# What restrictions do Bayesian games impose on the value of information?

Ehud Lehrer* and Dinah Rosenberg†

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**Abstract**

In a Bayesian game players play an unknown game. Before the game starts some players may receive a signal regarding the specific game actually played. Typically, information structures that determine different signals, induce different equilibrium payoffs. In zero-sum games the equilibrium payoff measures the value of the particular information structure which induces it. We pose a question as to what restrictions do Bayesian games impose on the value of information. We provide answers in two kinds of information structures: symmetric, where both players are equally informed, and one-sided where only one player is informed.

*School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel; e-mail: lehrer@post.tau.ac.il; home page: www.math.tau.ac.il/∼lehrer

†LAGA Institut Galilée, Université Paris 13, avenue Jean Baptiste Clément 93430 Villetaneuse, France; e-mail: dinah@math.univ-paris13.fr

# 1 Introduction

Markets or strategic interactions are typically not observable in full detail to the outside observer, may it be an econometrician or an analyst. Either the utilities of the agents or the actions available to them are unobservable. Frequently, only the outcome of the interaction is observable, if at all. The question arises as to what conditions the observable data should satisfy in order to be consistent with an underlying theoretical model. Stated differently, what restrictions on the outcomes of an interaction does the underlying model impose?

Afriat (1967) examined a situation where only finitely many observations of prices and consumption-bundles of an agent are available. Afriat's theorem (see also Varian, 1984) states that these observations may constitute a finite sample from a demand function induced by a continuous, concave and monotonic utility, if and only if a certain revealed preference condition is satisfied. Sonnenschein (1973), Debreu (1974) and Mantel (1974) examined functions that map prices to bundles. They questioned under what conditions such functions might convey the excess demand of a market with utility maximizing agents. It turns out that any function can be derived from rational individuals who maximize their utility.

This paper refers to strategic interactions and poses questions of a similar spirit. The exact specifications of the game played are unobservable to the outside observer. Only the payoffs received by the agents are knowable. In this case, what conditions should these payoffs satisfy in order to be consistent with the equilibrium paradigm of interactive models?

More specifically, consider a Bayesian game in which agents might receive information regarding the actual game played. As in Aumann (1974), we model the information structure in a Bayesian game by a partition of the state-space into disjoint cells: a player is informed of the cell containing the realized state. The information structure of the game obviously affects the behavior of the agents; it determines the equilibrium payoffs. Thus, in a given Bayesian game, any set of state-space partitions, one for each player, is associated with equilibrium payoffs of the induced (incomplete information) game. However, while all the details of the game, including the agents' action sets and the payoffs associated with any combination of agents' actions, are usually unobservable to the economist, the outcomes of the game frequently are.

The data available to the economist about the game includes all possible information structures and the payoffs associated with them. As in Afriat (1967) we look for conditions that data should satisfy in order to be consistent with a rational behavior of the agents in Bayesian games.

Another purpose of the paper is to study those properties essential to the functions that measures the value of information, as well as the role of information in Bayesian games and its effect on equilibrium payoffs. When the information structure changes typically the equilibrium payoffs also change. Specially interesting questions are: what is the extent to which information affects the outcome of the interaction; are there limitations on the way information affects the outcome; and whether the contribution of additional information should be related in any particular way to the information already available?

As a first step in studying the aforementioned questions, we restrict ourselves to zero-sum games. The main advantage of these games is that they have a unique equilibrium payoff $-$ the value. This implies that any information structure is associated with a unique equilibrium payoff rather than with multiple equilibrium payoffs. Furthermore, in zero-sum games the effect of getting more information is always positive: the equilibrium payoff cannot decrease as a result of receiving more information.

A Bayesian zero-sum game can be also perceived as a one-player decision problem under uncertainty when the decision-maker has a prior over her own payoff functions while she has no prior over the states nature may choose. Consider a decision-maker who takes a decision and then receives a payoff which depends also on the state nature chooses. Neither the payoff function nor the state of nature is known.

The payoff function reflects the decision-maker's own preferences, and therefore, she might have a prior over the possible payoff functions that may be relevant at the time the payoff is given. The state of nature, however, might be subject to complete ignorance: the decision-maker might have no assessment or hypothesis regarding the distribution of the states nature chooses. In such a situation a worst case analysis of nature's choice suggests that nature is malicious and it tries to minimize the decision-maker's payoff. Thus, in effect, the decision-maker plays a Bayesian zero-sum game against nature.

The value-of-information function of a Bayesian zero-sum game maps each possible information structure to the corresponding equilibrium payoff. We characterize those real-valued functions defined over the information structures that can be realized by an underlying Bayesian game, as value-of-information functions. That is, we specify the properties of functions over the state-space partitions that are necessary and sufficient for being value-of-information functions.

The issue of measuring the value of information has been previously addressed in the case of one decision-maker by Gilboa and Lehrer (1991). They characterized those functions that measure the value of information in optimization problems, where the decision-maker gets to know an equivalent class of states, rather than the realized state itself. In this paper we extend the model of Gilboa and Lehrer (1991) to zero-sum games and determine what kind of functions (of information) might measure the value of information. We answer this question in two polar cases: symmetric information in which the partitions of both players coincide and thus both obtain the same information about the state of nature; and one-sided information in which one player gets some information about the state of nature while the other does not.

