A Framework for Studying Economic Interactions (with applications to corruption and business cycles)

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Abstract

Most economic models implicitly or explicitly assume that interactions between economic agents are 'global' - in other words, each agent interacts in a uniform manner with every other agent. However, localized interactions between microeconomic agents are a pervasive feature of reality. What are the implications of more limited interaction? One set of mathematical tools which appears useful in exploring the economic implications of local interactions is the theory of interacting particle systems. Unfortunately, the extant theory mainly addresses the long-time behavior of infinite systems, and focuses on the issue of ergodicity; many economic applications involve a finite number of agents and are concerned with other issues, such as the extent of shock amplification. In this paper, I introduce a framework for studying local interactions that is applicable to a wide class of games. In this framework, agents receive shocks which are stochastically independent; payoffs depend both upon the shocks and the strategies of other agents. In finite games, ergodicity is straightforward to determine. In finite games which evolve in continuous time, the stationary distribution (if it exists) may be computed easily; furthermore, in this class of games, I prove that *any* stationary distribution may be attained by suitable choice of payoff functions using shocks which are distributed uniform on (0, 1). In systems in which all interactions are global, I prove that nonlinear behavior can arise even in the infinite limit (thus demonstrating that laws of large numbers can fail in systems characterized by interaction), despite the fact that the only driving forces are agent-level iid disturbances. Using numerical methods, I investigate the properties of the processes as one passes from discrete to continuous time, as one alters the pattern of interaction, and as one increases the number of interacting agents. In so doing, I provide further evidence that the existence of local interactions can change the aggregate behavior of an economic system in fundamental ways, and that the form of that interaction has important implications for its dynamic properties.

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1. Introduction

Local interactions are a pervasive feature of reality; examples include the spread of disease, the spread of information, competition in niche or geographical markets, or input-output linkages between firms. Yet relatively little work has studied their implications for the economy. This is surprising, given that such interactions are such a widespread phenomenon, and given that recent work has highlighted a number of their surprising implications. To take one example, local interactions can imply that persistent business cycles may be driven *entirely* by small, independent, agent-level (microeconomic) shocks (see Jovanovic, 1987; Horvath, 1997a; and Verbrugge, 1998). In other words, local interactions can sufficiently amplify completely idiosyncratic disturbances that the aggregate displays substantial volatility and persistence.[1] This is obviously of some interest to macroeconomists, since a growing empirical literature casts doubt on the (still predominant) hypothesis that aggregate fluctuations are driven by large, persistent aggregate shocks (e.g., Long and Plosser (1987), Cooper and Haltiwanger (1990, 1996), and Horvath and Verbrugge (1997)). Further, many of the aggregate or global interactions that are assumed in many theoretical models are more realistically thought of as local or limited (i.e., involve only a small number of other agents). Does restricting their range of interaction overturn the key results? This remains an open, and important, question. There is a growing literature exploring the implications of local interactions; however, most of this literature focuses on a particular issue, namely the long-time behavior of economies in which agents play two by two repeated games with myopic Darwinian or best-response behavior combined with noise or trembles. In this context, it has been shown that the ergodic distribution becomes concentrated on the risk-dominant Nash equilibrium (Kandori, Mailath and Rob, 1993; Haller and Outkin, 1998; *****), that local interactions can increase the number of potential long run equilibria (*****), and that local interactions *can* speed convergence to equilibrium (Ellison, 1993), though this need not be the case (Blume, 1995). (I review other related literature below.) The main focus of this paper is to develop a general framework that is applicable to a wide range of applications, and derive general implications, rather than making special assumptions on the nature of the game or the distribution of "noise".

In some part, the relative neglect of exploring the implications of local interactions is due to the complexity of modeling heterogeneous agents. Recently, interacting particle systems have been used as a modeling device to describe the implications of local (and aggregate) interactions between entities (Föllmer, 1974; Durlauf, 1991, 1992, 1993, 1994; Brock, 1993; Cooper, 1993;

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[1] The key to the amplification of the effects of these small shocks is the *interactions* of economic agents; when there exist local strategic complementarities between agents, laws of large numbers need not apply, and small disturbances can be substantially amplified.

Brock and Durlauf, 1995; Aoki, 1996; and Verbrugge, 1998).[2] Such systems appear to be quite promising in explaining a variety of observed phenomena, such as the volatility and persistence of aggregate time series, and may display interesting properties (such as nonergodicity). Further, such models may help us think more clearly about the effects of government policy, which is often conducted on the basis of macroeconomic or aggregate indicators. As many microeconomic states are compatible with the same macroeconomic state, the distributional, and probably even the aggregate, impact of policy actions will depend on the particular microeconomic state .. hence, policy will likely have a nonlinear effect on the economy.

Unfortunately, extant interacting particle systems theory is not tailor-made for economic applications, and most of the theory focuses on the long time behavior of infinite systems, chiefly resolving ergodicity. Though long-run ergodicity is, in some contexts, of interest to economists (e.g., Durlauf, 1993a), a great many other issues are important as well. For example, key questions in some contexts, such as business cycle analysis, are the variance and the degree of persistence exhibited by the economy. Persistence might matter in other contexts as well, such as the analysis of the degree of corruption in a society. In general ergodic systems, one might be interested in being able to determine the equilibrium distribution of the economy. Another key question is the degree to which the behavior of the economy changes as more and more agents are added, or as the nature or degree of interaction changes. Finally, the behavior of such systems over relevant time scales is of interest: nonergodic systems might never converge on a human time scale, whilst ergodic systems might become trapped in a metastable state for thousands of years, so that (on human time scales) they behave as if they were nonergodic.

Another major drawback of extant interacting particle systems models is that they tend to be unwieldy (but see the example in Verbrugge (1998)). One goal of this paper is to put forward a *tractable* class of interacting particle systems which appear to be ideally suited to exploring a variety of issues, i.e., a framework within which to study economic interaction. Herein, I explore two such issues in depth: corruption, and delay and business cycles. In each case, the economic behavior of each agent is summarized by a binary decision - in the first case, whether or not to act in a corrupt fashion this period, and in the second, whether or not to invest this period. The more general systems considered in this paper share the property that each economic agent (or entity) may be in (or select) one of two states (or strategies). This is not as restrictive as it first appears: there are many interesting economic situations in which optimal adjustment behavior by

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[2] There are a number of related studies which investigate the implications of local interactions in other contexts: cellular automata (Cowan and Miller, 1989; Bell, 1994), temporally ordered interaction behavior describing lock-in or informational cascades (Katz and Shapiro, 1986; Keisler, 1986; Arthur, 1989, 1990, and 1993; David, 1989,), random linkages between agents (Kirman, 1983; Kirman, Oddou, and Weber, 1986; Ioannides, 1988; Durlauf, 1993), and sandpile models (Bak, Chen, Scheinkman, and Woodford, 1994).

microeconomic units is not continuous or small, in which decision rules are of the threshold type, or which involve what amounts to binary states or decisions: lumpy investment, bank runs, choice of strategy, or entry or withdrawal from a market, to name a few. Further, some generality is obtained when one recognizes that this binary state could even refer to a particular *equilibrium* (in the case of multiple equilibria). The framework itself should be taken as illustrative of economies in which there are strong local interactions and nonlinear or lumpy adjustments. Note that the discrete time versions of this process are related to the discrete time contact processes used by Durlauf (1993), Brock (1993), and Brock and Durlauf (1995).[3] The continuous time versions are related to the continuous time process introduced in Verbrugge (1998), and encompass the stochastic Ising model.

Key theoretical findings are the following: First, ergodicity properties of finite systems are easily deduced. Ergodicity (or the lack thereof) is of crucial importance in some economic contexts; for example, in the context of economic growth, nonergodicity could imply that economies converge to different long run growth rates, while in the context of a model of corruption, nonergodicity might imply that societies converge to one of two long run equilibria, honest or corrupt. Using this result, it is straightforward to derive conditions under which, in the corruption model, the economy is nonergodic. As noted above, in some frameworks ergodicity or nonergodicity may be of little practical relevance. Using simulations, I demonstrate that both situations are likely to be relevant.

The second set of results are reminiscent of those in Jovanovic (1987), and have to do with systems in which each agent interacts with all other agents. First, for finite systems in which interaction is global, *any* strictly positive equilibrium distribution may be attained given the choice of a suitable probability law. Thus, it is clear that these systems can readily generate economically interesting behavior. Second, there exist probability laws governing interactions and transitions which give rise to systems whose variance does *not* fall to zero as the number of agents approaches infinity; in fact, it is straightforward to generate nonlinear and chaotic *aggregate* behavior, even though the only driving forces are agent-level idiosyncratic shocks.[4] Furthermore, there is no reason whatsoever to expect that the probability laws giving rise to such behavior are unusual or of "small measure"; in other words, aggregate volatility in the limit may be much more typical of interactive systems than has been previously believed. The proof also makes it clear that aggregate shocks might have nonlinear effects on the economy, and indeed that the effects of one-time shocks might persist indefinitely. This result is applied to the model of delay and cycles,

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[3] It is also closely related to stochastic cellular automata. It is well known that deterministic cellular automata may display endogenous noise and complex behavior, including chaotic behavior.

[4] An obvious consequence of this is that laws of large numbers do not apply to such systems.

showing conditions under which economies similar to Gale's will display nonlinear behavior in the infinite limit, though the only driving forces are independent, infinitesimal shocks.

A number of important questions remain. Do less than fully-interactive systems give rise to qualitatively similar behavior? This is a question we know a little about; there are many examples now which demonstrate that locally interacting systems can possess a richer stationary distribution. However, little more is known. Other questions abound as well. How does the behavior of discrete time systems compare to that of continuous time systems? What are the persistence properties of these systems? How does changing the pattern of interaction affect these properties? These questions are addressed via simulation. Obviously, simulations require the imposition of particular probability transition laws, and I investigate two different ones. It is well known that the behavior of interacting particle systems hinge critically on the exact probability law governing particle movements; thus the results based upon simulation evidence are much less general and merely suggestive. Nonetheless, large classes of phenomena may be studied using the range of systems investigated here, and I conjecture that many of the simulation results are robust.

