A Model of Financial Fragility

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Abstract

This paper presents a dynamic, stochastic game-theoretic model of financial fragility. The model has two essential features. First, interrelated portfolios and payment commitments forge financial linkages among agents. Second, iid shocks to investment projects' operations at a single date cause some projects to fail. Investors who experience losses from project failures reallocate their portfolios, thereby breaking some linkages. In the Pareto-efficient symmetric equilibrium studied, two related types of financial crises can occur in response. One occurs gradually as defaults spread, causing even more links to break. An economy is more fragile ex post the more severe this financial crisis. The other type of crisis occurs instantaneously when forward-looking investors preemptively shift their wealth into a safe asset in anticipation of the contagion affecting them in the future. An economy is more fragile ex ante the earlier all of its linkages break from such a crisis. The paper also considers whether fragility is worse for larger economies.

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> Financial fragility is, to a large extent, an unavoidable consequence of a dynamic capitalistic economy. Its fundamental sources … cannot be eliminated by government intervention, and attempts to do so may create more instability than they prevent. [Calomiris, 1995, p. 254]

> [T]he Great Depression, like most other periods of severe unemployment, was produced by government mismanagement rather than by any inherent instability of the private economy. [Friedman, 1962, p. 38]

> If [someone] would only fully specify any one financial-fragility model …, perhaps we could think more clearly about the potential scope of the argument. [Melitz, 1982, p. 47]

I. Introduction

An economy exhibits financial fragility if it possesses a propagation mechanism that allows small, common economic disturbances to have large-scale effects on the financial structure and thus on real activity. The notion of fragility originated with Fisher's (1933) and Keynes' (1936) theories of how the debt financing of investment activity that characterizes modern capitalism can be destabilizing. Despite the considerable controversy that has existed ever since over whether modern capitalist economies are inherently fragile, the literature on fragility has lacked a fully coherent model that could provide some insights.[1] The goal of this paper is to develop a model in which fragility is well defined and to explore the factors that determine how fragile an economy is and whether fragility worsens as an economy increases in size.

Two features seem essential to any model of financial fragility. First, the economic environment must drive agents to take actions that forge links between their financial positions and the positions of others. Second, the environment must drive agents to take actions that break those links, in some cases completely, and in others only to a limited extent. The model presented in Section II minimally embodies these features.

To capture the first feature, the model gives rise to interrelated portfolios and payment commitments that create financial linkages among agents. There are two types of agents: investors and entrepreneurs, possibly many of each. Investors choose investment strategies—rules

[1] There is a considerable literature on financial crises, which will be reviewed at the end of this section. Discussions of fragility are either absent from it or purely informal. Much of the literature on fragility is attributable to Minsky (e.g., 1977), but he does not present a formal model either.

determining whether they hold their wealth in a safe, noninterest-earning asset or make risky loans to entrepreneurs at the market interest rate. The assumptions on preferences and technology lead investors to choose portfolios that are linked in the sense that the return on an investor's portfolio depends on the portfolio allocations of other investors. These linkages collectively constitute the economy's financial structure. Because agents only know their own portfolios and not other agents', they do not know to whom they are linked.

Randomness enters the model through independent and identically distributed shocks to projects' operations. These shocks are allowed to occur only at a single date, and the economy's response to them is traced over time. The shocks cause come projects to fail, generating the second desired feature, namely that some linkages come undone. When projects fail, the projects' entrepreneurs default on their loans, which causes some investors to suffer losses on their portfolios. In response, the affected investors reallocate their portfolios. They choose to hold the safe asset rather than renew their loans, thereby breaking their financial links to other investors. Because of these portfolio reallocations, some entrepreneurs cannot fund the continued operation of their projects and thus default themselves. The process continues until a new steady state is reached, with no further defaults or portfolio reallocations.

Sections III and IV consider financial crises and fragility. A financial crisis is a breakdown of the economy's financial linkages, a collapse of all or part of the financial structure. The scenario described above is of one type of financial crisis that can occur in the model: actual defaults and losses spread as investors reallocate their portfolios in response to past losses incurred. These crises, studied in Section III, can arise if investors do not foresee the possibility of contagious defaults affecting them. They suggest evaluating fragility from an ex post perspective, that is, in terms of the severity of the crisis realized as a result of the shocks experienced. This approach to fragility seems natural and in fact is the one that has typically been taken. But the model teaches that the severity of the resulting crisis is random, so using an ex post approach will never yield a definitive conclusion about an economy's fragility. The next crisis realized can always be dramatically different from all past crises.

In Section IV, the problem of allowing investors to foresee contagious defaults is discussed, and the economy's financial crises and fragility are reconsidered. Allowing foresight proves to be difficult because if the state is too informative, investors can figure out much about the economy's linkages and contagion risk. They will condition on this additional information in forming expectations of the returns from various portfolios. This occurs even with a narrow definition of the state, but the resulting conditional expectations are tractable.

If investors have foresight, a second type of financial crisis is possible. It occurs when all investors simultaneously shift entirely from risky assets to the safe asset in anticipation of defaults possibly affecting them. Like the other type of crisis, this one is caused at root by fundamentals, namely the distribution of shocks, portfolio linkages, and the returns from various portfolios. This crisis also suggests evaluating fragility from an ex ante rather than an ex post perspective by asking how soon an economy's financial structure would collapse completely from such a crisis if shocks were to hit at some date.

Because one goal of this paper is to determine whether the economy is in some sense inherently fragile, attention is restricted to equilibria with foresight in which the economy has the best chance of surviving financial crises with its financial structure intact. In such equilibria, investors maintain diversified but risky portfolios as long as possible, given that other investors maintain theirs. Under the equilibrium strategies, investors maintain their portfolios if the amount of time that has passed does not exceed a state-dependent threshold. That threshold date is the first date at which the economy experiences a crisis caused by individuals simultaneously shifting to the safe asset. The earlier that date is, the more fragile is the economy. That date depends on the features of the environment that interact to determine which portfolios investors prefer: the utility function and degree of risk aversion, the discount factor, the rates of return on the various assets, the riskiness of the assets, and the degree of diversification possible. The use of an ex ante notion of fragility thus yields an unambiguous measure of fragility.

A related issue is how ex ante fragility depends on the size of the economy. A larger economy in the context of the model is one with more investment projects, entrepreneurs, and investors. As an economy increases in size, opportunities for portfolio diversification typically increase, which contributes to reduced fragility. But as portfolios become more diversified, they also become more interconnected, so the failure of a project somewhere in the economy can spread to more agents. It is shown in Section IV that for a sufficiently large economy, fragility worsens as the economy becomes even larger, holding fixed the degree of diversification. An implication of this finding is that studying the fragility of only small economies can be misleading. Section V modifies the economy of Section II to allow for greater diversification, holding fixed the degree of interconnectedness, and shows by way of an example that greater diversification reduces financial fragility. Section VI considers institutional responses to fragility, and Section VII presents some concluding remarks.

Since there is a large literature on financial crises, it is appropriate to discuss this paper's connection to that literature before moving on to describe the model more fully. One part of this literature generates crises from herd behavior—agents copy other agents' actions because they think the others have better information (e.g., Banerjee 1992, Bikhchandani et al. 1992, and Chari and Kehoe 1997). While limited information also is critical to the crises in this paper, here agents do what maximizes their expected utility given that other agents behave in ways that make crises least likely. A second subset of the literature on financial crises generates crises from asymmetric information between borrowers and lenders (e.g., Mishkin 1991). In that literature, borrowers are assumed to have private information about the investment projects they wish to operate, resulting in an adverse selection problem. In contrast, in this model expected and realized project returns are known to all agents. The only private information here concerns investors' portfolios: each investor knows his own portfolio but not those of other investors. With this limited knowledge, investors cannot determine to whom they are linked or their true exposure to contagion risk. A growing literature on evolutionary games also examines contagion. That literature typically posits an exogenous structure in which each agent strategically interacts with other agents positioned sufficiently close to him. Some types of behavior are shown to spread rapidly under certain behavioral rules (see, for example, Morris 1997 and the references contained therein). Agents in the evolutionary-game literature tend to be either myopic or boundedly rational, and thus only react once new behavior directly affects them. In contrast, this paper considers forward-looking and perfectly rational agents who anticipate the spread of defaults and so may take preemptive action to protect themselves. A fourth subset of the literature worth mentioning generates financial crises from extraneous randomness, or sunspots. This literature has had considerable success in explaining the Great Depression and the more recent Mexican debt crisis (see Cooper and Corbae 1997 and Cole and Kehoe 1996, respectively). Fragility certainly could be studied in a model with sunspot-driven crises, but here the focus is on crises driven by fundamentals, as in Atkeson and Ríos-Rull (1996), Allen and Gale (1996), and Kiyotaki and Moore (1997a,b).[2]

Of the entire financial crisis literature, the papers of Rochet and Tirole (1996), Kiyotaki and Moore (1997a), and the very recent Hiroshi, Green, and Yamazaki (1997) are the closest in spirit to this paper. In those papers as in this one, financial linkages exist and can be broken by routine economic shocks that propagate through the linkages. Rochet and Tirole posit a set of exogenous linkages among banks engaged in interbank lending. The failure of a single bank due to a small liquidity shock can lead to the closure of all banks in the economy. Kiyotaki and Moore model an economy with many small, entrepreneurial firms reliant on trade credit. Extensions of trade credit link the firms, and shocks to some firms' returns can cause defaults that can propagate and cause a generalized recession. Because the model is very general and its linkages endogenous, the pattern of linkages is not determined and the evolution of financial crises is not transparent. This is not the case in the Hiroshi, Green, and Yamazaki paper, which is very similar but focuses on payment commitments that arise from transfers of goods among just three or four exogenously linked traders. The salient feature common to these three models is that the agents are not anonymous: they know to whom they are lending and thus to whom they are linked.

This paper, in contrast, models well-known firms that issue debt in a perfectly competitive credit market and investors who are unknown to one another. The focus is on the financial linkages that arise endogenously despite considerable anonymity and on what happens as the economy gets large. An even more important distinction between this paper and the alternatives is that the latter are all silent on the subject of fragility, while this paper's goal is to define and characterize the fragility of an economy.

II. The Economy

The economy described here is very primitive, having only the features necessary to forge

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[2] More specifically, the model in this paper assumes that shocks—presumably real shocks—to projects' operations cause projects to fail at the initial date, and that insufficient funding causes projects to cease operation at later dates. The financial crises associated with these events do not require that projects fail for these reasons. Rather, financial crises will arise in the model as long as project failures, no matter what their cause, induce portfolio reallocations that spread through portfolio linkages. Thus, the model could be modified so that the financial crises studied could be generated by sunspots, coordination failures, or the ending of what Federal Reserve Board Chairman Alan Greenspan has called "irrational exuberance." See Lagunoff and Schreft (in progress).

financial linkages among agents. Specifically, the economy drives agents to choose minimally diversified portfolios whose returns depend on the portfolios of other agents. Payment commitments arise out of the asset purchases associated with the portfolio choices. Idiosyncratic shocks to some assets result in some payment commitments going unfulfilled, which causes losses on portfolios consisting of those assets. Holders of those portfolios reallocate their wealth to regain an optimal asset mix, and in the process undo some financial linkages.

Despite the simplicity of the portfolio-choice problem, the model permits numerous financial structures—patterns of linkages—to form. It turns out that the return on an agent's portfolio depends on the agent's position in the financial structure and on the portfolio's contents. Conveniently, each possible position falls into one of five categories, which makes solving for agents' investment strategies—their rules for how to allocate their portfolios—tractable. The remainder of this section describes the physical environment in detail, characterizes the financial structures that can arise, and discusses the situations in which agents can find themselves within any financial structure.

A. Agents, Preferences, Endowments, and Technologies

Time is discrete and continues forever, and at each date the economy is populated by k infinitely lived agents of each of two types: entrepreneurs and investors. At the initial date, $t = 0$, each entrepreneur has access to a risky investment project but has no funds with which to operate it. Funds are objects called "dollars," which may be money but need not be. Each investor has funds that he wishes to invest to support consumption in future periods, and he has access to a safe, noninterest-earning asset in which he can invest directly. He also has the option of making risky operating loans to entrepreneurs that offer the chance of a higher return.

