On the Shapley value of a minimum cost spanning tree problem*

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August 29, 2005

Abstract

We associate an optimistic coalitional game with each minimum cost spanning tree problem. We define the worth of a coalition as the cost of connection assuming that the rest of the agents are already connected. We define a cost sharing rule as the Shapley value of this optimistic game. We prove that this rule coincides with a rule present in the literature under different names. We also introduce a new characterization using a property of equal contributions.

Keywords: minimum cost spanning tree problems, Shapley value

1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*). Consider that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier.

*Latest version at: `http://webs.uvigo.es/vidalpuga/`. We thank Hervé Moulin and William Thomson for helpful comments. Financial support from the Ministerio de Ciencia y Tecnologia and FEDER through grant BEC2002-04102-C02-01 and Xunta de Galicia through grant PGIDIT03PXIC30002PN is gratefully aknowledged.

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Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network. This example appears in Dutta and Kar (2004). Bergantiños and Lorenzo (2004) studied a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supplier. Other examples include communication networks, such as telephone, Internet, or cable television.

The literature on *mcstp* starts by defining algorithms for constructing minimal cost spanning trees (*mt*). We can mention, for instance, the papers of Kruskal (1956) and Prim (1957).

Another important issue is how to allocate the cost associated with the *mt* between the agents. Bird (1976) and Dutta and Kar (2004) introduced two rules based on Prim's algorithm. Feltkamp, Tijs, and Muto (1994) introduced the Equal Remaining Obligation rule (*ERO*) based on Kruskal's algorithm.

Bird (1976) associated with each *mcstp* a coalitional game with transferable utility (*TU* game). According to Bird, the worth of a coalition is the cost of connection, assuming that the rest of the agents are not present. Since the Shapley value is a suitable solution concept for *TU* games, we can use it in *mcstp*. Kar (2002) gave additional arguments supporting the Shapley value of this *TU* game as a nice rule in *mcstp*. The core and the nucleolus of this *TU* game are studied in Granot and Huberman (1981, 1984).

In this paper, we associate with each *mcstp* a different *TU* game. We define the worth of a coalition as the cost of connection, assuming that the rest of the agents are already connected, and that connection is possible through them.

Both *TU* games compute the cost of connecting agents to the source. The former takes a pessimistic point of view because it assumes, given a coalition, that the rest of the agents are not connected. The latter takes an optimistic point of view because it assumes that the rest of the agents are already connected.

In general there is no relationship between the optimistic and the pessimistic *TU* game. However, it is possible to find a relationship in an interesting class of problems. A *mcstp* is *irreducible* if reducing the cost of any arc, the total cost of connection is also reduced. Given a *mcstp*, Bird (1976) defined the irreducible problem associated with it. In Bergantiños and Vidal-Puga (2005a) we defined the rule φ as the Shapley value of the

TU game associated with the irreducible problem. In Bergantiños and Vidal-Puga (2005b) we proved that φ coincides with ERO.

In this paper, we prove that, in irreducible problems, both TU games are dual (two TU games v, w are *dual* if $v(S) + w(N \setminus S)$ is constant for all S). Moreover, we define two rules in $mcstp$ using the optimistic TU game. The first rule is the Shapley value of the optimistic TU game. The second one is the Shapley value of the optimistic TU game associated with the irreducible problem.

We thus have four rules in $mcstp$ based on the Shapley value of an associated TU game. We prove that, in fact, we have two rules. The Shapley value of the optimistic TU game coincides with the Shapley value of the optimistic TU game associated with the irreducible form, and with the Shapley value of the pessimistic TU game associated with the irreducible form. The classical Shapley value (as defined by Kar (2002)) differs from these three.

Finally, we present a new characterization of φ using a property of equal contributions.

The paper is organized as follows. In Section 2 we introduce $mcstp$. In Section 3 we introduce the optimistic TU game and we study the relationship among the four Shapley values. In Section 4 we present the axiomatic characterization. In Appendix we give the proof of some of the results.