Key simulation results are the following. First, the behavior of discrete and of continuous time versions of the same process is remarkably similar; this is important because it suggests that the modeler's choice between discrete and continuous time may be made safely on the basis of convenience. Other properties depend upon the probability transition law, as follows. Systems with linear probability laws have a characteristic behavior: The variance of the aggregate series is usually minimized in a system devoid of interactions (the iid case), and tends to be maximized in fully interactive systems. However, the qualitative behavior is remarkably similar as one changes the interaction topology; in fact, the degree of persistence does not depend on the degree of interaction. Laws of large numbers hold for these linear systems. Systems with ENE probability laws (defined below) display more complicated behavior; for a given level of strategic complementarity and a given number of agents, there is a critical interaction topology (number of neighbors) at which the process becomes highly variable and persistent; away from this point, the variability and persistence drop off,[5] and variance falls approximately with N. However, near critical points, laws of large numbers can be greatly "postponed", in that (in their vicinity) variance does not fall with N.

There are a handful of literatures which have some relation to this paper, and the list of papers is too long to survey it completely. Some of the literature on conventions is quite closely related, as it seeks to explain the evolution of conventions; see (*****). Similarly, the literature on lock-in and informational cascades (see Arthur; Arthur and Lane; Banerjee; Bickchandani et.

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[5] Alternatively, given an interaction topology and a number of agents, there is a critical level of strategic complementarity, away from which variability and persistence drop off.

al.) The work of Blume, of Berninghaus and Schwalbe (*****), and of Haller and Outkin (1998) is fairly closely related, though the focus of these works is more narrow, typically involving a focus on two by two or N by N games, in addition to parametric restrictions on noise, which allows the closed-form solution of the ergodic distribution in some cases.

The outline of the remainder of this paper is as follows. Section 2 introduces the issue of corruption. Section 3 formally lays out the mode. Section 4 generalizes it to a class of interacting agent systems. Section 5 presents some theoretical results for the general case, which are applied to the model of corruption. Section 6 lays out the second model of the paper, a modification of Gale's (1996) model of delay and cycles, which applies results from section 5. Section 7 presents simulation evidence. Section 8 concludes. The first appendix, section 9, presents proofs of Propositions 1 and 2. The second appendix, section 10, reviews basic concepts relating to ergodicity and stationarity.

2. Corruption

Corruption is a widespread and ancient problem that is of particular importance to the study of economic development. In many countries, in fact, politics (and the associated prospect of abusing power) has been viewed as the principal route to wealth. While economists have no difficulty explaining the prevalence of corruption, numerous authors have noted[6] that one of its most puzzling aspects is its variability. Why does corruption vary so strongly across nations, and across regions of a country, in many cases despite strong similarities between them? A common explanation, popular among sociologists, is that (national or regional) social norms simply differ. But as Bardhan (1997) points out, a major problem with such explanations is that they can become near tautological. He suggests that a more satisfactory explanation should elucidate how otherwise similar countries or regions may settle with different social norms in equilibrium, and how a country may sometimes shift from one equilibrium into another. Economists have generally relied upon models of multiple equilibria to address this question.

There are several extant models of corruption which possess multiple equilibria, exploiting the idea that in many contexts (and for various reasons), the expected gain from corruption depends upon the number of other people who are expected to be corrupt.[7] Prevalent themes in the literature are that the costs of being corrupt decline as the probability of detection (or cooperation with an investigative body) by an honest peer declines, and that switching from one equilibrium to another might be accomplished by short-lived policy changes or exogenous shocks. The first systematic economic explanation of the varying levels of corruption in otherwise similar

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[6] See, for example, Cadot, 1987; Andvig and Moene, 1990; Andvig, 1991; Tirole, 1993; and Bardhan, 1997.

[7] See Lui (1986), Cadot (1987), Sah (1988), Andvig and Moene (1990), and Tirole (1993, 1996); for surveys of the literature, see Advig (1991) and Bardhan (1997).

economies is Lui (1986), whose analysis builds upon the idea that the effectiveness of repression is inversely related to the prevalence of corruption. Officials are not equally honest, in that some have a higher moral cost from each bribe than others. In his overlapping-generations model, a temporarily harsh anti-corruption policy may induce a jump of the economy from a high-corruption equilibrium to a low-corruption one. Sah (1987) presents an overlapping-generations model in which agents who are Bayesian learners with limited information face the problem of trying to learn the prevalence of (and hence, profitability of) corruption in the society. The Bayesian learning process implies a much slower movement from high-corrupt to low-corrupt states, so a successful anti-corruption campaigns will need to be sustained longer. The model of Tirole (1993, 1996) shares this property. He explores the importance of collective reputations; in his framework, a one-shot increase in corruption will result in corruption persisting indefinitely, while a one-shot crackdown in an economy which has experienced persistent corruption in the past will have no lasting effects. (Historically, some short-lived anti-corruption campaigns appear to have been quite successful, whilst others have had much less success.) Other models which generate multiple equilibria are Cadot (1987) and Andvig and Moene (1990). Both models focus on the possibility of being detected by other honest officials, and explicitly model both the demand for and supply of corrupt services.

A shortcoming of this literature is that while they can explain how multiple equilibria can be generated, none of these models explain how a *particular* equilibrium might be selected, though several discuss what it would take to move the economy from a particular existing equilibrium to another. This is akin to making a statement like "Either VHS or Betamax will dominate the market" without providing an explanation as to the process by which VHS managed to capture the market. Thus, at best theories which merely demonstrate the existence of multiple equilibria can only partially explain how two identical economies could, starting from identical initial conditions, end up in different equilibria (and to do so, they would need to rely upon exceptional equilibrium-selecting events, rather than upon natural (stochastic) dynamics of the system). Tirole (1993) asks the question: Why does history matter? In my view, it is not necessary to look to exceptional equilibrium-selecting events to explain divergent patterns across similar economies. Rather, the model below demonstrates that a sequence of small (independently distributed) shocks can completely determine the long run behavior of the economy, with no changes in exogenous parameters.

Why might small, independent shocks matter? Why wouldn't they simply cancel out? Because *interactions* between agents may imply correlated *actions*. Indeed, an aspect of corruption which has gone almost unrecognized by theorists is the localized aspect of interactions between potentially corrupt officials. Most models assume that, on a period by period (or decision by decision) basis, officials interact with others drawn at random from the population. Thus, interactions are global, in that characteristics of the entire population are what matter. In reality, most officials work closely with only a handful of other officials, and these interactions occur or repeat (with the same agents) over a period of time. Why would this matter? While an honest agent might well report dishonest behavior by a colleague, a dishonest or corrupt agent is less likely to do so. Even honest officials might not report corrupt behavior if corruption is sufficiently widespread. Honest officials in post-independent Ghana who reported corrupt activities by their colleagues, for example, frequently faced penalties of one form or another, including humiliation, arrest, and dismissal; see LeVine (1989). Social disapproval of corrupt activity might be lower in a local climate of corruption. Further, auditing of officials generally requires the cooperation of other officials, to supply evidence. If these colleagues are corrupt, they will be less likely to supply accurate information to the government, either because they themselves will accept bribes from the individual under investigation, or because they are worried about the possibility that the additional evidence will somehow help implicate themselves. Examples of such behavior abound; for example, there is plentiful evidence that for decades corrupt officials in China have formed coalitions which made it difficult to audit them (see Liu (1986) and his sources). Thus, what might well matter for a particular official is not the overall level of corruption in the economy, but only the level of corruption among those officials with whom he has frequent contact - for example, those in his immediate office, or the level of corruption by his immediate superior. It is this *localized* aspect of the expected returns to corruption that will be focused upon in this paper. Naturally, formally modeling the more realistic case of local interactions is more difficult than positing global interaction. My purpose is not to investigate the many facets of corruption; rather, I wish to explore the implications of local interactions in a tractable framework that has potential applicability to a wide range of phenomena.

3. Corruption: local interactions and nonergodicity

In this section, a model of corruption will be described and analyzed. This model will be generalized in the next section, and results germane to the general model will be presented.

The economy evolves in discrete time, and consists of an anti-corruption agency[8], an infinite number of private citizens, and a finite, odd number N^2 of infinitely lived risk-neutral government officials, indexed by (i, j), who discount the future at rate $0 < b < 1$. Each period, each official receives a wage of w, and (in addition to other duties) each officials is randomly paired with a member of the public. Each member of the public possesses a project. Citizens are heterogeneous with respect to the value of their projects; the return r_k associated with citizen k's project is assumed to be distributed independently and uniformly (1, 2). As projects cannot be

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[8] Such agencies have been used in Singapore and China, for example.

implemented without a permit issued by a government official, each such pairing represents an opportunity for the official to extract a bribe. Thus, at the beginning of every period each official must decide whether or not to act in a corrupt manner toward (i.e., whether or not to extort a bribe from) the citizen.[9] If officials choose to extort, they are assumed to be sufficiently skillful so as to capture the entire return from the project while leaving little hard evidence in the hands of the citizen. Officials in this economy need not fear exposure from ordinary citizens, only from their colleagues.

Officials are arranged on a two-dimensional lattice, and official (i, j) associates or interacts with officials $(i+1, j)$, $(i-1, j)$, $(i, j+1)$, and $(i, j-1)$, who hereafter will be termed (i, j)'s neighbors or colleagues.[10] Interaction occurs in the following manner: immediately after the beginning of each period (after corruption decisions for the period are made), if official (i, j) acts in a corrupt manner, each of his neighbors may observe this with exogenous probability q (i.e., each of (i, j)'s neighbors has probability q of getting her hands on some incriminating evidence). If detection occurs, this fact will be common knowledge to official (i, j) and to all detecting officials. Officials who observe a corrupt colleague have the option of reporting this to the anti-corruption agency, along with the evidence, in return for a personal reward R. These reporting officials also agree to cooperate in the investigation; thus, as will be made more precise below, they bear the risk that their own past indiscretions, if any, will come to light (in which case they forfeit the reward and face other penalties outlined below.)

Officials that are turned in by a colleague (along with evidence) are investigated by the anti-corruption agency, and must pay a cost L to avoid losing their jobs; this can be thought of as a large bribe, or alternatively as expenses associated with covering up evidence, hiring lawyers, or "cooking the books". At the end of period t, the behavior of each agent (i, j) during period t becomes commonly known. However, if official (i, j) is not "caught in the act", then sufficiently hard evidence to indict him would only become available if one of his colleagues were to cooperate in an audit, at a personal cost of c (associated with getting one's books in order, time spent meeting with investigators etc.).[11] However, officials who are currently under investigation may costlessly cooperate in audits of any of their neighbors. After one period, all hard evidence

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[9] These decisions are made jointly. Note that here, in contrast to many other formulations in the literature, officials can always choose to "reform" (or, conversely, to become corrupt).