In the case of symmetric information both players are equally informed, and after being informed they actually play another Bayesian game which is restricted to the states within the informed cell. Therefore, the value of the original Bayesian game is the expected value of the Bayesian game played aposteriori. In other words, the value of the Bayesian game is a weighted sum of the values of the restricted Bayesian games played after the players have been informed. This implies, in particular, that a value-of-information function of a symmetric information game should be additively separable. It turns out that this very condition characterizes all possible value-of-information functions: any additively separable function over partitions is a value-of-information function.

When the information is one-sided, refining the partition of the informed player increases her equilibrium payoff. Thus, any value-of-information function must be monotonic (with respect to refinement). Our conclusion concerning one-sided information states that, unlike the case of one-player decision problems, no further condition beyond monotonicity is required to characterize the value-of-information functions.

To summarize, in both types of information structures − the symmetric and the one-sided − the obvious necessary conditions (i.e., additivity in the symmetric case and monotonicity in the one-sided case) are also sufficient to guarantee consistency with the equilibrium paradigm of Bayesian games.

The paper is organized as follows. In Section 2 we present the model and the main issues treated by the paper. In Sections 3 and 4 we present the two main results: the characterizing of the value-of-information functions in symmetric and one-sided information structures. In Section 5 we prove these results. Section 6 reviews related literature and Section 7 is devoted to final comments.

# 2 The model

In this section we give a more formal content to the question asked in the introduction. We first define information structures and then model a bayesian game for each possible information structure. Given this game we define the notion of value of information and then characterize the functions that are value of information for some bayesian game.

## 2.1 Information structures

We consider an incomplete information game preceded by a phase in which the players may obtain some partial information about the exact game to be played. Before the game starts, a state of nature $k$ is drawn from a finite set $K$ according to a known probability $p$. None of the players is directly informed of the realized state $k$. The players receive signals that depends on $k$ through an information structure. This information structure is the main subject of this study and is to be distinguished from the uncertainty embedded in $p$ and $k$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of the state space $K$. The signal player $i$ receives about $k$ in the atom of $\mathcal{P}_i$ that contains $k$. Formally,

**Definition 1** *A* partitional information structure $I = (\mathcal{P}_1, \mathcal{P}_2)$ *consists of two* partitions of $K$*:* $\mathcal{S}_i$ *for player* $i$*,* $i = 1, 2$.

An information structure in general is a device that associates (random) private signals (provided to the players) with the payoff-relevant information. In the model discussed here, the payoff-relevant information is the state. It is clear that partitional information structures are a specific class of information structures. On the other hand, it can be shown that that any information structure can be modelled as a partitional information structure with a new set of parameters (see for instance, Lehrer and Rosenberg, 2003).

In this paper we focus on two specific kinds of information structures: symmetric information in which both players receive the same signal and one-sided information in which only one player receives information while the other does not.

**Definition 2** *A partitional information structure $I = (\mathcal{P}_1, \mathcal{P}_2)$ is* symmetric *if both partitions are equal. That is, $\mathcal{P}_1 = \mathcal{P}_2$.*

**Definition 3** *A partitional information structure $I = (\mathcal{P}_1, \mathcal{P}_2)$ is* one-sided *if only player 1 receives information. That is, if the partition $\mathcal{P}_2$ is trivial (i.e., contains only one set, $K$).*

## 2.2 The game

The bayesian game is defined by a finite state space $K$; a probability distribution over $K$, $p$; a finite actions set for each player, $A_1$ and $A_2$; and finally, a payoff function, $g_k$, defined on $A_1 \times A_2$ for each $k \in K$.

The game associated with the information structure $I = (\mathcal{P}_1, \mathcal{P}_2)$ is played as follows. Before the game starts a state of nature $k \in K$ is drawn according to the distribution $p$. None of the players observe $k$. However, player $i$ observes the cell of the partition $\mathcal{P}_i$ to which $k$ belongs. Then both players simultaneously choose an action $a_i \in A_i$ and get the payoff $g_k(a_1, a_2)$. Player 1 tries to maximizes the expected payoff while player 2 tries to minimize it.

This game can be put in a normal form. A pure strategy of player $i$ is a function, $\tau_i$, that associates an action in $A_i$ to each cell $B \in \mathcal{P}_i$. For each $B_i \in \mathcal{P}_i$ and each $a_i \in A_i$, $\tau_{iB_i}(a_i)$ denotes the probability that player $i$ plays action $a_i$ if he is informed of $B_i$. Let $B_i(k)$ denote the cell of $\mathcal{P}_i$ that contains $k$. The payoff corresponding to a pair of strategies $\tau_1$, $\tau_2$ is

$\sum_{k\in K} p(k) \sum_{a_1\in A_1} \sum_{a_2\in A_2} \tau_{1B_1(k)}(a_1)\tau_{2B_2(k)}(a_2)g_k(a_1,a_2)$. This game is a finite game and therefore has a value denoted by $v^{\mathcal{P}_1,\mathcal{P}_2}(p,(g_k)_{k\in K})$.