2 The minimum cost spanning tree problem

In this section we introduce minimum cost spanning tree problems.

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite set $N \subset \mathcal{N}$, let Π_N be the set of all permutations over N. Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i.e.* $Pre(i, \pi) = \{j \in N : \pi(j) < \pi(i)\}$. Given $S \subset N$, let π_S denote the order induced by π among agents in S.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we will work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly a *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given a *mcstp* (N_0, C), we define the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of g are called *arcs*.

Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$.

A *tree* is a network satisfying that for all $i \in N$ there is a unique path from i to the source. If t is a tree we usually write $t = \{(i^0, i)\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0.

Let $\mathcal{G}^N$ denote the set of all networks over N_0. Let $\mathcal{G}_0^N$ denote the set of all networks where every agent $i \in N$ is connected to the source, *i.e.* there exists a path from i to 0 in the network.

Given a *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well-known that an *mt* exists, even though it is not necessarily unique. Given a *mcstp* (N_0, C), we denote the cost associated with any *mt* t in (N_0, C) as $m(N_0, C)$.

Given an *mcstp*, Prim (1957) introduced an algorithm for solving the problem of connecting all agents to the source, such that the total cost of creating the network is minimal. The idea of this algorithm is quite simple: starting from the source we construct a network by consecutively adding arcs with the lowest cost and without introducing cycles.

Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1: Take an arc $(0, i)$ such that $c_{0i} = \min_{i \in N} \{c_{0i}\}$. If there are several arcs $(0, i)$ satisfying this condition, select any of them. Now, $S^1 = \{0, i\}$ and

$g^1 = \{(0,i)\}$.

Stage $p+1$: Assume that we have defined $S^p \subset N_0$ and $g^p \in \mathcal{G}^N$. We now define S^{p+1} and g^{p+1}. Take an arc (j,i) with $j \in S^p$ and $i \in N_o \setminus S^p$ such that $c_{ji} = \min\limits_{k \in S^p, l \in N_o \setminus S^p} \{c_{kl}\}$. If there are several arcs (j,i) satisfying this condition, select any of them. Now, $S^{p+1} = S^p \cup \{i\}$ and $g^{p+1} = g^p \cup \{(j,i)\}$.

This process is completed in n stages. We say that g^n is a tree obtained via Prim's algorithm. Notice that this algorithm leads to a tree, but that this is not always unique.

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source between them. We now briefly introduce some of the rules studied in the literature.

A (*cost allocation*) *rule* is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ for each *mcstp* (N_0, C) and $\sum\limits_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

Notice that we implicitly assume that the agents build an *mt*. As far as we know, all the rules proposed in the literature make this assumption.

A *coalitional game with transferable utility*, briefly a *TU game*, is a pair (N, v) where $v : 2^N \to \mathbb{R}$ satisfies that $v(\varnothing) = 0$. $Sh(N, v)$ denotes the Shapley value (Shapley, 1953) of (N, v).

A quite standard approach for defining rules in some problems is using *TU* games. We first associate with each problem a *TU* game. We then compute a solution for *TU* games (Shapley value, core, ...) in the associated *TU* game. Thus, the rule in the original problem is defined as the solution applied to the *TU* game associated with the original problem. This approach was already applied in *mcstp*.

Bird (1976) associated a *TU* game (N, v_C) with each *mcstp* (N_0, C). For each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

We define, in *mcstp*, the rule Sh^1 as the Shapley value of the associated *TU* game, *i.e.*

$$Sh^1(N_0, C) = Sh(N, v_C).$$

This rule was studied in Kar (2002).

A *mcstp* (N_0, C) is *irreducible* if reducing the cost of any arc, the cost of connecting agents to the source $(m(N_0, C))$ is also reduced. In Bergantiños and Vidal-Puga (2005a, Proposition 3.1) we prove that (N_0, C) is irreducible if and only if there exists an *mt* t in (N_0, C) satisfying the two following conditions:

$(A1)$ $t = \{(i_{p-1}, i_p)\}_{p=1}^{n}$ where $i_0 = 0$ (the source).

