Payoffs-dependent Balancedness and Cores

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Abstract

We provide a result for non-emptiness of the core in NTU games. We use a payoffs-dependent balancedness condition, based on transfer rate mappings. Going beyond the non-emptiness of standard core, existence of some refined solution is proved, including specific core allocations and equilibrium-core allocations in parameterized collection of cooperative games. The proofs borrow mathematical tools and geometric constructions from general equilibrium theory with non convexities. Applications to various extant results taken from game theory and economic theory are given.

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Key words: cooperative games, balancedness, non-emptiness, core concepts, parameterized collections of games.

1 Introduction

The core of an n-person cooperative game is the set of feasible outcomes that cannot be improved by any coalition of players. The stability of a model where social interactions are at stake is guaranteed when the core is non empty. Bondareva-Shapley's result states that the core of a TU game is non empty if and only if the game is balanced and Scarf's theorem states that the core of a NTU balanced game is non-empty.

Several improvements have been made since Scarf's result, but the balancedness conditions in the literature always rely on the same principle. Let us first

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consider a TU game. In such a game, within a coalition, transfers of utility can be made from one player to another at a constant one-to-one rate. Then, a transfer rate vector is naturally associated to each coalition. The vector is defined as the vector whose coordinates are 1 for the members of the coalition and 0 for the other players. Given these transfer rate vectors, a family of coalitions is balanced if the transfer rate vector of the grand coalition is positively generated by the transfer rate vectors of the coalitions in the family. Then the TU game is balanced if any feasible payoff for a balanced family is feasible for the grand coalition.

For NTU games, the transfer rate vector can be defined but it does not always correspond to a feasible transfer among the members of the coalition. Nevertheless, the balancedness condition remains sufficient for the non-emptiness of the core. Billera (1970) generalizes Scarf's result by considering transfer rate vectors with any positive coordinates. In this paper, we allow the transfer rates to depend on the payoffs instead of being constant. Consequently, the notion of balanced family of coalitions depends on a given payoff and the notion leads naturally to the definition of a payoffs-dependent balanced game. We will show how our notion incorporates previous notions, including: the standard notion of balancedness (Scarf, 1967); b-balancedness (Billera, 1970); balancedness for convex games (Billera, 1970); $(b, <)$-balanced condition (Keiding and Thorlund-Petersen, 1987).

Independently of our work, Predtetchinski and Herings (2002) define the class of Π-balanced games, which is identical to the class of payoffs-dependent balanced games. They proved that the condition is not only sufficient for non-emptiness of the core in NTU games but also necessary. Thus, they characterize core non-emptiness in NTU games, in the likeness of Bondareva-Shapley's result for the TU games.

In this paper, using the flexibility of the payoffs-dependent balancedness, we go beyond the non-emptiness of the core. Indeed, payoffs-dependent balancedness permits a number of results in the literature involving specific core allocations. We state in the first result of this paper, Theorem 3, that in any payoffs-dependent balanced game, there exists a core allocation such that an equilibrium condition is satisfied with respect to the transfer rate vectors of the family of coalitions, which are feasible for the core allocation. The result is very much in the spirit of Reny and Wooders (1996) key lemma, which exhibited a core allocation satisfying an equilibrium condition for credit/debit mappings. We deduce Reny and Wooders lemma as a corollary of Theorem 3, the transfer rate rules defining the payoffs-dependent balancedness will take into account the individual contributions of the agents within the different coalitions. Problems linked up with fair division schemes are briefly described thanks to this approach, namely the partnered core of Reny and Wooders (1996) and the average prekernel of Orshan and Zarzuelo (2000). As a particular case, Theorem 3 also states that any payoffs-dependent balanced game has a non-empty core, in the standard sense.

Lastly, we consider a parameterized collection of cooperative games. This is very much in the spirit of Ichiishi (1981) even if he does not explicitly use this abstract framework. The stake is the following: the payoffs sets, taken as set-valued mappings, depend on parameters which stand for an abstract environment; furthermore, one adds an equilibrium condition on the parameters, which is represented by a set-valued mapping depending on the parameters and the payoffs. We define an equilibrium-core allocation, which is a pair of a payoff and an environment, the payoff belonging to the core of the game associated to the environment, and, the environment being a fixed point of the equilibrium set-valued mapping. We prove the existence of equilibrium-core allocations, in Theorem 18, under our payoff-dependent balancedness condition. The existence of a Social Coalitional Equilibrium as stated in the benchmark work of Ichiishi (1981) is a consequence of our result [1]. Let us recall that Debreu (1952)'s Social Equilibrium is a particular Social Coalitional Equilibrium. In economies without ordered preferences, or in economies with increasing returns, Border (1984) or Ichiishi and Quinzii (1983) did already use the parametric framework as intermediate steps. We show how these results can be deduced from Theorem 18.

Since most of our reasonings are deeply impregnated by the theory of general equilibrium, the main results are naturally obtained through Kakutani's fixed point theorem [2]. We will also borrow mathematical tools and geometric constructions from literature on general equilibrium with non-convexities theory, see Bonnisseau and Cornet (1988, 1991) and Bonnisseau (1997). We explore shortly the connection between cooperative game theory and General Equilibrium with non convexities theory in Section 2.4. We show in the proof of Theorem 3, that a core allocation may actually be considered as an equilibrium of a two production set economy. Moreover, the prices of the production equilibria may be restated as transfer rate rules equilibrium satisfied by the core allocation. Finally, we keep in mind this duality, putting a light on the close relationship between the two key assumptions into both theories, namely balancedness, at stake in cooperative games, and the survival assumption in general equilibrium. Note that Shapley and Vohra (1991) did already quote similarities between the fixed point mappings they use to show the non-emptiness of the core, and, the fixed point mappings at stake in General

[1] Note that we limit ourself to a space of environment in a finite dimensional Euclidean space, whereas Ichiishi considers a locally convex Hausdorff topological vector space.

[2] One usually associates the question of non-emptiness of cores with KKMS covering theorems or Fan's coincidence theorems, but binding the concept of core with Kakutani's theorem makes sense due to its intimate link with Walrasian economies. See the discussion about these links in Ichiishi (1993, p.118-125).

Equilibrium theory [3]. However these authors did not investigate further.

The paper is organized as follows: in Section 2, Theorem 3 states the existence of specific core allocations in payoffs-dependent balanced games. Then, we show how one can deduce number of results involving balancedness. Section 3 is devoted to the model with parameters and its related topics. Under assumption of payoffs-dependent balancedness, we state a result for the existence of equilibrium-core allocation in parameterized collection of games, Theorem 18, covering Theorem 3. Quoted examples of applications will follow. In the body of the paper, the proofs consist mainly of geometric constructions which are of constant use, the proofs of Theorems and technical lemmas are given in Appendix. Except for some notations and basic assumptions given below, Sections 2 and 3 can be taken independently. We discuss the related literature and possible directions for future works in Section 4.

2 Core solutions in NTU Games

[4] *Notations.* Let N be the finite set of players and $\mathcal{N}$ be the non-empty subsets of N, i.e. the coalitions of players; for each $S \in \mathcal{N}$, we denote L_S the $|S|$-dimensional subspace of $\mathbb{R}^N$ defined by $L_S = \{x \in \mathbb{R}^N \mid x_i = 0, \forall i \notin S\}$, L_{S+} (L_{S++}) is the non negative orthant (positive orthant) of L_S; $\mathbf{1}$ is the vector of $\mathbb{R}^N$ whose coordinates are equal to $\mathbf{1}$, $\mathbf{1}^\perp$ is the hyperplane $\{s \in \mathbb{R}^N \mid \sum_{i \in N} s_i = 0\}$ and proj is the orthogonal projection mapping on $\mathbf{1}^\perp$; if $x \in \mathbb{R}^N$ then x^S is the projection of x into L_S; for each $S \in \mathcal{N}$, let $\Sigma_S = \text{co}\{\mathbf{1}^{\{i\}} \mid i \in S\}$, $m^S = \frac{\mathbf{1}^S}{|S|}$ and let Σ be the unit simplex $\Sigma = \Sigma_N$ ($\Sigma_{++} = \Sigma \cap \mathbb{R}^N_{++}$).

Game description. Each coalition $S \in \mathcal{N}$ has a feasible set of payoffs or utilities, denoted $V_S \subseteq \mathbb{R}^N$. For all $x \in \mathbb{R}^N$, $\mathcal{S}(x) = \{S \in \mathcal{N} \mid x \in \partial V_S\}$. W will denote the union of the payoffs sets $W := \cup_{S \in \mathcal{N}} V_S$. A cooperative game is denoted $(V_S, S \in \mathcal{N})$. $x \in \mathbb{R}^N$ is called a payoff.

Definition 1 *Let $(V_S, S \in \mathcal{N})$ be a game. A payoff x is in the core of the game if $x \in V_N \setminus int\, W$.*

It is worth noting that, in this formulation, the core concept only involves two sets: a feasible payoff for the grand coalition belongs to the core if this payoff lies on the boundary of the whole game. This formulation is crucial in the remainder of the paper, leading most of our geometric constructions.

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[3] See Vohra (1988) to convince oneself of this fact for Shapley–Vohra mappings.

[4] For any set-valued mapping Γ, Gr Γ will denote its graph. For any set $Y \subseteq \mathbb{R}^N$, co(Y), ∂Y and int Y will denote respectively its convex hull, boundary and interior.

We now posit two basic assumptions on the game. The payoffs sets of any coalition, ranked as cylinders of $\mathbb{R}^N$ for convenience, satisfy the free disposal assumption (H1), and the boundedness of the individually rational payoffs of any coalition of players (H2).

Assumption H1 (i) $V_{\{i\}}$, $i \in N$, and V_N are non-empty.
(ii) For each $S \in \mathcal{N}$, V_S is closed, $V_S - \mathbb{R}^N_+ = V_S$, $V_S \neq \mathbb{R}^N$, and, for all $(x, x') \in (\mathbb{R}^N)^2$, if $x \in V_S$ and $x^S = x'^S$, then $x' \in V_S$. [5]

Assumption H2 There exists $m \in \mathbb{R}$ such that for all $S \in \mathcal{N}$, for all $x \in V_S$, if $x \notin \text{int } V_{\{i\}}$ for all $i \in S$, then $x_i \leq m$.

2.1 The main theorem

Before stating the main result of this section, we note that, under Assumption H1, V_N and W satisfy the assumptions of Bonnisseau and Cornet (1988, Lemma 5.1. p.139). Therefore, there exist continuous mappings p_N from $\mathbb{R}^N$ to ∂V_N, p_W from $\mathbb{R}^N$ to ∂W, λ_N and λ_W from $\mathbf{1}^\perp$ to $\mathbb{R}$ such that $p_N(x) = \text{proj}(x) - \lambda_N(\text{proj}(x))\mathbf{1}$ and the same for W.

We are now in position to define the notion of payoffs-dependent balanced game.

Definition 2 *A game $(V_S, S \in \mathcal{N})$ is payoffs-dependent balanced (with $(\varphi_S)_{S\in\mathcal{N}}$ and ψ) if the following assertion is satisfied:*

There exist, for each $S \in \mathcal{N}$, a set-valued mapping φ_S from ∂V_S to Σ_S, which is upper semi-continuous with non-empty compact and convex values, and, a set-valued mapping ψ from ∂V_N to Σ which is upper semi-continuous with non-empty compact and convex values such that:

For each $x \in \partial W$, if $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$, then $x \in V_N$.

To give some intuition into this balancedness condition, consider the mappings $(\varphi_S)_{S\in\mathcal{N}}$ and ψ, as transfer rate rules. For any given efficient payoff x of the coalition S, $\varphi_S(x)$ defines a set of admissible payoff rate of transfer between agents within the coalition. Firstly, note that we define two transfer rate rules for the grand coalition, φ_N and ψ. The idea is to make a distinction between

[5] Assumption H1 implies that there exists, for each $i \in N$, $v_i \in \mathbb{R}$ such that $V_{\{i\}} = \{x \in \mathbb{R}^N \mid x_i \leq v_i\}$. We do not assume the non-emptiness of V_S. But if we put $V_S := \{x \in \mathbb{R}^N \mid x_i \leq v_i, \forall i \in S\}$ for the empty payoffs sets, then Assumptions H1 and H2 are satisfied, all payoffs sets are non-empty, and, the core is unchanged. We do not normalize the game as usually done. For instance, in Shapley and Vohra (1991), the game is normalized without loss of generality by imposing $v_i > 0$ for each $i \in N$.

the transfer rate rule of the coalition N for coalitional organization and that used for the stand alone grand coalition; our next result shed a light on the importance of such a distinction. Secondly, the condition of payoffs-dependent balancedness states roughly speaking that: at any given payoff $x \in \partial W$, the hyperplane game, induced by the transfer rates between the players, is balanced. We also point out that the condition needs only to hold on ∂W, the (*weakly*) efficient frontier of the game.