More specifically, at each date an entrepreneur's project yields a random return of $R(N)$ dollars per dollar invested, where N denotes the total number of dollars invested. Each project can be operated only if it has sufficient funding. For simplicity, the critical level of funding—the level at which projects pay the maximum return per dollar, $\overline{R}$, is taken to be \$2. Projects that have less than two dollars invested in them pay a gross return per dollar of zero.[3] Once a project has been

[3] Technologies with a minimum scale of operation have long been used in the literature on financial intermediation. See Diamond (1984).

insufficiently funded, it becomes inoperable at all future dates. When a project is overfunded, with more than two dollars invested in it, decreasing returns are realized and the project yields a return per dollar of $2\bar{R}/N$.

At date 0 only, there is a second way by which a project can become inoperable. Independently and identically distributed shocks can hit projects, causing them to fail and pay a zero return. A shock hits a project with probability p, $p \in (0,1)$. Once a project fails, it is forever inoperable. If a project succeeds at date 0, it pays a return that depends on the amount invested, as described above. In summary, then, a project's return per dollar satisfies

$$R(N) = \begin{cases} 0 \text{ with probability } p \text{ and } \dfrac{2\bar{R}}{N} \text{ with probability } 1-p & \text{at } t = 0 \text{ if } N \geq 2, \\ \dfrac{2\bar{R}}{N} & \text{at } t > 0 \text{ if } N \geq 2, \\ 0 & \text{at } t \geq 0 \text{ if } N < 2, \end{cases}$$

where $1 < \bar{R} < 1.5$.[4] An entrepreneur, however, has no funds to finance his project's operation, and thus must obtain loans from investors.[5]

Each investor is endowed with two dollars at the initial date and has preferences defined over wealth consumed, c, at each date. To fund consumption throughout his life, an investor chooses at the beginning of each date a portfolio consisting of safe and risky assets. The economy's single safe asset always yields a return of one dollar per dollar invested. The risky assets are loans to entrepreneurs, who are assumed to be price takers in the loan market. Without loss of generality, all loans are assumed to be for one period.[6]

The timing of economic activity, illustrated in Figure 1 below, is as follows. At the start of

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[4] Two comments are appropriate. First, the assumption that the shocks are iid differs from that often made in models of financial crises; typically shocks are perfectly correlated (e.g., Allen and Gale 1996). Since the objective here is to assess whether economies are inherently fragile, the assumption that shocks are iid is preferable because it gives the economy its best chance of avoiding being labeled "fragile." Second, the restriction of $\bar{R}$ to be less than 1.5 ensures that investors' interest earnings are always less than a dollar and thus that investors never find themselves with more than \$2 to invest.

[5] The entrepreneurs are uninteresting in this model. They simply take funds lent to them and operate projects. The model works identically if the entrepreneurs are omitted and investors fund and operate projects directly. The reason for having entrepreneurs in the model is to introduce indirect finance, debt, and default, which better matches the stories underlying the financial-fragility literature. However, the nonessential nature of the entrepreneurs indicates that debt and default are not essential to fragility.

[6] Nothing in the physical environment requires that loans be for only one period. Allowing long-term loans would not affect the model's results because all entrepreneurs' projects are identical ex ante, so investors who choose to lend for multiple periods have no incentive to change the set of entrepreneurs to whom they lend.

date 0, investors purchase a portfolio of assets with their initial \$2 endowment. After portfolio choices have been made, the iid shocks to investment projects are realized, causing some projects to fail. If an entrepreneur's project fails, the entrepreneur defaults on his loan payments and the investors who lent to him are repaid nothing.

$t = 0$ $t = 1$				
	Agents choose portfolio allocations.	The iid shocks to investment projects are realized.	Project returns are realized.	Consumption occurs.

$t \geq 1$ $t + 1$			
	Agents choose portfolio allocations.	Project returns are realized. Projects that were insufficiently funded become forever inoperable.	Consumption occurs.

Figure 1 The Timing of Economic Activity

At the beginning of all subsequent dates, investors can reallocate their portfolios. Those who experienced losses at the previous date will do so. Once portfolios are chosen, project returns are realized and any entrepreneurs whose projects had insufficient funding default. These defaults induce additional investor losses and, at the next date, more portfolio reallocations. At the end of each period, consumption occurs.

Based on this timing of economic activity, an investor's decision problem can be written formally as follows. A portfolio at date t is a triple $\left(a_{ht}, a_{jt}, a_{st}\right)$, where a_{ht} denotes dollars invested in risky project h at t, a_{jt} denotes dollars invested in risky project j, $j \neq h$, and a_{st} denotes dollars invested in the safe asset. An investor with \$2 can hold any of the following portfolios: an undiversified loan portfolio ((2,0,0) or (0,2,0)), a safe portfolio ((0,0,2)), a diversified loan portfolio ((1,1,0)), or a part-safe-part-risky portfolio ((1,0,1) or (0,1,1)). The gross return on portfolio $\left(a_{ht}, a_{jt}, a_{st}\right)$ is $r\left(a_{ht}, a_{jt}, a_{st}\right)$. At each date $t \geq 0$, then, an investor with beginning-of-period wealth W_t solves:

$$\max_{\left\{\left(a_{h\tau}, a_{j\tau}, a_{s\tau}\right), c_\tau\right\}} E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau), \quad 0 < \beta < 1,$$

subject to

$$a_{ht} + a_{jt} + a_{st} \le W_t \,,$$
$$W_{t+1} = W_t - \left(a_{ht} + a_{jt} + a_{st}\right) + r\left(a_{ht}, a_{jt}, a_{st}\right) - c_t \,,$$
$$a_{ht}, a_{jt}, a_{st} \in \{0,\$1,\$2\},\; c_t \ge 0,\; W_0 = \$2,$$

where $u(\cdot)$ is an increasing, strictly concave, and time-separable von Neumann-Morgenstern utility function with $u(0) = 0$. The first constraint simply states that the total amount invested at any date cannot exceed wealth available at the beginning of that date. The second condition describes the evolution of beginning-of-period wealth and reflects the fact that investors can either consume or reinvest their portfolio returns. The final set of conditions states that investments must be made in increments of $1, that consumption must be nonnegative, and that an investor's initial wealth is $2.

Preferences, the bounds on $\overline{R}$, and the indivisibility of dollars imply that an investor who realizes return $\overline{R}$ on a $1 risky loan chooses to consume only his interest earnings, $\overline{R} - 1$, and reinvest his principal (a dollar) at the next date. An investor who holds $1 in the safe asset, and thus earns a return of $1, chooses to consume the proceeds rather than reinvest them and consume nothing at the current date. An investor with $2 in the safe asset either consumes his entire $2 return immediately, or consumes $1 and reinvests $1 to fund consumption at the next date, depending on which yields higher utility. Once consumed, wealth is not available for future investment.

A comment is in order about the assumption that dollars are indivisible since, given the fixed wealth endowment, it limits loan size and thus the degree of diversification.[7] It is well known that investors are better off when more fully diversified, but that, in practice, investors' portfolios exhibit only limited diversification. Less is known about the costs of diversification.[8] In the model, fixing the extent to which investors can diversify also fixes the benefits investors can achieve through diversification, which allows attention to be focused on diversification's costs. Those costs are associated with the linkages among portfolios and the economy's fragility. One can ask, then, how the costs of holding diversified portfolios change as an economy increases in size. The first four sections of this paper take this approach. Section V considers the effect of greater diversification by increasing the wealth endowment while maintaining the indivisibility of dollars and the assumptions

[7] The assumption that dollars are indivisible is equivalent to an assumption that investments (i.e., loans) must be made in $1 increments.
[8] Krasa and Villamil (1992) show that limited diversification is optimal when there are monitoring or similar costs associated with holding diversified portfolios.

necessary to hold fixed the degree of interconnectedness.

B. Chains and Chain Structures

The solution to the investor's portfolio-choice problem is of critical concern here because the objective is to model linkages among investors' portfolios. The investor's problem as specified thus far, however, does not pin down which portfolio will be chosen. Three additional assumptions are needed regarding investors' preferences over various portfolio allocations. The first is

ASSUMPTION 1. At $t = 0$, an investor with \$2 prefers a *maximum-return diversified loan portfolio*, a portfolio (1,1,0) with both loans made to entrepreneurs who end up receiving full funding, to any other feasible portfolio.

Assumption 1 accomplishes two objectives: It defines a maximum-return diversified loan portfolio, and it implies that an investor who holds such a portfolio at the beginning of $t = 0$ chooses not to reallocate his portfolio before the shocks to projects are realized. Because the objective here is to study situations in which the iid shocks initiate the dismantling of portfolio linkages, in what follows it is also assumed that all investors start date 0 holding such portfolios. That is, each investor initially lends \$1 to each of two entrepreneurs, and each entrepreneur receives \$1 from each of two investors.[9] This, along with Assumption 1, prevents any portfolio reallocations from occurring at date 0 solely because investors are seeking portfolios offering higher returns. It thus gives the economy its best chance of avoiding being labeled as fragile.

With attention restricted in this manner, it follows that the portfolio linkages at the beginning of date 0 can be represented with closed chains. A *closed chain* is a set of entrepreneurs (and thus investment projects) and investors such that each entrepreneur's project is fully funded, each investor is fully invested and diversified, and investor portfolios are all linked directly or indirectly. Figure 2 below illustrates a closed chain.

[9] It is not tractable to consider economies in which entrepreneurs differ in the amount lent to them. Economies in which all entrepreneurs borrow the same amount in excess of \$2 differ from the economies studied here only in that the return on projects is lower and thus that investors' returns and consumption are lower.

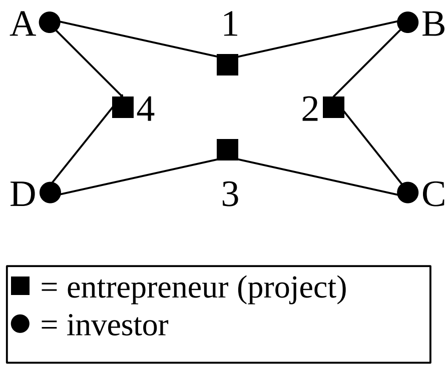

Figure 2 A Closed Chain

More precisely, the figure shows a closed *four-link* chain because the chain links each investor either directly or indirectly to four projects. Investor A, for example, is directly linked to the two investors nearest to him, investors B and D, because his portfolio contains loans to finance projects 1 and 4, projects to which B and D, respectively, also lent funds. He also is indirectly linked to the remaining investor, investor C, because although his portfolio has no assets in common with C's, C's portfolio has assets in common with B's and D's.

The iid shocks that occur at the initial date turn closed chains into open chains. An *open chain* is a set of entrepreneurs (projects) and investors such that at least one project is fully funded,

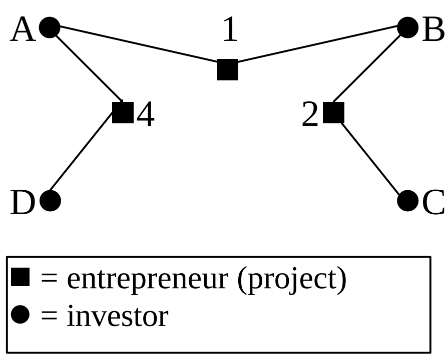

Figure 3 An Open Chain

each investor is fully invested but not necessarily diversified, and the investors' portfolios are all linked directly or indirectly. Figure 3 above depicts the open three-link chain that results when project 3 in Figure 2 fails.