## 2.3 Measuring the contribution of information

We now define the value of information in a bayesian game. Consider game with a state space $K$, payoff functions $(g_k)_{k\in K}$ and a distribution $p$ over $K$. The value-of-information function of this game is $V(I) = v^I(p,(g_k)_{k\in K})$. The main issue of this paper is to characterize the value-of-information functions. Formally, let $V(I)$ be a function over partitional information structures $I$. The question arises as to when this function is a value-of-information function of some game. If the properties of the functions that are values of information for some bayesian game are restrictive, it means that the bayesian model imposes restrictions on the way information is valued when it varies.

In the case of one decision maker this problem has been analyzed by Gilboa and Lehrer (1991). They characterize the functions of partitions defined on the set of partitions that are the value of information of finite games.

**Definition 4** *Let $V$ be a function defined over all the partitions of a finite set $K$. $V$ is* separately additive *if there is a function $v$, defined over subsets of $K$, such that for any partition $\mathcal{P}$, $V(\mathcal{P}) = \sum_{B\in\mathcal{P}} v(B)$.*

**Notation 1** *If $\emptyset \neq T \subseteq B \subseteq K$ and $(x_i)_{i\in B}$ is a vector, then $x(T)$ denotes $\sum_{i\in T} x_i$, and $x(\emptyset) = 0$.*

**Definition 5** *For $B \subseteq K$, the $B$-anti-core of $v$ is non empty if there is a vector $(x_i)_{i\in B}$, such that $x(T) \leq v(T)$ for every $T \subseteq B$.*

Gilboa and Lehrer (1991) showed that a function $V$ defined over all the partitions of a finite set $K$ is a value-of-information function of a one-player decision making problem with state space $K$ if and only if it has the following two properties.

(i) $V$ is separately additive: $V(\mathcal{P}) = \sum_{B\in\mathcal{P}} v(B)$; and
(ii) for any $B \subset K$, the $B$-anti-core of $v$ is non empty.
Moreover, the underlying probability distribution over $K$ can be any, as long as the support is the entire $K$ (i.e., any $k \in K$ is assigned a positive probability).
Condition (i) is clearly necessary in a one-player decision problem for the following reason. Let $\mathcal{P}$ be a partition and $B \in \mathcal{P}$. Define $v(B)$ as $\max_{a\in A_1} \sum_{k\in B} p(k)g_k(a)$. The value of the decision problem $V(\mathcal{P})$ has to be $\sum_{B\in\mathcal{P}} v(B)$. This requirement will be extended to the case of two-player zero-sum games with symmetric information.

In Sections 3 and 4 we study analogous questions in zero-sum games with symmetric and one-sided partitional information. Note that the one-player case is a particular case of a zero-sum game (player 2 has only one action). However, since the number of actions available to each player is not specified in the condition, there are more zero-sum games than one player decision problems and therefore more functions of partitions that can be a value of information of zero-sum game than of one-player decision problems. Thus, the conditions that characterize value-of-information functions of zero-sum games are weaker than those characterizing value-of-information functions of one-player decision problems.

# 3 The value of symmetric partitional information

In this section we focus on zero-sum games with symmetric partitional information.

**Definition 6** *A function $V$ defined over all the partitions of $K$ is a value-of-information function of a partitional symmetric information game if there* *is a distribution $p$ over $K$ and payoff functions $(g_k)_{k\in K}$ such that for any partition $\mathcal{P}$ of $K$, $V(\mathcal{P}) = v^{\mathcal{P},\mathcal{P}}(p,(g_k)_{k\in K})$.*

**Example 1** *Let $K$ be $\{1,2\}$. The payoff functions $g_1$ and $g_2$ are given by the matrices.*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \textit{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \textit{ respectively.}$$

*Suppose that the probability of state $k=1$ is $p$. If no player is informed of the state selected, the players actually play the game whose matrix is*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix}.$$

*The value of this game is $\frac{1-p}{2-p}$. On the other hand, if the players are informed of the game selected, then with probability $p$ the value of the game played is $0$ and with probability $1-p$ the value of the game played is $\frac{1}{2}$. Thus, the average of the Bayesian game is $\frac{1-p}{2}$.*

*To sum up, there are two possible partitional symmetric information structure: the trivial, $\mathcal{T}$, where no information about the state selected is being in given to the players, and the perfect one (that corresponds to the discrete partition), $\mathcal{D}$, where both players are fully informed of the state selected. The value-of-information function in this case is therefore given by , $V(\mathcal{T}) = \frac{1-p}{2-p}$ and $V(\mathcal{D}) = \frac{1-p}{2}$. One can see that the additional information given by $\mathcal{D}$ is harmful for player 1.*

We are now ready to characterize the functions over partitions that are values of information of zero sum games with symmetric information.