The first result of the paper is the following. Its proof is referred to Appendix.

Theorem 3 *Under Assumptions H1 and H2, in any payoffs-dependent balanced game (with $(\varphi_S)_{S\in\mathcal{N}}$ and ψ), there exists a core allocation x such that:*

$$\mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset.$$

2.2 About the proof

We briefly comment the main outline of the proof to emphasize the geometric intuition which leads our reasonings. We will need an abstract result of Bonnisseau and Cornet (1991, Theorem 1 p.67), to obtain Theorem 3 as a corollary of an existence result in General Equilibrium with non convexities theory.

Before recalling the general equilibrium result, we first posit some notations. C is a convex cone with a non-empty interior of $\mathbb{R}^N_{++} \cup \{0\}$ and $\mathbf{1}$ is an element of the interior of C. H_1 is the hyperplane defined by $\{x \in \mathbb{R}^N \mid x \cdot \mathbf{1} = 1\}$ and Δ is the non-empty, compact, convex set $H_1 \cap -C^\circ$ where C° is the negative polar cone of C. We only need a weak version with two sets [6] :

Theorem 4 (Bonnisseau–Cornet (1991)) *Let Y_1 and Y_2 two subsets of $\mathbb{R}^N$. For each $j = 1; 2$, let $\tilde{\varphi}_j$ be a set-valued mapping from ∂Y_j to Δ. We assume the following assertions:*

Assumption P Y_j is closed, non-empty and $Y_j - C = Y_j$, for each $j = 1; 2$.
Assumption PR $\tilde{\varphi}_j$ is upper semi-continuous with non-empty convex values, for each $j = 1; 2$.
Assumption BL There exists $\alpha_j \in \mathbb{R}$ such that for all $y_j \in \partial Y_j$, for all $p \in \tilde{\varphi}_j(y_j)$, $p \cdot y_j \geq \alpha_j$, for each $j = 1; 2$.
Assumption B For each $t \geq 0$, $A_t = \{(y_1, y_2) \in Y_1 \times Y_2 \mid y_1 + y_2 + t\mathbf{1} \in C\}$ is bounded.

[6] We also note at this point that $\Sigma \subset \mathrm{int}\ \Delta$ and the Assumption of free disposal with the set $\mathbb{R}^N_+$ implies free disposal with the cone C, that is $Y_j - \mathbb{R}^N_+ = Y_j$ implies $Y_j - C = Y_j$.

Assumption S For each $t > 0$, for each $(p,(y_1,y_2)) \in \Delta \times \partial Y_1 \times \partial Y_2$, if $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ and $y_1 + y_2 + t\mathbf{1} \in C$, then $p \cdot (y_1 + y_2 + t\mathbf{1}) > 0$.

Then there exists $(y_1, y_2, p) \in \partial Y_1 \times \partial Y_2 \times \Delta$ such that $y_1 + y_2 \in C$ and $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$.

In the original statement of Assumption S, one has $t \geq 0$. In the case of pure production economies, one needs only to consider a positive number t. Indeed, when Assumption S does not hold for $t = 0$, then the conclusion of the result obviously holds true. To do the link precisely with Bonnisseau and Cornet (1991, Theorem 1 p.67), the reader must consider $C = X = \mathbb{R}^N_+$. It is an easy matter to check that the proof works with a general convex cone C [7]. The use of the cone C is technical, it is necessary to show that Assumption S holds true, see Appendix, proof of Lemma 26 Claim 29.

The proof of Theorem 3 relies on the explicit construction of two (*non convex*) production sets and pricing mappings. Then, one derives the existence of our core allocation from an equilibrium, which exists thanks to Theorem 4. Roughly speaking [8], we prove the existence of a general equilibrium for the production sets $Y_1 = V_N$ and $Y_2 = -(\text{int } W)^c$ and pricing rules relying on the transfer rate mappings $(\varphi_S)_{S \in \mathcal{N}}$ and ψ.

In the body of the proof of Lemma 26 (Claim 3), we show that Assumption S holds true under the requirements of Theorem 3. More precisely, from the construction we made and the condition of payoffs-dependent balancedness (the unique line of arguments where we need it), we derive the survival assumption (Assumption S). Surprisingly (or not ?), the argument binds intimately the most questionable assumptions of general equilibrium and cooperative games theories, respectively the survivance and the balancedness.

2.3 Application 1: the case where $\psi = \varphi_N$

Theorem 3 provides a result for non-emptiness of the core (*in the standard sense*) in NTU games. Indeed, consider the statement of Theorem 3 when $\psi = \varphi_N$. It states that any payoffs-dependent balanced game has a non-empty core.

In this section, we focus on this particular case of Theorem 3, we show how it covers number of results involving balancedness. The proofs of the corollaries consist only in defining the "right" mappings $(\varphi_S)_{S \in \mathcal{N}}$. We also highlight, with

[7] See Bonnisseau and Jamin (2003).

[8] Assumptions B and BL enforce us to modify slightly the payoffs sets of the game $(V_S, S \in \mathcal{N})$.

an example, that the class of payoffs-dependent balanced games is strictly larger than the class of balanced games of extant literature.

Remark 5 *Independently, Predtetchinski and Herings (2002) prove the following result: under Assumptions H1 and H2, the core of the game is non-empty if and only if the game is payoffs-dependent balanced.*

It follows from the previous remark that, for non-emptiness of core in NTU games, the class of payoffs-dependent balanced games is maximal. Consequently, obtaining as corollaries most of the results of the literature on balancedness is not surprising. Actually, this section is mainly devoted to the manipulation of the payoffs-dependent balancedness condition.

2.3.1 Discussion on balancedness

There are mainly three versions of balancedness in the cooperative game theory literature, Scarf (1967), Billera (1970) and Keiding and Thorlund-Petersen (1987). The third one is the weakest. Clearly, the statement of Theorem 3 for constant mappings, $(\varphi_S)_{S\in\mathcal{N}}$, encompasses itself the three versions of balancedness. Indeed, we show that the transfer rate mappings $(\varphi_S)_{S\in\mathcal{N}}$ at stake can be seen as a family of b-balancedness coefficients for payoffs taken on the boundary of the coalitional payoffs. Therefore, one defines the notion $(\partial - b)$-balancedness. Secondly, we turn to the characterization given by Keiding and Thorlund-Petersen (1987), which is slightly clarified thanks to the more readable notion of $(\partial - b)$-balancedness.

A family of coalition $\mathcal{F} \subset \mathcal{N}$ is balanced if for each $S \in \mathcal{F}$, there exists $\lambda_S \in R_+$ such that $\sum_{S\in\mathcal{F}} \lambda_S \mathbf{1}^S = \mathbf{1}$. The game is balanced if for any balanced family of coalition $\mathcal{F} \subset \mathcal{N}$, $\cap_{S\in\mathcal{F}} V_S \subset V_N$.

The next generalized version of the balancedness use the notion of b-balancedness, firstly defined in Billera (1970). For each $S \in \mathcal{N}$, let $b_S \in L_{S+} \setminus \{0\}$. A family of coalition $\mathcal{F} \subset \mathcal{N}$ is b-balanced if for each $S \in \mathcal{F}$, there exists $\lambda_S \in I\!R_+$ such that $\sum_{S\in\mathcal{F}} \lambda_S b_S = b_N$. The game is b-balanced if for any b-balanced family of coalition $\mathcal{F} \subset \mathcal{N}$, $\cap_{S\in\mathcal{F}} V_S \subset V_N$.

Lastly, one is led to the most refined definition of $(\partial - b)$-balanced games. The game is said to be $(\partial - b)$-balanced if for any b-balanced family of coalitions $\mathcal{F} \subset \mathcal{N}$, $\partial W \bigcap (\cap_{S\in\mathcal{F}} V_S) \subset V_N$.

Corollary 6 *For each $S \in \mathcal{N}$, let $b_S \in L_{S+} \setminus \{0\}$. The core of the game is non-empty if it is $(\partial - b)$-balanced and satisfies Assumptions H1 and H2.*

Proof of Corollary 6. We show that the game is payoffs-dependent balanced. For each $S \in \mathcal{N}$, let $b_S \in L_{S+} \setminus \{0\}$. For each $S \in \mathcal{N}$, φ_S is the constant mapping which associates $\frac{1}{\sum_{i\in S} b_{Si}} b_S$ to each $x \in \partial V_S$. Let $x \in \partial W$ such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(x) \neq \emptyset$. This means that there exist $\lambda_S \in I\!R_+$, $S \in \mathcal{S}(x)$, which satisfy $\sum_{S\in\mathcal{S}(x)} \lambda_S \frac{1}{\sum_{i\in S} b_{Si}} b_S = \frac{1}{\sum_{i\in N} b_{Ni}} b^N$. Consequently, $\sum_{S\in\mathcal{S}(x)} \lambda_S \frac{\sum_{i\in N} b_{Ni}}{\sum_{i\in S} b_{Si}} b_S = b_N$, which implies that the family $\mathcal{S}(x)$ is b-balanced. Then, the $(\partial - b)$-balancedness implies that $x \in V_N$. $\square$

From the previous result, one can deduce Scarf (1967), Billera (1970) and Keiding and Thorlund-Petersen (1987) results.

Assumption WH2 There exists $m \in I\!R$ such that for all $x \in V_N$, if $x \notin \text{int } V_{\{i\}}$ for all $i \in N$, then $x \leq m\mathbf{1}$.

In Scarf (1967), the author proves that the core is non-empty in balanced games under Assumptions H1 and WH2. We remark that, under Assumptions H1 and WH2, if the game is balanced then Assumption H2 is satisfied. Indeed, let $S \in \mathcal{N}$, we remark that the family $\{S, (\{i\})_{i\notin S}\}$ is a balanced family. Now let $x \in V_S$ such that $x_i \geq v_i$, $i \in S$. Let x' defined by $x'_i = x_i$, $i \in S$ and $x'_i = v_i$, $i \notin S$. From Assumption H1, $x' \in V_S \bigcap (\cap_{i\notin S} V_{\{i\}})$. From the balancedness of the game, $x' \in V_N$ and clearly, $x'_i \geq v_i$, $i \in N$. Consequently, from Assumption WH2, $x'_i \leq m$, $i \in N$, which implies $x_i \leq m$, $i \in S$. Thus, Scarf's result is obtained as a corollary of Theorem 3.

In Billera (1970), the author proved that the core of the game is non-empty in b-balanced game (he also assumes that $b_{Ni} > 0$ for all $i \in N$) and under Assumption H1 and H2.

The third notion is the $(b, <)$-balancedness, due to Keiding and Thorlund-Petersen (1987). We let the reader check that $(\partial - b)$-balancedness is weaker than the $(b, <)$-balancedness, so that the result of Keiding and Thorlund-Petersen (1987, Theorem 2.1 p.277) is also obtained as a corollary. These authors advance the idea that some dominated payments should not be taken into account when examining the non-emptiness of the core, then a procedure of elimination is proposed. This intuition will be made useless in our approach since the irrelevant payoffs are straightaway disregarded by considering the efficient payoffs on the whole game, ∂W. However, the notion of $(b, <)$-balanced games leads Keiding and Thorlund-Petersen (1987) to a characterization for non-emptiness of the core in the class of weakly $(b, <)$-balanced games. As done before, one can clarify their result by replacing the notion of $(b, <)$-balancedness by $(\partial - b)$-balancedness.