[10] To ensure symmetry, the lattice is embedded on a torus, so that the right edge of the lattice is joined to the left edge, . Thus, official (N, j) interacts with official $(1, j)$ (among others), official $(j, 1)$ interacts with official (j, N) (among others), and so on.

[11] Lippmann (1989) notes that it is typically much harder to prove corruption than it is to become aware of it; for example, there is a huge gap between scandalous political behavior and specifically indictable offenses. As noted in Section 2, it is difficult to collect enough evidence to indict someone without the cooperation of his colleagues.

completely decays, so that an official who has been "clean" for one period can no longer be indicted, regardless of his or her past indiscretions.

As noted above, officials who turn in their colleagues run the risk of having their own past behavior investigated. For example, suppose official (*k*, *l*) acted corruptly in period *t-1*, and catches the official (*k+1, l*) in the act in period *t*, then turns the evidence over to the anti-corruption unit. Since investigators must go over records, etc., official (*k*, *l*) must pay a cost l_1 associated with paying off witnesses, hiding damaging documents, and so on, in order to avoid making himself the target of an investigation.[12] Further, if the official (*k+1, l*) under investigation in turn agreed to cooperate with investigators in investigating the official (*k*, *l*), it is assumed that sufficient evidence of indiscretion will be found to warrant the forfeiture of (*k, l*)'s reward; in fact, official (*k*, *l*) must pay an additional cost l_2 to avoid losing *his* job. (Officials who turn in colleagues with evidence, but who are themselves turned in by one of their colleagues, forfeit the reward as well.) Conversely, suppose official (*k+1, l+1*), someone who acted *honestly* last period, had been the colleague who caught official (*k+1, l*) in the act. Since she does not have any recent record of corrupt behavior, she is in no danger of being investigated herself.

Each official interacts with her neighbors over an infinite number of periods. To analyze equilibrium behavior, we must first consider the stage game. The timing of events is as follows: at the beginning of the period, each pair of officials decides whether to act corruptly or not; next, detections of wrongdoing (if any) occur; then, further negotiation, trades of evidence, or bribery (between officials) is allowed; next, officials may report colleagues (with or without evidence); then, counter-accusations (without evidence) may occur; then all costs are assessed, all bribe revenue comes in, how each official decided to behave vis a vis the public becomes generally known, and finally the period ends.

First, what are the interesting Nash equilibria of the stage game? Starting at the bottom of the tree: No one has an incentive to report corrupt colleagues without evidence, unless they themselves are currently under investigation. However, once a corrupt official has been turned in, it is costless for her to cooperate in a "tit-for-tat" investigation of a recently corrupt colleague, and we may safely assume that an (unmodeled) revenge motivation would ensure that this would in fact occur. Given this behavior, a recently corrupt colleague, faced with the real possibility of a "tit-for-tat" investigation, has no incentive to turn in the evidence in the first place, as this would result in *losses* of $l_1 + l_2$, rather than a gain of *R*. Hence, a corrupt official does not fear detection by any recently corrupt colleague, since such a colleague would never turn in the evidence. (It is clear that in a situation in which two neighboring officials collect incriminating evidence on each

[12] Note that it is assumed that previous, rather than current, corrupt behavior is what matters for this purpose. This assumption is made to clarify the analysis and to focus on the dynamics of equilibrium selection.

other, and no other officials detect either one, it is a dominant strategy for both to trade evidence and forget the whole thing). Conversely, officials who have been caught in the act by a recently honest colleague have no such lever, as a "tit-for-tat" investigation would not impact the payoff of the reporting colleague. Facing this situation, as long as the corrupt official has been detected by only one colleague, he has an incentive to pay a bribe to his detector to avoid being turned in; specifically, he would be willing to bribe the detector anything less than L (in exchange for the evidence). To highlight the potential policy role of changes in R, it will hereafter be assumed that $R \geq L$; thus, a recently honest official who has obtained hard evidence of wrongdoing has no static incentive to accept a bribe from the corrupt official. This will be assumed in the sequel.[13]

This brings us to the top of the tree. An official will clearly decide to behave in a corrupt manner if the expected gains exceed the expected losses (i.e., the expected costs associated with being detected). What are these expected losses, if the above Nash behavior is assumed? This depends upon the number of (i, j)'s neighbors who were honest last period. Denoting this number by q_{ij}, this is computed as

$$\left[1-(1-q)^{q_{ij}}\right]L$$

which is the probability that none of (i, j)'s honest neighbors detect him in the act, times the cost if such detection occurred.

The expected gain from acting in a corrupt manner is simply the bribe revenue plus wages. However, the expected gain from acting in an honest manner exceeds wages minus conformity costs, since being honest has an expected second benefit: *next* period, there is a chance that the official will detect wrongdoing by one of his neighbors and thus be able to collect a reward R. The expected benefit depends upon b, R, q, and upon the probability of each neighbor acting corruptly next period. We will denote official (i, j)'s beliefs regarding the probability that neighbor office $(i, j+1)$ will act corruptly next period by $p_{i,j+1}$, ignoring for the moment how such beliefs are formed. Then the expected benefits of acting honestly this period are given by

$$w + \frac{bqR}{4} \sum_{k,l \in Ne(i,j)} p_{k,l}$$

where $Ne(i, j)$ denotes the neighbors of official (i, j).

Period-by-period play of the Nash behavior described above is clearly a candidate subgame-perfect equilibrium for the game. Folk theorems inform us that, particularly with low discount rates, there are other equilibria of the game. However, of the set of possible equilibria, period-by-period Nash is the most interesting, for the following reasons. First, it is of interest to

[13] This may be justified by assuming that the government understands this and consciously sets $R > L$. Alternatively, we may assume that $R < L$, and (by convention) a bribe B which satisfies $L \geq B \geq R$ changes hands in this situation. Most results follow if one substitutes B for R in formulas below; however, in this case, small increases in R would clearly not impact behavior.

find out whether, devoid of strategic considerations on the part of the players, we might still observe "coalitions" forming (or "cooperative" behavior emerging) spontaneously as the result of repeated play. Indeed, under parameter restrictions given below, non-strategic Nash behavior can give rise to divergent equilibrium outcomes (e.g., all players playing honest forever, and all players playing corrupt forever); thus, simple Nash behavior is extremely interesting in and of itself. (The fact that such divergent outcomes are possible given simple Nash behavior also suggests that little insight would be gained by considering other supergame equilibria.) Second, Nash behavior places minimal sophistication requirements on the players. Third, supergame equilibria require complicated social coordination, which in this context would be costly, difficult, and might break down under mild perturbations of the game (such as the replacement of players). Finally, a goal of this paper is to describe how particular equilibrium outcomes might plausibly evolve; selecting a supergame equilibrium is fairly arbitrary and leaves no role for history, in addition to requiring strong assumptions about the sophistication of players and the existence of complicated pre-game social interaction.

To complete the analysis of this game, we must return to the issue of the determination of the $p_{i,j}$ - i.e., the beliefs that agents have regarding the number of their colleagues who will act corruptly *next* period. To solve these explicitly would be a difficult task indeed. Fortunately, one can go a long way without a full solution: though one cannot completely specify the transition dynamics without them, it is possible to prove (for particular parameter restrictions) that this economy is nonergodic. Specifically, there are multiple long run equilibria, of which two are particularly interesting: one in which all agents act corruptly each period, and the other in which all agents act honestly each period. This result is important, as it highlights how two identical economies might end up with very different long run levels of corruption, and demonstrates the role of history in determining which equilibrium results. Further, I will show that there exist policy interventions which will ensure that the economy ends up in the long run equilibrium typified by honest behavior.

The information official (i, j) needs to make his (corruption) decision consists of the number of neighboring officials who acted honestly last period, and the current potential bribe revenue (i.e., r_{ij}). From this information, expected costs and benefits to corrupt behavior can be calculated, and each official can place bounds on the expected benefits to honest behavior. Specifically, the expected benefits to acting honestly must lie between

$$w+\beta qR$$

and

$$w$$

with the lower expression reflecting "pessimistic" expectations (no corrupt colleagues to observe and fink on) and the upper expression reflecting "optimistic" expectations (all colleagues acting corruptly next period). Under myopic expectations, for example, this value would be given by

$$w + \frac{qbqR}{4}$$

Analysis of equilibrium behavior is completely summarized by the function $P\left(ij \text{ plays honest} \middle| q_{ij} = k\right)$, since this economy is Markov, and this function completely describes transitions (and hence, the evolution of the economy). If all (*i, j*)'s neighbors were honest last period, then (*i, j*)'s net expected benefit to acting corruptly this period is given by

$$w + r_{ij} - \left[1-(1-q)^4\right]L$$

If $\left[1-(1-q)^4\right]L \geq 2$, then (*i, j*) will never act corruptly in this situation - $P\left(ij \text{ plays honest} \middle| q_{ij} = 4\right) = 1$. This implies that, under this parameter restriction, the outcome in which all agents behave honestly is a closed set which is an equilibrium. Conversely, suppose that all (*i, j*)'s neighbors were corrupt last period. Then (*i, j*)'s net expected benefit to acting corruptly this period is given by

$$w + r_{ij}$$

while her net expected benefit to acting honestly this period is (at most) given by

$$w + \beta qR$$

If $\beta qR \leq 1$, then (*i, j*) will never act honestly in this situation - $P\left(ij \text{ plays honest} \middle| q_{ij} = 0\right) = 0$. (This implies that an outcome in which all agents behave corruptly is a closed set which is an equilibrium.) Now suppose that two of (*i, j*)'s neighbors were honest last period. Then (*i, j*)'s net expected benefit to acting corruptly is given by

$$w + r_{ij} - \left[1-(1-q)^2\right]L$$

while his net expected benefit to acting honestly, *NBH*, satisfies

$$w + bqR \geq NBH \geq w$$

If

$$2 - \left[1-(1-q)^2\right]L > bqR \quad \text{and} \quad 1 - \left[1-(1-q)^2\right]L < 0$$

there exist realizations of r_{ij} such that (*i, j*) will act corruptly, and other realizations of r_{ij} such that official (*i, j*) will act honestly; in other words, $1 > P\left(ij \text{ plays honest} \middle| q_{ij} = 2\right) > 0$. The set of parameters which satisfy all the above inequalities is not empty.[14]