An additional assumption is needed on portfolio preferences to determine the implications of a closed chain becoming open. That assumption is

ASSUMPTION 2. At each $t > 0$, an investor with \$1 prefers holding \$1 in the safe asset to holding a single \$1 loan to one entrepreneur forever.[10]

Assumption 2 states that investors who at any date $t > 0$ find themselves with only \$1 will, at the next opportunity, shift their remaining principal (\$1 each) to the safe asset. Consequently, if, say, project 3 fails from a shock at the initial date, then its entrepreneur will default on his loans from D and C, failing to repay even the principal of the loan. The losses to D and C will lead them, given Assumption 2, to shift out of risky assets and into the safe asset.[11] The remaining entrepreneurs to whom D and C had made loans, those operating projects 4 and 2, respectively, will then find their projects insufficiently funded. Since insufficiently funded projects (those run with less than \$2) yield a return of zero, the entrepreneurs operating projects 4 and 2 will default on the loans they received. This means that investors A and B will incur losses solely as a result of the portfolio reallocations of investors D and C. Their losses will drive them to reallocate their own portfolios, leaving project 1, to which they had both lent funds, with insufficient funding.

This example illustrates how the returns to each investor in a chain are affected by the portfolio allocations of all other agents in the chain. It also shows that the linkages in an open chain break over time as investors rationally adjust their portfolios in response to actual defaults experienced. This process results in open chains disintegrating with certainty in finite time.

One further assumption on portfolio preferences will prove useful:

ASSUMPTION 3. At each $t > 0$, an investor with \$2 prefers shifting to a safe portfolio ((0,0,2)) over shifting to a part-safe-part-risky portfolio (for example, (1,0,1)) forever.

Strict concavity of the utility function by itself implies that a maximum-return diversified loan portfolio is preferred to an undiversified loan portfolio. This, along with Assumption 1, means that

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[10] The outcome where the risky loan is held forever is the best one possible, the outcome yielding the investor the highest possible lifetime return from the portfolio. If the return on \$1 in the safe asset dominates that outcome, then it must dominate any other outcome associated with a single \$1 loan.

[11] This demand for the safe asset matches Keynes' (1937) description of liquidity preference.

in solving an investor's portfolio-choice problem at dates $t > 0$, one need only compare the expected lifetime utility of maintaining a diversified loan portfolio to that of holding a safe portfolio. In addition, Assumption 3 along with Assumption 1 ensures that at all dates investors who hold risky assets do so in a way that links their portfolios to those of other investors.

There do exist preferences and parameter values consistent with an investor's objective function and Assumptions 1 through 3. The proof appears in Appendix A because it depends on the solution to the investor's problem at dates $t > 0$, which is the subject of the latter half of the paper.

The concept of chains leads naturally to the concept of a chain structure for the economy. The economy's *chain structure* at any date is the combination of open and closed chains that reflects the prevailing portfolio linkages. That is, it is a partitioning of investors and entrepreneurs (with their projects) into chains. The *initial chain structure*, that which exists early in date $t = 0$, consists *solely* of closed chains. In an economy with four entrepreneurs and four investors, for example, two initial chain structures are possible. One consists of a single closed chain in which all investors are linked to all projects, as shown in Figure 2. The other consists of two closed chains, each with two investors linked to two projects, as shown in Figure 4. Any other structures must have at least one degenerate

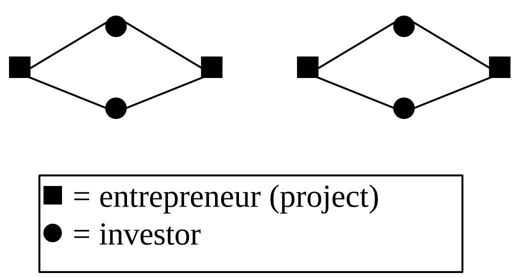

Figure 4 An Initial Chain Structure for the *k* = 4 Economy

chain consisting of one investor linked to a single project. Such structures are inconsistent with optimizing initial portfolios because the utility function is strictly concave.

This raises the question of what determines the initial chain structure. Nothing in the physical environment itself determines which investors hold which risky assets, so all chain structures with investors holding maximum-return diversified loan portfolios are equally likely. The realized initial chain structure is taken to be the outcome of a random draw from the set of all chain structures in which investors hold such portfolios.[12]

At any date after the initial date, the economy's chain structure will be a mix of closed and open chains. The mix depends on the shocks realized at the initial date and the strategic portfolio choices of investors at subsequent dates.

C. Information and Communication

The information structure is a particularly important feature of the economy, as will become obvious later in the paper. All agents in the economy know at date 0 how many and which projects are potentially operable (i.e., they know k) and that the shocks at date 0 are iid. They also know at all dates the function determining project returns, $R(N)$. Portfolio decisions, however, are always anonymous: investors know their own portfolio allocations, but not those of other agents. That is, they know to whom they lent funds but not to whom other investors lent funds. In particular, they do not know the identities of the others who made loans to the entrepreneurs to whom they lent funds.[13] Finally, they know the realized returns on their own investment projects at all dates.

In addition to having limited information, agents also are assumed to have limited ability to communicate and thus to overcome the information restrictions. Limited communication ensures that agents remain anonymous and cannot join with other agents to share risk or prevent defaults.

An implication of the informational features of the economy is that investors do not know the chain structure of the economy, which type of chain—closed or open—they are in, or their location within their chain. In particular, if they are in an open chain, they do not know how close they are to the endpoints and thus to an impending default.[14]

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[12] The initial chain structure could come about in other ways that also are consistent with investors having a flat prior over structures with only maximum-return diversified loan portfolios. For example, investors could move sequentially, each randomly selecting two distinct projects in which to invest $1, without observing the previous selections made. If a choice would result in $3 being invested in a particular project, then the investor would select again. This approach to assigning investors to projects requires a coordinating institution that verifies how much is invested in each project after each choice. Thus, it is comparable to having a coordinator look over all possible chain structures and reassign investors to projects until all chain structures consist solely of maximum-return diversified loan portfolios.

[13] Thus, this model is more applicable to large, modern economies in which investors have considerable anonymity, rather than small, village economies.

[14] The information problem facing investors when allocating their portfolios is similar to that facing individuals when choosing sexual partners when sex involves the risk of catching a sexually transmitted disease. Individuals may or may not know their current and past sexual partners, and they have at most limited accurate information about their partners' partners. Thus, they do not know the other people to whom they are connected, either directly or indirectly, through a string of past sexual encounters.

D. Investor Types

While investors are identical with respect to their portfolios before any shocks are realized, they are heterogeneous after the shocks hit. An investor’s type at the beginning of any date after the shocks hit depends on his current portfolio and location in the economy’s chain structure. Conveniently, each portfolio-location possibility in the chain structure is associated with an investor being in one of five situations, which makes solving for investor strategies tractable.

One possibility is for an investor to have \$2 invested in the safe asset. This “safe investor” receives a total portfolio return of \$2. He can either consume the \$2 return immediately, or consume only \$1 and reinvest the other \$1 to fund future consumption. He chooses the strategy that yields the highest lifetime utility.

The second possibility, illustrated in Figure 5 below, is for an investor to be on an end of an open chain. This “endpoint investor” has already lost \$1 from the default of one loan, but still holds

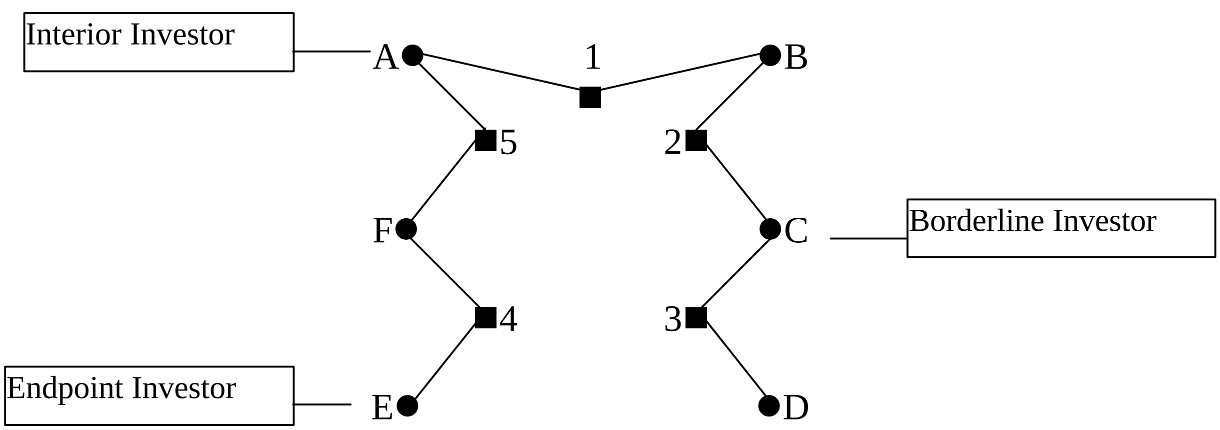

Figure 5 Some Investor Types

a \$1 loan to one entrepreneur. By Assumption 2, the investor will shift his remaining \$1 to the safe asset and receive a return of \$1. The indivisibility of dollars means that the investor either can consume nothing and reinvest his entire \$1 return in the safe asset, or can consume the entire \$1 and never invest or consume again. Since $b < 1$, the investor chooses the latter option, which yields utility, in present value, of $bu(1)$.

Third, the investor could be one of the diversified agents closest to the ends of an open chain.

This "borderline investor" has a diversified portfolio with one loan to a project that ends up fully funded and one to a project that ends up insufficiently funded. This portfolio yields a return of $\bar{R}$ and current utility of $u(\bar{R}-1)$. At the next date, this investor is an endpoint investor.

The fourth possibility is for an investor to be one of the diversified agents far from the ends of an open chain. This "interior investor" holds a portfolio of loans to two projects that end up fully funded and thus receives the return $2\bar{R}$ and current-period utility of $u(2\bar{R}-2)$. The investor with this portfolio, however, faces the certain breakdown of his chain's linkages over time. Specifically, an interior investor in an open r-link chain at t will be in an open $(r-2)$-link chain at $t+1$, since both projects currently on the chain's ends will be insufficiently funded and thus inoperable next period. Ultimately the investor will be in the situation of a borderline investor.

Finally, an investor could be in a closed chain. This investor holds the same portfolio as an interior investor but will never have to reallocate his portfolio because of actual loan defaults since his chain was not hit by any shocks at $t = 0$. A "closed-chain investor," therefore, can earn utility $u(2\bar{R}-2)$ for as long as he continues to be fully invested and fully diversified.

At any date, safe investors and endpoint investors know their types, which are determined completely by their current portfolios. Their types, in turn, give them dominant investment strategies. For example, there is no question that an endpoint investor, once invested in the safe asset, will immediately consume his entire return and have nothing to invest at subsequent dates.

In contrast, the other investors do not know their types because they do not know their locations within the chain structure. They only know that they hold diversified portfolios, not whether they are borderline, interior, or closed-chain investors. In what follows, then, they all will be referred to as "diversified investors." They can calculate, however, the probability of being a particular type, and their strategies will depend on these probabilities and the expected returns of the associated portfolios.

E. Financial Crises and Fragility: An Introduction

It remains to explore the financial crises that can arise and their implications for fragility. A financial crisis is a breakdown of the economy's financial linkages, a collapse of all or part of the chain structure, and arises from actions taken in response to shocks realized. To study financial crises, this paper looks at how the shocks at $t = 0$, when the economy is in a steady state, work their way through the economy until another steady state is reached. In a steady state, all investors' portfolio allocations remain unchanged, the projects in operation and their returns remain unchanged, and thus investors' wealth and consumption remain unchanged. The larger the share of the chain structure that collapses during the transition to the new steady state, the more severe the crisis initiated by the shocks. An economy that experiences a complete collapse of its chain structure has no portfolio links intact in the new steady state. In contrast, an economy that experiences only a partial collapse has some closed chains remaining in the new steady state, but fewer than at the initial date. The model generates two nonexclusive types of financial crises, and thus two ways to characterize fragility, depending on whether investors can foresee the possibility of defaults spreading to them.