The game is weakly $(\partial - b)$-balanced if there exists a sequence $\{V^\tau\}_{\tau=1}^\infty$ of $(\partial - b)$-balanced games such that: $V_N = V_N^\tau$ for all τ, and, for all $S \in \mathcal{N}$ the sequence $\{V_{S*}^\tau\}_{\tau=1}^\infty$ converges to the set V_{S*} in the topology induced by Hausdorff metric on the set of non-empty compact sets of $I\!R^N$, where $V_{S*} \in L_S$ denotes the individually rational set for the coalition S, those for which $x \notin V_{\{i\}}$, $i \in S$. To prove the following result, it suffices to duplicate the arguments of Keiding and Thorlund-Petersen (1987, Proof of Theorem 5.1. p.286), furthermore, the procedure they proposed is shortened in several points.

Corollary 7 *Under H1,H2, the core of the game is non-empty if and only if there exists a weakly $(\partial - b)$-balanced game (V', N) such that $V_N = V'_N$ and $V_S \subseteq V'_S$, $S \in \mathcal{N}$.*

2.3.2 An example

We end the discussion on balancedness by considering an example. The following 3-player game with a non-empty core is not $(\partial - b)$-balanced. Let $N = \{1, .., 3\}$, and define:

$V_{\{i\}} = \{x \in I\!R^3 \mid x_i \leq 1\}$ for all $i = 1; ..; 3$,

$V_{\{ij\}i \neq j} = (\{x \in I\!R^3 \mid x_i \leq 1\} \cup \{x \in I\!R^3 \mid x_j \leq 1\}) \cap \{x \in I\!R^3 \mid x_i \leq 2; x_j \leq 2\}$,

$V_{\{123\}} = \{x \in I\!R^3 \mid \sum_{i=1}^3 x_i = 3\} - I\!R^3_+$.

Then, the core is non-empty and reduced to the point $(1, 1, 1)$, the game satisfies Assumption H1 and H2 (take $m = 2$).

Proposition 8 *The game is not $(\partial - b)$-balanced.*

Proof of Proposition 8. Consider the two points lying outside V_N: $(1, 2, 1) \in \partial W$ verifying

$$\mathcal{S}((1, 2, 1)) = \{\{1\}, \{3\}, \{12\}, \{13\}, \{23\}\},$$

and $(1, 1, 2) \in \partial W$ verifying

$$\mathcal{S}((1, 1, 2)) = \{\{1\}, \{2\}, \{12\}, \{13\}, \{23\}\}.$$

Now, remark that, one of the families: $C_1 = \{\{1\}, \{3\}, \{23\}\} \subseteq \mathcal{S}((1, 2, 1))$ and $C_2 = \{\{1\}, \{2\}, \{23\}\} \subseteq \mathcal{S}((1, 1, 2))$ must be b-balanced. Indeed, if $b^i_{123} = 0$ for some $i = 1; ..; 3$, it is obvious, if $b^3_{\{23\}} = 0$, then C_1 is clearly b-balanced, and similarly, if $b^2_{\{23\}} = 0$, C_2 is b-balanced. For the remaining cases, suppose that C_1 and C_2 are not b-balanced. Then, from C_1 one gets that $\frac{b^2_{\{123\}}}{b^2_{\{23\}}} \times b^3_{\{23\}} > b^3_{\{123\}}$, and, from C_2, $\frac{b^3_{\{123\}}}{b^3_{\{23\}}} \times b^2_{\{23\}} > b^2_{\{123\}}$ a contradiction. Consequently, one asserts that the game cannot be $(\partial - b)$-balanced for any $b_S \in L_{S+} \setminus \{0\}$, $S \in \mathcal{N}$. $\square$

The previous game provides an example of a game, which is not $(\partial - b)$-balanced (hence, neither balanced for the previous versions), but, for which

Theorem 3 applies. Indeed the counter-example does not hold any more with respect to the payoffs-dependent balancedness condition. To prove this, let φ_S, $S \in \mathcal{N}$, defined on ∂V_S as follows: $\varphi_S(x) = \{t_S \in \Sigma_S \mid t_S \cdot x = 1\}$. The set-valued mappings φ_S, $S \in \mathcal{N}$, have convex values, which are non-empty since $(1,1,1) \in \partial V_S$, for all $S \in \mathcal{N}$, and from the free disposal assumption. Furthermore, it is routine to check that these set-valued mappings are upper semi-continuous [9].

Suppose that $x \in \partial W$, such that $\mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(p_N(x)) \neq \emptyset$. Then, there exist, for all $S \in \mathcal{S}(x)$, $\lambda_S \in \mathbb{R}_+$, $b_S \in \varphi_S(x)$, and, $b_N \in \varphi_N(p_N(x))$, such that $\sum_{S\in\mathcal{S}(x)} \lambda_S = 1$ and $\sum_{S\in\mathcal{S}(x)} \lambda_S b_S = b_N$. Suppose now, that $N \notin \mathcal{S}(x)$, it implies that $x = p_N(x) + \alpha\mathbf{1}$ with $\alpha > 0$. Therefore, for all $b \in \Sigma$, $b \cdot x > b \cdot p_N(x)$. But, $b_N \cdot x = (\sum_{S\in\mathcal{S}(x)} \lambda_S b_S) \cdot x = \sum_{S\in\mathcal{S}(x)} \lambda_S b_S \cdot x = \sum_{S\in\mathcal{S}(x)} \lambda_S = 1$ and $b_N \cdot p_N(x) = 1$, a contradiction. Hence, the game is payoffs-dependent balanced and the existence of the point $(1,1,1)$ lying in the core can be proved through Theorem 3. This is not surprising, remember the result of Predtetchinski and Herings (2002) in Remark 5.

At this point it is worth noting that the transfer rate rules $(\varphi_S)_{S\in\mathcal{N}}$ have been taken as constant in the proof of Corollary 6. An actual payoffs-dependent condition is given below, it cannot be handled by the previous constant versions.

2.3.3 Convex games

We now consider a case involving convexity in payoffs sets, and, for which Billera (1970) gives a necessary and sufficient condition for non-emptiness of the core. He uses the notion of the support function. For all $S \in \mathcal{N}$, σ_S denotes the support function of V_S, that is, the mapping from $\mathbb{R}^N$ to $\mathbb{R} \cup \{+\infty\}$ defined by $\sigma_S(p) = \sup\{p \cdot v \mid v \in V_S\}$.

Corollary 9 (Billera (1970)) *The core of the game is non-empty if Assumptions H1 and H2 are satisfied, if V_N is convex and if for all $S \in \mathcal{N} \setminus N$, there exists $b_S \in \mathbb{R}^N \setminus \{0\}$ such that $\sigma_S(b_S)$ is finite and for all $b \in$ cone$\{b_S \mid S \in \mathcal{N} \setminus N\}$, $\sigma_N(b) \geq \max\{\sum_{S\in\mathcal{N}\setminus N} \lambda_S \sigma_S(b_S) \mid \forall S \in \mathcal{N} \setminus N, \lambda_S \geq 0,\ \sum_{S\in\mathcal{N}\setminus N} \lambda_S b_S = b\}$.*

Remark 10 *The condition is necessary if all payoffs sets are convex. Note also that a TU game enters in the class of convex games à la Billera [10]. In*

[9] We remark that the mappings $(\varphi_S)_{S\in\mathcal{N}}$ can be seen as the exact analogues of average cost pricing rules up to a translation (we consider the point $(1,1,1)$ instead of $(0,0,0)$). An example of such rules is given in Bonnisseau and Cornet (1988, Corollary 3.3 p.130).

[10] In TU case, there exists a payment $v_S \in \mathbb{R}$ for each coalition, in other terms

this case, $\sigma_S(b_S)$ is finite if and only if b_S is positively proportional to $\mathbf{1}^S$, which leads back to the standard balancedness. Hyperplane games are particular cases of convex games as well.

Proof of Corollary 9. It suffices to prove that the game is payoffs-dependent balanced. For each $S \in \mathcal{N} \setminus N$, we let φ_S be the constant mapping which associates $\frac{1}{\sum_{i \in N} b_{Si}} b_S$ to each $x \in \partial V_S$ and we let $\varphi_N(x) = N_{V_N}(x) \cap \Sigma$, where $N_{V_N}(x)$ is the normal cone of convex analysis to V_N at x. From the convexity of V_N and Assumption H1, the set-valued mapping φ_N has convex values and it is upper semi-continuous. From Assumption H1, $b_S \in L_{S+}$, $S \in \mathcal{N}$, since $\sigma_S(b_S)$ is finite and V_S is a cylinder. Let $x \in \partial W$ such that $\mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(p_N(x)) \neq \emptyset$. Suppose that $N \notin \mathcal{S}(x)$, from the definition of φ_S, this implies that there exists $b \in N_{V_N}(p_N(x)) \cap \mathrm{cone}\{b_S \mid S \in \mathcal{S}(x)\}$. Note that $b \cdot p_N(x) = \sigma_N(b)$. Remark also that, if x does not belong to V_N, then $x = p_N(x) + \alpha \mathbf{1}$ with $\alpha > 0$. Consequently, $b \cdot x > b \cdot p_N(x) = \sigma_N(b)$. On the other hand, for all $S \in \mathcal{S}(x)$, $b_S \cdot x \leq \sigma_S(b_S)$. Since $b \in \mathrm{cone}\{b_S \mid S \in \mathcal{S}(x)\}$, there exists $\lambda_S \geq 0$, $S \in \mathcal{S}(x)$, such that $b = \sum_{S \in \mathcal{S}(x)} \lambda_S b_S$. From our assumption, one has $\sigma_N(b) \geq \sum_{S \in \mathcal{S}(x)} \lambda_S \sigma_S(b_S) \geq (\sum_{S \in \mathcal{S}(x)} \lambda_S b_S) \cdot x = b \cdot x$. Therefore, it leads to a contradiction which proves that $x \in V_N$. $\square$

2.4 Application 2: the case where $\psi \neq \varphi_N$

We now consider the case where the mapping ψ can differ from the mapping φ_N, so we make full use of Theorem 3. The statement of Theorem 3 allows us to pick up a particular element of the core for which an equilibrium holds between the transfer rates rules.

As application, we will deduce a powerful result, Corollary 11, due to Reny and Wooders (1996). We apply Corollary 11 to demonstrate the existence of solutions concepts closely related to fair division schemes, namely the partnered core and the core intersected with the average prekernel. In Corollary 11, ψ will depend on the cooperative commitments of each player in all the coalitions. The partnerships properties have been firstly described and, later, carried out in other fields by Reny and Wooders.

2.4.1 Partnerships and average prekernel

The use of the mapping ψ is made more concrete in the following result, obtained as a corollary of Theorem 3. Once again, the proof consists in taking suitable mappings $(\varphi_S)_{S \in \mathcal{N}}$ and ψ.

$V_S = \{x \in I\!R^N \mid \sum_{i \in S} x_i \leq v_S\}$.

Corollary 11 (Reny-Wooders (1996)) *Let $(V_S, S \in \mathcal{N})$ be a ∂-balanced game satisfying H1 and H2. Suppose that for each pair of players i and j, there is a continuous mapping c_{ij}: $\partial W \to I\!R_+$ such that c_{ij} is zero on $V(S) \cap \partial W$ whenever $i \notin S$ and $j \in S$. Then there exists a core allocation x such that, for each $i \in N$, $\eta_i(x) := \sum_{j \in N}(c_{ij}(x) - c_{ji}(x)) = 0$.*

Remark 12 *The original result is stated for balanced games, it is slightly improved by considering ∂-balanced games (Scarf's balancedness only on the boundary of the feasible sets).*

It makes sense to interpret the mappings c_{ij} as credit/debit mappings. Then, one can see $\eta_i(x)$ as the measure of the grand coalition's net indebtness to i or as i's net credit against the grand coalition. If $c_{ij} = 0$ for each $i, j \in N$ then one gets Scarf's result. For this refined balanced game result, we provide a direct and intuitive proof. The set-valued mapping ψ will take into account individual contributions in the payoffs of the grand coalition, then ψ^i stands for the cooperation index of the agent i. One is led to show the existence of a constant index among the agents.