$(A2)$ Given $i_p, i_q \in N_0$, $p < q$, then $c_{i_p i_q} = \max_{p<r\leq q} \{c_{i_{r-1} i_r}\}$.

Given a *mcstp* (N_0, C), Bird (1976) defined the irreducible form (N_0, C^*) associated with (N_0, C). We define the rule Sh^2 as the Shapley value of the *TU* game associated with the irreducible form, *i.e.*

$$Sh^2 (N_0, C) = Sh (N, v_{C^*}).$$

In Bergantiños and Vidal-Puga (2005b) we proved that φ coincides with the Equal Remaining Obligations rule (Feltkamp *et al.*, 1994).

3 The optimistic approach

We associate an "optimistic" *TU* game (N, v_C^+) with each *mcstp* (N_0, C). Next we define two rules based on the Shapley value of the optimistic game. Sh^3 is defined as the Shapley value of the optimistic game associated with C. Sh^4 is the Shapley value of the optimistic game associated with the irreducible form C^*. The main result of this section says that Sh^2, Sh^3, and Sh^4 coincide.

Given $S \subset N$, Bird (1976) defined the worth of coalition S, $v_C(S)$, as the minimal cost of connecting all agents of S to the source, assuming that agents in $N \backslash S$ are out. This is a pessimistic approach because agents in $N \backslash S$ want to be connected to the source.

Alternatively, we can take an optimistic approach. We can define the worth of coalition S, $v_C^+(S)$, as the minimal cost of connecting all agents of S to the source, assuming that agents of $N \setminus S$ are already connected and agents in S can connect to the source through them.

In many problems it is possible to associate two *TU* games: a pessimistic game and an optimistic game. An example could be queuing problems, where a set of agents stands to receive a service. No two agents can be served simultaneously. Each agent has a constant per unit of time waiting cost. A queue has to be organized, but monetary compensations may be set up for those who have to wait. Maniquet (2003) defined the worth of a coalition S as the sum of its waiting cost in an efficient queue if they had the power to be served before agents in $N\backslash S$. Maniquet is taking an optimistic approach. Later, Chun (2004) defined the worth of a coalition S as the sum of its waiting cost in an efficient queue, assuming that members of S are served after the members of $N\backslash S$. Of course, Chun is taking a pessimistic approach.

Given a *mcstp* (N_0, C), $S, T \subset N$, $S \cap T = \varnothing$, $\left(S_0, C^{+T}\right)$ is the *mcstp* obtained from (N_0, C) assuming that agents in S have to be connected, agents in T are already connected, and agents in S can connect to the source through agents in T. Formally, $c_{ij}^{+T} = c_{ij}$ for all $i, j \in S$ and $c_{0i}^{+T} = \min\limits_{j \in T_0} c_{ji}$ for all $i \in S$.

We now associate a *TU* game $\left(N, v_C^+\right)$ with each *mcstp* (N_0, C). For each $S \subset N$,

$$v_C^+ (S) = m \left(S_0, C^{+(N\backslash S)}\right).$$

Notice that given $S \subset N$, $v_C^+ (S)$ is the minimal cost of connecting all the agents of S to the source assuming that the agents of $N \setminus S$ are already connected.

Example 1. Let (N_0, C) such that $N = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 10 & 100 \\ 10 & 0 & 2 \\ 100 & 2 & 0 \end{pmatrix}.$$

We now compute v_C and v_C^+.

S	$v_C(S)$	$v_C^+(S)$
{1}	10	2
{2}	100	2
{1,2}	12	12

This example shows that v_C and v_C^+ are different in general.