III. Fragility without Foresight about Contagion

The simplest approach to assessing fragility involves assuming that investors do not foresee contagious defaults. Without foresight, investors have no reason to reallocate their portfolios at any date after the shocks hit unless they personally experience a default. In equilibrium, they just wait, like sitting ducks, until defaults spread to them.[15] The outcome is one type of financial crisis: an *actual-default crisis*. As its name suggests, this crisis arises from investors rationally reallocating their portfolios in response to losses they have incurred from actual defaults.

The discussion in Section II of the unraveling of open chains was an analysis of an actual-default crisis. Any shocks that hit a chain cause investor losses, which lead to rational portfolio reallocations, which lead to some projects having insufficient funding, further defaults and investor losses, and further portfolio reallocations. As defaults spread, diversified investors one by one become endpoint investors and shift their remaining wealth into the safe asset. Even if all investors continue to hold diversified portfolios until they actually experience a default, any chain hit by a shock at the initial date will collapse with certainty in finite time.

The actual-default crisis suggests one characterization of fragility. An economy is more

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[15] More precisely, this is an equilibrium with subjectively rational agents who have incorrect beliefs about their exposure to contagious defaults.

fragile *ex post* the more severe the actual-default crisis that occurred in response to shocks that were realized. The notion of ex post fragility seems to be the one economists have used historically. In the literature on financial fragility, an economy's fragility is taken to depend on the number and severity of crises the economy experienced in the past. This literature has yielded no clear conclusions about fragility because there is no consensus about what constitutes a crisis or what constitutes a severe crisis (see, for example, Kindleberger 1989 and Schwartz 1986). The analysis above suggests that no clear conclusions will ever be reached by analyzing ex post fragility because the actual-default crises that generate that fragility are purely random events. Their likelihood depends on the economy's initial chain structure and the number and distribution of shocks realized, all of which are determined randomly.

If, for example, the economy initially consists of a single closed *k*-link chain, then any shocks that hit the economy must hit that chain, causing all its linkages to ultimately break. How fast the chain collapses depends on the number and distribution of shocks. If there are few shocks and they hit adjacent projects, the chain survives longer than if there are many shocks uniformly distributed around the chain.

Alternatively, a chain structure consisting of many chains, each with two people and two projects, as in Figure 4, is the least susceptible to actual-default crises. It allows maximum diversification with minimum risk of defaults spreading. With this structure, the investors in any chain hit by a shock immediately become endpoint investors, and their chain collapses. But investors in chains not hit by shocks are forever safe from actual-default collapse. As a result, the best-case scenario with this chain structure is when there are few shocks and they hit a few chains. The worst-case scenario is when there are many shocks distributed uniformly across chains. Thus, this chain structure gives the economy its best chance of surviving with some of its linkages intact, assuming of course that investors continue to hold diversified portfolios until they suffer losses from actual defaults.

IV. Fragility with Foresight about Contagion

The analysis of the no-foresight case naturally leads one to wonder what would happen if investors could foresee the threat of contagious defaults. With foresight, a second type of crisis—an *anticipated-default crisis*—becomes possible in addition to the actual-default crisis. In an anticipated-default crisis, investors simultaneously shift completely from risky assets to the safe asset as protection against loss from expected defaults, even if they have not experienced a default themselves. Since all diversified investors have the same information at *t*, if it is rational for one to shift to the safe asset, then it is rational for them all to do so. The implication is that an anticipated-default crisis involves an instantaneous collapse of all remaining chains, whether open or closed.

Unlike actual-default crises, anticipated-default crises occur by choice, not by chance. Anticipated-default crises occur as a strategic response to the mere possibility of defaults occurring. They arise when investors decide not to put themselves at risk of incurring losses from future defaults. Nevertheless, actual- and anticipated-default crises are related and endogenous equilibrium outcomes. An actual-default crisis leads to the complete collapse of an economy's chain structure *only* if every chain is hit by at least one shock at the initial date *and* if investors strategically choose to hold diversified loan portfolios long enough for all chains to unravel completely.

Anticipated-default crises suggest a second characterization of fragility. An economy is more fragile *ex ante* the sooner its chain structure is expected to collapse if shocks hit at some date. Although ex ante fragility is not the notion of fragility that instinctively comes to mind or that has been used historically, it seems the more appropriate notion. When speculating about an economy's fragility, one is really asking from the perspective of some initial date what would happen if shocks were to hit the economy at some later date. The model studied here suggests that the only clear-cut answer to that question comes from looking at when an anticipated-default crisis occurs, since such crises are not random events. The sooner such a crisis occurs, the more fragile is the economy.

A. The Problem with Modeling Foresight

The existing literature on contagion (e.g., Morris 1997) has not allowed agents to have foresight. It turns out that modeling foresight regarding contagion is very difficult. The reason, at least for this model, is that if investors have too much information about the current and past state of the economy, they can figure out too much about the chain structure and distribution of shocks and would condition on this additional information in calculating the expected utilities from various portfolios. Such calculations appear to be intractable.

This particular problem arises if the state is taken to be the initial number of projects (k) and the number of projects that remain at the beginning of each subsequent date up through the present. For example, if an investor knows that k = 10, t = 3, eight projects remained at the beginning of date 1, six remained at date 2, and six currently remain, then he can figure out much about his exposure to contagious defaults. Specifically, he can determine that only two adjacent projects were hit by shocks at date 0, that those projects were in a closed 4-link chain, and that that chain collapsed as a result of portfolio reallocations at t = 2, leaving him safe from contagious defaults at later dates.

Even if the state is taken to consist of the initial number of projects and the number of projects currently remaining, but not the entire historical path of projects remaining, an investor can rule out many possibilities. For example, if an investor knows k = 6, t = 2, and that four projects remain, then he knows that at t = 0 either one or two projects were hit by shocks in a single closed 2-link chain. The reason is that if any larger chain had been hit by even one shock at date 0, then two projects would have failed at date 1 from contagion, leaving three or fewer projects at date 2. Thus, the investor knows that any contagion that had started is over and that the initial chain structure had to have contained at least one closed 2-link chain.

For simplicity, then, this paper takes the state to be simply the number of projects at the initial date, k. An investor knows the state and the current date. In the context of this model, this assumption is reasonable because endowments and the assumptions on portfolio preferences imply that an investor selects a portfolio of risky assets once, at the initial date. At all subsequent dates, a diversified investor must decide whether to maintain his existing diversified loan portfolio, which has been paying him the maximum return, or to shift all his funds to the safe asset. Even with this simple treatment of the state, investors can rule out certain outcomes (e.g., they know that they will never be in an open 6-link chain if only four projects existed initially), but it is tractable to condition on such information.

B. An Equilibrium with Foresight

Given this specification of the state, a strategy for diversified investor i may be formally defined as a sequence $f = \{f_t^i\}$, where if $f_t^i(k)$ = 1, then the investor remains diversified at time t if there were k projects initially, while if $f_t^i(k)$ = 0, then the investor shifts entirely to holding only the safe asset.[16] This leads to the following definition of equilibrium:

DEFINITION. A *symmetric equilibrium* is an identical strategy $f_t^{i*}(k) = f_t^*(k)$ for each diversified investor i that maximizes i's expected utility given his correct forecast $f_t^{h*}(k)$, $\forall\ h \neq i$.

A few comments about this definition are in order. First, there may exist asymmetric equilibria—equilibria in which some investors behave differently than others in some states. But because all investors have the same preferences and information in each state, any two investors who differ in their strategies f^i in state k at some date t must be indifferent between remaining diversified and shifting to the safe asset. This indifference, however, will not generally hold except for very special (nongeneric) parameters. In this sense, such equilibria are knife-edged.

Second, there is a class of symmetric equilibria with the property that each investor shifts to holding safe assets at some date t solely because he believes that others will do the same. Such equilibria (e.g., sunspot equilibria) represent coordination failures and are driven by strategic behavior analogous to that which arises in any coordination game with Pareto-dominated equilibria. While expectations do play a role in the traditional fragility literature (e.g., Minsky 1977), crises are also driven in that literature by economic fundamentals. This paper looks only at crises caused by fundamentals: actual project failures and loan defaults fuel actual-default crises, and the anticipation of these events generates anticipated-default crises. Neither type of crisis arises from investors reallocating their portfolios only because they think others will do so, although they would be collectively better off not reallocating. Since coordination-failure equilibria are not particularly relevant for understanding fragility based on fundamentals, the remainder of the paper works with a refinement of the symmetric equilibrium defined above that excludes equilibria resulting from sunspots or other coordination failures.

Under the refined notion of equilibrium, a diversified investor shifts entirely to holding the safe asset at t when there were k projects initially, even if all other diversified investors maintain their portfolios, as long as the lifetime expected utility from shifting, denoted v^S, exceeds that from

[16] Recall that Assumption 3 makes it undesirable for an investor to have \$1 in one risky asset and \$1 in the safe asset. Thus, even though such behavior is feasible, it is excluded from consideration in constructing strategies.

remaining diversified, denoted $v_t(k)$.[17] Otherwise (i.e., when $v_t(k) \geq v^S$) a diversified investor maintains his portfolio. The expected lifetime utility $v_t(k)$ is calculated assuming that all other diversified investors maintain their portfolios until they experience an actual default. That is, a diversified investor calculates the expected lifetime utility from remaining diversified by considering what happens if he and all other investors continue to hold their portfolios and allow the existing open chains to unravel. This equilibrium can be called a *maximal sustainable equilibrium* because it preserves the economy's financial structure as long as possible and thus gives the economy its best chance of avoiding being labeled as fragile. It follows that it is the unique Pareto-efficient equilibrium in the class of symmetric equilibria studied. It is formalized in the following definition:

DEFINITION. A *maximal sustainable equilibrium* is a symmetric equilibrium such that for state k and date t, $f_t^*(k) = 1$ iff $v_t(k) \geq v^S$.

This is indeed an equilibrium because (1) all investors are identical ex ante, having identical utility functions over wealth, and (2) it is a best response for each investor to remain fully diversified given that he expects others to do the same.

C. Expected Utilities

Solving for an equilibrium requires calculating $v_t(k)$. To do so, an investor must consider what can happen to him in each possible chain, given his knowledge of the state k, the date, and his investments' continued success to date. That is, he must determine the probability of being in either a closed or an open chain and the expected utilities from each type of chain, conditional on k and the fact that he is still diversified.

The expected utility from being in an open chain weights the possibility of a diversified agent being either an interior or a borderline investor. At the beginning of any date $t \geq 1$, an open r-link chain has $r-1$ diversified investors. Two of these investors are borderline investors, so the probability of being a borderline investor is $2/(r-1)$. The probability of being an interior investor is

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[17] As explained in Section II, the lifetime utility from a portfolio with \$2 invested in the safe asset is the maximum of $u(2)$ and $u(1)+\beta u(1)$, and so is not state-dependent.

then 1–2/(r–1). Thus, using the expected utilities defined in Section II of being a borderline or interior investor, the expected utility from being in an open r-link chain may be defined recursively by

$$\mathrm{p}(r)=\begin{cases}\dfrac{2}{r-1}\left(u(\overline{R}-1)+\mathrm{b}u(1)\right)+\left(1-\left(\dfrac{2}{r-1}\right)\right)\left(u(2\overline{R}-2)+\mathrm{b}\mathrm{p}(r-2)\right) & \text{if } r\geq 3,\\ 0 & \text{if } r<3.\end{cases}$$

In contrast, closed-chain investors have not had any shocks hit projects in their chain. Thus, if they remain diversified permanently, they enjoy the lifetime utility of $\overline{\mathrm{p}}\equiv u(2\overline{R}-2)/(1-\mathrm{b})$. The utility $\overline{\mathrm{p}}$ also is precisely the limit of the expected utility from remaining in an open r-link chain as r approaches infinity: $\overline{\mathrm{p}}=\lim_{r\to\infty}\mathrm{p}(r)$. The reason is that the larger an open chain is, the longer an interior investor can stay on the interior. The interior investor in an infinitely large open chain thus receives the expected utility of a closed-chain investor.