**Theorem 1** *Let $V$ be a function defined over all the partitions of $K$ then $V$ is the value of information of a game with symmetric partitional information if and only if $V$ is separately additive. Moreover, if $V$ is separately additive then for any probability distribution on $K$ with full support there are payoff functions $(g_k)_{k\in K}$ such that for any partition $\mathcal{P}$ over $K$, $V(\mathcal{P}) = v^{\mathcal{P},\mathcal{P}}(p,(g_k)_{k\in K})$.*

Note that as in the case of one decision maker it is easy to prove that additivity is a necessary condition. Indeed, for a fixed partition $\mathcal{P}$, there is one different mixed action for each player for each atom of the partition. Therefore $V(\mathcal{P})$ can be written as the sum for all atoms $B$ of the partitions of functions $v(B)$ where $v(B)$ is the value of the matrix game with action sets $A_1$ and $A_2$ and payoff functions $\sum_{k \in B} p(k) g_k(\cdot, \cdot)$.

Therefore, the main contribution of this theorem is to state that no further condition beyond additivity is needed for a function to be the value of information of a zero-sum game with symmetric information. This means that in a games with symmetric information the impact of information can be literally unlimited (as long as additivity is preserved). Information may have a positive or a negative contribution, and it may alternate arbitrarily between having positive and negative effects, as the information increases. Furthermore, the marginal contribution of additional information may be arbitrarily small or large. In other words imposing a bayesian model does not impose additional restrictions on the impact of information on the outcome of an interaction. This means that if information is symmetric, bayesianism cannot be rejected as a model on the grounds of the impact of additional information on the value of a zero sum game.

The proof of Theorem 1 is postponed to section 5.

# 4 One-sided information structures

In this section we discuss the case where one player, typically the maximizer, receives some information about the state selected, while the other player receives no information. Formally, player 1 will be informed of the cell of partition $\mathcal{P}$, while the other player will be informed of the trivial partition, $\mathcal{T}$.

**Definition 7** *A function $V$ defined over all the partitions of $K$ is a* value-of-information function of a game with partitional one-sided information *if* *there is a distribution $p$ over $K$ and payoff functions $(g_k)_{k\in K}$ such that for any partition $\mathcal{P}$ over $K$, $V(\mathcal{P}) = v^{\mathcal{P},\mathcal{T}}(p,(g_k)_{k\in K})$.*

**Definition 8** *A function $V$ from the set of partitions of a finite set $K$ to the real numbers is said to be* monotonic *if for two partitions $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P}$ is a refinement of $\mathcal{P}'$ (i.e., any $B' \in \mathcal{P}'$ is a union of atoms $B \in \mathcal{P}$), then $V(\mathcal{P}) \geq V(\mathcal{P}')$.*

**Example 2** *Recall Example 1 and consider one-sided partitional information. When the information is trivial, then the value, as in Example 1 is $\frac{1-p}{2-p}$. However, when player 1 is fully informed of the state and player 2 obtains no information, then the game actually played is*

$$\begin{pmatrix} 1 & 0 \\ p & 1-p \\ 1-p & 0 \\ 0 & 1-p \end{pmatrix}.$$

*The value of this game is $\frac{1}{2}$ if $p \leq \frac{1}{2}$ and $1-p$ if $p > \frac{1}{2}$. Note that this game is the one-sided partitional information corresponding to the discrete partition $\mathcal{D}$.*

*We conclude by writing the value-of-information function of this one-sided partitional information: $V(\mathcal{T}) = \frac{1-p}{2-p}$ and $V(\mathcal{D}) = \frac{1}{2}$ if $p \leq \frac{1}{2}$ and $V(\mathcal{D}) = 1-p$ if $p > \frac{1}{2}$. Note that $V$ is monotonic, since $\mathcal{D}$ refines $\mathcal{T}$ and indeed, $\frac{1}{2} \geq \frac{1-p}{2-p}$ for $p \leq \frac{1}{2}$ and $1-p \geq \frac{1-p}{2-p}$ for $p > \frac{1}{2}$.*

It is clear that in zero-sum games when only one player receives additional information, the value increases. Thus, the value-of-information functions of games with one-sided partitional information must be monotonic. It turns out, as the following theorem states, that monotonicity is not only necessary but also sufficient for being a value-of -information function of a game with one-sided partitional information. As in the symmetric case, there is no restriction (as long as monotonicity is preserved) on the possible impacts of information on the outcome of an interaction for different payoff functions, and the study of the impact of information on the value cannot help in accepting or rejecting the bayesian model as an explanatory model..

**Theorem 2** *A function $V$ from the set of partitions of a finite set $K$ is value-of-information function of a partitional one-sided information game if and only if it is monotonic.*

The proof of this theorem will be given in the next section.

# 5 Proofs of the theorems

## 5.1 The proof of Theorem 1.

We first prove that if $V$ is a value of information function of a zero-sum game with symmetric information then it has to be additive. Recall that since each player knows the set of the partition $\mathcal{P}$ to which $k$ belongs, the strategies $\tau_1$ and $\tau_2$ of player 1 and player 2 are functions from the sets of the partition to probabilities over $A_1$ and $A_2$ respectively. We will denote for $B \in \mathcal{P}$, $\tau_{1B}$ (resp. $\tau_{2B}$) the mixed action corresponding to the information $B$. Therefore

$$\begin{aligned} V(\mathcal{P},\mathcal{P}) &= v^{\mathcal{P},\mathcal{P}}(p,(g_k)_{k\in K}) \\ &= \max_{(\tau_{1B})_{B\in\mathcal{P}}} \min_{(\tau_{2B})_{B\in\mathcal{P}}} \sum_{B\in\mathcal{P}} \sum_{k\in B} p(k) \left( \sum_{\substack{a_1\in A_1\\ a_2\in A_2}} \tau_{1B}(a_1)\tau_{2B}(a_2) g_k(a_1,a_2) \right) \\ &= \sum_{B\in\mathcal{P}} h(B), \end{aligned}$$

where $h(B) = \max_{\tau_{1B}} \min_{\tau_{2B}} \sum_{k\in B} p(k) \left( \sum_{\substack{a_1\in A_1\\ a_2\in A_2}} \tau_{1B}(a_1)\tau_{2B}(a_2) g_k(a_1,a_2) \right)$. Thus, $V$ is additive.

