g is the player of the highest priority in T (this g is unique). Thus, $(T, x) = (S^g, x^g) \in C^g$. Then $v_h(T \cup S, x) > v_h(T, x) \geq \bar{v}_h$ for all $h \in T$ and $v_h(T \cup S, x) > v_h(S, \mu) \geq \bar{v}_h$ for all $h \in S$. By monotonicity, $x \in \phi(T \cup S)$. This means $(T \cup S, x) \in D^g$ and $(T, x) \notin C^g$; this is a contradiction.

Since every coalition is adjacent to another, all coalitions must consume μ. Note that there exists $S \in 2^N$ such that $\phi(S) \neq \varnothing$. So, there is $x \in \phi(N)$

and $x \neq \mu$. Moreover, $v_i(N, x) > v_i(\hat{a}(i)) \geq \bar{v}_i$ for all $i \in N \backslash r$. This means $(N, x) \in D^r$; a contradiction. ∎

So far, we have shown that $\hat{a}$ is well-defined. The next lemma shows that $\hat{a}$ has the separation property and, then, it is a Tiebout equilibrium.

Lemma 6. *$\hat{a}$ satisfies the separation property (G).*

Proof. Since $\hat{a}$ consists of connected coalitions, there is a unique linking edge ij of two adjacent coalitions S and T. Suppose $i \in S$, $j \in T$, S consumes x, and T consumes y. Without loss of generality, suppose i has a higher priority than j. First, $y \in \phi(T \cup \{i\})$ and $(T \cup \{i\}, y) \in D^i$, so we have $v_i(\hat{a}(i)) \geq v_i(T \cup \{i\}, y) > v_i(\hat{a}(j))$ because of $\hat{a}(i) \in C^i$ and the network effect. Second, suppose $v_j(\hat{a}(i)) \geq v_j(\hat{a}(j))$; then, $v_j(\hat{a}(i)) \geq \bar{v}_j$. By monotonicity, $x \in \phi(S \cup \{j\})$. Let g be the player of the highest priority in S. Then, $(S \cup \{j\}, x) \in D^g$ which means $(S, x) \notin C^g$; a contradiction. So, $v_j(\hat{a}(j)) > v_j(\hat{a}(i))$. Finally, suppose there is $h \in M(ij)$ such that $v_h(\hat{a}(j)) > v_h(\hat{a}(i))$, then $j, h \in \{k \in N \mid u_k(y) - u_k(x) > f(|S|) - f(|T|)\}$. Since i is on the path linking j, h, the above set is not connected and this violates connected support. So, there is no $h \in M(ij)$ such that $v_h(\hat{a}(j)) > v_h(\hat{a}(i))$. By the same argument, there is no $h \in M(ji)$ such that $v_h(\hat{a}(i)) > v_h(\hat{a}(j))$. ∎

By Lemma 1, $\hat{a}$ is a Tiebout equilibrium. Next we show that no connected (G) coalition can block. Suppose there is a pair (S, x) such that $x \in \phi(S)$ and $v_i(S, x) > v_i(\hat{a}(i))$ for all $i \in S$ and S is connected. Then, $v_i(S, x) > v_i(\hat{a}(i)) \geq v_i(S^i, x^i)$ for all $i \in S$. Suppose g is the player of the highest priority in S; then $S \in N^g$, $(S, x) \in D^g$, and $v_g(S, x) > v_g(S^g, x^g)$; a contradiction. Consequently, a well-defined allocation $\hat{a}$ is in the core by Proposition 1. ∎

The following example illustrates how the algorithm works. Core allocations in general are not efficient. The constructed allocation may have a better chance of achieving efficiency since it allows some players to choose the most preferred coalitions and public goods. However, as shown in the following example, it can still be inefficient when monotonicity and connected support are satisfied. In the example, the decision-maker is indifferent between two public goods, and the

coalition forgoes a chance to improve efficiency.

Example 1. Consider a three-player game: $N = \{1\ 2\ 3\}$, $X = \{x\ y\ z\}$, $\phi(1) = \{y\}$, $\phi(2) = \phi(3) = \{z\}$, $\phi(1\ 3) = \phi(2\ 3) = \{y\ z\}$, and $\phi(1\ 2) = \phi(N) = X$. Their utility functions are the following: $f(n) = n/5$

$$u_1(x) = 2,\ u_1(y) = 2,\ u_1(z) = 1,$$
$$u_2(x) = 3,\ u_2(y) = 2,\ u_2(z) = 1,$$
$$u_3(x) = 3,\ u_3(y) = 1,\ u_3(z) = 2.$$

Link players according to their labels $1 - 2 - 3$; thus, connected support is satisfied. Let 1 be the root. We have $C^3 = ((\{3\}, z))$, $C^2 = ((\{2\ 3\}, y))$, and $C^1 = ((\{1\ 2\}, x), (\{1\ 2\}, y))$. Take $(S^1, x^1) = (\{1\ 2\}, y)$, and $(S^3, x^3) = (\{3\}, z)$, we have allocation $(\{\{1\ 2\}, \{3\}\}, (y, z))$. This is not efficient since $\{1\ 2\}$ can switch to x and make player 2 strictly better off. ∎

3 Non-monotonic feasible sets

Although different preference structures are used, the monotonicity assumption appears or is implied in all previous results of public good models (Guesnerie and Oddou 1981, Greenberg and Weber 1986, 1993, and Demange 1994). Monotonicity looks benign but there are no positive results for the core without assuming it. In this section, two stability notions weaker than the core are examined.