It remains to calculate the probabilities of open and closed chains. The following events can be defined. C_{rt} denotes the event that a closed r-link chain exists at date t, while N_{rt} denotes the event that an open r-link chain exists at t. $E_t(k)$ represents the event that an investor holds a diversified loan portfolio at the beginning of date t, before portfolio reallocations can take place, when state k prevails. With these definitions, the expected lifetime utility from remaining diversified is

$$v_t(k)=\sum_{r=2}^{k}\left[P\left(C_{rt}\middle|E_t(k)\right)\overline{\mathrm{p}}+P\left(N_{rt}\middle|E_t(k)\right)\mathrm{p}(r)\right],\quad t\geq 1.$$

The summation is over r from 2 to k because no chains with fewer than two links will arise given the concavity of utility, and none larger than k is feasible.

To compute $v_t(k)$, it remains to find $P\left(C_{rt}\middle|E_t(k)\right)$ and $P\left(N_{rt}\middle|E_t(k)\right)$ for each $t > 0$ and k. Both depend on $q(r,k)$, the probability of an investor belonging to a closed r-link chain initially, before the shocks hit at t = 0. For a fixed k,

$$q(r,k)=\frac{\dbinom{k-1}{r-1}}{\displaystyle\sum_{n=1}^{k-1}\binom{k-1}{n}-\binom{k-1}{k-2}},\quad r=2,\ldots,k-2,k\,.$$

By concavity again, no agent will invest in risky assets without diversifying, so there will be no initial closed r-link chains for r = k–1 or r = 1. Therefore, $q(k–1,k) = q(1,k) = q(0,k) = 0$. And naturally, $q(r,k) = 0$ if $r > k$.

Using the expression for $q(r,k)$, the probabilities of being in a particular type of chain at any date $t \geq 0$ can be determined. Clearly, at date 0, the probability of an investor's being in a closed r-link chain conditional on holding a diversified loan portfolio in state k at the start of the period, $P\left(C_{r0}|E_0(k)\right)$, is $q(r,k)$. And for dates $t \geq 1$, an investor can be in a closed chain only if he was in such a chain at date 0 and no shocks hit the chain. Thus, for any $t \geq 1$, the *nonnormalized* conditional probability of a closed r-link chain, denoted $\overline{P}\left(C_{rt}|E_t(k)\right)$, is $q(r,k)(1-p)^{r-2}$ if $2 \leq r \leq k$ and zero otherwise (normalized probabilities will be constructed below). The reason is that $(1-p)^{r-2}$ is the probability that none of the $r-2$ projects in the chain other than those in the investor's portfolio, which are known to have survived, were hit by shocks at $t = 0$. The probability $\overline{P}\left(C_{rt}|E_t(k)\right)$ is independent of t.

It remains to consider the probabilities of open chains. Since no shocks have occurred as of the beginning of date 0, the nonnormalized probability of an investor's being in an open r-link chain at date 0, conditional on his holding a diversified loan portfolio in state k at the beginning of the period, denoted $\overline{P}\left(N_{r0}|E_0(k)\right)$, is zero. But shocks have occurred by the beginning of date $t = 1$, so an investor could find himself in an open r-link chain at date $t \geq 1$. There are two routes by which this could happen. One possibility is that the open r-link chain came from a closed m-link chain at date 0, with $m = r+1$. That is, a single shock could have hit a closed m-link chain at date 0, creating an open chain. The probability of a closed m-link chain at date 0 is $q(m,k)$, and there are $r-1$ ways that r adjacent projects, two of which are in the investor's portfolio, can be chosen from the initial $r + 1$. Thus, the probability of being in an open r-link chain at $t = 1$ that came from an initial closed m-link chain, with $m = r + 1$, is $(r-1)q(m,k)p(1-p)^{r-2}$.

Alternatively, the open r-link chain at $t = 1$ could have come from a closed m-link chain at date 0, with $m > r + 1$. In this case, at least two distinct projects in the original closed chain must have been hit by shocks and failed at the initial date, but only the two projects whose failures created the ends of the open chain are relevant for the chain's evolution. Thus the probability of being in an open r-link chain that arose in this manner is $\sum_{m=r+2}^{k}(r-1)q(m,k)\,p^2(1-p)^{r-2}$.

It does not matter by which of these routes an investor finds himself in an open r-link chain at t because the same utility is associated with each case. The probabilities of the subcases can be summed, then, to get $\overline{P}\left(N_{r1}|E_1(k)\right)$, the nonnormalized total conditional probability that at the start of date 1 an investor resides in an open r-link chain, given that there were k projects initially and he still holds a diversified loan portfolio:

$$\overline{P}\left(N_{r1}|E_1(k)\right)=(r-1)(1-p)^{r-2}\left[p^2\sum_{m=r+2}^{k}q(m,k)+pq(r+1,k)\right].$$

In calculating the probability of open chains at later dates, the possibility of contagious defaults also must be considered. Because of contagion, chains at dates $t > 1$ correspond one-to-one with certain open chains at date $t = 1$. For this reason, it is useful to first calculate $\overline{P}\left(N_{n1}|E_t(k)\right)$, the normalized probability that a diversified investor started date 1 in an open n-link chain, conditional on his portfolio having remained intact until the beginning of some later date t, given state k. Clearly, $\overline{P}\left(N_{n1}|E_t(k)\right) = 0$ if $n \geq k$ since at date 1 there could have been no open chains with k or more links when there were only k projects at date 0. Likewise, $\overline{P}\left(N_{n1}|E_t(k)\right) = 0$ if $n < 2 + 2(t - 1)$ because at the start of $t = 1$ any chain the investor could have resided in must have been large enough to contain the two projects in the investor's portfolio and all projects that would default from contagion between dates 1 and t (i.e., two per date for each date up through t–1). The probability of all other open n-link chains at date 1, those of sizes $2 + 2(t-1) \leq n \leq k - 1$, conditional on event $E_t(k)$, is positive and equal to $\overline{P}\left(N_{n1}|E_1(k)\right)$.

It follows that $\overline{P}\left(N_{rt}|E_t(k)\right)$, the nonnormalized probability of a diversified investor being in an open r-link chain, conditional on his still holding a diversified loan portfolio in state k at date t, satisfies $\overline{P}\left(N_{rt}|E_t(k)\right)=\overline{P}\left(N_{n1}|E_t(k)\right)$, where $n = r + 2(t-1)$.

Since each investor holding risky assets must be in either an open or a closed chain, the possibilities considered above are exhaustive. They can be normalized to sum to one by dividing them by $\Lambda_t(k)$, the total probability weight attached to all r-link chains, whether open or closed, in state k at date t:

$$\begin{aligned}\Lambda_t(k) &\equiv \sum_{r=2}^{k}\overline{P}\left(C_{rt}|E_t(k)\right)+\sum_{r=2}^{k}\overline{P}\left(N_{rt}|E_t(k)\right)\\ &= \sum_{r=2}^{k}\overline{P}\left(C_{rt}|E_t(k)\right)+\sum_{r=2+2(t-1)}^{k-1}\overline{P}\left(N_{r1}|E_t(k)\right)\\ &= \sum_{r=2}^{k}\overline{P}\left(C_{rt}|E_t(k)\right)+\sum_{r=2t}^{k}\overline{P}\left(N_{r1}|E_1(k)\right).\end{aligned}$$

It follows that the probabilities of belonging to closed and open r-link chains, respectively, are

$$P\left(C_{rt}|E_t(k)\right)=\begin{cases}\overline{P}\left(C_{rt}|E_t(k)\right)/\Lambda_t(k) & \text{if } 2\leq r\leq k\\ 0 & \text{otherwise}\end{cases}$$

and

$$P\left(N_{rt}|E_t(k)\right) = P\left(N_{n1}|E_t(k)\right) = \begin{cases} \overline{P}\left(N_{n1}|E_1(k)\right)/\Lambda_t(k) & \text{if } 2t \le n \le k-1 \text{ and } n = r+2(t-1) \\ 0 & \text{otherwise.} \end{cases}$$

This completes the expression for $v_t(k)$.

D. Characterizing Ex Ante Fragility

To characterize ex ante fragility, the date at which an anticipated-default crisis occurs must be determined. It is clear that once the shocks hit at the initial date, some closed chains become open chains. It is also clear that these chains shrink in size over time. Without information on how many chains are left, investors still holding diversified portfolios may initially view the risk of defaults spreading to them as increasing over time because the probability of being in smaller open chains is increasing.[18] Thus, it is reasonable to expect there to be a first date at which investors will choose to shift to the safe asset. This idea is formalized in the following proposition:

PROPOSITION 1. Let f^* be a maximal sustainable equilibrium. There exists a discount factor $\hat{\beta}$ such that if $\beta \ge \hat{\beta}$, then there is a time $\tau(k)$ for each k that satisfies

$$f_t^*(k) = 1 \quad \text{if} \quad t < \tau(k), \text{ and}$$
$$f_t^*(k) = 0 \quad \text{if} \quad t \ge \tau(k).$$

Moreover, $\tau(k) \le k/2$.

By this result, if investors are sufficiently patient ($\beta \ge \hat{\beta}$), a date $\tau(k)$ exists that is the earliest date at which all diversified investors view themselves to be at sufficient risk of impending defaults that $v_t(k) < v^S$ and they all simultaneously switch to the safe asset. For $t < \tau(k)$, all diversified investors choose to remain fully diversified. The role of patience is to ensure that closed chains are sufficiently preferable to open ones. After all, if β is close to zero, then an investor who likely is in the interior of an open chain does not care about defaults because they only affect him in subsequent periods. This interpretation suggests the following result:

COROLLARY. An economy is *increasingly fragile* the smaller is $\tau(k)$. In particular, if $\tau(k) = 1$, the

[18] At some point, of course, they will view the risk of contagious defaults as decreasing because they know they are almost certainly in a closed chain.

economy suffers an anticipated-default crisis immediately, while if $\tau(k) = \infty$, then investors never preemptively switch to the safe asset and thus remain diversified until they experience a default.

For the economy's financial structure to remain intact for some intermediate amount of time (i.e., for $1 < \tau(k) < \infty$), $\tau(k)$ must have some initial monotonicity in t. That is, it is necessary that $v_1(k) \geq v^S$ and $v_t(k)$ strictly decreases over some range $t = 1,\ldots,t'$. Since two projects fail per open chain due to contagious defaults at all dates $t \geq 1$, no contagion can continue beyond $t = (k–1)/2$. Thus, all investors remaining diversified at subsequent dates know that they must be in a closed chain, which means that $t' < (k–1)/2$.