Assume now that $V$ is an additive function on partition, we want to prove that it is a value of information function. In order to prove this result we will use the following proposition (theorem ???) from Lehrer and Rosenberg (2003).

**Proposition 1** *Let $f$ be any polynomial from the set of probability distributions over $K$ to the reals. There exist two finite sets $A_1$ and $A_2$, and a function $g_k$ from $A_1 \times A_2$ to the reals, for each $k \in K$, such that the value of the game with the action sets $A_1$ and $A_2$ and the payoff function $\sum_{k \in K} p_k g_k(\cdot, \cdot)$ is $f(p)$, for any $p$. This game is called a game with no information.*

Take $p$ any probability distribution with full support on $K$. For any subset $B$ of $K$ we denote by $p_B$ the conditional probability on $B$ namely $p_B(k) = p(k)/p(B)$ if $k \in B$ (and 0 otherwise).

Let $f$ be a polynomial defined on $\Delta(K)$ such that for any subset $B$ of $K$, $f(p_B) = h(B)/p(B)$. Such a polynomial exists (note that for $B$ different from $B'$, $p_B$ is different from $p_{B'}$). Now Proposition 1 implies that there are payoff functions $(g_k)_{k \in K}$ such that the value $u$ of the game with no information and payoffs $(g_k)_{k \in K}$ satisfies $u = f$.

For these payoffs we therefore have proven that

$$\begin{aligned} V(\mathcal{P}, \mathcal{P}) &= \sum_{B \in \mathcal{P}} h(B) = \sum_{B \in \mathcal{P}} f(p_B) p(B) = \sum_{B \in \mathcal{P}} u(p_B) p(B) \\ &= \sum_{B \in \mathcal{P}} p(B) \max_{\tau_{1B}} \min_{\tau_{2B}} \sum_{k \in B} p_B(k) \left( \sum_{\substack{a_1 \in A_1 \\ a_2 \in A_2}} \tau_{1B}(a_1) \tau_{2B}(a_2) g_k(a_1, a_2) \right) \\ &= \sum_{B \in \mathcal{P}} \max_{\tau_{1B}} \min_{\tau_{2B}} \sum_{k \in B} p(k) \left( \sum_{\substack{a_1 \in A_1 \\ a_2 \in A_2}} \tau_{1B}(a_1) \tau_{2B}(a_2) g_k(a_1, a_2) \right), \end{aligned}$$

which is the desired result. ■

## 5.2 Proof of Theorem 2

The proof of Theorem 2 makes use of the following proposition.

**Refer to the probabilities of the various one-player decision problems in the proof.**

**Proposition 2** *A function $V$ from the set of partitions of a finite set $K$ is value-of-information function of a partitional one-sided information game if and only if it is a minimum of finitely many value-of-information functions of one-player decision making problems.*

**Proof.** Let $V$ be the value-of-information function of the game $G$ with $K$ being its state space. We prove that it is the minimum of finitely many value of information functions for one player decision making problems. Consider the following auxiliary multi-stage game, $\bar{G}$. At the beginning player 2 announces a mixed strategy, then a state is chosen with respect to the prior distribution $p$ and player 1 is informed of the cell of the appropriate partition that contains this state. Finally, player 1 takes an action and an action of player 2 is selected according to the mixed strategy previously announced.

Obviously the values of $\bar{G}$ and $G$ coincide. Moreover, the optimal strategies in both games also coincide. Denote by $y_{\mathcal{P}}$ an optimal strategy of player 2 in the one-sided information game induced by the partition $\mathcal{P}$.

Consider a fixed partition $\mathcal{P}$. In $\bar{G}$, after player 2 announces $y_{\mathcal{P}}$, player 1 actually faces a one-player decision problem, denoted $D_{\mathcal{P}}$. $D_{\mathcal{P}}$ is defined by the state space $K$ and some payoff functions. Denote by $D_{\mathcal{P}}^{\mathcal{Q}}$ the one-player decision problem $D_{\mathcal{P}}$ when the partitional information is induced by $\mathcal{Q}$. The value of this problem is denoted by $U_{\mathcal{P}}^{\mathcal{Q}}$. Note that $V(\mathcal{P})$ coincides with $U_{\mathcal{P}}^{\mathcal{P}}$.