How does the level of corruption evolve in this economy? Suppose that the above parameter restrictions are satisfied. Over time, localized "cooperative" play arises endogenously,

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[14] For example, $q = .1$; $L = R = 5.81$. Note that the conditions stated are sufficient, not necessary.

even though all players are essentially behaving in a non-cooperative fashion: there arise clusters of agents who behave honestly, and clusters of agents who behave corruptly, which may be thought of as coalitions. Agents in the "interior" of such clusters (i.e., completely surrounded by agents playing strategy *i*) will play strategy *i* themselves with probability 1. The play of agents "on the border" of such clusters depends upon the particular opportunities they face: given a large extortion opportunity, they will behave corruptly, but given only a small extortion opportunity, they will behave honestly. Over time, clusters grow and merge to form larger clusters, or shrink and collapse.[15] If, given the formation of beliefs, $P\left(ij \text{ plays honest} \middle| q_{ij} = 1\right) = 0$ and $P\left(ij \text{ plays honest} \middle| q_{ij} = 3\right) = 1$[16], then two types of long run run equilibria are possible: one in which all agents play the same strategy, and another in which there are "bands" of agents (with the width of the band at least two players wide, and the heighth of the band the entire span of the torus, or vice versa) playing the same strategy, alongside such bands populated by players playing the opposite strategy. If, given the formation of beliefs, the above condition is not satisfied, then the only type of long run equilibrium possible is one in which all agents in the economy play the same strategy. The particular sequence of random draws of project returns will ultimately determine whether the economy converges to a state in which all players are behaving honestly, all players are behaving corruptly, or (under the assumptions above) one in which players are arranged in "bands". Once the economy reaches one of these states, it will remain in that state forever. This nonergodicity result will be stated formally as a corollary to Lemma 2.

There is a clear policy implication of this model. Raising the level of *R* sufficiently will cause "all-corrupt" to cease being a closed state, since it will mean that $P\left(ij \text{ plays honest} \middle| q_{ij} = 0\right) > 0$ (and hence, $P\left(ij \text{ plays honest} \middle| q_{ij} = k\right) > 0 \quad k \in \{0,\ldots,4\}$). The outcome "all honest", however, remains a closed state. These facts imply that the economy cannot converge to the outcome "all corrupt", or to an outcome in which there are bands of agents playing corruptly.

The returns to corrupt behavior have a strong localized aspect: corrupt officials who interact with each other frequently have been observed to form coalitions in order to escape detection; corrupt officials would have difficulty keeping their behavior hidden if officials with whom they interacted with frequently were honest; honest officials have been known to suffer losses when those officials they interacted with frequently were corrupt; and so on. This model

[15] There may be other transitory patterns, such as a local checkerboard pattern (with each official on "black" squares playing corrupt every even period and honest every odd period, and vice versa for officials on "white" squares), but these cannot span the entire economy nor persist indefinitely, since the number of officials is odd.

[16] This would be true under myopic expectations for the parameters listed in footnote (*****), for sufficiently large β.

demonstrates how coalitions and widespread corruption might arise "spontaneously" as a result of this sort of local interaction, and how corrruption is a contagious phenomenon. Further, it provides an explanation of how an otherwise identical economy might converge to a completely different equilibrium, in which no corrupt coalitions form and all officials behave honestly. Thus, the model describes how a particular long run equilibrium of the economy might be selected endogenously (rather than imposing a particular supergame equilibrium ex ante, or remaining silent on the issue), and how small shocks, combined with correlated actions, determine the asymptotic behavior, rendering the economy path-dependent. The welfare implications are not formally worked out, but it would not be difficult to put more structure on the model and demonstrate that widespread corruption has adverse welfare consequences. Finally, the model has a clear policy implication: raising the reward for "turning in" a colleague sufficiently will guarantee that the economy will not converge to an all-corrupt state.

4. A framework for studying interaction

The model of corruption presented above is a member of a more general class of models, which will now be described. Consider a population of *N* individuals. Individual *x*'s strategy at time *t* is $h_{x,t}$, whose support is $\{0,1\}$.[17] The state space for these economies is then given by $\Xi := \{0,1\}^N$. Let $h_t \in \Xi$ denote the state of the system at *t*. For each agent *x*, define a set of 'neighbors' *N*(*x*); it is assumed that these are the agents that *x* interacts with, in the sense that only the current strategies played by these agents enter into *x*'s "reaction function".[18] Frequently the set of neighbors will be defined by associating each agent on a d-dimensional integer lattice (embedded on an appropriately dimensioned analogue of a torus to ensure symmetry).[19,20] At event time t, each agent x in a subset of agents *J* receives the opportunity to change his

[17] This binary state may refer to strategies, states, equilibria, or any other economically relevant variable which can take on one of two possible values; see Durlauf and Brock (1995) for a number of interesting binary choices, and Verbrugge (1998) which demonstrates that this binary restriction is not nearly as restrictive as it initially appears.

[18] This might be due to the information structure of the game, or due to a payoff structure in which *x*'s returns don't depend upon the strategies of agents not in *N*(*x*).

[19]We would like to ensure that there are no 'edges', i.e., to ensure that every agent has the same number of neighbors. In one dimension, one simply links up the ends of the set of points, forming a circle. In two dimensions, one embeds the integer lattice on a torus, which is a doughnut-shaped manifold. These conditions are referred to as 'periodic boundary conditions' in the physics literature.

[20] Systems with neighborhood structures associated with *d*-dimensional lattices are stochastic cellular automata; systems with more general interaction patterns may be interpreted as neural networks. A handful of prior studies have modeled particular economic applications using cellular automata (or their stochastic analogues); see Schelling (1969, 1971), Bhargava and Mukherjee (1994), Hegelsmann (1996), and Page (1997) for their use in economic contexts. Keenan and O'Brien (1993) and Föllmer (1994) are perhaps the closest predecessors to the present work, in that they explicitly attempt to avoid simply positing behavioral functions and exploring their implications. No prior work has made a formal connection between this class of games and neural networks, although there is work in progress by Outkin and Haller (1998), who place further restrictions on the games considered here, and consequently may derive some sharper analytical results.

strategy.[21] His decision rule r_x is assumed to depend upon the current strategies played by those in his "neighborhood" $N(x)$, and upon an agent-specific shock $v_x \in \Omega$: $r_x: \Xi \times \Omega \to \{0,1\}$. This "reaction function" will be summarized by a probability (transition) law $P_x(h_t)$; without loss of generality, let $P_x(h)$ refer to the probability that agent x chooses strategy 1, given that the current configuration of the economy is h. (In the sequel, I will typically suppress time subscripts, when there is no danger of confusion.) Obviously we require $0 \leq P_x(h) \leq 1 \;\; \forall h, x$. Further, it is clear that $P_x(h) \equiv P_x\left(h_{N(x)}\right)$. Note that the process is Markovian: only the current state of the system matters.

In the present paper, I will specialize to the case in which the net effects of one's neighbors' states can be summed up by a single number $q \in [0,1]$, a weighted average of the states of one's neighbors, and that $P_x\left(h_{N(x)}\right) \equiv P(q) \;\; \forall x$[22]. In other words, any particular agent reacts to the 'average' influence of his neighbors, and the probability law governing transitions is common across agents (in mathematical terms, the transition behavior is translation invariant). *Conditional on* q, these probabilities are independent; that is, even though two agents x, y may possess a common q,

$$P\left(\{h_x(t+1) = 1|q\} \cap \{h_y(t+1) = 1|q\}\right) = P\left(h_x(t+1) = 1|q\right)P\left(h_y(t+1) = 1|q\right).$$

Four things should be noted. First, a wide class of games may be mapped into this framework; for example, games incorporating best-response dynamics are straightforward to model. However, nothing precludes more sophisticated behavior, as long as each agent's current action depends only upon his own shock and the current (or just prior) strategies played by his neighbors. Second, nothing in the definition implies that the neighborhood structure must be in any way symmetric or reflexive. Third, "negative" interactions (or strategic substitutes) between agents are allowed by admitting negative weights; one simply must assure that $q \in [0,1]$. Fourth, in some contexts one can incorporate a notion of stickiness by simply weighting one's own current state more heavily. This generality is important, as we would like to be able to investigate implications of stochastic neighborhood structures and linkages in finite systems; see Follmer (1974), Kirman (1983), Kirman, Oddou, and Weber (1986), Ioannaides (1990), and Haller (1990) for some theoretical results and discussion. The special case of independent agents simply makes the probability law independent of everything, i.e. $P_x(h) = p$. I will be primarily interested in symmetric transition laws, i.e. transition laws which satisfy $P(q) = 1 - P(1-q) \;\; \forall q$. One property

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[21] In the sequel, $J=N$ in discrete time, and $J=x$ in continuous time.

[22]This precludes the analysis of stochastic Ising models; there, $P_x(h) \equiv P(h_x, q) \;\; \forall x$, i.e. the transition law for x depends upon its own state in a nonlinear fashion. One of the probability laws I investigate, however, gives rise to qualitatively similar behavior.

of such systems is that they have no 'bias'.[23] What will be investigated in the sequel is how the behavior of finite economies change as the neighborhood structure changes: as the dimensionality is increased/as the number of neighbors increases, as interactions move from being uniformly positive to random and possibly negative, and as the influence of global interaction is added.

It is worth pointing out that the vast bulk of what is known about interacting particle systems concerns a handful of particular processes, and that the long-time behavior of such systems depends sensitively on the particular transition laws specified. As noted previously, the necessity of developing simulation evidence for some results forces me to choose particular transition laws. Thus, the simulation evidence I present should be taken as suggestive, not definitive.

4.1 Discrete time process

For this class of processes, *J* consists of the entire system, and events occur at integer times; in other words, at every integer time *t*, *every* agent *x* chooses his strategy according to the probability law *P*. Formally, h is a function on Z^+ which takes values in Ξ. Note that if *N* is finite, the entire process is then a finite state Markov chain, with a transition matrix of dimension $2^N \times 2^N$ whose entries are implicitly defined by $P_x(h)$. Hence many theoretical results (ergodicity/existence of an invariant measure, upper bounds on transition time to invariant measure (if it exists), etc.) are readily deduced..

