On characterization of a class of convex operators for pricing insurance risks

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Abstract

The properties of risk measures or insurance premium principles have been extensively studied in actuarial literature. We propose an axiomatic description of a particular class of coherent risk measures defined in Artzner, Delbaen, Eber, and Heath (1999). The considered risk measures are obtained by expansion of TVar measures, consequently they look like very interesting in insurance pricing where TVar measures is frequently used to value tail risks.

KEYWORDS: Risk measures, premium principles,Choquet measures distortion function,TVar .

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1 Introduction

In recent years the axiomatic approach to risk measures has been an important and very active subject. Risk measures can be characterized by axioms that may be different for various applications of actuarial and financial interest.In fact depending on where it is used a risk measures should take into account different probabilities quantities such as variability, tail behaviour or skewness.
In actuarial science the premium principles are considered risk measures, , and the insurers are often interested to measure the upper tails of distibution functions. Risk measures according to the last meaning are considered and studied in [1],[6],[9]. In particular in [1] there is the definition of coherent risk measure which represents a landmark in the following developments, see [3],[6],[8].
We propose an axiomatic approach based on a minimal set of properties

which characterizes an actuarial risk measure likewise in [5]and [8]. In particular we show that the considered actuarial risk measures have a Choquet integral representation with respect to a distorted probability.
As it is well known distortion risk measures introduced in the actuarial literature by Wang [6] and they belong to an important class of risk measures that include Value at Risk at level α, V_α, and Tail Value at Risk at level α, $TVaR_\alpha$. We give an axiomatic characterization of a class of risk measures; our representation guarantees us to achieve risk measures which are all coherent (see Theorem 2) and which are a convex combination of $TVaR_\alpha, \alpha \in [0,1]$.
The paper is organized as follow. In section 2 we propose and discuss the properties of actuarial risk measures. In section 3 we recall some basic facts of Choquet expected utility and we present its most important features in connection with the properties of actuarial risk measure of section 2.
Finally in section 4 we obtain the integral representation result for actuarial risk measures and the characterization as convex combination of $TVaR_\alpha$.

2 Properties of insurance premium functional

We consider an insurance contract in a specified time period $[0,T]$. Let Ω be the state space and $\mathcal{F}$ the event σ-field at the time $T$and we denote by S_X the survival function corresponding to F_X.. Let Π be a probability measure on $\mathcal{F}$. We consider an insurance contract described by a non-negative random bounded variable X, $X : \Omega \to \mathbb{R}$ where $X(\omega)$ represents its payoff at time T if state ω occurs. In actuarial applications a risk is represented by a nonnegative random variable.
If X is a random variable we denote by F_X the distribution function of X i.e. $F_X(x) = \mathbf{P}(\omega \in \Omega : X(\omega) \leq x), \quad x \in \mathbb{R}$.
Frequently an insurance contract provides a franchise and then it is interesting to consider the values ω such that $X(\omega) > a$: in this case the contract pays for $X(\omega) > a$ and nothing otherwise. Then it is useful to consider also the random variable

$$(X - a)_+ = \max(X(\omega) - a, 0) \tag{2.1}$$

Let L be a set of nonnegative random variables such that L has the following property:

i) $\quad aX, \quad (X-a)_+, \quad (X-(X-a)_+) \in L \qquad \forall X \in L$, and $a \in [0,+\infty)$.

We observe that the assumption **i)** does not require that L is a vector space.
We denote the insurance prices of the contracts of L by a functional P where

$$H : L \to \widetilde{\mathbb{R}} \tag{2.2}$$

We consider some properties that it is reasonable to assume for a insurance functional price H.

(**P1**) $H(X) \geq 0$ for all $X \in L$.

Property (**P1**) is a very natural requirements.

(**P2**) If $c \in [0, +\infty)$ then $H(c) = c$.

Property (**P2**) implies that when there is no uncertainty, there is no safety loading.

(**P3**) $H(X) \leq \sup_{\omega \in \Omega} X(\omega)$ for all $X \in L$.

This is just a natural price condition for any customer who wants to underwrite an insurance policy.

(**P4**) $H(aX + b) = aH(X) + b$ for all $X \in L$ such that $aX + b \in L$ with $a, b \in [0, +\infty)$.

This is a linearity property.

(**P5**) $H(X) = H(X - (X - a)_+) + H((X - a)_+)$ for all $X \in L$ and $a \in [0, +\infty)$.

This condition splits into two comonotonic parts a risk X (see for example [3]), and permits to identify the part of premium charged for the risk with the reinsurance premium charged by the reinsurer.

(**P6**) If $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$ for $X, Y \in L$ then $H(X) \leq H(Y)$.

This condition states that the price of the larger risk must be higher.