First, *individually stable allocation* requires that no player can join another coalition and makes every member of the new coalition strictly better off (Dréze and Greenberg 1980, Bogomolnaia and Jackson 2002). *Even though preferences over public goods are intermediate and also single-peaked, there may not exist an individually stable allocation when feasible sets are not monotonic.* This is demonstrated in Example 2.

Player's preferences over public goods are *single-peaked* if there is a linearly order $>$ on X such that for all $i \in N$, there exists $x_i \in X$ such that for all $y, y' \in X$, $x_i > y > y'$ and $y' > y > x_i$ respectively implies $u_i(x_i) > u_i(y) > u_i(y')$.

Players' have *intermediate preference (G)* over public goods if there is a tree graph G on N such that for any pair $x, y \in X$, the sets $\{i \in N \mid u_i(x) > u_i(y)\}$ and $\{i \in N \mid u_i(x) \geq u_i(y)\}$ are connected on G.

A feasible allocation a is an *individually stable allocation* if there is no $i \in N$, $S \subset \Pi_a \cup \{\emptyset\}$, and $x \in \phi(S \cup \{i\})$ such that $v_j(S \cup \{i\}, x) > v_j(a(j))$ for all $j \in S \cup \{i\}$.

Example 2. Consider a three-player game: $N = \{1\ 2\ 3\}$, $X = \{x\ y\ z\ \alpha\ w\}$, $\phi(\{1\}) = \phi(\{2\}) = \phi(\{3\}) = \{\alpha\}$, $\phi(\{1\ 2\}) = \{x\}$, $\phi(\{2\ 3\}) = \{y\}$, $\phi(\{1\ 3\}) = \{z\}$, and $\phi(N) = \{w\}$. For simplicity, let the network effect be negligible in the sense that it will not tip over preferences over public goods.[6] Their utility functions are the following:

$$u_1(x) > u_1(y) > u_1(z) > u_1(\alpha) > u_1(w)$$
$$u_2(y) > u_2(z) > u_2(x) > u_2(\alpha) > u_2(w)$$
$$u_3(z) > u_3(y) > u_3(x) > u_3(\alpha) > u_3(w)$$

Preferences are intermediate if players are linked according to their labels $1 - 2 - 3$. Preferences are single-peaked as well if public goods are ordered as $x > y > z > \alpha > w$. Player 1 will stay alone with α in allocation $(\{\{1\ 2\ 3\}\}, (w))$. Player 1 will join 2 with public good x in allocation $(\{\{1\}, \{2\}, \{3\}\}, (\alpha\ \alpha\ \alpha))$. Player 2 will join 3 with public good y in allocation $(\{\{1\ 2\}, \{3\}\}, (x\ \alpha))$. Player 3 will join 1 with public good z in allocation $(\{\{2\ 3\}, \{1\}\}, (x\ \alpha))$. Player 1 will join 2 with public good x in allocation $(\{\{1\ 3\}, \{2\}\}, (z\ \alpha))$. There is no individually stable allocations. ∎

Another way to relax the core is to reduce a coalition's ability to exclude members. Tiebout equilibrium treats memberships differently from the core: Core allows a blocking coalition to form if every member can be better off. This means that a coalition can exclude members; one cannot join a coalition if her arrival makes others worse off. On the other hand, Tiebout equilibrium allows an individual to join a coalition freely. It is possible to make existing members worse off when joining a coalition. For example, one can move into a congested community and reduce the welfare of its residents. This means that Tiebout equilibrium does not allow coalitions to exclude individual members. The above observation motivates the *no-exodus equilibrium*, which requires that when a coalition deviates

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[6]For example, in a game with finite X, we can let $\bar{u} = \min_{i \in N,\ x \in X} u_i(x)$ and $f(|N|) = \bar{u}/|N+1|$. Notice that these examples still work if the network effect is absent.

with a public good, they have to include all players who prefer the new alternative. It captures the idea of free migration into a new country which welcomes all newcomers who dislike the status quo. The no-exodus equilibrium is equivalent to the core when feasible sets are monotonic. This is a notion specific to public good models that does not apply to payoff functions.

Definition 1. A feasible allocation a is a *no-exodus equilibrium* if there is no public good $x \in X$ such that $x \in \phi(E)$ where $E = \{i \in N \mid v_i(E, x) > v_i(a(i))\}$.