Proof of Corollary 11. Firstly, notice that, for each $x \in \partial W$, $\eta(x) \in \mathbf{1}^{\perp}$. Then, put for each $x \in \partial W$: $\eta_*(x) = \max_{i \in N} \mid \eta_i(x) \mid$ and let $\tilde{\eta}_i$, $i \in \mathcal{N}$, be mappings from ∂W to R_+: $\tilde{\eta}_i(x) := \frac{1}{\eta_*(x)+1} \frac{\eta_i(x)}{|N|}$. The idea of the proof is to include in a suitable way the net credit and normalized mapping $\tilde{\eta}$ into ψ. Define for each $x \in \partial V_S$, $S \in \mathcal{N}$, and each $x' \in \partial V_N$:

$$\varphi_S(x) = m^S \text{ and } \psi(x') = m^N - \tilde{\eta}(p_W(x'))$$

It appears clearly that, for each $S \in \mathcal{N}$, φ_S is valued into Σ_S, and, ψ is valued into Σ. Furthermore, the mappings are all upper semi-continuous with non-empty compact and convex values. It stems from the continuity of the mappings c_{ij} and the normalization we choose.

Lemma 13 *If there exists $x \in \partial W$ such that* $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$, *then $\psi(p_N(x)) = m^N$.*

We provide in Appendix the detailed proof of Lemma 13. Let $x \in \partial W$, then from Lemma 13 the condition $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$ says that the family $\mathcal{S}(x)$ is balanced and since the game is ∂-balanced one deduces that the game is payoffs-dependent balanced. Now, applying Theorem 3, there exists x in the core of the game and such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$. Noticing that $x = p_N(x) = p_W(x)$ and using once again Lemma 13, this implies that $\tilde{\eta}(x) = 0$, that is $\eta(x) = 0$, so Corollary 11 is proved. $\square$

We briefly recall two applications of the last result, namely the partnered core and the average prekernel intersected with the core, we refer the reader respectively to the works of Reny and Wooders (1996) and Orshan et al. (2001).

A payoff $x \in \partial W$ is said to be partnered if the collection $\mathcal{S}(x)$ satisfies, for all $i, j \in N$, $P_i \subset P_j \Rightarrow P_j \subset P_i$ where $P_i = \{S \in \mathcal{S}(x) \mid i \in S\}$. Then, the concept of partnered core stands for the set of feasible payoffs, which cannot be improved upon by any coalition with an additional requirement of no asymmetric dependencies. Reny and Wooders (1996) apply Corollary 11 to suitable mappings c_{ij} such that x is in the partnered core (the core intersected with the set of partnered outcomes). Indeed, Bennett and Zame (1988) have exhibited suitable mappings, satisfying the requirements of Corollary 11, which are: $c_{ij}(x) = \min\{\text{dist }(x_S, V(S)) \mid S \in \mathcal{N} \text{ and } i \notin S \ni j\}$ for each $x \in \partial W$ where dist is the Euclidean distance [11] .

As a second direct application, one can also prove the existence of an element lying in the core intersected with the average prekernel (former bilateral consistent prekernel) as defined in Orshan and Zarzuelo (2000), see also Serrano and Shimomura (2001). The average prekernel is the consistent extension of the usual prekernel at stake in TU games. The result can be deduced from Corollary 11 by considering suitable credit mappings. Indeed, the average prekernel may be rewritten as the set of elements $x \in \partial V_N$ such that $\sum_{j \in N}(c_{ij}(x) - c_{ji}(x)) = 0$ for some credit mappings satisfying the requirements of Corollary 11 (in this context, $c_{ij}(x)$ can be seen as the weighted surplus of agent i with respect to agent j at the point x). Orshan et al. (2001) have shown the non-emptiness of the core intersected with the average prekernel in ∂-separating games, here the result is improved by considering the larger class of ∂-balanced games [12] .

3 Parameterized collection of NTU games

This section intends to unify the literature, which uses explicitly or implicitly parameterized collection of games. To take into account the environment and the possible interactions between players payments, we introduce here a

[11] If the core of the game is not tight, the partnership property holds automatically. Indeed if we take an element in the core such that $x \notin V_S$ for each $S \in \mathcal{N} \setminus N$, which exists if the core is not tight, then $P_i[x] = P_j[x] = \{N\}$ for each $i, j \in N$, where $P_i[x]$ denotes the partners of agent i in x, hence the core is partnered. The statement of Theorem 3 is in the same spirit since in the case of non tight core one can get: $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} = \varphi_N(x)$ and thus $\varphi_N(x) \cap \psi(x) \neq \emptyset$ that is: $m^N = \varphi_N(x) = \psi(x) = m^N - \tilde{\eta}(x)$, then $\eta(x) = 0$.

[12] Note that a ∂-separating game is ∂-balanced. To define the average prekernel, we need to introduce two more assumptions on the game, namely non-levelness and smoothness.

canonical version of parameterized collection of games and equilibrium-core allocation. The stake is the following: the players receive payoffs in function of an element $\theta \in \Theta$ which stands for a given abstract environment. For any given $\theta \in \Theta$ and a payoff $(x_1, ..., x_n) \in I\!R^N$, a set-valued mapping G restricts the set of possible parameters to $G(\theta, x)$. The mapping G will guarantee the equilibrium. Then, we define an equilibrium-core allocation, which is a pair of an environment and a payoff, the payoff belonging to the core of the game associated to the environment, and, the environment being a fixed point of the equilibrium set-valued mapping.

The model is closely linked up to the model of Ichiishi (1981), who defines a Social Coalitional Equilibrium. We show that it is a corollary of our main result. Another example of application is provided by Border (1984) for the core of an economy without ordered preferences. Note that these results originally used the same mathematical tool in their proofs. These authors have applied Fan's coincidence theorem to exhibit the non-emptiness of the core at stake.

Furthermore the general framework could allow us to investigate some other topics of economic theory (see Section 4 for a discussion). For instance, Ichiishi and Idzik (1996) have shown the non-emptiness of the incentive compatible core in incomplete information framework, using Ichiishi's Social Coalitional Equilibrium existence.

Obviously the parametric framework encompasses the case of constant payoffs sets with respect to the environment, then the results of Section 2 are all covered by Theorem 18 presented below.

3.1 Equilibrium-core allocations

We need some new notations since the payoffs are taken as set-valued mappings. We consider a parameterized collection of cooperative games which is defined as follows: for each parameter $\theta \in \Theta$, each coalition has a feasible set of payoffs $V_S(\theta)$, which is a subset of $I\!R^N$. Moreover, there exists a global feasible set of payoffs $V(\theta)$. Here the definition of the game is slightly modified since we consider an additional set $V(\theta)$, which possibly differs from $V_N(\theta)$. This distinction is made to take into account works where V will point out the feasible set of the whole economy as in Border (1984). The distinction is also made more concrete in Ichiishi (1981) where V denotes the set of all feasible allocations in coalition structure, see also the justification given in the seminal paper of Boehm (1974), where V can differ from V_N due to costs of forming a coalition [13].

[13] As quoted in Section 2.4, one naturally associates the transfer rate rule ψ to the set V.

Equilibrium will be given by a set-valued mapping G from $\Theta \times I\!R^N$ to Θ. We again denote by $W(\theta) = \cup_{S \in \mathcal{N}} V_S(\theta)$ the union of the payoffs sets. For any given parameter $\theta \in \Theta$, the NTU game is denoted $(V(\theta), V_S(\theta)_{S \in \mathcal{N}})$. Let $\mathcal{S}_\theta(y)$ be the set $\{S \in \mathcal{N} \mid y \in \partial V_S(\theta)\}$. Formally, the parameterized collection of cooperative games is summarized by:

$$\text{For all } S \in \mathcal{N},\ V_S : \Theta \to I\!R^N,\ V : \Theta \to I\!R^N \text{ and } G : \Theta \times I\!R^N \to \Theta.$$

Definition 14 *An equilibrium-core allocation is a vector $(\theta^*, x^*) \in \Theta \times I\!R^N$ such that:*

$$x^* \in \partial V(\theta^*) \setminus \mathit{int}\, W(\theta^*) \text{ and } \theta^* \in G(\theta^*, x^*).$$

The assumptions on the game are the following. They stand for the former H1, H2. Actually we just add continuous dependencies with respect to the environment.

Assumption PH0 Θ is a non-empty, convex, compact subset of an Euclidean space. G is a non-empty upper semi-continuous set-valued mapping with convex values.

Assumption PH1 (i) The set-valued mappings $V_{\{i\}}$, $i \in N$, and V are non-empty valued. V_S, $S \in \mathcal{N}$, and V are lower semi-continuous set-valued mappings with closed graph.

(ii) For each $\theta \in \Theta$, $V_S(\theta)$, $S \in \mathcal{N}$, and $V(\theta)$ satisfy Assumption H1(ii).

Assumption PH2 For all $\theta \in \Theta$ there exists $m(\theta) \in I\!R$ such that, for all $S \in \mathcal{N}$, $x \in V_S(\theta)$, if $x \notin \text{int } V_{\{i\}}(\theta)$, for all $i \in S$, then $x_i \leq m(\theta)$. For all $\theta \in \Theta$, for all $x \in V(\theta)$ such that $x_i \notin \text{int } V_{\{i\}}$, for all $i \in N$, then $x \leq m(\theta)\mathbf{1}$.

Remark 15 *Assumption H1 implies that there exist functions v_i, $i \in N$, from Θ to $I\!R$, such that, for each $\theta \in \Theta$, $V_{\{i\}}(\theta) = \{z \in R^N \mid z_i \leq v_i(\theta)\}$. Bonnisseau (1997, Lemma 3.1. p.217) states that if a set-valued mapping M from Θ to $I\!R^N$ is lower semi-continuous with non-empty values, has a closed graph and satisfies Assumption H1(ii) for all $\theta \in \Theta$, then there exists a continuous mapping λ from $\Theta \times \mathbf{1}^\perp$ to $I\!R$ such that, for all $(\theta, s) \in \Theta \times \mathbf{1}^\perp$, $s - \lambda(\theta, s)\mathbf{1} \in \partial M(\theta)$. For V, let p_V and λ_V be the continuous mappings defined respectively on $\Theta \times I\!R^N$ and $\Theta \times \mathbf{1}^\perp$ such that: $p_V(\theta, x) = \mathit{proj}(x) - \lambda_V(\theta, \mathit{proj}(x))\mathbf{1} \in \partial V_N(\theta)$. We define similarly the mappings p_W and λ_W associated to W.*

Definition 16 *A parameterized collection of games $((V(\theta), V_S(\theta)_{S \in \mathcal{N}}), \theta \in \Theta)$ is payoffs-dependent balanced if the following assertion is satisfied:*

There exist set-valued mappings φ_S from $Gr\, \partial V_S$ to Σ_S, $S \in \mathcal{N}$, which are upper semi-continuous with non-empty compact and convex values, and a set-valued mapping ψ from $Gr\, \partial V$ to Σ, which is upper semi-continuous with non-empty compact and convex values, such that:

For each $(\theta, x) \in Gr\, \partial W$*, if* $\operatorname{co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset$*, then* $x \in V(\theta)$*.*

Remark 17 *To do the link with the literature, the notion of payoffs-dependent balancedness is a generalization of the balancedness in the sense of Boehm (1974), which says that (in our parameterized framework), for a given* $\theta \in \Theta$*, the game* $(V(\theta), V_S(\theta)_{S \in \mathcal{N}})$ *is balanced if for all balanced family* $\mathcal{F}$*,* $\cap_{S \in \mathcal{F}} V_S(\theta) \subset V(\theta)$*. We add payoffs-dependent balancedness to this definition.*

The third result of the paper is the following.

Theorem 18 *Under Assumptions PH0, PH1 and H2, in any payoffs-dependent balanced parameterized collection of games, there exists an equilibrium-core allocation* (θ^*, x^*) *such that:*

$$\operatorname{co}\{\varphi_S(\theta^*, p_W(\theta^*, x^*)) \mid S \in \mathcal{S}_{\theta^*}(p_W(\theta^*, x^*))\} \bigcap \psi(\theta^*, x^*) \neq \emptyset.$$

The proof, referred to Appendix, follows the geometric construction given in the proof of Theorem 3.

3.2 Two applications of parameterized collection of cooperative games

In this section, the different works we review use an alternative version of Boehm's condition (see Remark 17), where the balancedness relies on feasible allocations of the economy and no more on the outcomes of the game. In any case, we recover a balancedness on the game and the constructions of the balancing mappings φ_S and ψ are unchanged (see Section 2.3.1). In the following, note that we do not need any payoffs-dependencies in the balancedness condition. Roughly speaking, for any given parameter, the games, $(V(\theta), (V_S(\theta))_{S \in \mathcal{N}})$, are balanced with constant transfer rules, so one mostly focuses on the role of the parameterization in these models. The two applications, we proposed here, will clarify the usefulness of the mapping G, which is explicitly given in each case. Elementary proofs are provided thanks to Theorem 18.