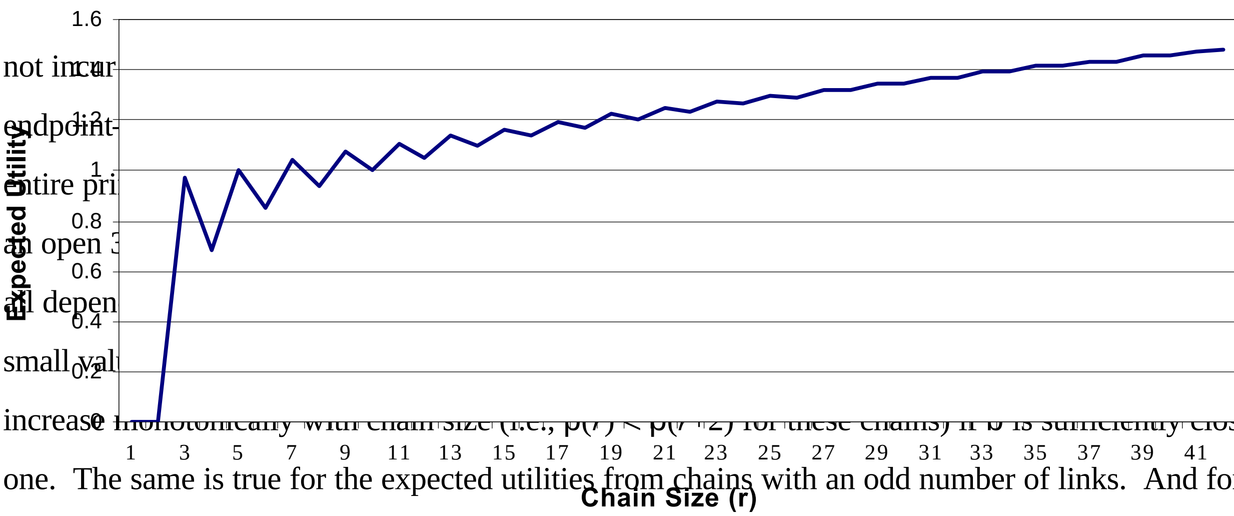
It need not be the case that $v_t(k)$ is strictly decreasing for small t. The reason depends both on the probabilities of the chains and on the expected utilities of various chains. The effect of chain probabilities on $v_t(k)$ is complicated. The reason is that, as time goes by, the probability of investors being in the largest open chains goes to zero. As shown above, $P\left(N_{rt} | E_t(k)\right) = P\left(N_{n1} | E_t(k)\right)$, where $n = r + 2(t–1)$. Since the largest open chain at the beginning of $t = 1$ is of size $k – 1$, open r-link chains at date t can have no more than $k – 2t + 1$ links. In particular, an open $(k – 1)$-link chain is possible at $t = 1$ but not at $t = 2$. Hence, the probability mass assigned to $\pi(k – 1)$ at $t = 1$ is reassigned to the expected utility of other chains at $t = 2$. There are two countervailing effects. First, some of the probability mass is assigned to $\bar{\pi}$, the utility from being in a closed chain. That is, as time passes, it becomes more likely that an investor who remains diversified belongs to a closed chain. This is the best possible scenario from the investor's perspective. Second, some of the mass is reassigned to the expected utility from smaller open chains. This is bad for the investor because contagious defaults reach him sooner in smaller open chains. Thus, to have $v_1(k) > v_2(k)$, relatively less probability weight must be reassigned to $\bar{\pi}$ than to the expected utilities $\pi(r)$ for small r.

The movement of the expected utilities of chains is itself complicated, however. The utility from being in a closed chain is constant over time and the same as the utility from an infinitely large open chain. But the expected utility from an open chain is not monotonically increasing in chain size.[19] This is because an investor's position in an open chain has a large impact on his expected

[19] Since open chains shrink in size over time, if the expected utility from an open chain was monotonically increasing in chain size, then at least over some intermediate period the investor's expected utility from remaining diversified might be decreasing. Ultimately, the expected utility from remaining diversified would have to increase, however, since it must converge to the expected utility from being in a closed chain.

utility, and in small chains an investor can remain on the interior for fewer periods. In particular, a diversified investor in an open 3-link chain is necessarily a borderline investor. If he waits for defaults to spread to him, he will lose one of his investments in the current period, which will lead him to protect his remaining wealth ($1) at the next date. In contrast, a diversified investor in an open 4-link chain has a 2/3 chance of being a borderline investor and a 1/3 chance of being an interior investor. As an interior investor, if he waits for contagious defaults to affect him, he will

Figure 6 Expected Utilities from Open r-Link Chains



not incur ... endpoint- ... his entire pri ... rom an open 3 ... nks all depen ... or small val ... increase monotonically with chain size (i.e., p(*r*) < p(*r*+2) for these chains) if b is sufficiently close to one. The same is true for the expected utilities from chains with an odd number of links. And for sufficiently large open chains, the weight on p(3) or p(4) is so small that the expected utility of an open chain is monotonically increasing in chain size. Figure 6, above, illustrates p(*r*) for the preferences and parameter values used in the appendices. The cycles are apparent, and the function is strictly increasing for $r > 39$.

To make more concrete the calculation of chain probabilities, portfolio expected utilities, and the threshold date, subsection F below presents examples for economies of various sizes.

E. Fragility and Economy Size

These results raise the question of how fragility depends on the size of the economy. It is reasonable to expect that as an economy gets larger (i.e., as the number of entrepreneurs, investors, and investment projects, as reflected by the parameter *k*, increases), holding fixed the degree of diversification, it also becomes more fragile. In other words, t(*k*) should be decreasing in *k*. The next result shows that, for sufficiently large economies, financial fragility indeed worsens as the economy increases in size.

PROPOSITION 2. There exists a k' such that, $\forall\, k \geq k'$, $\tau(k)$ is decreasing in k.

Proof. See Appendix B.

The intuition behind proposition 2 relies on three observations. First, the average size of a closed chain before the shocks hit at $t = 0$ is approximately $k/2$. This average chain size comes from summing each chain size by its probability: $\sum_{r=2}^{k} q(r,k)r$. Second, as $k \to \infty$, the total probability of being in a closed chain after the shocks hit at $t = 0$ approaches zero because the average number of shocks realized—kp—is increasing in k. Thus, the larger is k, the more likely it is that an investor is in an open chain after the shocks hit. Third, the projects that are hit by shocks are as likely to be distributed around a closed chain one way as they are any other way. On average, then, shocks are uniformly distributed around a chain. Combining these three observations yields the average size of an open chain after the shocks have hit at the initial date. All such open chains are formed from shocks hitting closed chains. The average number of shocks hitting the average-size initial chain—a closed ($k/2$)-link chain—is $kp/2$ because there are only two chains of average size, and half of the uniformly distributed shocks must hit half the chains. The average size of the open chains that result is then $(k/2)/(kp/2)$, or $1/p$, which is independent of k. Thus, as k increases, the likelihood at $t = 0$ of being in an open chain after the shocks have hit also increases, while the average size of an open chain stays the same. This implies that as economies become larger, they also become more fragile.

F. Examples

Some numerical examples for economies of various sizes nicely illustrate the results presented in this section. In the examples to follow, preferences and parameter values are the same as those shown in Appendix A to be consistent with an investor's objective function and Assumptions 1 through 3: $u(c)=(c-\alpha)^{1-\sigma}$ for $c>\alpha$ and $u(c)=0$ otherwise, $\sigma = 0.20$, $\alpha = 0.03$, $\overline{R} = 1.05$, $\beta = 0.95$, $p = 0.01$.

Examples for the $k = 4$ and $k = 10$ economies make clear how investors calculate $v_t(k)$. The results for the $k = 4$ economy are summarized in Table 1. The probabilities of closed chains at the

r	$q(r,4)$	$\overline{P}(C_{rt}\|E_t(4))$ for all t	$\overline{P}(N_{r1}\|E_1(4))$	$P(C_{r1}\|E_1(4))$	$P(N_{r1}\|E_1(4))$	$P(C_{r2}\|E_2(4))$	$\overline{P}(N_{r2}\|E_2(4))$, $P(N_{r2}\|E_2(4))$
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.75	0.75	0.000025	0.75	0.000025	0.75375	0.0
3	0.0	0.0	0.00495	0.0	0.00495	0.0	0.0
4	0.25	0.245025	0.0	0.245025	0.0	0.24625	0.0
				$\Lambda_1(4)=1.0$		$\Lambda_2(4)=0.995025$	

Table 1 Probabilities of Chains for the *k* = 4 Economy

initial date (the $q(r,k)$ values) are consistent with the discussion in Section II of possible chain structures for this economy. Columns 3 and 4 of the table show the nonnormalized conditional probabilities of closed and open chains, while columns 5 through 8 show the corresponding normalized conditional probabilities. Open chains are possible at date 1 (column 6), but not at date 2 (column 8). No diversified investor can be in an open chain at date 2 because the largest open chain that occurs with positive probability at date 1 is a 3-link chain. By date 2, that chain will have lost enough links that the only person remaining in it will be an endpoint investor. $\Lambda_2(4)$ spreads the probability weight that had been assigned to open chains at date 1 across closed chains at date 2.

The expected utility from remaining diversified in the $k = 4$ economy is shown in the third column of Table 2 below. It initially exceeds the expected utility from shifting to the safe asset (column 2) and converges to the expected utility from being in a closed chain by date 2. That result corresponds to the zero probability of open chains as of date 2.

t	v^S for all k	v_t(k) for k=4	v_t(k) for k=10	Prob. Of Residing in Closed Chain, k=10	Prob. Of Residing in Open Chain, k=10	Prob. Open r-Link Chain, $r > 5$, k=10	Prob. Open r-link Chain, $r \leq 5$, k=10
1	1.9031	2.374	2.33	0.9661	0.0338	0.0122	0.0216
2	1.9031	2.38	2.328	0.9703	0.0297	0.0002	0.0295
3	1.9031	2.38	2.354	0.9875	0.0124	0.0	0.0124
4	1.9031	2.38	2.38	1.0	0.0	0.0	0.0
5	1.9031	2.38	2.38	1.0	0.0	0.0	0.0
6	1.9031	2.38	2.38	1.0	0.0	0.0	0.0
7	1.9031	2.38	2.38	1.0	0.0	0.0	0.0
8	1.9031	2.38	2.38	1.0	0.0	0.0	0.0
9	1.9031	2.38	2.38	1.0	0.0	0.0	0.0
10	1.9031	2.38	2.38	1.0	0.0	0.0	0.0

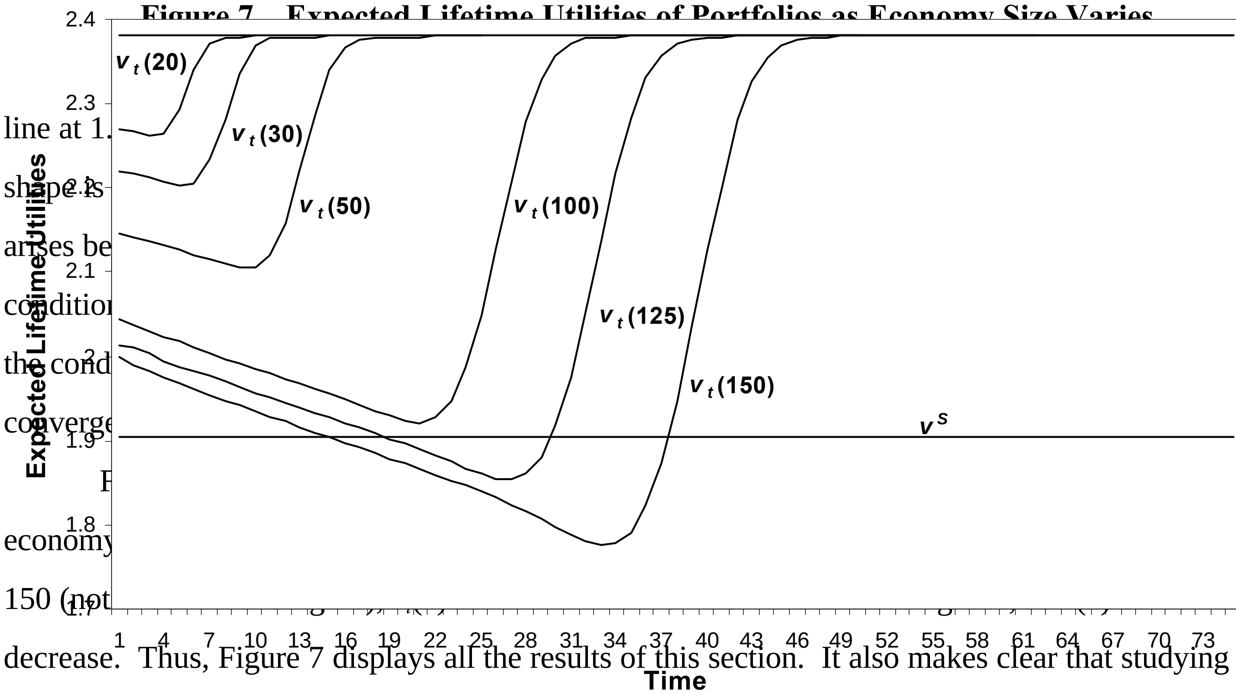
Table 2 Analysis of k = 10 Economy and Comparison to k = 4 Economy

Table 2 also compares the results for the $k = 4$ economy to those for the $k = 10$ economy. The table shows that $t(k) = \infty$ for both economies. That result is not surprising. It obtains partly because the expected lifetime utility from continuing to hold a diversified portfolio at any date is calculated conditional on the investor's knowing that he entered the period with a diversified portfolio. In a small economy, an investor's portfolio contains loans to a large share of the economy's projects. Also contributing to the result is an investor's knowledge that contagious defaults cause a minimum of two projects per period to fail. That means that at each date 50 percent of all projects fail from contagion in the $k = 4$ economy, and 20 percent do so in the $k = 10$ economy. Thus, if an investor in those economies is holding a diversified portfolio at the start of any date $t \geq 1$, then there is a very good chance that he is in a closed chain. And he knows that he is definitely in a closed chain if $t \geq (k{-}1)/2$. For the $k = 10$ economy, that happens as early as date 4.