(**P7**) If $E(X-a)_+ \leq E(Y-a)_+$ for all $a \in [0, +\infty)$ then $H(X) \leq H(Y)$ for all $X, Y \in L$.

In other words our functional price H respects the stop-loss order. We remember that stop-loss order considers the weight in the tail of distributions; when other characteristics are equals, stop-loss order select the risk with less heavy tails.

(**P8**) $H(X + Y) \leq H(X) + H(Y)$ for all $X, Y \in L$ such that $X + Y \in L$.

This property require the premium for the sum of two risks is not greater that the sum of the individual premiuum; otherwise the buyer would simply insure the two risks separately.

(**P9**) $H(aX + (1-a)Y) \leq aH(X) + (1-a)H(Y)$ for all $X, Y \in L$ and $a \in [0,1]$such that $aX + (1-a)Y \in L$.

Convexity means that diversification does not increase the total risk.In the insurance context this property allows for pooling -of -risks effects.

(**P10**) The price, $H(X)$, of the insurance contract X depends only on its distribution F_X.

Frequently this hypothesis is assumed in literature, see for example [8]. The property (**P10**) says that it is not the state of the world to determine the price of a risk, but the probability distribution of X assigns the price to X. So risks with identical distributions have the same price.

Finally, we present a continuity property that is usual in characterizing certain premium principles.

(**P11**) $\lim_{n\to+\infty} H(X - (X-n)_+) = H(X)$ for all $X, Y \in L$.

3 Choquet pricing of insurance risks

The development of premium functionals based on Choquet integration theory has gained considerable interest in recent years when there is ambiguity on the loss distribution or when there is correlation between the individual risks the traditional pricing functionals may be inadequate to determine the premiums that cover the risk.
Capacities are real-valued functions defined on 2^Ω that generalize the notion of probability distribution. Formally a capacity is a normalized monotone function.

Definition 1. *A function* $\upsilon : 2^\Omega \to I\!R^+$ *is called a capacity if*

i) $\upsilon(\emptyset) = 0$ *and* $\upsilon(\Omega) = 1$.

ii) if $A, B \subseteq 2^\Omega$ *and* $A \subseteq B$ *then* $\upsilon(A) \leq \upsilon(B)$.

A capacity is called convex if $\upsilon(A)+\upsilon(B) \leq \upsilon(A\cup B)+\upsilon(A\cap B)$*if* $A, B \subseteq 2^\Omega$.

We will assume here $\upsilon(\Omega) = 1$ as usual although this is not necessary.

We introduce a non-additive integral operator. As is well known the Choquet integral has been extensively applied in the context of decision under uncertainty. For the properties of nonadditive integration we refer to [2].

Definition 2. *Let υ a capacity $\upsilon : 2^\Omega \to I\!R^+$ and X a random variable defined on $(\Omega, \mathcal{F})$ then the Choquet integral of X respect to υ is*

$$\int_\Omega X d\upsilon = \int_0^{+\infty} \upsilon\{\omega : X(\omega) > x\} dx \qquad (3.1)$$

We give now the representation theorem for the functional H which satisfies some properties of the list above.

Theorem 1 (Modified Greco Theorem) Let L be a set of nonnegative random variables such that L has property i) . Suppose that a premium principle $H : L \to \widetilde{\mathbb{R}}$ satisfies the properties (**P1**),(**P2**), (**P5**), (**P6**) ,(**P8**) and(**P11**) .
Then there exists a convex capacity $\upsilon : 2^\Omega \to \mathbb{R}$ such that for all $X \in L$

$$H(X) = \int_\Omega X d\upsilon = \int_0^{+\infty} \upsilon\{\omega : X(\omega) > x\} dx \qquad (3.2)$$

Proof By the representation theorem and prop. 1.2 of [4] there exists a capacity υ such that $H(X) = \int_\Omega X d\upsilon$. It is well known that υ is convex if and only if H is subadditive.

Remark By the properties of Choquet integral it is easy to prove that the functional L in Theorem 1 satisfies also properties (**P3**) ,(**P4**) and(**P9**) .

4 Distortion risk measures

In this paragraph we report some well known risk measures and present the distortion functions measure for some of them. Distorted probability measure was introduced for calculation of insurance premium for nonnegative risks in [6].