3.2.1 Ichiishi's Social Coalitional Equilibrium

The Social Coalitional Equilibrium of Ichiishi (1981) is the benchmark work in the framework of parameterized collection of games. Furthermore, the general formulation of the Equilibrium, where the agents can realize a coalition structure, encompasses Debreu (1952)'s Social Equilibrium and the usual core as special cases. Some applications of this seminal result are reviewed in Ichiishi (1993).

A coalition structure is a partition of N. Let $\mathcal{P}$ be a non-empty collection of coalition structure, a member of $\mathcal{P}$ is denoted P. Players will play cooperative games parameterized by the elements of a set Θ. The model is the following: each player has a parameter set Θ_i ($\Theta_S = \prod_{i\in S}\Theta_i$, $\Theta = \Theta_N$). For each $S \in \mathcal{N}$, let F^S be a mapping from Θ into Θ_S. Preference relation of each player i in a coalition S is represented by a utility function; v_S^i : Gr $F^S \to I\!R$.

A Social Coalitional Equilibrium of a society is a pair of a parameter $\theta^* \in \Theta$ and admissible coalition structure $P^* \in \mathcal{P}$, such that: (i) For each $D \in P^*$, $\theta^{D*} \in F^D(\theta^*)$. (ii) It is not true that there exists $S \in \mathcal{N}$ and $\theta' \in F^S(\theta^*)$ such that $v_S^i(\theta^*, \theta') > v_{D(i)}^i(\theta^*, \theta^{*D(i)})$ for every $i \in S$, where $D(i) \in P^*$ and $i \in D(i)$.

Corollary 19 (Ichiishi(1981)) *There exists a Social Coalitional Equilibrium if: (1) For every $i \in N$, Θ_i is a non-empty, convex compact subset of an Euclidean space. (2) For every $S \in \mathcal{N}$, F^S is a lower and upper semi-continuous set-valued mapping with non-empty values. (3) For every $S \in \mathcal{N}$, v_S^i is continuous on Gr F^S. (4) For every $\theta \in \Theta$ and every $v \in I\!R$, if there exists a balanced collection $\mathcal{B}$ such that for each $S \in \mathcal{B}$ there exists $\theta(S) \in F^S(\theta)$ for which $v_i \leq u_S^i(\theta, \theta(S))$ for each $i \in S$, then there exist $P \in \mathcal{P}$ and $\theta'^D \in F^D(\theta)$ for every $D \in P$ such that $v_i \leq u_D^i(\theta, \theta'^D)$ for all $i \in D \in P$. (5) For every $\theta \in \Theta$, and for every $v \in I\!R^N$, the set*

$$\bigcup_{P\in\mathcal{P}} \left\{\theta' \in \Theta \mid \forall D \in P, \theta'^D \in F^D(\theta) \text{ and } v \leq (v_{D(i)}^i(\theta, \theta'^{D(i)}))_{i\in N}\right\}$$

is convex.

In the original paper of Ichiishi, the strategy sets are taken as Hausdorff topological vector spaces, we limit here the corollary within the Euclidean spaces.

Proof of Corollary 19. For each $S \in \mathcal{N}$, let us define:

$$V_S(\theta) = \left\{u \in I\!R^N \mid \exists\theta' \in F^S(\theta),\ u_i \leq v_S^i(\theta, \theta'),\ i \in S\right\}$$

$$\tilde{V}_S(\theta) = \left\{u \in \bar{V}_S(\theta) \mid \forall i \in N \setminus S, u_i = 0\right\}$$

$$V(\theta) = \bigcup_{P\in\mathcal{P}} \sum_{D\in P} \tilde{V}_D(\theta)$$

And let $G(\theta, x)$ be equal to

$$\bigcup_{P\in\mathcal{P}} \left\{\theta' \in \Theta \mid \forall D \in P, \theta'^D \in F^D(\theta) \text{ and } p_V(\theta, x) \leq (v_{D(i)}^i(\theta, \theta'^{D(i)}))_{i\in N}\right\}.$$

Consider the parameterized collection of games defined above, we show that it meets the requirements of Theorem 18. We begin with the condition of payoffs-dependent balancedness. For each $S \in \mathcal{N}$, let φ_S be the constant mapping equals to m^S and $\psi = \varphi_N$. Let (θ, x) be in Gr ∂W such that $\mathrm{co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset$. As seen before, one easily checks that the family $\mathcal{S}_\theta(x)$ is balanced. For each $S \in \mathcal{S}_\theta(x)$, there exists $\theta(S) \in F^S(\theta)$ for which $x_i \leq u^i_S(\theta, \theta(S))$, $i \in S$. Then, from (4), there exist $P \in \mathcal{P}$ and $\theta'^D \in F^D(\theta)$ for every $D \in P$ such that $x_i \leq u^i_D(\theta, \theta'^D)$, $i \in D \in P$. It states that $x \in \sum_{D \in P} \tilde{V}_D(\theta) \in V(\theta)$.

Ichiishi proved the continuity of the set-valued mappings V_S and V (Ichiishi, 1981, Proof of Lemma, step 1, p.372) so PH1 clearly holds true. G is non-empty from the definitions of p_V and V, and, convex valued from (5). It suffices to prove that G has a closed graph to imply that it is upper semi-continuous. It is straightforward, from the finiteness of $\mathcal{P}$ and (2), (3), so PH0 holds. (1) and the continuity of v^i_S guarantees that PH2 is satisfied.

We apply Theorem 18, then there exists (θ^*, x^*) such that $\theta^* \in G(\theta^*, x^*)$ and $x^* \in \partial V(\theta^*) \setminus \mathrm{int}\, W(\theta^*)$. Hence, there exists $P \in \mathcal{P}$ such that for all $D \in P$, $\theta^{*D} \in F^D(\theta^*)$ and $p_V(\theta^*, x^*) \leq (v^i_{D(i)}(\theta^*, \theta^{*D(i)}))_{i \in N}$, furthermore $x^* = p_V(\theta^*, x^*)$. Necessarily $(v^i_{D(i)}(\theta^*, \theta^{*D(i)}))_{i \in N} \in \partial V(\theta^*) \setminus \mathrm{int}\, \cup_{S \in \mathcal{N}} V_S(\theta^*)$, satisfying the requirement of Social Coalitional Equilibrium. □

Ichiishi and Quinzii (1983) use a variant of Corollary 19 to prove the non-emptiness for economies with increasing returns. The authors split the parameter set into an abstract parameter set and action sets for each individual. Moreover, they do not need the agents realize a coalition structure, they only use the benchmark partition N. Therefore, $V = V_N$, and, the feasibility condition in the definition of Social Equilibrium must hold only on the coarsest coalition N. Using our own materials, one can prove directly the result as stated in Ichiishi and Quinzii (1983, Lemma A p.406). Furthermore, since we make no distinction between V and V_N, the conclusion of Theorem 18 gives a core allocation equilibrium (θ^*, x^*) such that $\mathrm{co}\{\varphi_S(\theta^*, x^*) \mid S \in \mathcal{S}_{\theta^*}(x^*)\} \cap \psi(\theta^*, x^*) \neq \emptyset$ (indeed $p_W(x^*) = x^*$). And, it is true for any mappings $(\varphi_S)_{S \in \mathcal{N}}$ and ψ satisfying the standard balancedness condition, especially for the ones defined in Section 2.4.1.

Consequently, one could have combined results from Section 2.4 with the solution concept of Social Equilibrium to show, for instance, the existence of a Social Equilibrium with partnerships. Indeed, by considering the condition resting on the mappings $(\varphi_S)_{S \in \mathcal{N}}$ and ψ and under Assumptions given in Ichiishi and Quinzii (1983, Lemma A p.406), one could prove that there exists a Social Equilibrium such that the social outcome is a partnered outcome, from Corollary 11. Though the concept of partnered structure is much weaker than any partition structure, it avoids the use of the ad hoc Assumption (4) in Corollary 19. Then, the emerging structure is obtained as the outcome of an endogenous process. To the best of our knowledge, such solutions have not been explored any further in this parameterized framework.

Remark 20 *In a standard cooperative game, one can also define solution concepts dealing with coalition structure. Using Theorem 18 when the parameter set is reduced to a point, one provides a sufficient condition for non-emptiness of the core with a coalition structure, in NTU games, if V is the super-additive cover of the game $(V_S, S \in \mathcal{N})$ defined by $\cup_{P\in\mathcal{P}} \sum_{S\in P}\{x \in V_S \mid x_i = 0,\ i \in N \setminus S\}$.*

3.2.2 Core allocations for non-ordered preferences

One can also recover Border (1984)'s result from our abstract result following the reasonings of the previous section, and, the generalization given by Kajii (1992) which is also briefly described. Border proves the non-emptiness of the core of an economy where the agents preferences are non ordered. One exhibits a parameterized collection of games such that the core of the economy is exactly the equilibrium allocation stated in Definition 14.

Let Ξ_i, $i \in N$, be the payoffs set of agent i, $\Xi_S = \prod_{i\in S} \Xi_i$ and $\Xi = \Xi_N$. For each $S \in \mathcal{N}$, let F^S be the feasibility mapping from Ξ into Ξ_S and denote by $\Theta \in \Xi$ the set of all jointly feasible allocations. Preference relation of each player is represented by a set-valued mapping P_i from Ξ_i into Ξ_i.

An element $\xi \in \Xi$ is said to be in the core if : (i) $\xi \in \Theta$. (ii) There is no $S \in \mathcal{N}$ and $\xi' \in F^S(\xi)$ satisfying $\xi'_i \in P_i(\xi_i)$ for all $i \in S$.

Corollary 21 (Border (1984)) *The core is non-empty if: (1) For each i, Ξ_i is a non-empty convex subset of a Euclidean space. (2) For each $S \in \mathcal{N}$, $F^S : \Xi \to \Xi_S$ is a lower and upper semi-continuous set-valued mapping with compact values and F^i, $i \in N$, is non-empty valued. (3) Θ is compact and convex. (4) For each i, P_i has an open graph in $\Xi_i \times \Xi_i$ and $\xi_i \notin \mathrm{co}P_i(\xi_i)$. (5) The game is balanced: for all $\xi' \in \Xi$, for any balanced family β with the balancing weights λ_β, if there exist $(\xi^B)_{B\in\beta}$ such that $\xi^B \in F^B(\xi')$, $B \in \beta$, then $\xi \in \Theta$ where $\xi_i = \sum_{B\in\beta, i\in B} \lambda_B \xi^B_i$.*

Remark 22 *As Scarf (1967), Border only assumes compacity on the set of all jointly feasible allocations. The reasonings of Section 2.3.1 to get WH2 from H2 will overcome this difficulty.*

Proof of Corollary 21. Without a loss of generality, we can assume that $P_i(\xi_i)$ is convex. Using a well known trick, define pseudo-utility functions $v_i : \Xi_i \times \Xi_i \to I\!R$, $i \in N$, as follows: $v_i(\xi'_i, \xi_i) = \mathrm{dist}[(\xi_i, \xi'_i), (\mathrm{Gr}\ P_i)^c]$. The convexity of $P_i(\xi_i)$ implies that v_i is quasi-concave in its first argument (see Border's Appendix).

A parameterized collection of games is defined on the compact set Θ. For each $\xi \in \Theta$, for each $S \in \mathcal{N}$, let $V_S(\xi) = \{u \in I\!R^N \mid \exists \xi' \in F^S(\xi),\ u_i \le v_i(\xi'_i, \xi_i),\ i \in$

$S\}$, and, $V(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in \Theta,\ u_i \leq v_i(\xi'_i, \xi_i),\ i \in N\}$.