4.2 Continuous time process

This process is closely related to the stochastic Ising model.[24] In this case, h is a now a function on [0, ∞) which is right-continuous, which has left-limits, and which takes values in Ξ. It is conveniently described as a Markov jump process (in fact, a birth-death process), with transitions governed by the functions $P_x(h)$. This process may be described as follows. At random times t_{xi}, a strategy revision opportunity arrives for agent *x*; she chooses strategy 1 with probability $P_x(h_t)$. The stochastically independent times $t_{xi} - t_{xi-1}$ between her *i*th and *i*-1st strategy revision opportunities are independently distributed exponentially with parameter l. In between times, she may not revise her choice. Thus, *J* consists of a single player.

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[23] The behavior of biased systems may also be of interest; an arbitrarily small amount of bias can completely alter the global probability measure in the case of infinite systems and can essentially make the system respond to and amplify tiny shocks; this property is exploited heavily in Brock (1993) and Brock and Durlauf (1995).

[24] For the stochastic Ising model, infinitesimal transition rates are specifically chosen so that the process has as time-reversible equilibria the Gibbs states of the (static) Ising model. Gray (1986) points out that, remarkably, most of what is known about this model is derived from already known facts about Gibbs states. Once one deviates slightly from the small set of appropriate transition rates, virtually all the known facts about such systems become open questions. Gray goes on to propose a class of models (a class of transition rates) whose behavior is conjectured to be similar. This larger class of models is also encompassed by the general framework outlined above.

With finite N, one may implicitly define the transition matrix using $P_x(h)$, and use it to generate a jump process. Thus, many properties of these systems are readily deduced.

5. Theoretical results

The first lemma presents sufficient conditions for ergodicity of the processes, and follows from standard results in Markov process theory.

Lemma 1

Suppose $N<\infty$. If $0 < P_x(h) < 1 \quad \forall h \in \Xi$, then both the discrete time and the continuous time processes are ergodic and stationary. Furthermore, the invariant distribution will be attained in the limit, regardless of the initial distribution of the process.

Notice that there is no assumption about the neighborhood structure. Naturally, this result can be strengthened in obvious ways: for example, one need only ensure that the global transition matrix Π governing the transitions of h has enough positive entries to ensure that $\Pi^M(i,j) > 0 \quad \forall i,j$ for some M.

Lemma 2 gives the simplest sufficient conditions for nonergodicity, and also follows from standard Markov process theory:

Lemma 2

Suppose $N<\infty$. If all interactions are positive, $P_x(q=0)=0$, and $P_x(q=1)=1$, then both the discrete time and continuous time processes are nonergodic.

Note that this result does not state that the economy can only converge to one of the states $(\tilde{0})$ and $(\tilde{1})$; depending on the structure of interaction, other long run distributions may be possible. In fact, a necessary condition for these to be the only closed sets is that the neighborhood structure "fully communicates", i.e. each agent directly or indirectly (via an extended chain of interactions of neighbors with neighbors) interacts with every other agent. It is also true that the neighborhood structure can matter, as can whether the total number of agents is even or odd. For example, in discrete time and two-dimensional interaction, a two-cycle in which there is a checkerboard pattern of strategies (where red denotes strategy 0, and black denotes strategy 1), alternating with its mirror image, is a possible equilibrium; however, this equilibrium is impossible with an odd number of players.

A corollary of Lemma 1 is that, under the parameter restrictions on the corruption model of Section 3 which are given below, the economy is nonergodic and path-dependent: the particular sample path of shocks will determine the long run equilibrium level of corruption in the economy.

Corollary

If $\left[1-(1-q)^4\right]L \geq 2$, $bqR < 1$, and $0 > 1-\left[1-(1-q)^2\right]L > bqR - 1$, then the corruption economy of Section 3 is nonergodic and path-dependent.

Proof

All interactions are positive, since if (*i*, *j*) has more neighboring agents honest last period increases the probability that (*i*, *j*) will be honest this period. Under these conditions, $P_x(q=0) \equiv P\left(x \text{ plays honest} \middle| q_{ij}=0\right)=0$, and $P_x(q=1) \equiv P\left(x \; plays \; honest \middle| q_{ij}=4\right)=1$. Thus, the economy is nonergodic. Since $1 > P_x(q=\tfrac{1}{2}) > 0$, there exist sample path realizations of shocks so that, starting from initial conditions in which all agents possess exactly two neighbors who behaved honestly last period, either all-corrupt or all-honest are possible long run equilibria.

■

The next lemma is easy to verify, and follows immediately from a related argument is given in Blume (1995).

Lemma 3

Suppose *P*( ) is monotonically increasing, $P_x(q=0)=0$, and $P_x\left(q=1\right)=1$. If interactions are positive and global, i.e. *N*(*x*) consists of the entire population for every agent *x*, then the only limit configurations are "symmetric" configurations, in which each agent plays the same strategy.

Let $\bar{h} := \frac{\sum_x h_x}{N}$; this is the mean of the system. The next lemma states that for *non-*interacting processes of this form, $\bar{h}$ must converge to ½ as $N \to \infty$. This is a simple consequence of the strong law of large numbers.

Lemma 4

The non-interacting processes satisfy

$$\bar{h} \xrightarrow[N\to\infty]{} ½ \text{ with probability 1.}$$

Proposition 1, which is a result previously reported (in part) by Föllmer (1994), presents a result concerning the evolution of the economy for the maximally-interacting infinite discrete-time process when the transition probability is monotone. In this case, the economy moves deterministically, and there might be multiple possible limits; if so, the one to which the process will move in the limit will entirely depend upon initial conditions. When the transition probability is not monotone, complex behavior can arise, and the law of large numbers need not apply: economies composed of a countably infinite number of agents, each of which is receiving independent shocks, may feature substantial volatility. A brief sketch of the argument is given in Appendix 2.

Proposition 1

Consider a globally interacting discrete time process. If $P(q)$ is continuous, symmetric, monotone, and $\exists q' > ½$ such that $P(q') > q'$, then in the limit as $N \to \infty$, $\bar{h}$ will, with probability 1, converge to a member of a set which is composed of at least three distinct real numbers. The limit to which $\bar{h}$ converges will (deterministically) depend upon the initial configuration h. If $P(q)$ is continuous but is not monotone, cycles and chaotic behavior can occur.

How might one interpret a nonmonotonic $P(q)$ function? Note that, as long as $P(q)$ lies above the line $P \equiv ½$ for $q > ½$, this may (loosely) be interpreted as strategic complementarity in the sense that if the weighted average of *x*'s neighbors' behavior is above ½, *x* is more likely to select strategy 1. Viewed in this light, it is easy to see a nonmonotonic $P(q)$ function is nothing extraordinary: the strength of the strategic complementarity simply varies over q. For example, there may be coexisting (and competing) positive and negative externalities (increasing returns of some sort in conjunction with congestion externalities), with the negative externality remaining the weaker influence but gaining in dominance over some regions of q.

Notice that under conditions which give rise to cycles and/or chaos, the *entire infinite system* is undergoing large movements. The variance of the system is *not* converging to 0 in the limit … despite the fact that each individual agent is receiving an *independent* disturbance.

The next proposition, reminiscent of a theorem in Jovanovic (1987), is a statement about the attainability of *any* symmetric equilibrium (stationary) distribution with strictly positive support for the maximally-interacting (finite) continuous-time process. One can implement any such distribution by selecting an appropriate $P(q)$ function. Though this is a statement about the steady-state distribution, and not about dynamics within the steady state, it also indicates the substantial scope for interacting systems which are driven purely by idiosyncratic noise to display economically interesting behavior.

For this result, it is convenient to redefine the state variable to be the number *k* of agents who are in the +1 state, i.e.,

$$k \in (0,1,\ldots,N)$$

where *N* is the total number of agents in the system. We restrict $0 < P(q) < 1 \; \forall q$, where q is defined to be the *fraction* of agents in state 1, i.e. $q := \frac{k}{N}$; this restriction ensures that the Markov process is stationary and ergodic. With this definition of the state, the dynamic evolution of the process obeys the following transition function:

$$Q(k,l)=\begin{cases}\left(1-\frac{k}{N}\right)P\left(\frac{k}{N}\right) & l=k+1\\ \left(1-\frac{k}{N}\right)\left[1-P\left(\frac{k}{N}\right)\right]+\frac{k}{N}P\left(\frac{k}{N}\right) & l=k\\ \frac{k}{N}\left[1-P\left(\frac{k}{N}\right)\right] & l=k-1\\ 0 & else\end{cases}$$

where use has been made of the fact that the idiosyncratic disturbances are distributed uniform $[0,1]$. Let t_i denote the time of the *i*th jump; then if $h(t)$ denotes the Markov process, $h(t_i)$ is a (discrete time) Markov chain, sometimes referred to as the embedded chain. The invariant distribution of $h(t_i)$ is the same as that of h_t, a fact which might prove useful in more complex contexts. Since the process is reversible, the equilibrium distribution $m(k)$ satisfies

$$l\,Q(k,l)m(k)=l\,Q(l,k)m(l)$$

Thus,

$$m(k)=\frac{(m(1)/m(0))\ldots(m(k)/m(k-1))}{1+\sum_{l=1}^{N}(m(1)/m(0))\ldots(m(l)/m(l-1))}$$

where

$$\frac{m(k+1)}{m(k)}=\frac{Q(k,k+1)}{Q(k+1,k)}=\frac{(1-\frac{k}{N})P(\frac{k}{N})}{(\frac{k+1}{N})\left(1-P(\frac{k+1}{N})\right)}$$

Proposition 2

Let $N<\infty$ be even. Let $m(k)$ be a symmetric distribution with $m(k)>e\ \forall k$. Then there exists a symmetric probability transition law $P\left(\frac{k}{N}\right)$ with $0<P\left(\frac{k}{N}\right)<1\ \forall k$ such that the continuous time process with transition law $P(\cdot)$ has stationary distribution $m(\cdot)$.

Proof

See Appendix 1.