Since $y_{\mathcal{P}}$ is an optimal strategy of player 2 in the game with one-sided information induced by the partition $\mathcal{P}$, $U_{\mathcal{P}}^{\mathcal{P}} \leq U_{\mathcal{P}}^{\mathcal{Q}}$ for any partition $\mathcal{Q}$. Thus, for any $\mathcal{P}$, $V(\mathcal{P}) = \min_{\mathcal{Q}} U_{\mathcal{P}}^{\mathcal{Q}}$, which completes the proof of necessity.

As for sufficiency, suppose that $V$ is the minimum of finitely many values of one-player decision making problems: $D_1, .., D_n$. That is, if $U_i(\mathcal{P})$ denotes the value of $D_i$ when the information is induces by $\mathcal{P}$, then $V(\mathcal{P}) = \min_{1\leq i\leq n} U_i(\mathcal{P})$. We need to show a zero-sum game whose value is $V$.

Consider the following multi-stage game, $G$. Player 2 chooses a whole number from $1, ..., n$ ,say $r$, then a state $k$ is chosen, player 1 is informed of the cell containing this state, and finally player 1 takes an action, say $a$. The payoff of player 1 is the payoff that corresponds to the action $a$ and the state $k$ in the decision problem $D_r$.

Note that for any partition $\mathcal{P}$, the value of $G$ when the information is induced by $\mathcal{P}$ is $\min_{1 \leq i \leq n} U_i(\mathcal{P})$. Thus, the value of information of $G$ coincides with $V$, as desired. ■

**Definition 9** *Let $\mathcal{F}$ be an algebra of subsets of $K$. That is, $\mathcal{F}$ consists of subsets of $K$ and it is closed under unions and intersections. Let $v$ be a real function defined over $\mathcal{F}$. We say that the* anti-core of $(v, \mathcal{F})$ is not empty, *if for every $A \in \mathcal{F}$ there is a vector $x_A$ such that $x_A(A) = v(A)$ and $x_A(B) \leq v(B)$ for every $B \subseteq A$ such that $B \in \mathcal{F}$.*

**Remark 1** *Suppose that $\mathcal{F}$ is the set of all subsets of $K$. The anti-core of $(v, \mathcal{F})$ is not empty implies that the $B$-anti-core of $v$ is not empty for every $B \subseteq K$.*

**Lemma 1** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two algebras of subsets of $K$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Assume that the anti-core of $(v, \mathcal{F}_1)$ is not empty. Then, for every constants $c_B$, $B \in \mathcal{F}_2 \setminus \mathcal{F}_1$ there is $u$ defined on $\mathcal{F}_2$ which coincides with $v$ on $\mathcal{F}_1$ and satisfies $u(S) \geq c_S$ for every $S \in \mathcal{F}_2 \setminus \mathcal{F}_1$, such that the anti-core of $(u, \mathcal{F}_2)$ is not empty.*

**Proof.** Suppose that the algebra $\mathcal{G}_2$ refines the algebra $\mathcal{G}_1$. We say that $\mathcal{G}_2$ is generated from $\mathcal{G}_1$ by splitting an atom of $\mathcal{G}_1$ into two subsets, if there is an atom $A$ of $\mathcal{G}_1$, and a partition of $A$ into two subsets $B$ and $B'$ that belong to $\mathcal{G}_2$, such that any set $C \in \mathcal{G}_2$ can be written as $C = C_1 \cup C_2$ with $C_1 \in \mathcal{G}_1$ and $C_2 \in \{B, B', \emptyset\}$.

Without loss of generality we can assume that $\mathcal{F}_2$ is generated from $\mathcal{F}_1$ by splitting an atom of $\mathcal{F}_1$ into two subsets. This is so because when $\mathcal{F}_2$ refines

$\mathcal{F}_1$, any cell of $\mathcal{F}_1$ is a union of cells of $\mathcal{F}_2$. Thus, by finitely many successive splits of sets into two subsets one can generate $\mathcal{F}_2$ from $\mathcal{F}_1$. Therefore, if the lemma is proven for any two algebras such that the first is generated from the second by splitting an atom in the second into two sets, one can apply it successively and obtain the desired result for any two algebras that one refines the other.

Let $B \in \mathcal{F}_2 \setminus \mathcal{F}_1$ be a set that does not contain any set from $\mathcal{F}_1$. That is, $B$ is a proper subset of $A \in \mathcal{F}_1$ (i.e., $B$ is a result of splitting $A$ into two subsets). Thus, the sets of $\mathcal{F}_2$ are of the type $D \cup E$, where $D \in \{B, A \setminus B, \emptyset\}$ and $E \in \mathcal{F}_1$.

Since the anti-core of $(v, \mathcal{F}_1)$ is not empty, for every $A \in \mathcal{F}_1$ there is a vector $x_A$ that satisfies the conditions described in Definition 9. For $D = B, A \setminus B$, set $d_D = \max_{A;\ D \subseteq A \text{ and } A \in \mathcal{F}_1} x_A(D)$ and let $b_D > d_D$. Define $u$ as follows: $u$ coincides with $v$ on $\mathcal{F}_1$; $u(D) = b_D$ for $D = B, A \setminus B$ and finally, for $D \cup E$, where $D = B, A \setminus B$ and $E \in \mathcal{F}_1$, $u(D \cup E) = u(D) + u(E)$.