Remark also that $\xi \in \Theta$ is in the core if and only if $0 \in V(\xi) \setminus \mathrm{int}\ \cup_{S \in \mathcal{N}} V_S(\xi)$. Put:

$$G(\xi, x) = \{\xi' \in \Theta \mid p_V(\xi, x) \leq (v_i(\xi'_i, \xi_i))_{i \in N}\}\,.$$

Consider the game above. We provide in Appendix the detailed and rather technical proof that the assumptions of Theorem 18 are all fulfilled. Then, there exists a bundle (θ^*, x^*) such that $x^* \in \partial V(\theta^*) \setminus \mathrm{int}\ W(\theta^*)$ and $\theta^* \in G(\theta^*, x^*)$. Hence, $x^* = p_V(\theta^*, x^*) \leq (v_i(\theta^*_i, \theta^*_i))_i = 0 \in \partial V(\theta^*)$ and $\theta^* \in \Theta$. That is to say that $0 \in \partial V(\theta^*) \setminus \mathrm{int}\ W(\theta^*)$ and $\theta^* \in \Theta$, as required. $\square$

This model has been carried out more generally. Kajii (1992), proposes a generalization of both Border's result and of Scarf's α-core non-emptiness result (Scarf, 1971). The same construction as in the previous proof can be applied to show Kajii's result. The difference comes from the fact that preferences are interdependent, that is, the mappings P^i are defined from Ξ into Ξ. Consequently the pseudo utility mappings are defined on $\Xi \times \Xi$ but still verify quasi-concavity in their first variables (Kajii, 1992, p.196).

In this setting, a coalition S blocks a feasible allocation $\xi \in \Theta$ if there exists $\xi' \in F^S(\xi)$ such that for all ξ'' with $\xi''_i = \xi'_i$ all $i \in S$, one has $\xi'' \in P_i(\xi)$. Then, the payoffs sets are naturally defined as: $V_S(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in F^S(\xi)$ such that for all $\xi''_i = \xi'_i, i \in S,\ u_i \leq v_i(\xi'', \xi),\ i \in S\}$, for all $S \in \mathcal{N}$ and $V(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in \Theta\ \ u_i \leq v_i(\xi', \xi),\ i \in N\}$, these mappings satisfy the expected properties of continuities as in Border's setting. We obtain the result of Kajii (1992, Corollary p.201) (he additionally assumes that $F^N(\xi) = \Theta$ and $\Xi_i = \Theta_i$) if we posit:

$$G(\xi, x) = \bigcap_{i \in N} \left\{\xi' \in \Theta \mid p^i_V(\xi, x) \leq v_i(\xi', \xi)\right\}.$$

The mapping G is an upper semi-continuous set-valued mapping with convex values as a finite intersection of upper semi-continuous set-valued mappings with convex values. G has non-empty values since, for all $(\theta, x) \in \Theta \times \mathbb{R}^N$, $p_V(\theta, x) \in V(\theta)$ by definition. Therefore the parameterized collection of games meets the requirements of Theorem 18.

4 Further developments

We discuss the literature which could be submitted to a similar treatment. In non-convex economies, the most achieved results for non-emptiness of cores are deeply relying on elasticities conditions of the demand functions (Ichiishi and Quinzii, 1983) which raise some basic difficulties in terms of interpretation. These works consist mostly in recovering some convexity properties, as far as possible, to draw nearer to the notion of distributive sets introduced by Scarf (1986), then, they comprehend the non convexity thanks to convexifying assumptions. New developments could follow from Theorem 18 within a non-convex environment, indeed the payoffs-dependent balancedness enlarges the geometric possibilities to get a non-empty core. The negative result (Scarf, 1986, Theorem 5 p.426) delimits, however, the range of new results. Direct approaches for the core are inspired by Florenzano (1989), one should restate payoffs-dependent balancedness into her framework where no cooperative game structure is defined. Here, the balancing weights should rest on the fundamentals of the economy.

In addition, one can cite related topics partially evoked in this paper. For the games in parametric form: the incentive cores in asymmetric information, see Ichiishi and Idzik (1996) and Ichiishi and Radner (1999), both using the seminal result of Ichiishi (1981); the α-core, as seen before with Scarf (1971) and Kajii (1992), see also Yannelis (1991) where feasibility constraints are incorporated. We have shown that such recent works can receive a positive treatment by computations of equilibrium-core allocations. Note however that, in both research fields, there exist robust counterexamples of empty cores (see respectively Forges et al. (2002) and Holly (1994)).

Core allocations with additional requirements are, in particular, linked up to the fair division schemes. Indeed, as quoted by Reny and Wooders (1996), the notion of partnered collections of sets is closely related to the concept of kernel (prekernel) at stake in NTU games. We have clearly illustrate this point in Section 2.4.1 showing the connection between the partnered core and core intersected with average prekernel. Besides, the notion of partnership gave also rise to a literature on the side of covering theorems as developed by Reny and Wooders (1998), see also Ichiishi and Idzik (2002). Lastly, Page and Wooders (1996) extended the notion of partnership to competitive equilibrium and cores in economies. Very recently, Herings et al. (2003) define a notion of Social Stable core allocations, for which power indexes are equally shared out among the coalitions, Social Stable core allocations are also very much in the spirit of core solution with equilibrium for credit/debit mappings.

5 Appendix

5.1 *Proof of Theorem 3*

Let $Y_2 = -(\text{int } W)^c$ where $(\text{int } W)^c$ denotes the complementary of the interior of the set W. Note that Y_2 is bounded above by $-v$. Let $\tilde{\varphi}_2$ be the set-valued mapping from ∂Y_2 to Σ defined by

$$\tilde{\varphi}_2(y_2) = \text{co}\,\{\varphi_S(-y_2) \mid S \in \mathcal{S}(-y_2)\}\,.$$

Lemma 23 *For all $(y_2, p) \in Gr\ \tilde{\varphi}_2$ and such that $y_{2i} < -m$ for some $i \in N$, then $p_i = 0$.*

Proof of Lemma 23. Let $y_2 \in \partial Y_2$ such that $y_2 \notin \{-m\mathbf{1}\} + I\!R^N_+$. Let $i \in N$ such that $y_{2i} < -m$. Then, for all $S \in \mathcal{S}(-y_2)$ we show that $i \notin S$. Indeed, recalling that $-y_2 \geq v$, Assumption H2 states that if $-y_2 \in V_S$, then $-y_{2j} \leq m$ for all $j \in S$. Thus, $i \notin S$. Consequently, for all $S \in \mathcal{S}(-y_2)$, for all $p \in \varphi_S(-y_2)$, $p_i = 0$ since φ_S takes its values in Σ_S. Hence, for all $p \in \tilde{\varphi}_2(y_2)$, $p_i = 0$. $\square$

Since Y_2 is bounded above by $-v$ and from Assumption H2, the set $Y_2 \cap (\{-m\mathbf{1}\} + I\!R^N_+)$ is compact, therefore there exists $\rho > 0$ such that $\text{proj}(y) \in B_{\mathbf{1}^\perp}(0, \rho)$ for all $y \in Y_2 \cap (\{-m\mathbf{1}\} + I\!R^N_+)$, where m is the upper bound chosen in H2.

Define $Y_1 = \{p_N(s) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)\} - I\!R^N_+$, we remark that, for all $y_1 \in \partial Y_1$, if $\text{proj}(y_1) \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)$ then $y_1 \in \partial V_N$. Put $\underline{y} \in \text{int } Y_1$.

Lemma 24 *There exists a continuous mapping c from ∂Y_1 to Σ_{++} such that $c(y_1) \cdot (y_1 - \underline{y}) \geq 0$ for all $y_1 \in \partial Y_1$.*

Proof of Lemma 24. Since Y_1 satisfies the free disposal condition and $\underline{y} \in \text{int } Y_1$, for all $y_1 \in \partial Y_1$, there exists a vector $p \in \Sigma_{++}$ such that $p \cdot (y_1 - \underline{y}) > 0$. Define the set valued mapping Γ from ∂Y_1 to Σ_{++} as $\Gamma(y_1) = \big\{p \in \Sigma_{++} \mid p \cdot (y_1 - \underline{y}) > 0\big\}$, this set-valued mapping is non-empty valued from the argument above. It is an easy matter to check that it has open graph and convex values. One gets the existence of a continuous selection of Γ applying a weak version of Michael's selection theorem. $\square$

Let $\tilde{\varphi}_1$ be the set-valued mapping from ∂Y_1 to Σ defined by:

$$\tilde{\varphi}_1(y_1) = \begin{cases} \psi(y_1) & \text{if } \|\text{proj}(y_1)\| < \rho \\ \text{co}\{\psi(y_1), c(y_1)\} & \text{if } \|\text{proj}(y_1)\| = \rho \\ c(y_1) & \text{if } \|\text{proj}(y_1)\| > \rho \end{cases}$$

Lemma 25 *For all $(y_1, y_2) \in \partial Y_1 \times \partial Y_2$ and $p \in \Sigma$ such that $proj(y_1) = -proj(y_2)$ and $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$, one has $p \in \psi(y_1)$ and $y_1 \in \partial V_N$.*

Proof of Lemma 25. We first prove that $\|\text{proj}(y_1)\| \leq \rho$. Indeed, if it is not true, then $\varphi_1(y_1) = c(y_1) \in \Sigma_{++}$ and, from Lemma 23 and the choice of ρ, since $\|\text{proj}(y_2)\| = \|\text{proj}(y_1)\| > \rho$, one has $\varphi_2(y_2) \notin \Sigma_{++}$. But, this contradicts $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$. Now, the above remark implies that $y_1 \in \partial V_N$. If $p \notin \psi(y_1)$, $\|\text{proj}(y_1)\| = \rho$ and $p \in \Sigma_{++}$. The same argument leads again to a contradiction. $\square$

Given these materials, one can state the following technical lemma. The proof consists of the verifications of Theorem 4 assumptions with respect to the construction above.

Lemma 26 *Y_1,Y_2 and $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ satisfy the requirements of Theorem 4.*

Proof of Lemma 26. We check with the three following claims that the assumptions of Theorem 4 hold true for the sets Y_1 and Y_2 and the mappings $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$.

Claim 27 *Y_1 and $\tilde{\varphi}_1$ satisfies Assumptions P, PR and BL.*

Proof of Claim 27. Y_1 clearly satisfies Assumption P, Assumption PR is also clearly satisfied from the definition of the set-valued mapping $\tilde{\varphi}_1$ and the continuity of the function c.

Assumption BL also holds true. Indeed, there exists $\alpha \in I\!R$ such that for all $s \in \bar{B}(0, \rho)$ and for all $p \in \Sigma$, $p \cdot (s - \lambda_N(s)\mathbf{1}) \geq \alpha$ and $p \cdot \underline{y} \geq \alpha$. Thus one gets that $p \cdot y_1 \geq \alpha$ for all $(y_1, p) \in \text{Gr } \tilde{\varphi}_1$, using Lemma 24. $\square$

Claim 28 *Y_2 and $\tilde{\varphi}_2$ satisfies Assumptions P, PR and BL.*

Proof of Claim 28. One easily checks that Y_2 is closed and that it satisfies the free-disposal assumption so P is satisfied. $\tilde{\varphi}_2$ has obviously convex compact values from the assumption of payoffs-dependent balancedness. $\tilde{\varphi}_2$ has non-empty values since $y_2 \in \partial Y_2$ implies that $-y_2 \in \partial V_S$ for at least one $S \in \mathcal{N}$. Since Σ is compact, it suffices to show that the set-valued mapping $\tilde{\psi}_2$ defined by $\tilde{\psi}_2(y_2) = \cup_{S \in \mathcal{S}(-y_2)} \varphi_S(-y_2)$ has a closed graph in order to prove that $\tilde{\varphi}_2$ is upper semi-continuous. Let (y_2^ν, p^ν) a sequence of $\partial Y_2 \times \Sigma$ which converges to (y_2, p) and such that $p^\nu \in \tilde{\psi}_2(y_2^\nu)$ for all ν. From the definition of $\mathcal{S}$, for ν large enough, $\mathcal{S}(-y_2^\nu) \subset \mathcal{S}(-y_2)$. Consequently, for all ν large enough, there exists $S^\nu \in \mathcal{S}(-y_2)$ such that $p^\nu \in \varphi_{S^\nu}(-y^\nu)$. Since $\mathcal{S}(-y_2)$ is a finite set, there exists a subsequence such that S^ν is constant equal to S. Since φ_S is upper semi-continuous, this implies that $p \in \varphi_S(-y_2) \subset \tilde{\psi}_2(y_2)$. It ends the proof.