The discussion above also indicates that one can readily compute the equilibrium distribution of a fully interactive continuous time process, once a particular $P(\cdot)$ is specified.

6. Deferred investment, multiple steady states, and cycles

Cooper and Johri (1997) locate evidence for the existence of intertemporal strategic complementarities in the US economy. Recently, a number of authors have highlighted the implications of intertemporal strategic complementarities associated with investment (Acemoglu and Scott (1997), Gale (1996), and Ruiz (1998); those authors (and others) suggest numerous sources for such complementarities, such as R&D spillovers or various forms of learning by doing. Below, I demonstrate that a minor modification of the framework of Gale (1996) demonstrates how aggregate fluctuations can arise in infinite economies, even though the only driving forces are idiosyncratic shocks. Following Gale, in this example, interactions are intertemporal (ruling out the possibility of perfect coordination and self-generating bursts of activity), and global (in the sense that each agent interacts with *all* previously living agents).

Time is discrete. At each date, a countable infinity of agents enters the economy and lives for two periods. The utility function of agent i born at date t is given by

$$U_i(c_{i,t}, c_{i,t+1}) = c_{i,t+1}$$

While young, agents have the option to search for innovations which may be converted into profitable investment opportunities the next period. Search is assumed to be costless, so all searching agents search at maximum intensity. An agent searching at maximum intensity is assumed to discover an innovation with probability one; agents who do not search do not discover any innovations. Innovations are not equally profitable; the gross return $r_{i,t}$ on the innovation discovered by agent i is a random variable, distributed iid across agents and uniform on [0, 1].

Innovations require investment in physical capital. Following the full depreciation case in Gale (1996), entrepreneurs may benefit from innovations for only one period. However, in this framework entrepreneurs with innovations in hand have no incentive to delay; their only decision is whether to exploit the opportunity or not. Agents who do not exploit their innovations earn a rate of return zero (one may think of these agents being involved in home production). The profitability of production is assumed to be a function both of the gross return on the innovation and (following Gale) of the general level of prior investment in the economy; that is, if y_t is the fraction of agents who invested in period t, and $p_{i,t+1}$ is the profit to date t innovator i, then

$$p_{i,t+1} = r_{i,t} - C(y_t)$$

where $C(y_t)$ is a positive, continuous function which maps [0, 1] into [0, 1] which is described in more detail below.

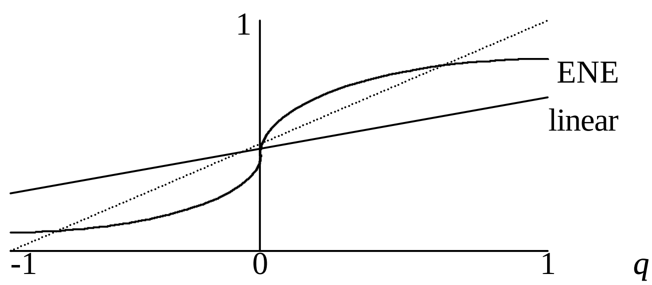
The function $C(y_t)$ is assumed to reflect strategic complementarities, in the sense that if more than half of the agents in the economy invested last period, the incentive (or net return) for any entrepreneur to invest is greater than if less than half of the agents in the economy invested last period. Loosely speaking, if a large fraction of agents invested last period, costs are "low" so that the incentive to invest this period is "high" and (given the distribution of gross returns) a large fraction of agents will invest this period. $C(\ )$ is assumed to lie above ½ for $y \in (½,1]$, and to lie below ½ for $y \in [0,½)$; it is in this sense that it incorporates strategic complementarities: when over half of the agents invested last period, over half of the agents will invest this period.

Proposition 1 above demonstrates that this economy displays deterministic dynamics and may possess multiple steady states. Dynamics are deterministic because despite the uncertainty at the micro level, there is no macro uncertainty: the fraction of agents which possess innovations whose return exceeds $C(y_t)$ is given by $(1 - C(y_t))$, and this in turn determines y_{t+1}. The proposition shows that if $C(\ )$ is symmetric about ½ and $\exists y' \in (½,1]$ s.t. $C(y') < y'$, then the economy has at least three distinct steady states, with initial conditions determining the particular steady state attained in the limit.

The lack of aggregate randomness does not preclude aggregate fluctuations. If *C*( ) is not assumed to be monotonically decreasing, Proposition 1 demonstrates that complex behavior can arise. If *C*( ) possesses a minimum at fraction $½<f_1<1$ and rises sharply enough after that (reflecting, for example, intertemporal congestion effects such as increasing costs in the construction industry), this economy may exhibit nonlinear and possibly chaotic behavior. (The same will be true if *C*( ) possesses a maximum at fraction $½>f_2>0$, rising sharply before it.) Note that in this case, the law of large numbers does not apply; the economy is composed of an infinite number of agents, each of which is receiving independent shocks, yet the aggregate displays substantial volatility.

7. Numerical results

For these simulations, I consider two monotonic $P(q)$ functions,[25] both parameterized by a number $½< a \leq 1$. For convenience, I relabel the possible states as {-1, +1}. The first $P(q)$ function is linear and satisfies $P(1) = a$ (and hence $P(-1) = 1 - a$). The second is given by $P(q) = ½ + (a - ½)q^{1/3}$; I will refer to this as the ENE ("effectively nonergodic") probability law, so denoted because systems with this law may *behave* nonergodically (over shorter time scales) even if the system is in fact ergodic. When $a < 1$, both processes are ergodic, and when $a = 1$, both processes are nonergodic. These $P(q)$ functions are depicted below.



Each simulation is run ten times (with ten different random number seeds); in each, random initial conditions are utilized, then the process is run for 400 time periods to shed the effects of initial conditions, and finally the mean of the process is recorded for the next 1000 time periods. For each simulation run, the sample variance of levels, the sample variance of first differences, and the sample 1st autocorrelation are computed, and the histogram of the sample is constructed. I investigate how these change as the neighborhood structure changes (in particular, as the

[25] With θ reverting to its original definition.

dimensionality and interconnectedness of the system increases), as the size of the system increases, as the parameter a is varied, and compare the discrete- and continuous-time analogues. I investigate both nonstochastic neighborhood structures (agents associated with lattice points on *d*-dimensional lattices, which interact with other agents that are a distance 1 away), as well as stochastic neighborhood structures (agents interact with other agents that have been randomly selected). In generating the stochastic neighborhood for each simulation run, I select the number of other agents a particular agent will interact with from a (truncated) exponential distribution, then select that number of agents uniformly from the set of all agents (selection is with replacement). Note that with stochastic neighborhoods, being a neighbor is not reflexive: *b* might be *a*'s neighbor, but not vice versa. Further, with stochastic neighborhoods there is positive probability that the system is not fully connected, in the sense that there may one or more groups of agents of size $m<N$ such that none of the *m* agents in this set have a neighbor in its complement. In such cases, of course, each group's behavior is independent of the state of rest of the economy. I illustrate the results in a series of graphs; discussion will be mostly informal. Note that when I refer to 'the variance' of a series, I am referring to the variance of the mean.

7.2.A The behavior of systems with the linear probability law

{Still to do: compute equilibrium distribution for fully interactive case} Aside from the $a=1$ case, systems with the linear probability law have a characteristic behavior that varies relatively little as *N* increases and as one passes from discrete to continuous time. Figures 1-6 all report results on discrete time processes (discrete and continuous time analogues are compared in Figure 7). Figures 1-3 illustrate how the variance, the variance of first differences, and first autocorrelation of this process scale as the number of agents increases, holding constant the neighborhood structure and the parameter a. Each data point on each figure is an average over 10 simulation runs. To illustrate how variance (or variance of first differences) scales in *N*, Figures 1 and 2 plot the value *kNM*, where *M* refers to the measurement (variance or variance of first differences), *N* is the number of agents, and *k* is a constant that is chosen so that the average value is 1. Evidently these measurements fall with *N* (i.e., 'obey the LLN'), as the graphs are flat at 1.

The main message in Figure 3 is that the autocorrelation of this process is *not* a function of the number of agents. Note that even large fully interactive systems display high autocorrelation, in stark contrast to non-interactive systems. The intuition for this is the following: In finite systems, there is always the chance that the system will wander away from ½. After it has done so, it will *not* immediately jump back to ½; instead, it will move back gradually.[26] A non-interactive system always tends to move back to ½ immediately, regardless of its prior position.

Figures 4-6 illustrate how variance, variance of first differences, and first autocorrelation change for discrete-time systems with a linear probability law as the neighborhood structure (degree of interaction) changes, for various *N*. Each data point is the average of 10 simulation runs; included are one standard deviation bands. For systems with this probability law and nonstochastic neighborhood structures, there is some tendency for variance to increase (modestly) in the degree of interaction, but this is not always monotonic.[27] Given this, it is interesting that variance does *not* appear to be significantly affected by changing the (average) number of neighbors in randomly-selected neighbors systems. Not surprisingly, variance of first-differences is maximized in a non-interactive system; but more interestingly, for the nonstochastic neighborhood cases, it is minimized in systems that are minimally interactive (one neighbor). First autocorrelation, surprisingly, is not affected by the degree of interaction (once some level of interaction is allowed).

One of the most important findings is that the behaviors of the discrete-time and continuous-time processes are surprisingly similar. Note in Figure 7 that, though the continuous time process is more variable, the qualitative behavior of the variance as connectivity increases is quite similar, with minor qualitative differences for the case of minimal, and full, interaction. The behavior of variance of first-differences is also very similar, with a qualitative difference at full interaction. The continuous time process is more persistent, as one would expect.

The variance and persistence of these processes increase strongly in a . This is illustrated in Figure 8, which depicts how the variance, variance of first differences, and persistence change as a is increased. Figure 9 sheds some light on this; it depicts histograms of two continuous time fully-interactive processes as a is increased from .95 to .995. The histogram covers 10,000 iterations. In both cases, activity is concentrated around 0. However, the mean of the $a = .995$ process ranges much more widely. With linear probability laws such that $a < 1$, both the discrete time and continuous time processes are ergodic and effectively ergodic. Figure 10 depicts the stationary distribution of the continuous time fully-interactive system, for several choices of a and *N*; its message is the same as that of the empirical distributions given in Figure 9, and further suggests that these systems settle down into their stationary distributions rather rapidly.