We say that two *mcstp* (N_0, C) and (N_0, C') are *tree-equivalent* if there exists a tree t such that, firstly, t is an *mt* for both (N_0, C) and (N_0, C') and secondly, $c_{ij} = c'_{ij}$ for all $(i, j) \in t$.

In the next theorem we give some results about v_C^+.

Theorem 1. (a) If (N_0, C) is irreducible, then

$$v_C (S) + v_C^+ (N \setminus S) = m (N_0, C)$$

for all $S \subset N$.

(b) If (N_0, C) and (N_0, C') are tree-equivalent, then $v_C^+ = v_{C'}^+$.

Proof. See Appendix.

Theorem $1(a)$ says that (N, v) and (N, v^+) are dual games in irreducible problems. This result it is not true when (N_0, C) is not an irreducible problem. In Example 1, $v_C(\{2\}) = 100$, $v_C^+(\{1\}) = 2$, and $m(N_0, C) = 12$.

In Section 2 we defined two rules in *mcstp* based on the Shapley value of the pessimistic game: $Sh^1(N_0, C) = Sh(N, v_C)$ and $Sh^2(N_0, C) = Sh(N, v_{C^*})$.

We now introduce two rules in *mcstp* based on the Shapley value of the optimistic game. For all *mcstp* (N_0, C), we define

$$\begin{aligned} Sh^3(N_0, C) &= Sh\left(N, v_C^+\right) \text{ and} \\ Sh^4(N_0, C) &= Sh\left(N, v_{C^*}^+\right). \end{aligned}$$

If we compute the four rules in Example 1 we obtain

Rule	Agent 1	Agent 2
$Sh^1(N_0, C)$	-39	51
$Sh^2(N_0, C)$	6	6
$Sh^3(N_0, C)$	6	6
$Sh^4(N_0, C)$	6	6

In this example $Sh^2(N_0, C) = Sh^3(N_0, C) = Sh^4(N_0, C)$. We now prove that this is true in general.

Theorem 2. For all *mcstp* (N_0, C),

$$Sh^2(N_0, C) = Sh^3(N_0, C) = Sh^4(N_0, C).$$

Proof. Let (N_0, C) be a *mcstp*.

In Bergantiños and Vidal-Puga (2005a, Remark 3.1) we proved that C and C^* are tree-equivalent. Under Theorem $1(b)$, $v_C^+ = v_{C^*}^+$. Thus, $Sh^3(N_0, C) = Sh^4(N_0, C)$.

Under Theorem $1(a)$, $v_{C^*}(S) + v_{C^*}^+(N \setminus S) = m(N_0, C)$ for all $S \subset N$. Since $v_{C^*}(N) = v_{C^*}^+(N) = m(N_0, C^*)$ and for all $i \in N$ the Shapley value of a TU game (N, v) can be expressed as

$$Sh_i(N, w) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \left(v\left(Pre(i, \pi) \cup \{i\}\right) - v\left(Pre(i, \pi)\right)\right),$$

it is not difficult to conclude that $Sh^2(N,C) = Sh^4(N,C)$. ■

Because of Theorem 3 we can define the rule φ as

$$\varphi(N_0, C) = Sh(N, v_{C^*}) = Sh\left(N, v_C^+\right) = Sh\left(N, v_{C^*}^+\right).$$

4 An axiomatic characterization

Myerson (1980) introduced the property of balanced contributions in *TU* games. Next property is inspired by Myerson's property.

We say that a rule ψ satisfies *Equal Contributions* (EC) if for all $i, j \in N$, $i \neq j$,

$$\psi_i(N_0, C) - \psi_i\left((N \setminus \{j\})_0, C^{+j}\right) = \psi_j(N_0, C) - \psi_j\left((N \setminus \{i\})_0, C^{+i}\right).$$

EC says that the impact of the connection of agent j on agent's i cost coincides with the impact of the connection of agent i on agent's j cost.

Next theorem characterizes φ as the only rule satisfying EC.