Assumption BL also holds true. Indeed, for all $(y_2, p) \in \text{Gr } \tilde{\varphi}_2$, from Lemma 23, one has $p \cdot y_2 \geq \sum_{i \in N, y_{2i} \geq -m} p_i y_{2i} \geq -m \sum_{i \in N, y_{2i} \geq -m} p_i = -m \sum_{i \in N} p_i = -m$. $\square$

Claim 29 *Assumptions B and S are satisfied.*

Proof of Claim 29. Assumption B is satisfied since Y_2 is bounded above by $-v$ and Y_1 is also bounded above since $\{p_N(s) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)\}$ is a compact set.

Assumption S holds true. If it is not the case, there exists $t > 0$, $(y_1, y_2) \in \partial Y_1 \times \partial Y_2$ and $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ such that $y_1 + y_2 + t\mathbf{1} \in C$ and $p \cdot (y_1 + y_2 + t\mathbf{1}) = 0$. Since $p \in \mathbb{R}^N_+ \setminus \{0\}$ and $C \subset \mathbb{R}^N_{++} \cup \{0\}$, one deduces that $y_1 + y_2 + t\mathbf{1} = 0$. Let $s_1 = \mathrm{proj}(y_1)$ and $s_2 = \mathrm{proj}(y_2)$. Clearly, $s_1 = -s_2$. Then one can apply Lemma 25 which states that $y_1 \in \partial V_N$ and $p \in \psi(y_1)$.

Let $x = -y_2$, thus $x \in \partial W$. $s_1 = -s_2$ implies that $p_N(x) = y_1$ consequently $\mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$. Thus, since the game is payoffs-dependent balanced, one has $x \in V_N$. But $y_1 = x - t\mathbf{1}$ contradicts the fact that $y_1 \in \partial V_N$ from the free disposal property of V_N. $\square$

From the previous claims, the conclusion of Lemma 26 is satisfied. $\square$

Theorem 4 implies that there exists a vector $(y_1, y_2, p) \in \partial Y_1 \times \partial Y_2 \times \Sigma$ such that: $y_1 + y_2 \in C \subset \{0\} \cup \mathbb{R}^N_{++}$, and, $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$.

We first show that $y_1 + y_2 = 0$. If it is not true, $-y_2 \ll y_1$. But $y_1 \in Y_1 \subset V_N$, and, thus, $-y_2 \in \mathrm{int}\ V_N \subset \mathrm{int}\ W$, which contradicts that $-y_2 \in (\mathrm{int}\ W)^c$. Since $\mathrm{proj}(y_1) = -\mathrm{proj}(y_2)$, applying Lemma 25, we get $p \in \mathrm{co}\{\varphi_S(y_1) \mid S \in \mathcal{S}(y_1)\} \cap \psi(y_1) \neq \emptyset$ and $y_1 \in \partial V_N$. Since $y_1 = -y_2 \in \partial W$, $y_1 \notin \mathrm{int}\ V_S$ for all $S \in \mathcal{N}$. As was to be proved, y_1 satisfies the conclusion of Theorem 3. $\square$

5.2 Proof of Lemma 13.

Suppose for some $x \in \partial W$ that $\mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$. Then there exists some non-negative λ_S for each $S \in \mathcal{S}(x)$ such that: $\sum_{S \in \mathcal{S}(x)} \lambda_S = 1$, $x = p_W(p_N(x))$ and $m^N - \sum_{S \in \mathcal{S}(x)} \lambda_S m_S = \tilde{\eta}(p_W(p_N(x)))$ $(*)$. To end the proof, we show that : $\eta^* := \tilde{\eta}(x) = 0$. Putting $M := \{m \in N \mid \eta^*_m = \max_{i \in N} \eta^*_i\}$, we only need to show that $\sum_{i \in M} \eta^*_i \leq 0$ since $\eta^* = \tilde{\eta}(x) \in \mathbf{1}_\perp$. If $i \in M$ and $j \in N \setminus M$, that is $\eta^*_j < \eta^*_i$, then, from $(*)$, there exists $R \in \mathcal{S}(x)$ such that $R \ni j$ and $i \notin R$. Therefore, $c_{ij}(x) = 0$ from the definition of the mapping c_{ij}. Using this argument in the following lines extracted from Ichiishi and Idzik (2002) [14], we get, denoting by $t(x)$ the positive normalization constant for x:

$$
\begin{aligned}
\sum_{i \in M} \eta^*_i &= \sum_{i \in M} \tilde{\eta}_i(x) = t(x) \sum_{i \in M} \sum_{j \in N} (c_{ij}(x) - c_{ji}(x)) \\
&= t(x) \Big(\sum_{i \in M} \sum_{j \in M} (c_{ij}(x) - c_{ji}(x)) + \sum_{i \in M} \sum_{j \in N \setminus M} (c_{ij}(x) - c_{ji}(x))\Big) \\
&= 0 + t(x) \sum_{i \in M} \sum_{j \in N \setminus M} (-c_{ji}(x)) \leq 0.
\end{aligned}
$$

Consequently $\eta^* = 0$. $\square$

[14] They provide a new proof of an extension of the KKMS theorem proposed by Reny and Wooders (1998), which was also exhibiting the partnership property.

We first introduce some uniform bounds with respect to the parameter set. From the lower semi-continuity and closed graph assumptions of the set valued $V_{\{i\}}$, it is immediate to see that the mappings v_i, $i \in N$, are continuous. Let us denote $v = \min\{v_i(\theta) \mid \theta \in \Theta,\ i \in N\}\mathbf{1}$. The bound $m(\theta)$ given in Assumption PH2 can also be chosen continuous since the set-valued mapping V_S, $S \in \mathcal{N}$, are lower semi-continuous with closed graph. Let $m = \max\{m(\theta) \mid \theta \in \Theta\}$. These elements exist since Θ is compact.

We define the set-valued mapping Y_2 from Θ into $\mathbb{R}^N$ by:

$$Y_2(\theta) = -\text{int } (W(\theta)^c).$$

Note that Y_2 is lower semi-continuous with a closed graph, and, for all $\theta \in \Theta$, $Y_2(\theta) - \mathbb{R}^N_+ = Y_2(\theta)$ and $Y_1(\theta) \neq \mathbb{R}^N$. Let $\tilde{\varphi}_2$ be the set-valued mapping from Gr ∂Y_2 into Σ defined by:

$$\tilde{\varphi}_2(\theta, y_2) = \text{co}\left\{\varphi_S(\theta, -y_2) \mid S \in \mathcal{S}_\theta(-y_2)\right\}.$$

Lemma 30 *Let $(\theta, y_2) \in$ Gr ∂Y_2, if $y_{2i} < -m$ for some $i \in N$, then $p_i = 0$ for all $p \in \tilde{\varphi}_2(\theta, y_2)$.*

Proof of Lemma 30. We apply Lemma 23 to the set-valued mappings $\tilde{\varphi}_2(\theta, .)$ and the set $Y_2(\theta)$. $\square$

Since $Y_2(\theta)$ is uniformly bounded above by $-v$, there exists ρ such that $\text{proj}(\theta, y_2) \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)$ for all $(\theta, y_2) \in$ Gr ∂Y_2 such that $y_2 \in (\{-me\} + \mathbb{R}^N_+)$. Let us define the set-valued mapping Y_1 from Θ into $\mathbb{R}^N$ by:

$$Y_1(\theta) = \left\{p_V(s, \theta) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)\right\} - \mathbb{R}^N_+.$$

Since p_V is continuous, note that Y_1 is lower semi-continuous with a closed graph, and, for all $\theta \in \Theta$, $Y_1(\theta) - \mathbb{R}^N_+ = Y_1(\theta)$ and $Y_1(\theta) \neq \mathbb{R}^N$. Then, the compacity of Θ implies the existence of two real numbers α_1 and β_1 such that for all $y_1 \in \{z_1 \in \partial Y_1(\theta) \mid \|\text{proj}(z_1)\| \leq \rho,\ \theta \in \Theta\}$, $\alpha_1 \mathbf{1} \leq y_1 \leq \beta_1 \mathbf{1}$. Note also that, for all $\theta \in \Theta$, for all $y_1 \in \partial Y_1(\theta)$, if $\|\text{proj}(y_1)\| \leq \rho$, then $y_1 \in \partial V(\theta)$. Let us choose $\underline{y}' \in \text{int } Y_1(\theta)$ for all $\theta \in \Theta$. Such element exists since every element strictly inferior to $\alpha_1 \mathbf{1}$ satisfies this condition.

Lemma 31 *There exists a continuous mapping c from Gr ∂Y_1 to Σ_{++} such that $c(\theta, y_1) \cdot (y_1 - \underline{y}') \geq 0$ for all $(\theta, y_1) \in$ Gr ∂Y_1.*

Proof of Lemma 31. Define a mapping Γ' on Gr ∂Y_1 such that $\Gamma'(\theta, y_1) = \{p \in \Sigma_{++} \mid p \cdot (y_1 - \underline{y}') > 0\}$ and use the arguments given in the proof of Lemma 24. $\square$

Let $\tilde{\varphi}_1$ be the set-valued mapping from Gr ∂Y_1 into Σ defined by:

$$\tilde{\varphi}_1(\theta, y_1) = \begin{cases} \psi(\theta, y_1) & \text{if } \|\text{proj}(y_1)\| < \rho \\ \text{co}\{\psi(\theta, y_1), c(\theta, y_1)\} & \text{if } \|\text{proj}(y_1)\| = \rho \\ c(\theta, y_1) & \text{if } \|\text{proj}(y_1)\| > \rho \end{cases}$$

Lemma 32 *There exists $\alpha \in I\!R$ such that, for all $(\theta, y_1, y_2) \in$ Gr $(\partial Y_1 \times \partial Y_2)$, $(p_1, p_2) \in \tilde{\varphi}(\theta, y_1) \times \tilde{\varphi}_2(\theta, y_2)$, one has $p_1 \cdot y_1 + p_2 \cdot y_2 \geq \alpha$.*

Proof of Lemma 32. For all $(\theta, y_2) \in$ Gr ∂Y_2, for all $p \in \tilde{\varphi}_2(\theta, y_2)$, from Lemma 30, one has $p \cdot y_2 \geq \sum_{i \in N, y_{2i} \geq -m} p_i y_{2i} \geq -m \sum_{i \in N, y_{2i} \geq -m} p_i = -m \sum_{i \in N} p_i = -m$ since $p \in \Sigma$. Secondly, for all $(\theta, y_1) \in$ Gr ∂Y_1, $p \in \tilde{\varphi}_1(\theta, y_1)$, if $\|\text{proj}(y_1)\| \leq \rho$ then $p \cdot y_1 \geq p \cdot \alpha_1 \mathbf{1} = \alpha_1$; if $\|\text{proj}(y_1)\| > \rho$, from Lemma 31, $p \cdot y_1 = c(\theta, y_1) \cdot y_1 \geq c(\theta, y_1) \cdot \underline{y}' \geq \min\{q \cdot \underline{y}' \mid q \in \Sigma\}$. Hence $p \cdot y_1$ is bounded below, which proves the result. $\square$

Since the values of Y_1 and Y_2 are respectively uniformly bounded above by $\beta_1 \mathbf{1}$ and $-v$, there exists a convex and compact set $\bar{B} \in (\mathbf{1}^\perp)^2$ such that: $B(0, \rho) \times B(0, \rho) \subset \bar{B}$ and for all $(\theta, y_1, y_2) \in$ Gr $(\partial Y_1 \times \partial Y_2)$ such that $y_1 + y_2 - \alpha \mathbf{1} \in I\!R^N_{++} \cup \{0\}$, $(\text{proj}(y_1), \text{proj}(y_2)) \in \text{int } \bar{B}$.

Finally, using again (Bonnisseau, 1997, Lemma 3.1 p.217), one introduces the continuous mappings λ_1 and λ_2 from $\Theta \times \mathbf{1}^\perp$ to $I\!R$ associated to Y_1 and Y_2. We fix $\eta > 0$ arbitrary, let Σ_η be the set $\left\{p \in I\!R^N \mid \sum_{i \in N} p_i = 1;\ p_i \geq -\eta,\ i \in N\right\}$.

Let F be the set-valued mapping from $\Theta \times B \times \Sigma_\eta \times \Sigma^2$ into itself. $F = \prod_{j=1}^4 F_j$.