However, what happens when a increases from .99 to 1.0? We know that in this case, the process is nonergodic; does nonergodicity (with this linear probability law) matter? The answer is, yes. Even for $N = 1024$, given enough interaction, these processes *will* quickly converge to

[26] This would be true even in the infinite limit. In that case, however, the system would never depart from ½ in the first place.
[27] Details of the interaction structure can clearly matter; this will be commented upon later.

either +1 or -1 (i.e., *all* agents +1, or all -1) if they are nonergodic. However, Figures 9 and 10 make it clear that there exist probability laws such that nonergodicity would be largely irrelevant. Consider, for example, a probability law of the form:

$$P(q) = \begin{cases} 5q - 4 & q \in [.9,1] \\ 5q + 5 & q \in [-1,-.9] \\ .5 & \text{else} \end{cases}$$

Even after 10,000 units of time, a fully interactive process would be unlikely to converge to either +1 or -1.

Thus, we have a reasonably complete picture of the qualitative behavior of these systems when there is a probability law of the linear form suggested above. The behavior of the system with the ENE probability law is significantly different, however.

7.2.B The behavior of systems with the ENE probability law

The proof of Proposition 1 makes clear that we might expect effectively nonergodic behavior with the ENE law if the system is large enough (since in the limit, the system is both nonergodic and effectively nonergodic). Indeed, this is the case. Even relatively small systems may mimic the nonergodic behavior proved in Proposition 1 for the infinite N case; a fully-interactive (discrete- or continuous-time) system as small as 81, with a =.7 and the ENE probability law, begins to display the beginnings of effective nonergodicity, spending most of its time in the intervals (-.50, -20) or (20, 50).[28] The qualitative shift in behavior is apparent upon comparison of the ENE processes shown below in Figures 11-13 with the behavior shown in Figures 4-7. It is evident that for such systems, given a fixed N and a , there may be a critical interaction level (or range) at which the behavior of the system changes markedly, exhibiting substantially more variance and persistence, and that this critical number of neighbors varies from case to case. We will see below that, past this point, these systems *tend to* behave effectively nonergodic: they become ‘trapped’ in a *relatively* high (or low) activity state. Interestingly, behavior is not completely straightforward to describe, and this illustrates the previous assertion that details of the interaction structure matter: the N = 256 case, for example, does not have a unique maximal interaction level for a given level of a . Examination of the histograms of these processes indicates that, across runs (with different random number seeds, but the same level of interaction and a), it is sometimes the case that the system is effectively nonergodic (discussed below), while other runs end up being tightly concentrated about 0. The reason for the complexity of the variance graphs in some cases is that different interaction topologies admit different attractors. For example, in the two-dimensional case with N even, there is an attractor which is characterized

[28] See Figure 13 for some illuminating histograms.

by cycling between two states: one in which every other agent is in a +1 state, and the other in which all agents reverse their orientation. This attractor does not exist for N odd. As the purpose of this study is to determine the general behavior of a class of systems, no attempt was made to exhaustively study any particular case.

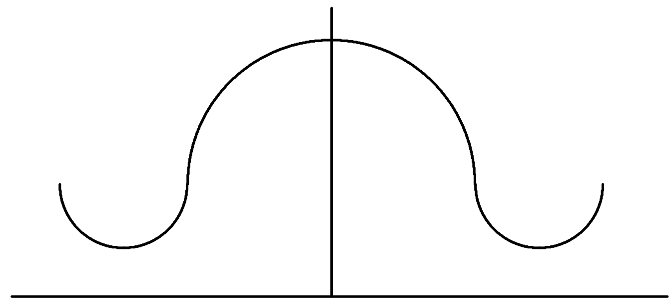
Note that in these systems as well, continuous time processes behave similarly to their discrete time analogues. (The point of the nongergodicity threshold may be slightly different, however; c.f. the N = 81, $a = .8$ case.) Comparing Figure 13 to Figure 11, it is evident that persistence tends to be maximized when variance is maximized.

Figure 14 presents histograms of ENE processes as the interaction structure changes, for N = 729 and two different values of a. Each run covers 10,000 time periods. As the number of neighbors increases, the mean of the system ranges more and more widely. However, once a critical threshold level of interaction is reached, the system begins to behave effectively nonergodically, being 'trapped' (in these cases) in a high-activity state. Figure 15 tells the same story; it depicts the stationary distributions of several fully-interactive continuous time processes.

Figures 16-18 illustrate how variance, variance of first differences, and first autocorrelation change for particular ENE systems, holding the neighborhood structure (degree of interaction) fixed, as N increases. Each data point on each figure is an average over 10 simulation runs. To illustrate how variance (or variance of first differences) scales in N, Figures 16 and 17 plot the value kNM, where M refers to the measurement (variance or variance of first differences), N is the number of agents, and k is a constant that is chosen so that the initial value is 1. Figure 16 indicates that in some cases, variance scales with N (graph is flat at 1), while in other cases, variance clearly does not fall at rate N (i.e., 'the LLN is postponed'). Note the critical point at N = 128 in the $a = .7$, full interaction graph; after this point, the system starts to behave nonergodically and displays substantially less variance. Variance of first differences falls at rate N regardless. Finally, the first autocorrelation of these processes hinges crucially upon distance to critical points: near such points, autocorrelation is high (even for large systems), but once past such points, autocorrelation falls off rapidly.

The following figure is very helpful in explaining what is going on. With an ENE probability law with a large a, there are two macroeconomic states that are locally stable (each of these, of course, consists of a multitude of microeconomic states). The process will move quickly to one or the other. Once there, the mean first passage time to the other locally stable state becomes exceedingly large as a rises: the "hill" or barrier between the two states grows higher and higher.

Figure 19: "Energy Barriers"



An ergodic process will (eventually) roam over the entire state space (there are no *infinite* barriers), *but* the barrier may be high enough so that the process is *effectively* nonergodic (it will take a long, long time before the process moves from one valley to the other). By the same token, such systems are effectively path-dependent: the sample path of the process determines the metastable region of the state space to which the process converges. Systems as small as $N = 64$ display effectively nonergodic behavior, given a large enough value of a and enough interaction.

As a process approaches effective nonergodicity, it becomes *much* more variable. Once it passes into effective nonergodicity, its variance once again begins to subside, and the process becomes effectively trapped in either a high- or low-activity state. This qualitative behavior is very common in interacting agent systems; it has been shown that, for example, the two-dimensional infinite Ising model displays *infinite* variance exactly at its critical point. It is a fairly safe conjecture that the variance of these processes is maximized 'on the edge' of effective nonergodicity.

8. Conclusion

What are the implications of local interactions? Economists are beginning to explore this issue in more earnest; as a result, interacting particle systems are becoming increasingly important tools for economists. Yet very little is known about many of the interesting properties of such systems; in particular, how does the behavior of finite systems compare to that of infinite systems? How different are discrete-time systems from analogous continuous-time systems? To what extent is nonergodicity important for such systems? How do the variance of levels and the persistence of these systems change as the size of the system increases, as the level of interconnectivity increases, and as the probability laws governing micro behavior are varied? All of these are crucial questions for economists who desire to use such models to understand the aggregate behavior of interacting economic agents, and these questions are the focus of this paper.

This paper introduces a framework for studying economic interaction, and derives some key theoretical and numerical results. In this framework, for finite economies, resolving ergodicity is straightforward.[29] In finite continuous time versions, computing the stationary distribution (if it exists) is simple; furthermore, I show that any stationary distribution may be attained by these models. A recent strand of the literature on business cycles attempts to explain a significant proportion of aggregate volatility by appealing to interactions between microeconomic agents. (Horvath and Verbrugge (1997), for example, show that industry-level shocks which feed through inter-industry linkages can account for the bulk of the variance in aggregate industrial output.) A key result in this paper strongly supports this view: Under relatively weak conditions, aggregate variability can survive in the infinite limit, *even though the only driving forces are independent agent-specific shocks.* Put differently, laws of large numbers need not hold for interacting economies, so there is no a priori reason to disregard the effects of disaggregate shocks in attempting to explain aggregate movements.

The numerical results in the paper are also important. This paper demonstrates that interactions can dramatically amplify the effects of shocks; further, they induce substantial persistence in the time series of aggregate activity. Even controlling for the existence of strategic complementarity, the micro specification of behavior is has a big influence on the medium- and long-term behavior of the economy. Economies with linear probability laws have a much more regular behavior, devoid of critical points. When the probability laws have some curvature, however, critical points exist, around which both variability and persistence are maximized (and, as noted, laws of large numbers can fail for such laws). As one would expect, the behavior of randomly interlinked economies is much more regular and smooth. Somewhat surprisingly, passing from discrete to continuous time makes little difference on the qualitative behavior of such systems; this is convenient, as the modeler may freely select the one which is most convenient for addressing the issue at hand, without biasing the results.

Whether or not a particular economic system may be usefully modeled via this framework depends upon the number of interacting agents, the existence of a mapping between decision rules and two states, and the types of interaction. Global interaction and/or bias can greatly increase the variance of the process. The probability law that summarizes micro interaction must be chosen with care, since its influence upon dynamics is so significant. For example, with linear probability laws, unless small fluctuations in $\bar{h}$ translate into much larger variations in the economic variable of interest, large interacting agent systems may not generate sufficient variance.

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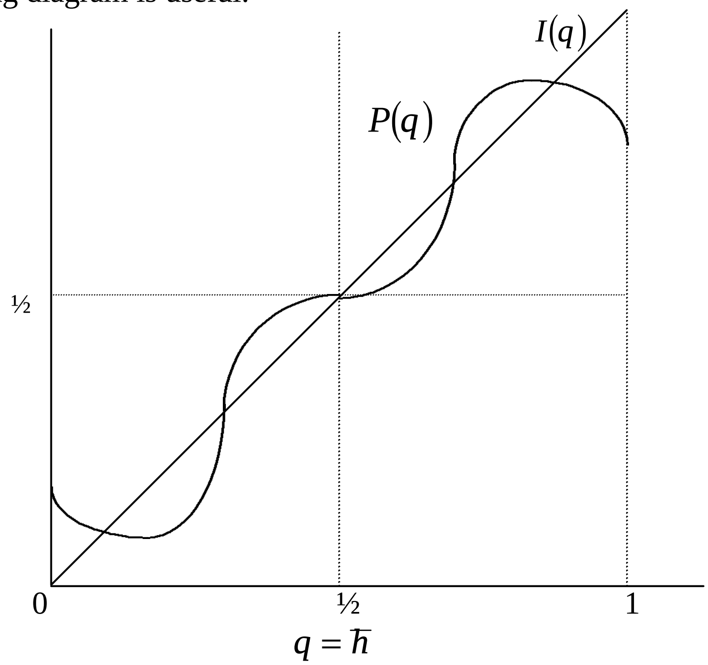
29 Though as is pointed out, resolving the question of ergodicity in either finite or infinite interacting particle systems might not be of much practical interest, since in nonergodic cases, the passage time out of the transient set of states may be extremely large, while in ergodic cases, the system may behave like a nonergodic process over time scales of interest.