When an exodus coalition E forms with a feasible public good x, it allows all players who prefer (E, x) to the status quo to join. This is in contrast with the core, where a blocking coalition can form with some other players who want to join left out.

Obviously, the no-exodus equilibrium contains the core since an exodus coalition is also a blocking coalition. When feasible sets are monotonic, the two notions are equivalent since the same public good provided by a blocking coalition will still be feasible if more people join in. The following example illustrates the difference between them: *when feasible sets are not monotonic, the core may be empty while a no-exodus equilibrium exists.*

Example 3. Consider a three-player game: $N = \{1\ 2\ 3\}$, $X = \{x\ y\ z\ w\}$, $\phi(N) = \phi(\{1\}) = \phi(\{2\}) = \phi(\{3\}) = \{w\}$, $\phi(\{1\ 2\}) = \{y\}$, $\phi(\{2\ 3\}) = \{z\}$, and $\phi(\{1\ 3\}) = \{x\}$. For simplicity, let the network effect be negligible as in the previous example. Their utility functions are the following:

$$u_1(x) > u_1(y) > u_1(z) > u_1(w)$$
$$u_2(x) > u_2(y) > u_2(z) > u_2(w)$$
$$u_3(z) > u_3(x) > u_3(y) > u_3(w)$$

Note that preferences are intermediate if players are linked according to their labels $1 - 2 - 3$. Preferences are single-peaked as well if public goods are ordered as $y > x > z > w$. Allocations with $\{\{1\ 2\ 3\}\}$ and with $\{\{1\}, \{2\}, \{3\}\}$ are blocked by $\{1\ 2\}$ with y; allocation $(\{\{1\ 2\}, \{3\}\}, (y\ w))$ is blocked by $\{1\ 3\}$ with x; allocation $(\{\{1\ 3\}, \{2\}\}, (x\ w))$ is blocked by $\{2\ 3\}$ with z; allocation $(\{\{1\}, \{2\ 3\}\}, (w\ z))$ is blocked by $\{1\ 2\}$ with y. Even through connected support is also satisfied, the core is empty. However, $(\{1\ 2\ 3\}, (w))$, $(\{\{1\}, \{2\}, \{3\}\}, (w\ w\ w))$, $(\{\{1\ 2\}, \{3\}\}, (y\ w))$ are no-exodus equilibria. ∎

The following example shows that *there may not exist a no-exodus equilibrium when feasible sets are not monotonic even though preferences over public goods are intermediate and single-peaked.*

Example 4. Consider a three-player game: $N = \{1\ 2\ 3\}$, $X = \{x\ y\ z\ w\}$, $\phi(\{1\}) = \{z\}$, $\phi(\{2\}) = \{y\}$, $\phi(\{3\}) = \{x\}$, $\phi(\{1\ 2\}) = \{x\}$, $\phi(\{2\ 3\}) = \{w\}$, $\phi(\{1\ 3\}) = \{y\}$, and $\phi(N) = \{z\}$. For simplicity, let the network effect be negligible. Their utility functions are the following:

$$u_1(x) > u_1(w) > u_1(y) > u_1(z)$$
$$u_2(w) > u_2(x) > u_2(y) > u_2(z)$$
$$u_3(y) > u_3(z) > u_3(w) > u_3(x)$$

Preferences are intermediate if players are linked according to their labels $1 - 2 - 3$. Preferences are single-peaked if public goods are ordered as $x > w > y > z$. Players $\{1\ 2\}$ will form an exodus coalition with x in allocations $(\{\{1\},\{2\},\{3\}\},(z\ y\ x))$, $(\{\{1\ 2\ 3\}\},(y))$, and $(\{\{1\ 3\},\{2\}\},(y\ y))$. $\{2\ 3\}$ will form an exodus coalition with w in allocation $(\{\{1\ 2\},\{3\}\},(x\ x))$. $\{1\ 3\}$ will form an exodus coalition with y in allocation $(\{\{1\},\{2\ 3\}\},(z\ w))$. ∎

4   Concluding Remarks

Utility functions that contain public and private goods and coalition members were proposed by Drèze and Greenberg (1980) but did not attract many followers. Subsequent works investigate public good models and pure hedonic models separately. This paper uses a simple hedonic utility function and focuses on the network effect in local public goods. Players have preferences over coalition size as well. We derive characterizing properties for the core and obtain its nonemptiness.

With the network effect, each coalition is composed of players with similar preferences (they are connected on a tree structure). Moreover, in a core allocation, every player prefers own coalition to another. This means core allocations are envy-free. The existence of a core allocation is obtained in several papers, but efficiency in public good models, however, is not guaranteed and not characterized. More

There seems to be no results in public good models without assuming monotonic feasible sets. A weaker stability notion, individually stable allocations, may not

exist even with strong preference restrictions. We also examine the no-exodus equilibrium which contains the core and is equivalent to the core when feasible sets are monotonic. It is relaxed from the core by not allowing coalitions to exclude members. Still, we fail to obtain positive results. Whether stability can be obtained without monotonicity requires further research.

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