In addition, Table 2 shows that $v_t(10)$ falls and then rises and converges to 2.38, the expected utility of a closed chain. The reason is that the probability of being in a large open chain (column 7, with "large" arbitrarily defined to be more than five links) decreases by 0.012 from date 1 to date 2, while the total probability of being in an open chain (column 6) decreases only by 0.0041. The implication is that the probability weight formerly given to large open chains is spread between small open chains and closed chains. The probability of small open chains (column 8) increases by 0.0079, and the probability of closed chains (column 5) increases by 0.0042. Since the probability of small open chains rises more than the probability of closed chains from date 1 to date 2, and the expected utility from smaller open chains is lower than that from larger ones (illustrated for the same preferences and parameter values in Figure 6 above), the expected utility from remaining diversified decreases from date 1 to date 2.

Figure 7 below illustrates how t(*k*) changes with the size of the economy. The horizontal

Figure 7 – Expected Lifetime Utilities of Portfolios as Economy Size Varies



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decrease. Thus, Figure 7 displays all the results of this section. It also makes clear that studying examples for small economies is instructive but also somewhat misleading because small economies have different fragility properties than large ones.

V. Diversification

Thus far this paper has taken the degree of diversification as given and examined the impact of greater financial interconnectedness on financial fragility. A more challenging task is to take the degree of interconnectedness as given and explain the impact of greater diversification on fragility. While studying the effect of diversification in general is quite difficult, insight can be gleaned from the following example, which is based on a modified version of the model considered thus far.

Greater diversification can be introduced most simply by assuming that each investor is endowed with three instead of two indivisible dollars and adding

ASSUMPTION 4. At $t = 0$, a maximum-return diversified loan portfolio, one with three $1 loans to distinct entrepreneurs who end up fully funded, is preferred to any other feasible portfolio.

Thus, investors are more diversified in this modified economy in that they hold 33 percent of their wealth in a loan to a particular project versus 50 percent in the economy studied above. The degree of financial linkage can be held fixed by maintaining Assumptions 1 through 3, holding k constant, and assuming that portfolios are such that chains are still symmetric.

With these assumptions, the initial chain structure when $k = 4$, for example, will necessarily be a single closed four-link chain (the chain structure shown in Figure 4 is ruled out now because it does not allow sufficient diversification). This structure could look as illustrated in Figure 8 below. Each investor is *directly* linked to the project on his left and the two closest projects on his right.

It is readily apparent, given the assumptions on preferences, that the single closed four-link chain structure is less subject to financial crisis when there is greater diversification because only a

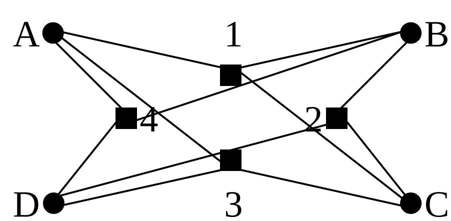

Figure 8 The Effect of Greater Diversification on a Closed Four-Link Chain

subset of the possible realizations of shocks can initiate an actual-default crisis.[20] With an actual-default crisis less likely, an investor's probability of being in an open chain after the initial shocks is lower, making an anticipated-default crisis less likely too. Specifically, if only one project in the economy, say project 4, is hit by a shock, then investors A, B, and D suffer portfolio losses. By Assumptions 2 and 3, however, they will keep their remaining wealth invested as is. Since no portfolio reallocations occur, no losses spread to investor C. And the chain remains closed, as illustrated in Figure 9, despite the failure of a project. Thus, for either type of financial crisis to

[20] This assumes that projects are exactly fully funded with either $2 or $3 invested and insufficiently funded with less than $2 invested in them.

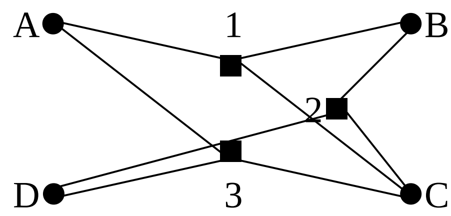

Figure 9 With Greater Diversification, a Closed Chain Can Remain Closed after a Shock Hits

occur in this economy with greater diversification, at least one investor must at the initial date suffer losses on at least two of the loans he made. Greater diversification, then, holding fixed the linkage structure, economy size, and preference assumptions governing when an investor reallocates, makes an economy less financially fragile.

In general, as an economy increases in size, one would expect the degree of diversification and the degree of financial linkage to increase. Which dominates, the benefits of greater diversification or the costs of greater interconnectedness, remains the subject of future research.

VI. Institutional Responses to Fragility

Some comments on the model's policy implications are appropriate. The equilibria examined here are Pareto efficient in the class of symmetric equilibria, but financial crises do occur, so fragility is costly. This model is very primitive, abstracting from any institutions or government policy that might help overcome crises. For example, mutual funds and financial intermediaries are obvious candidates for financial institutions that would naturally arise to enhance risk sharing in this economy. These institutions can be introduced into the model simply by reinterpreting the individual investors as mutual funds. Likewise, the entrepreneurs could be reinterpreted as sectors. None of the model's results, however, would be affected by these changes. Unless all sectors and all investors share risk, financial crises are still possible, and the economy is still potentially fragile.

Alternatively, the government could intervene to reduce fragility. In fact, as the opening quotation from Friedman illustrates, some economists argue that financial crises represent a failure of government policy. As Schwartz (1986, p. 12) puts it, "A real financial crisis occurs only when institutions do not exist, when authorities are unschooled in the practices that preclude such a development, and when the private sector has reason to doubt the dependability of preventive arrangements." One often-prescribed intervention is for the government to function as a lender of last resort to prevent financial crises. If the government controls the supply of objects known as dollars in this model, then it can serve as a lender of last resort, lending to entrepreneurs who otherwise would be insufficiently funded. The government can serve this role even if it has the same information as private agents about the chain structure and spreading of defaults; all that is necessary is for the government to be known to all agents and able to broadcast announcements. Given such capabilities, if the government announces at the beginning of $t = 1$ that it will serve as a lender of last resort, it can bring about a Pareto-optimal allocation.[21] The reason is that there are no incentive-compatibility problems here: entrepreneurs have no incentive to misrepresent themselves to obtain extra funds to finance additional or riskier investments. In a more general model, however, such incentive problems would arise, and it is not clear if it would be optimal for the government to serve as a lender of last resort starting at $t = 1$.

Another possibility is for the government to sell insurance against investment losses from contagious defaults. Investors could buy such insurance at dates $t > 0$. They could pay the premiums out of the interest income ($\bar{R} - 1$ per dollar invested) that they otherwise would have consumed. The funds raised from insurance sales would keep projects with insufficient private-sector funding in operation. Again, there would be no incentive-compatibility problems with providing such insurance because of the simple structure of the entrepreneur's problem. How high the premium would be will depend on the solution to the investor's consumption problem, which in turn will depend on how risk averse investors are. Economies with an insurance equilibrium should be less fragile than those without because the insurance essentially allows complete diversification: it connects each investor to the government, and through the government to all other investors in the

[21] The role for a government lender of last resort could be viewed as an artifact of the absence of outside investors from the model. But that view presumes that there is a financial system outside the model from which new investors could come to take the place of those who have experienced losses. That is, it presumes that the model economy's financial linkages are limited by some boundaries (e.g., national boundaries). There is no reason for that to be the case. In fact, if the model is to reflect the financial fragility of a country's economy, then it must reflect all the financial linkages of that country's residents. In a world with global financial markets, it becomes more appropriate to interpret the model as that of a global economy, and the lender of last resort as some international entity that makes up for the absence of any outside investors and the market incompleteness caused by the assumptions of limited information and communication.

economy. The insurance, of course, is just a type of tax-transfer scheme that brings about a Pareto-superior outcome. Solving for an insurance equilibrium remains the subject of future research.

VII. Conclusion

This paper presents a model in which agents' financial positions are linked through the diversified portfolios they hold and the payment commitments that emerge from credit market activity. Shocks to the economy cause some entrepreneurs to default on their payment commitments and thus some investors to suffer losses on their portfolios. Because of the portfolio linkages, defaults can spread through the financial system, allowing the shocks to have an impact far beyond their place of origin. If investors do not foresee defaults spreading to them, then a financial crisis can occur in which the contagion runs its course, destroying some linkages. This equilibrium suggests characterizing fragility in terms of the severity of the resulting crisis. This approach seems natural and is the one typically used, but since the severity of crises is random in the model, the approach cannot yield definitive conclusions about fragility. If, instead, investors have foresight about contagion, then a financial crisis can occur in which all investors seek safer portfolios as protection against the mere possibility of future defaults. This equilibrium suggests characterizing fragility in terms of the speed with which the entire financial structure collapses. The sooner the collapse occurs, the more fragile is the economy. The model yields a date at which such a collapse unambiguously occurs. That date depends on the features of the environment that jointly determine which portfolios investors prefer. For sufficiently large economies, the date at which total financial collapse occurs decreases as the economies increase further in size, with the degree of diversification fixed. In contrast, greater diversification, holding fixed the degree of financial linkage, can reduce fragility. Various institutional responses to fragility are considered. In particular, if the government can control the supply of dollars, it can overcome fragility by serving as a lender of last resort because no incentive-compatibility problems exist.

The model is extremely simple and makes some highly specialized assumptions to generate minimally diversified and linked portfolios while keeping the investor's problem manageable. If the only concern was with portfolio allocations, these assumptions would indeed be quite restrictive. But here the concern is with fragility that arises from financial linkages and the propagation mechanism. Even with the minimally diversified and linked portfolios generated here, myriad financial structures are possible, which makes analyzing the economy quite complicated. In a way, then, the model might not be restrictive enough.

Finally, several specialized features of the model deserve comment. One such feature is the assumption that shocks occur only at the initial date. A related specialized feature is the assumption that projects can die but not regenerate. If new projects could enter the economy over time, investment cycles would arise. For the model to characterize such investment cycles, however, it would have to explain why one financial, or chain, structure forms instead of another. The model is silent on this issue. This omission is of some consequence, for it rules out feedback between the economy and the chain structure.[22] An endogenous chain structure could be introduced into the model here by allowing investors to have priors over the possible chain structures and to update those priors based on the realization of financial crises over time. After an extended period without a crisis, for example, investors might begin to put more probability weight on the more resilient chain structures. This, in turn, could encourage them to choose more risky portfolios, thereby forging more links to other investors and making the financial structure more fragile. This process could continue until a crisis occurs and the chain structure collapses.

Appendix A

This appendix proves that there exist preferences and parameter values consistent with an investor's objective function and Assumptions 1 through 3, so that investors holding maximum-return diversified loan portfolios at the beginning of $t = 0$ do not reallocate their portfolios at $t = 0$.

Clearly the expected lifetime utility at any $t \geq 1$ from holding a $1 loan depends on the risk of loss due to contagion. This loss is bounded above by the utility from holding a $1 loan in the best situation, when the loan is held forever without the investor's incurring a loss on it.

Assumptions 2 and 3 make use of this fact. Assumption 2 states that the lifetime utility from investing $1 in the safe asset exceeds that from holding a $1 loan forever: $u(1) > u(\overline{R} - 1)/(1 - \beta)$.