We know that the macroeconomy is composed of many randomly interlinked firms. The implications of interactions are only beginning to be explored. Horvath (1997a,b) and Verbrugge (1998) demonstrated that, due to such interactions, aggregate fluctuations can arise from purely independent small shocks. This paper has introduced a new framework for analyzing interaction that may prove useful in future models of interactions, and has provided some evidence on its qualitative behavior, thus beginning to fill a huge gap in the extant literature.

9. Appendix 1 - Proofs

Proof of Proposition 1 (Sketch)

The following diagram is useful:



$P(q)$ is bounded between 0 and 1, must pass through the point (½, ½), and satisfy the symmetry property. By independence and the law of large numbers, in the infinite limit, the fraction of agents whose state next period is 1, if this period $q := \hbar = q''$, will be given by $P(q'')$; in other words, the evolution of the environment is given by the iteration of the map *P*. Hence if $\exists q' > ½$ such that $P(q') > q'$, there exists $\hat{q} \in (q',1]$ such that $P(\hat{q}) = \hat{q}$. The above diagram summarizes the dynamics: when $P(q) > q$, i.e. when $P(q)$ lies above the 45° line in the diagram, q increases, and when $P(q)$ lies below the line, q decreases. Clearly there might be numerous limits to which the system might converge; any intersection of $P(q)$ and the 45° line is a steady state. Under the assumptions of the result, there must be at least two such points in addition to the point ½.

If $\exists \breve{q} > ½$ such that $P(\breve{q}) > \breve{q}$ and $P'(\breve{q}) < 0$, one can easily construct examples in which cycles and chaos occur. If $P'(\frac{1}{2}) < 0$, there are an infinity of period two cycles which the economy might be in, depending upon initial conditions.

■

Proof of Proposition 2

Since $m(\cdot)$ is symmetric, $m(\tfrac{N}{2}+l)/m(\tfrac{N}{2}-l)=1\ \forall l\in\{0,\ldots,\tfrac{N}{2}\}$. Using the fact that $\dfrac{m(k+1)}{m(k)}=\dfrac{(1-\frac{k}{N})P(\frac{k}{N})}{(\frac{k+1}{N})\left(1-P(\frac{k+1}{N})\right)}$, and that $m(k)/m(l)=\left(m(k)/m(k+1)\right)\ldots\left(m(l-1)/m(l)\right)$, one can easily show that $P(\cdot)$ is symmetric. Hence it suffices to determine $P\left(\frac{k}{N}\right)$ for $k\in\{\frac{N}{2},\ldots,N\}$.

Set P(½) = ½. To save on notation, denote $g_k:=P\left(\frac{k}{N}\right)$. Note that

$$\frac{m(\frac{N}{2})}{m(\frac{N}{2}+1)}=\frac{(\frac{N}{2}+1)\left(1-g_{\frac{N}{2}+1}\right)}{(\frac{N}{2})(\frac{1}{2})}\ \Rightarrow\ \frac{1}{2}\frac{\frac{N}{2}}{(\frac{N}{2}+1)}\frac{m(\frac{N}{2})}{m(\frac{N}{2}+1)}=1-\left(1-g_{\frac{N}{2}+1}\right)$$

As the second term is bounded above by 1 and below by 1/*N*, and the second term is bounded above by $(1-e)$ and below by e, one may use $m(\cdot)$ to determine $g_{\frac{N}{2}+1}$, which must be strictly between 0 and 1. Once g_k $\left(N>k\geq\frac{N}{2}\right)$ is known (and is strictly between 0 and 1), one may iterate in the above fashion to find g_{k+1}, and show in the same way that it too must be strictly between 0 and 1.

■

10. Appendix 2: Stationarity and ergodicity

The fundamentals of ergodic theory are exposited in Petersen (1989); Lamperti (1977) provides closer links to the theory of stochastic processes.

Stationarity refers to the idea that the finite-dimensional distributions of the process are invariant under translations of time. Ergodicity refers to the long time average behavior of stochastic processes. A necessary condition for ergodicity is that the process visits all regions of its state space that have positive measure. However, neither this nor stationarity is sufficient to ensure ergodicity.

Let (X, $\mathcal{F}$, m) be a complete probability space. Let $T: X \to X$ be a bijective (one to one and onto) map such that T and T^{-1} are both measurable and $m\left(T^{-1}E\right) = m\left(E\right)\ \ \forall E \in \mathcal{F}$. Then T is called a measure-preserving transformation.

Given an measurable real function f on (X, $\mathcal{F}$, m), with a measure-preserving transformation T, then the stochastic process defined by $Y_n\left(w\right) := f\left(T^n w\right)$ for $w \in X$, $n \geq 0$ is stationary. The converse is 'almost true'; any stationary stochastic process corresponds to (has the same joint distributions as) a process (X', $\mathcal{F}'$, m', T') with a measure-preserving transformation.

The orbit $\left\{T^n x\right\}_{n=-\infty}^{\infty}$ of a point $x \in X$ represents a single complete history of the system. $\mathcal{F}$ is thought of as the family of observable events, with the T-invariant measure m specifying the *time independent* probabilities of their occurrences. A measurable real function f on (X, $\mathcal{F}$, m) can be thought of as a 'measurement' made on the system. $f\left(x\right)$, $f\left(Tx\right)$, $f\left(T^2 x\right)$ may be thought of as the values of some interesting variable measured at consecutive periods of time, beginning with the world in initial state x.

Two important and related questions that may be asked of a stochastic process (X, $\mathcal{F}$, m, T') with $f \in L^1$(X, $\mathcal{F}$, m):

1. Does the following limit exist in some sense?

$$\frac{1}{N}\sum_{k=1}^{N-1} f\left(T^k x\right) \xrightarrow[N\to\infty]{?} \bar{f}\left(x\right)$$

This is a question about the long term time average. The ergodic theorem states that this convergence must occur for ergodic processes, both a.e. and in L^1.

2. Does the time average of the process coincide with its ('space') mean?

$$\frac{1}{N}\sum_{k=1}^{N-1} f\left(T^k x\right) = \int_X f dm \quad a.e.$$

Ergodic processes satisfy this condition as well (in fact, property 2. holds $\Leftrightarrow$ process is ergodic).

An immediate consequence of property 2 is the following. For a measurable set E and a point $x \in X$, define the mean sojourn time of x in E by

$$\bar{I}_E(x) := \frac{1}{N}\sum_{k=1}^{N-1} I_{(E)}(T^k x)$$

where $I_{(\ )}$ is the indicator function. Then an ergodic process satisfies $\bar{I}_E(x) := m(E)$ $a.e.$

Recall the definition of stationarity: A stationary stochastic process on (X, $\mathcal{F}$, m) satisfies

$$m\left(w: x_{t_1}(w) \in B_1, \ldots, x_{t_m}(w) \in B_m\right) = m\left(w: x_{t_1+t}(w) \in B_1, \ldots, x_{t_m+t}(w) \in B_m\right)$$

$$\forall \text{ measurable subsets } B_1, \ldots, B_m,\ \forall\ t_1, \ldots, t_m,\ \forall t\ .$$

Not every stationary process is ergodic. For example, let (X, $\mathcal{F}$, m) be ((0,1), $\mathcal{B}$(0,1), m), where m is the uniform distribution on $(0,1)$, and T is the identity map, so that $Y_n(w) = w$. This process is stationary, but not ergodic.

For finite state Markov processes, stationarity and ergodicity are closely related; stationarity does not quite imply ergodicity, but if the Markov process is irreducible, then stationarity is equivalent to ergodicity.

Ergodicity suffices to ensure that laws of large numbers are applicable, but not to ensure central limit theorems. For this, one needs additional mixing conditions.

As noted in the introduction, knowing that a process is ergodic doesn't provide us with all the information we would like to know. Nothing in the definition of ergodicity depends upon timescale. It might take much longer than the age of the universe for an ergodic system to do a good job of 'covering' its state space. For all practical purposes, there may be regions of the state space which are never visited. In both ergodic and nonergodic systems, there may be regions of the state space which are very unlikely to be escaped from before the sun is projected to burn out. In such cases, the process might be effectively ergodic on that subregion (which might be a set of transient states).

Let me be a bit more formal. Fix a stochastic process Y_t. A set C of states is <u>closed</u> if passage out of it is impossible, i.e., $P\left(Y(t+s) = x' \middle| Y(t) = x\right) = 0\ \ \forall s, \forall x \in C, \forall x' \notin C$. A set C of states is <u>irreducible</u> if it is closed, and all the states communicate, i.e. $\exists s$ s.t. $P\left(Y(t+s) = x' \middle| Y(t) = x\right) > 0\ \ \forall x, x' \in C$. The entire state space of a stochastic process may be decomposed into disjoint classes: closed irreducible sets $C(1), C(2), \ldots$, and a set of transient states D. The sets $C(1), C(2), \ldots$ are often called the <u>ergodic classes</u> of the process. The process on such a set $C(k)$ may be regarded as a stochastic process on its own, with irreducible state space $C(k)$.

Suppose Y_t is ergodic, so that its entire state space is irreducible, but that it has passed into a set of states (subregion) Q at time t, from which the probability of 'escape' over time interval $(t+t)$ is extremely small. If t is 'large', then the system is 'effectively' nonergodic. Q may be thought of as a 'metastable' state. Most of the interesting phenomena in our world are such metastable phenomena. Of course, nonergodic systems may also possess metastable (as well as stable) subregions. In practice, it may be difficult or impossible to tell whether a given process is ergodic.

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