Note that if $b_D$, $D = B, A \setminus B$ are large enough, then $u(S) \geq c_S$ for every $S \in \mathcal{F}_2 \setminus \mathcal{F}_1$, as desired. It remains to show that the anti-core of $(v, \mathcal{F}_2)$ is not empty.

Fix $A \in \mathcal{F}_1$. If $C \subseteq A$ and $C \notin \mathcal{F}_1$, then by the definition of $d_C$ and since $b_C > d_C$, $u(C) > x_A(C)$. If however $C \subseteq A$ and $C \in \mathcal{F}_1$, then $u(C) = v(C) \geq x_A(C)$.

Now fix $A \in \mathcal{F}_2 \setminus \mathcal{F}_1$. $A = D \cup E$, where $D = B, A \setminus B$ and $E \in \mathcal{F}_1$. Define $x_A$ as follows. On the set $E$, $x_A$ coincides with $x_E$, while on $D$ the restriction of $x_A$ is an arbitrary vector whose sum is $u(D)$. Let $D \cup C \subseteq A$, where $C \subseteq E$ is in $\mathcal{F}_1$. Then[1], $x_A(D \cup C) = x_A(D) + x_A(C) \geq u(D) + v(C) = u(D) + u(C) = u(D \cup C)$ which completes the proof that the anti-core of $(u, \mathcal{F}_2)$ is not empty. ■

**Notation 2** *Denote by $\mathcal{A}(\mathcal{P})$ the algebra generated by $\mathcal{P}$.*

[1]Recall Notation 1.

**Proof of Theorem 2.** Let $V$ be a monotonic function defined over the set of partitions of a set $K$. We prove that it is a value of information. By Proposition 2 of Gilboa and Lehrer (1991) it is sufficient to show: (i) there are $v_1, ..., v_n$ where $v_i$ is such that for any $B \subset K$, the $B$-anti-core of $v_i$ is non empty; and (ii) for any partition $\mathcal{P}$, $V(\mathcal{P}) = \min_i \sum_{A \in \mathcal{P}} v_i(A)$.

For any partition $\mathcal{P}$ we will find $v_{\mathcal{P}}$ whose $B$-anti-core is non empty for every $B \subseteq K$, $V(\mathcal{P}) = \sum_{A \in \mathcal{P}} v_{\mathcal{P}}(A)$ and $\sum_{A \in \mathcal{P}} v_{\mathcal{P}}(A) \leq \sum_{A \in \mathcal{P}} v_{\mathcal{Q}}(A)$ for any partition $\mathcal{Q}$. This will imply the result.

Fix a partition $\mathcal{P}$ and define $v_{\mathcal{P}}(A)$ for $A \in \mathcal{P}$ so that $\sum_{A \in \mathcal{P}} v_{\mathcal{P}}(A) = V(\mathcal{P})$. Extend the definition of $v_{\mathcal{P}}$ to $\mathcal{A}(\mathcal{P})$ in a linear fashion. Note that this can be done in a unique way since any element of $\mathcal{A}(\mathcal{P})$ can be written in a unique way as a union of cells of $\mathcal{P}$. Moreover, if $\mathcal{P}$ refines $\mathcal{Q}$, then $V(\mathcal{P}) = \sum_{A \in \mathcal{P}} v_{\mathcal{P}}(A) = \sum_{A \in \mathcal{Q}} v_{\mathcal{P}}(A) \geq V(\mathcal{Q})$. The last inequality is by monotonicity of $V$.

Since $v_{\mathcal{P}}$ is linear on $\mathcal{A}(\mathcal{P})$, the anti-core of $(v_{\mathcal{P}}, \mathcal{A}(\mathcal{P}))$ is not empty. This is so for the following reason. Fix $A \in \mathcal{P}$ and let $x_A$ be any $|K|$ dimensional vector with two properties. First, the support of $x_A$ is $A$ (i.e., all the coordinates out of $A$ are zeros); and second, $x_A(A) = v_{\mathcal{P}}(A)$. Define $x = \sum_{A \in \mathcal{P}} x_A$. Note that for any $B \in \mathcal{A}(\mathcal{P})$, $x(B) = \sum_{k \in B} \sum_{A \in \mathcal{P}} x_A(k) = \sum_{A \in \mathcal{P}} x_A(A \cap B) = \sum_{A \in \mathcal{P} \text{ and } A \subseteq B} x_A(A)$ (because $\mathcal{P}$ is a partition of $K$). Therefore, $x(B) = \sum_{A \in \mathcal{P} \text{ and } A \subseteq B} v_{\mathcal{P}}(A) = v_{\mathcal{P}}(B)$ (the last equality is due to the linearity of $v_{\mathcal{P}}$ on $\mathcal{A}(\mathcal{P})$). Thus, for $B \in \mathcal{A}(\mathcal{P})$, $x$ satisfies $x(B) = v_{\mathcal{P}}(B)$ and the anti-core of $(v_{\mathcal{P}}, \mathcal{A}(\mathcal{P}))$ is not empty.