Thus, if Assumption 2 is satisfied, then holding $1 in the safe asset dominates holding a $1 loan for

[22] The idea that there is feedback between the financial structure and the business cycle that makes the economy inherently unstable is laid out in the financial-instability hypothesis, articulated frequently by Minsky (e.g., 1975).

any amount of time.

The reasoning underlying Assumption 3 is more complicated because the assumption addresses what happens if an individual with a maximum-return diversified loan portfolio, one with two \$1 loans to two distinct projects, shifts to holding a part-safe-part-risky portfolio. Since an investor has no incentive to prefer one project to another, it has been assumed that if he continues to lend, then he lends to the same entrepreneurs as in the past. However, in the situation considered in Assumption 3, the question arises of which loan the investor continues to hold. Since the assignment of investors to projects is random in the model, an investor contemplating a shift to a part-safe-part-risky portfolio is equally likely to pull out of either of his two loans and move his dollar to the safe asset. Thus, in contemplating such portfolio shifts, an investor must consider the possibility that his co-investor in a project might be reallocating similarly and choose to shift his \$1 to the safe asset. Unless both investors in a project continue to invest in that particular project, the project's entrepreneur will end up with insufficient funding and default. Thus, Assumption 3 is satisfied if $\max\{u(2),(1+\mathrm{b})u(1)\} > 0.5\left[u(\overline{R})+\mathrm{b}u(1)\right]+0.5\left[u(1)\right]$, where the first expression is the maximum lifetime utility from holding the safe asset; the first term in brackets to the right of the inequality is the utility from the part-safe-part-risky portfolio in the event that the investor's co-investor also renews his loan; and the second term in brackets is the utility from that portfolio when the other investor does not renew his loan.

The structure of the model also makes clear that the expected lifetime utilities after date 0 have a recursive structure, although the expected lifetime utilities at date 0 are not recursive. The investor nevertheless solves a dynamic optimization problem at date 0. Let $U_t(\mathrm{c}|(a_h,a_j,a_s))$ denote the maximum lifetime utility from date t, given that the investor holds portfolio (a_h,a_j,a_s) at t. Then the solution to the investor's portfolio-choice problem at date 0 maximizes the expected single-period utility from date 0 plus the discount value from behaving optimally in the recursive game that begins at date 1. That is, the expected lifetime utility at t = 0 of holding the maximum-return diversified loan portfolio at 0 and behaving optimally thereafter is (using Assumption 2)

$$E_0U_0(c|(1,1,0)) = p^2u(0)+2p(1-p)\left[u(\overline{R}-1)+\mathrm{b}u(1)\right]+(1-p)^2\left[u(2\overline{R}-2)+\mathrm{b}\max\{v_1(k),v^S\}\right]$$
$$\geq 2p(1-p)\left[u(\overline{R}-1)+\mathrm{b}u(1)\right]+(1-p)^2\left[u(2\overline{R}-2)+\mathrm{b}v^S\right],$$

where $v_1(k)$ and v^S are defined as in Section IV. For the maximum-return diversified loan portfolio to be the solution to the investor's choice problem at t = 0, $E_0 U_0\left(c|(1,1,0)\right)$ must exceed all of the following, which also embody Assumption 2:

$$E_0 U_0\left(c|(0,0,2)\right) = \max\left\{u(2),(1+\beta)u(1)\right\} \equiv v^S,$$

$$E_0 U_0\left(c|(2,0,0)\right) = p^2 u(0) + (1-p)^2\left[u\left(2\bar{R}-2\right) + \beta \max\left\{v_1(k), v^S\right\}\right],$$ [23]

$$E_0 U_0\left(c|(1,0,1)\right) = pu(1) + (1-p)\left[u\left(\bar{R}\right) + \beta u(1)\right].$$

Note that $E_0 U_0\left(c|(2,0,0)\right) \le E_0 U_0\left(c|(1,1,0)\right)$ necessarily.

Let preferences and parameter values be as follows: $u(c)=(c-\alpha)^{1-\sigma}$ for $c>\alpha$ and $u(c)=0$ otherwise, σ = 0.20, α = 0.03, $\bar{R}$ =1.05, β = 0.95, p = 0.01. Then Assumption 2 is satisfied: 0.9759 > 0.87. Also,

$$E_0 U_0\left(c|(1,1,0)\right) \ge 1.9465 \ge E_0 U_0\left(c|(2,0,0)\right),$$

$$E_0 U_0\left(c|(0,0,2)\right) = \max\left\{1.7202, 1.9031\right\} = 1.9031, \text{ and}$$

$$E_0 U_0\left(c|(1,0,1)\right) = 1.829.$$

Thus, at t = 0 the lowest expected lifetime utility if an investor holds a maximum-return diversified loan portfolio at that date dominates the expected lifetime utility from all other portfolios. In addition, Assumption 3 is satisfied: 1.9031 > 1.4595. This means that the critical assumption used in deriving the lifetime expected utility from holding a maximum-return diversified portfolio—namely that an investor at future dates compares a diversified portfolio to the safe portfolio—is valid for at least some preferences and parameter values. ■

Appendix B

PROOF OF PROPOSITION 2. Recall that $\bar{\pi} = \pi(\infty) \equiv \lim_{r\to\infty} \pi(r)$, and that all the expected utilities in increments of two are ranked $\pi(r) < \pi(r+2)$. In particular, this implies

$$\pi(2) < \pi(4) < \pi(6) < \cdots \pi(2r) < \cdots \pi(\infty)$$

and

[23] Technically, $v_1(k)$ is the lifetime expected utility from date 1 on of holding a diversified portfolio, not an undiversified portfolio. However, given strict concavity, as long as there is some risk of default from insufficient funding at future dates, an investor prefers being diversified to being undiversified. Thus, the expected lifetime utility at t = 0 of holding an undiversified loan portfolio is written incorporating the result that the agent will prefer a diversified loan portfolio to an undiversified portfolio at all later dates. In fact, for the preferences and parameter values given in what follows, v^S=1.9031, while the maximum utility from holding an undiversified loan portfolio (that from holding it forever with no chance of loss) is 1.476.

$$\mathrm{p}(1) < \mathrm{p}(3) < \mathrm{p}(5) <\cdots \mathrm{p}(2r+1) <\cdots \mathrm{p}(\infty).$$

To use a first-order-stochastic-dominance property, it must be shown that any left tails of the distributions of both even-numbered expected utilities ($\mathrm{p}(2r)$) and odd-numbered expected utilities ($\mathrm{p}(2r+1)$) are separately increasing in k. Specifically, it must be shown that there is some k' such that if $k \geq k'$, then

$$\forall\ j \leq k, \quad \sum_{r=2}^{j} P\left(N_{2rt} | E_t(k)\right) \quad \text{is increasing in } k \tag{1}$$

and

$$\forall\ j \leq k, \quad \sum_{r=2}^{j} P\left(N_{2r+1\,t} | E_t(k)\right) \quad \text{is increasing in } k\,. \tag{2}$$

The proof that (1) holds is as follows; the argument for (2) is completely analogous. Fix some j with $j \leq k$. Observe that

$$\begin{aligned}\sum_{r=2}^{j} P\left(N_{rt} | E_t(k)\right) &= \frac{\sum_{r=t}^{j+2(t-1)} \overline{P}\left(N_{2r1} | \left(E_1(k)\right)\right)}{\Lambda_t(k)} \\ &= \frac{\sum_{r=t}^{j+2(t-1)} \overline{P}\left(N_{2r1} | \left(E_1(k)\right)\right)}{\sum_{r=2t}^{k} \overline{P}\left(N_{r1} | \left(E_1(k)\right)\right) + \sum_{r=2}^{k} \overline{P}\left(C_{rt} | E_t(k)\right)} \\ &= \frac{\sum_{r=t}^{j+2(t-1)} (2r-1)p(1-p)^{2r-2}\left[q(2r+1,k) + p\left(\sum_{m=2r+2}^{k} q(m,k)\right)\right]}{\sum_{r=2t}^{k} (r-1)p(1-p)^{r-2}\left[q(r+1,k) + p\left(\sum_{m=r+2}^{k} q(m,k)\right)\right] + \sum_{r=2}^{k} \overline{P}\left(C_{rt} | E_t(k)\right)}.\end{aligned}$$

Dividing both numerator and denominator by $\sum_{n=1}^{k-1} \binom{k-1}{n} - \binom{k-1}{k-2}$ gives

$$\frac{\sum_{r=t}^{j+2(t-1)} (2r-1)p(1-p)^{2r-2}\left[\binom{k}{2r+1} + p\left(\sum_{m=2r+2}^{k} \binom{k}{m}\right)\right]}{\sum_{r=2t}^{k} (r-1)p(1-p)^{r-2}\left[\binom{k}{r+1} + p\left(\sum_{m=r+2}^{k} \binom{k}{m}\right)\right] + \sum_{r=2}^{k} (1-p)^{r-2}\binom{k}{r}}$$

$$= \frac{\sum_{r=t}^{j+2(t-1)} (2r-1)p(1-p)^{2r-2}\binom{k}{2r+1} + \sum_{r=t}^{j+2(t-1)} (2r-1)p^2(1-p)^{2r-2}\left(\sum_{m=2r+2}^{k} \binom{k}{m}\right)}{\sum_{r=2t}^{k} (r-1)p(1-p)^{r-2}\binom{k}{r+1} + \sum_{r=2t}^{k} (r-1)p^2(1-p)^{r-2}\left(\sum_{m=r+2}^{k} \binom{k}{m}\right) + \sum_{r=2}^{k} (1-p)^{r-2}\binom{k}{r}}\,. \tag{3}$$

Since $\binom{k}{r} \equiv \dfrac{k!}{(k-r)!\,r!}$, as $k \to \infty$ one can take $r < k/2$ without loss of generality. Therefore,

$\binom{k}{r} = o\left(k^{r+1}\right)$.[24] In particular, the rate of divergence is fastest at precisely $r = k/2$, in which case $\binom{k}{k/2} = o\left(2^{k/2+1}\right)$.

Now consider the numerator of expression (3). Observe that

$$\sum_{r=t}^{j+2(t-1)} (2r-1)p(1-p)^{2r-2}\binom{k}{2r+1} = o\left(k^{j+2(t-1)+1}\right) \quad \text{and}$$

$$\sum_{r=t}^{j+2(t-1)} (2r-1)p^2(1-p)^{2r-2}\left(\sum_{m=2r+2}^{k}\binom{k}{m}\right) = o\left(2^{k/2+1}\right). \tag{4}$$

As for the summands in the denominator of (3),

$$\sum_{r=2t}^{k} (r-1)p(1-p)^{r-2}\binom{k}{r+1} = o\left(\left(2(1-p)\right)^{k/2+1}\right)$$

$$\sum_{r=2t}^{k} (r-1)p^2(1-p)^{r-2}\left(\sum_{m=r+2}^{k}\binom{k}{m}\right) = o\left((1-p)2^{k/2+1}\right) \tag{5}$$

$$\sum_{r=2}^{k} (1-p)^r\binom{k}{r} = o\left(\left(2(1-p)\right)^{k/2+1}\right).$$

It turns out that the highest-order terms in both (4) and (5) increase at the same rate, $2^{k/2}$. This implies that (3) converges as $k \to \infty$. Just as important is the fact that the lesser-order terms in (4) increase faster than the lesser-order terms in (5). This implies that (3) converges from below. Hence, for some k' sufficiently large, (3) is increasing over the interval $[k',\infty)$. Thus, the left tail of the distribution of the expected utilities ($\mathsf{p}(2r)$) is increasing. One can conclude, then, that $P\left(\cdot \middle| E_t(k)\right)$ first-order stochastically dominates $P\left(\cdot \middle| E_t(k+1)\right)$ if $k \geq k'$, and so $v_t(k)$ is decreasing in k over the interval $[k',\infty)$.■

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[24] For any function f defined on some variable x, $f = o(x)$ if $\lim_{x\to\infty} \frac{f(x)}{x} = 0$. That is, f increases at rate x.

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