If X is a random variable the quantile reserve at 100αth percentile or Value at Risk is

$$V_\alpha(X) = \inf\{x \in \mathbb{R} |\, F_X(x) \geq \alpha\} \qquad \alpha \in (0,1) \tag{4.1}$$

A single quantile risk measure of a fixed level α does not an information about the thikness of the upper tail of the distribution function of X, so that other measures are considered.
In particular we consider the Tail Value at Risk at level α, $TVaR_\alpha(X)$, is defined as:

$$TVaR_\alpha(X) = \frac{1}{(1-\alpha)} \int_\alpha^1 V_\alpha(X) d\alpha \qquad \alpha \in (0,1) \tag{4.2}$$

It is known, that given a non negative random variable X, for any increasing function f, with $f(0) = 0$ and $f(1) = 1$, i.e. a distortion function, we can define a premium principle

$$H(X) = \int_0^{+\infty} (1 - f(F_X(t))dt = \int_0^{+\infty} g(S_X(t))dt = \int_\Omega X dv \tag{4.3}$$

with $g(x)$= $1 - f(1-x)$ and $v = fo\Pi$.

Remark All distortion premium principles with g concave or f convex satisfy the properties $(\mathbf{P1}) - (\mathbf{P11})$,.
The quantile Value at Risk is not a convex risk measure, while TailVar is a convex risk measure.
In fact, $TVaR_\alpha$ can be obtained by where f is the function defined as follows:

$$f(u) = \begin{cases} 0 & u < \alpha, \\ \frac{(u-\alpha)}{(1-\alpha)} & u \geq \alpha \end{cases} \tag{4.4}$$

5 Representation of a class of premium functionals

Before of deriving the representation theorem, we show a preliminary result.

Proposition If f is a continuous increasing convex function, defined on $[0,1]$ then exists a probability measure μ on $[0,1]$ such that

$$f(x) = \int_0^1 \frac{(x-\alpha)_+}{(1-\alpha)} d\mu(\alpha) \tag{5.1}$$

for $\alpha \in [0,1]$.

Proof Given f a continuous increasing convex function with f(0)=0 then exists a non negative measure ν on $[0,1]$ such that

$$f(x) = \int_0^x (x-\alpha) d\nu(\alpha). \tag{5.2}$$

We can write

$$f(x) = \int_0^1 (x-\alpha)_+ d\nu(\alpha). \tag{5.3}$$

Then a probability measure μ on $[0,1]$ exists such that

$$f(x) = \int_0^1 \frac{(x-\alpha)_+}{(1-\alpha)} d\mu(\alpha), \qquad \alpha \in [0,1]. \tag{5.4}$$

Theorem 2 Let L be a set of nonnegative random variables such that L has property i) and such that for every $a \in [0,+\infty)$, $I_a \in L$. Suppose that a premium principle $H : L \to \widetilde{\mathbb{R}}$ satisfies the properties (**P1**),(**P2**), (**P5**), (**P7**) ,(**P10**) and(**P11**) .
Then there exists a probability measure m on $[0,1]$ such that:

$$P(X) = \int_0^1 TVaR_\alpha(X) dm(\alpha) \tag{5.5}$$

Proof It is easy to prove that property (**P7**) imply property (**P6**). Then by the proof of Theorem 1 we can conclude that there exists a capacity $\upsilon : 2^\Omega \to \mathbb{R}$ such that for all $X \in L$

$$H(X) = \int_\Omega X d\upsilon \tag{5.6}$$

Since H has the comonotonic additivity property,[3], moreover H verifies (**P7**) then H is subadditive, [**?**], then the capacity is convex. By property (**P10**)an by the fact that for every $a \in [0,+\infty)$, $I_a \in L$ we have that there exists a convex increasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$ i.e. f such that

$$P(X) = \int_0^{+\infty} (1 - f(F_X(t)) dt \tag{5.7}$$

From Proposition 1 follows that a probability measure $m(\alpha)$ exists such that f can be represented

$$f(x) = \int_0^1 \frac{(x-\alpha)_+}{(1-\alpha)} dm(\alpha) \tag{5.8}$$

for $\alpha \in [0,1]$, and $f(1) = 1$.

Then interchanging the integrals for the Fubini Theorem for every $X \in L$:

$$\begin{aligned} P(X) &= \int_0^{+\infty} (1 - f(F_X(t))dt = \int_0^{+\infty} [1 - \int_0^1 \frac{(F_X(t)-\alpha)_+}{(1-\alpha)} dm(\alpha)]dt = \\ &= \int_0^{+\infty} dt \int_0^1 [1 - \frac{(F_X(t)-\alpha)_+}{(1-\alpha)}]dm(\alpha) = \\ &= \int_0^1 dm(\alpha) \int_0^{+\infty} dt[1 - \frac{(F_X(t)-\alpha)_+}{(1-\alpha)}] = \\ &= \int_0^1 dm(\alpha) TVaR_\alpha \end{aligned} \tag{5.9}$$

Then the results obtained for the class of insurance functional prices seems interesting both because the class of functionals is determined from few natural properties and these functional prices follow closely linked together to a well known risk measure as $TVaR_\alpha$, $\alpha \in [0, 1]$. Moreover we point out that the most important properties for a functional price follow easily from the obtained representation.

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