Theorem 3. φ is the only rule satisfying EC.

Proof. We first prove that φ satisfies EC.

For all $i \in N$, we denote $N^{-i} = N \setminus \{i\}$ and $N_0^{-i} = N_0 \setminus \{i\}$.

Given a *TU* game (N, v), Myerson (1980) proved that the Shapley value satisfies

$$Sh_i(N, v) - Sh_i\left(N^{-j}, v\right) = Sh_j(N, v) - Sh_j\left(N^{-i}, v\right)$$

for all $i, j \in N$, $i \neq j$.

Take $i, j \in N$, $i \neq j$. Under Claim 1 of the proof of Theorem 1, $v_C^+(S) = v_{C^{+j}}^+(S)$ for all $S \subset N^{-j}$. Since $\varphi_i(N^{-j}, C^{+j}) = Sh_i\left(N^{-j}, v_{C^{+j}}^+\right)$, we have $\varphi_i(N^{-j}, C^{+j}) = Sh_i\left(N^{-j}, v_C^+\right)$.

Applying Myerson's result to the *TU* game $\left(N, v_C^+\right)$, we obtain that φ satisfies EC.

We now prove the uniqueness. Let ψ be a rule satisfying EC. We prove that $\psi = \varphi$ by induction on $|N|$. If $|N| = 1$ it is trivial. Assume that $\psi = \varphi$ when $|N| \leq \alpha - 1$. We prove that $\psi = \varphi$ when $|N| = \alpha$.

Given $i, j \in N$, by simplicity, we write $\varphi_i = \varphi_i(N_0, C)$, $\psi_i = \psi_i(N_0, C)$, $\varphi_i^{+j} = \varphi_i\left(N_0^{-j}, C^{+j}\right)$, and $\psi_i^{+j} = \psi_i\left(N_0^{-j}, C^{+j}\right)$.

Since ψ satisfies EC,

$$\sum_{j \in N \setminus \{i\}} \psi_i - \sum_{j \in N \setminus \{i\}} \psi_i^{+j} = \sum_{j \in N \setminus \{i\}} \psi_j - \sum_{j \in N \setminus \{i\}} \psi_j^{+i}.$$

Thus,

$$n\psi_i = m(N_0, C) + \sum_{j \in N \setminus \{i\}} \psi_i^{+j} - \sum_{j \in N \setminus \{i\}} \psi_j^{+i}.$$

Since φ also satisfies EC,

$$n\varphi_i = m(N_0, C) + \sum_{j \in N \setminus \{i\}} \varphi_i^{+j} - \sum_{j \in N \setminus \{i\}} \varphi_j^{+i}.$$

Under the induction hypothesis, for all $i, j \in N$, $\psi_i^{+j} = \varphi_i^{+j}$ and $\psi_j^{+i} = \varphi_j^{+i}$. Thus, $\varphi_i = \psi_i$ for all $i \in N$. ■

Kar (2002) characterized Sh^1 as the only rule satisfying *Efficiency*, *Absence of Cross Subsidization*, *Group independence*, and *Equal Treatment*.

A rule ψ satisfies *Equal Treatment* (ET) if given (N_0, C) and (N_0, C') such that $c_{lk} = c'_{lk}$ for all $(l, k) \neq (i, j)$,

$$\psi_i(N_0, C) - \psi_i(N_0, C') = \psi_j(N_0, C) - \psi_j(N_0, C').$$

ET says that if the cost between agents i and j changes, both agents must win (or loss) the same.

Under Theorem 3, Sh^1 does not satisfy EC. Next example shows that φ does not satisfy ET.

Example 2. Let (N_0, C) be such that $N = \{1, 2\}$,

$$C = \begin{pmatrix} 0 & 5 & 14 \\ 5 & 0 & 10 \\ 14 & 10 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 5 & 14 \\ 5 & 0 & 12 \\ 14 & 12 & 0 \end{pmatrix}.$$

Making some computations we obtain that $\varphi(N_0, C) = (5, 10)$ and $\varphi(N_0, C') = (5, 12)$. Nevertheless, $Sh^1(N_0, C) = (3, 12)$ and $Sh^1(N_0, C') = (4, 13)$.