$F_1(\theta, (s_1, s_2), p, (p_1, p_2)) = G(\theta, y_1)$

$F_2(\theta, (s_1, s_2), p, (p_1, p_2)) = \{\sigma \in B \mid \sum_{i=1}^2 (p - p_i) \cdot \sigma_i \geq \sum_{i=1}^2 (p - p_i) \cdot \sigma'_i,\ \forall \sigma' \in B\}$

$F_3(\theta, (s_1, s_2), p, (p_1, p_2)) = \{q \in \Sigma_\eta \mid (q - q') \cdot (y_1 + y_2) \leq 0,\ \forall q' \in \Sigma_\eta\}$

$F_4(\theta, (s_1, s_2), p, (p_1, p_2)) = (\tilde{\varphi}_1(\theta, y_1), \tilde{\varphi}_2(\theta, y_2))$

where for $i = 1; 2$, $y_i = s_i - \lambda_i(\theta, s_i)\mathbf{1}$.

Lemma 33 *The mapping F satisfies Kakutani's fixed point theorem conditions.*

Proof of Lemma 33. F is a set valued mapping from a non-empty, convex, compact set into itself. Actually, it suffices to verify for F_4 that the assumptions of Kakutani's fixed point theorem are satisfied since the others components meet obviously the expected conditions.

By construction, $\tilde{\varphi}_1$ is a non-empty, convex valued and upper semi-continuous. $\tilde{\varphi}_2$ is obviously convex valued and it has non-empty values since $(\theta, y_2) \in$ Gr ∂Y_2 implies that $-y_2 \in \partial V_S(\theta)$ for at least one $S \in \mathcal{N}$. Since Σ is compact, it suffices to show that the set-valued mapping $\tilde{\psi}_2$ defined by $\tilde{\psi}_2(\theta, y_2) = \cup_{S \in \mathcal{S}_\theta(-y_2)} \varphi_S(\theta, -y_2)$ has a closed graph in order to prove that $\tilde{\varphi}_2$ is upper semi-continuous. Let $(\theta^\nu, y^\nu, p^\nu)$ a sequence of Gr $\partial Y_2 \times \Sigma$ which converges to (θ, y, p) and such that $p^\nu \in \tilde{\psi}_2(\theta^\nu, y^\nu)$ for all ν. From the definition of $\mathcal{S}$, for ν large enough, $\mathcal{S}_{\theta^\nu}(-y_2^\nu) \subset \mathcal{S}_\theta(-y_2)$. Indeed, it is not true, since $\mathcal{N}$ is a finite set, there exists $S \in \mathcal{N}$ and a subsequence (θ^ν, y^ν) such that for ν large enough $-y_2^\nu \in \partial V_S(\theta^\nu)$, $-y_2 \notin \partial V_S(\theta)$. Since V_S is a lower semi-continuous set-valued mapping with a closed graph and $V_S(\theta) - \mathbb{R}^N_+ = V_S(\theta)$, the set-valued mapping $\theta \to \partial V_S(\theta)$ has a closed graph. Since $-(y_2^\nu)$ converges to $-y_2$, one gets a contradiction. Consequently, for all ν large enough, there exists $S^\nu \in \mathcal{S}_\theta(-y_2)$ such that $p^\nu \in \varphi_{S^\nu}(\theta^\nu, -y_2^\nu)$. Since $\mathcal{S}_\theta(-y_2)$ is a finite set, there exists a subsequence such that S^ν is constant equal to S. Since φ_S is upper semi-continuous, this implies that $p \in \varphi_S(\theta, -y_2)$, which is included in $\tilde{\psi}_2(\theta, y_2)$ since $S \in \mathcal{S}_\theta(-y_2)$. $\square$

From the previous lemma, there exists $(\theta^*, (s_1^*, s_2^*), p^*, (p_1^*, p_2^*))$ such that, if, for $i = 1; 2$, $y_i^* = s_i^* - \lambda_i(\theta^*, s_i^*)\mathbf{1}$:

$$\theta^* \in G(\theta^*, y_1^*) \tag{1}$$

$$(s_1^*, s_2^*) = (\mathrm{proj}(y_1^*), \mathrm{proj}(y_2^*)) \text{ and } (y_1^*, y_2^*) \in \partial Y_1(\theta^*) \times \partial Y_2(\theta^*) \tag{2}$$

$$\sum_{i=1}^{2} (p^* - p_i^*) \cdot s_i^* \geq \sum_{i=1}^{2} (p^* - p_i^*) \cdot \sigma_i' \text{ for each } \sigma' \in B \tag{3}$$

$$(p^* - q') \cdot (y_1^* + y_2^*) \leq 0 \text{ for each } q' \in \Sigma_\eta \tag{4}$$

$$(p_1^*, p_2^*) \in (\tilde{\varphi}_1(\theta^*, y_1^*), \tilde{\varphi}_2(\theta^*, y_2^*)) \tag{5}$$

We now exhibit from the above equations an element satisfying the conclusion of Theorem 18 [15]. Let $\gamma^* = -p^* \cdot (y_1^* + y_2^*)$, remark that $p^* \cdot (y_1^* + y_2^* + \gamma^* e) = 0$ and $\gamma^* \leq -\alpha$. Indeed, $p^* \cdot \sum_{i=1}^2 y_i^* = p^* \cdot (\sum_{i=1}^2 s_i^* - \lambda_i(\theta^*, s_i^*)e)$. From (3) with $\sigma' = 0$, one gets: $p^* \cdot \sum_{i=1}^2 y_i^* \geq \sum_{i=1}^2 p_i^* \cdot s_i^* - \lambda_i(\theta^*, s_i^*) = \sum_{i=1}^2 p_i^* \cdot y_i^* \geq \alpha$ from Lemma 32.

From (4), for each $q' \in S_\eta$, $q' \cdot (y_1^* + y_2^* + \gamma^* \mathbf{1}) = q' \cdot (y_1^* + y_2^*) + \gamma^* \geq p^* \cdot (y_1^* + y_2^*) + \gamma^* = 0$. Therefore, $y_1^* + y_2^* + \gamma^* \mathbf{1} \in \{0\} \cup \mathbb{R}^N_{++}$ and it follows that $(s_j^*) \in \mathrm{int}\, \bar{B}$ by construction of the set $\bar{B}$. Then $p^* = p_1^* = p_2^* \in \Sigma$, from (3), since the maximum of a linear function is interior only if it is a null mapping. $p^* \in \Sigma$ implies $y_1^* + y_2^* + \gamma^* \mathbf{1} = 0$. From (2) that means $s_1^* = \mathrm{proj}(y_1^*) = -\mathrm{proj}(y_2^*) = -s_2^*$.

It remains to show that $y_1^* \in \partial V_N(\theta^*)$ and $p^* \in \psi(\theta^*, y_1^*)$. The argument is exactly the same as the one in the proof of Lemma 25.

Let $\xi^* = -y_2^*$ and $x^* = y_1^*$. It implies that $\xi^* \in \partial W(\theta^*)$ and therefore, from $x^* - \xi^* + \gamma^* \mathbf{1} = 0$, it follows that $p_W(\theta^*, x^*) = \xi^*$, or, equivalently, $p_V(\theta^*, \xi^*) = x^*$. From (5), $p^* \in \psi(\theta^*, p_V(\theta^*, \xi^*)) \cap \mathrm{co}\{\varphi_S(\theta^*, \xi^*) \mid S \in \mathcal{S}_{\theta^*}(\xi^*)\}$. So we deduce from the condition of payoffs-dependent balancedness that $\xi^* \in V(\theta^*)$. Since $\xi^* \in \partial W(\theta^*) \cap$

[15] Bonnisseau (1997) used a similar argument to show the existence of a general equilibrium with externalities.

$V(\theta^*)$, $x^* = p_V(\theta^*, \xi^*) \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$. Using (1), one can say that (θ^*, x^*) is an equilibrium-core allocation, and, $p^* \in \psi(\theta^*, x^*) \cap \text{co}\{\varphi_S(\theta^*, p_W(\theta^*, x^*)) \mid S \in \mathcal{S}_{\theta^*}(p_W(\theta^*, x^*))\}$ as was to be proved. $\square$

5.4 Proof of Corollary 21.

Assumption PH0: G is convex valued from the quasi-concavity of v_i with respect to the first variable, non-empty since from the definition of V_S, for all $\theta \in \Theta$, for all $x \in \partial V_N(\theta)$, there exists $\theta' \in \Theta$ such that $x \leq (v_i(\theta'_i, \theta_i))_i$. G is clearly an upper semi-continuous set-valued mapping from the continuity of the mappings v_i and p_N.

Assumption PH1: Since F^i is non-empty valued for all $i \in N$ and from the balancedness Assumption, taking the balanced family $(\{i\}, i \in N)$ one can prove the non-emptiness of Θ. Now, the lower-semi continuity and closed graph assumption of the set-valued mappings V_S, $S \in \mathcal{N}$, are proved.

(*l.s.c.*) For all $\theta^\nu \in \Theta$ a sequence converging to $\theta \in \Theta$, we show that, for all $x \in V_S(\theta)$, there exists a sequence (x^ν) converging to x with $x^\nu \in V_S(\theta^\nu)$ for ν large enough. Since $x \in V_S(\theta)$, there exists $\theta' \in F^S(\theta)$ such that $x_i \leq v_i(\theta'_i, \theta_i)$, $i \in S$. Since F^S is lower semi-continuous, there exists a sequence (θ'^ν) converging to θ' with $\theta' \in F^S(\theta^\nu)$ for ν large enough. Then, from the continuity of the mapping v_i, one has $v_i(\theta'^\nu_i, \theta^\nu_i)$ tends to $v_i(\theta'_i, \theta_i)$, $i \in S$. Let T be a subset of S such that for each $i \in T$ one has $x_i = v_i(\theta'_i, \theta_i)$. Now, it suffices to take $x^\nu_i = v_i(\theta'^\nu_i, \theta^\nu)$, $i \in T$, and $x^\nu_i = x_i$, $i \in S \setminus T$. This ends the proof.

(*closed graph*) Let (θ^ν) be a sequence converging to θ, and show that if $x^\nu \in V_S(\theta^\nu)$ converges to $x \in I\!R^N$, then $x \in V_S(\theta)$. For all $\nu \geq 0$, there exists $\theta'^\nu \in F^S(\theta^\nu)$ such that $x^\nu_i \leq v_i(\theta'^\nu_i, \theta^\nu)$. Since F^S is upper-semi continuous with compact values, $F^S(\Theta)$ is compact. Then, the sequence (θ'^ν) remains in a compact. So taking a subsequence if we need to, one can say that (θ'^ν) tends to an element $\theta \in F^S(\theta)$. Taking the limit and from the continuity of the mappings v_i, one gets $\theta' \in F^S(\theta)$ such that $x_i \leq v_i(\theta'_i, \theta)$ for all $i \in S$, that is to say that $x \in V_S(\theta)$, as was to be proved.

Remark now, that from a well known argument relying on the quasi-concavity of the functions $v_i(., \xi_i)$, the balancedness condition given in (5) is equivalent the balancedness (in Boehm's sense) of the game $(V(\theta), V_S(\theta)_{S \in \mathcal{N}})$ for each $\theta \in \Theta$. This fact is used in the two paragraphs below.

We check that Assumption PH2 holds true. Let $S \in \mathcal{N}$. The family $\{S, (\{i\})_{i \notin S}\}$ is a balanced family. Let $(\theta, x) \in \text{Gr } V_S$, since $F^i(\theta)$ is non-empty, there exist $\xi^i \in F^i(\theta)$, $i \notin S$. Let x' be defined by $x'_i = x_i$, $i \in S$ and $x'_i = v_i(\xi^i, \theta_i)$, $i \notin S$. Clearly, $x' \in V_S(\theta) \cap (\cap_{i \notin S} V_{\{i\}}(\theta))$. From the balancedness of the game, $x' \in V(\theta)$. Consequently, from the compactness of Θ and the continuity of v_i, $i \in N$, there exists $m(\theta)$ such that $x' \leq m(\theta)\mathbf{1}$, hence $x_i \leq m(\theta)$, $i \in S$.

The parameterized collection of games is payoffs-dependent balanced. Indeed, for each $S \in \mathcal{N}$, we let φ_S be the constant mapping equal to m^S and $\psi = \varphi_N$. Let (θ, x) be in Gr ∂W, such that $\mathrm{co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset$, one can easily show that the family $\mathcal{S}_\theta(x)$ is balanced. Then, from (5), one deduces that $x \in V(\theta)$, as required. $\square$

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