We extend $v_{\mathcal{P}}$ to the set of all the subsets of $K$. On every partition $\mathcal{Q}$, $v_{\mathcal{P}}$ should satisfy the linear inequality $\sum_{A \in \mathcal{Q}} v_{\mathcal{P}}(A) \geq V(\mathcal{Q})$. Consider the following set of linear inequalities with the variables $c_A$, $A \subseteq K$. $\sum_{A \in \mathcal{Q}} c_A \geq V(\mathcal{Q})$, for every partition $\mathcal{Q}$; $c_A = v_{\mathcal{P}}(A)$ if $A \in \mathcal{A}(\mathcal{P})$. This set of inequalities can be written as a set of inequalities with the set of variables $c_A$, $A \subseteq K$ and $A \notin \mathcal{A}(\mathcal{P})$, where all the inequalities are of the type "greater than or equal to" and the coefficients are either 0 or 1. Such a system has a solution.

Moreover, if $(c_A)_{A \subseteq K \text{ and } A \notin \mathcal{A}(\mathcal{P})}$ is a solution then, $(c_A + f_A)_{A \subseteq K \text{ and } A \notin \mathcal{A}(\mathcal{P})}$, is also a solution, whenever $f_A \geq 0$.

Now fix a solution $(c_A)$ and use the previous lemma with $\mathcal{F}_1 = \mathcal{A}(\mathcal{P})$ and $\mathcal{F}_2$ be the set of all subsets of $K$. We obtain $v_{\mathcal{P}}$ that coincides with $v_{\mathcal{P}}$ on $\mathcal{P}$. Moreover, it satisfies $v_{\mathcal{P}}(A) \geq c_A$ and therefore, it satisfies the set of the inequalities defined in the previous paragraph. Thus, $\sum_{A \in \mathcal{Q}} v_{\mathcal{P}}(A) \geq V(\mathcal{Q}) = \sum_{A \in \mathcal{Q}} v_{\mathcal{Q}}(A)$ for every $\mathcal{Q}$. Finally, the anti-core of $(v_{\mathcal{P}}, \mathcal{F}_2)$ is non-empty. Thus, by Remark 1 it completes the proof that a monotonic function is a value-of-information function.

We prove now the inverse direction: if $V$ is a value of information it has to be monotonic. Note that if $\mathcal{P}$ is a refinement of $\mathcal{Q}$, then in a one-sided information game induced by $\mathcal{P}$ player 1 has more strategies than in the game with information induced by $\mathcal{Q}$. Indeed, for any strategy $\tau$ of player 1 in the game with information structure $\mathcal{Q}$ denote by $\tau_B$ the action prescribed by $\tau$ when the state chosen is in the cell $B \in \mathcal{P}$. Define the following strategy of player 1 in the game with the information structure $\mathcal{P}$. When the state chosen is in $C \in \mathcal{P}$, where $C \subseteq B \in \mathcal{P}$, play according to $\tau_B$.

Since the set of strategies of player 2 is the same under both information structures, the value is higher in the game with information structure $\mathcal{P}$ than in the game with information structure $\mathcal{Q}$. This proves monotonicity. ∎

# 6 Related Literature

Most of the existing literature that relates to the role of information in interactive models compares different information structures. Blackwell (1951, 1953) initiated this trend when he characterized in the context of one-player decision problems when one information structure always provides at least as high payoff as another information structure. Gossner and Mertens (2001) compared different information structures in zero-sum games and Lehrer and Rosenberg (2003) did it in long-run repeated zero-sum games. Gossner (2000) compared the sets of correlated equilibrium distributions induced by different information structures. Gossner (2003) showed that the case where a player has in one game more strategies that in another can be interpreted as having more information.

Hirshleifer (1971) noted that in economic situations additional information does not necessarily imply greater payoffs for the agent. When the game is non-zero-sum, players might prefer dropping payoff-relevant information. This might happen when the equilibrium payoffs of the better informed player are lower than her equilibrium payoffs before receiving the additional information. This phenomenon is exemplified in Kamien et al. (1990). Bassan et al. (1999) introduced conditions that guarantee that getting more information always improves all players' payoffs. Neyman (1991) pointed out that a player might prefer not receiving information because other players would know that he was receiving this information.

# 7 Final Remarks

## 7.1 Non zero-sum games

In this paper we characterize the functions that are value of information functions for zero sum games. In the non zero sum case one could define for a game the value of information correspondence that associates to each information structure the set of corresponding Nash equilibrium payoffs. Then characterizing the set of Nash equilibrium correspondences even for symmetric or one sided information is an open problem.

## 7.2 Games with two-sided information

In this work we focused on the two polar cases of symmetric and one sided information. It would be interesting to characterize the functions $V$ of pair of partitions $(\mathcal{P}_1, \mathcal{P}_2)$ for which there is a $p$ and sets of actions and payoff functions $(g_k)_{k\in K}$ such that $V(\mathcal{P}_1, \mathcal{P}_2) = v^{\mathcal{P}_1,\mathcal{P}_2}(p, (g_k)_{k\in K})$.

### 7.3 General information structures

In this paper we restricted ourselves to games in which the information structure is defined by a pair of partitions. One could more generally define the value of information as a function of general information structures (namely functions from $K$ to probability distributions over a finite set of signals) and ask which functions are value of information functions.

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