We have two rules for *mcstp* based on the Shapley value of an associated coalitional game: Sh^1 and φ. Both rules are very different, as we can see in the examples. The rule Sh^1 is defined through the pessimistic TU game. The rule φ can be defined through the pessimistic TU game and both optimistic TU games.

One may wonder which is the fairest rule (Sh^1 or φ)? We strongly believe that φ is a more suitable rule in *mcstp*. See Bergantiños and Vidal-Puga (2005a) for a detailed discussion about this issue.

There exist many problems where authors propose rules through optimistic TU games and pessimistic TU games. We conclude the section comparing *mcstp* with bankruptcy problems and queuing problems.

In bankruptcy problems the Shapley value of the pessimistic TU game and the Shapley value of the optimistic TU game coincide. See, for instance, Thomson (2003). The reason is that both games are dual, like in irreducible *mcstp*.

In queuing problems the Shapley value of both games are different, like in general *mcstp*. Maniquet (2003) studied the Shapley value of the optimistic game. He provided several axiomatic characterizations. Chun (2004) studied the Shapley value of the pessimistic game, which he called the *reverse rule*. He provided axiomatic characterizations of the reverse rule. These characterizations are obtained by changing some properties in Maniquet's characterization by their "reverse".

5 Appendix

We prove Theorem 1.

(a) Assume, without loss of generality, that $t = \{(i-1, i)\}_{i=1}^n$ is the tree associated with C satisfying $(A1)$ and $(A2)$. Take $S = \left\{i_1, ..., i_{|S|}\right\}$ where $i_{p-1} \leq i_p$ for all $p = 2, ..., |S|$.

For each $p = 1, ..., |S|$ we define:

$$S^p = \{i \in N : i_{p-1} < i < i_p\}.$$

If $p = 1$ we take $i_0 = 0$. Moreover,

$$S^{|S|+1} = \left\{i \in N : i_{|S|} < i \leq n\right\}.$$

Thus, $S^p \subset N \backslash S$ for all $p = 1, ..., |S|+1$, $\bigcup_{p=1}^{|S|+1} S^p = N \backslash S$, and $S^p \cap S^q = \emptyset$ for all $p \neq q$. Notice that $S^p = \varnothing$ is also possible.

We know that

$$v_C^+(N \setminus S) = m\left((N \setminus S)_0, C^{+S}\right) = c\left((N \setminus S)_0, C^{+S}, t'\right)$$

where t' is obtained following Prim's algorithm. We now compute t'.

Assume that $i \in S^p$, $j \in S^q$, and $p < q$. Thus, $i < i_p \leq i_{q-1} < j$. Since t satisfies $(A2)$,

$$c_{0i}^{+S} = \min_{k \in S_0} \{c_{ik}\} = \min\left\{c_{i_{p-1}i}, c_{ii_p}\right\} \leq c_{ii_p}.$$

Moreover, $c_{ij}^{+S} = c_{ij}$. Since t satisfies $(A2)$, $c_{ii_p} \leq c_{ij}$. Hence, $c_{0i}^{+S} \leq c_{ij}^{+S}$. This means that we can construct t' such that there is no direct link between agents in S^p and S^q.

Thus, t' can be expressed as $\bigcup_{p=1}^{|S|+1} t'_{S_0^p}$ where $t'_{S_0^p}$ is an mt computed following Prim's algorithm in $\left(S_0^p, C^{+S}\right)$. When $S^p = \emptyset$, we take $t'_{S_0^p} = \emptyset$. Hence,

$$v_C^+(N \setminus S) = \sum_{p=1}^{|S|+1} c\left(S_0^p, C^{+S}, t'_{S_0^p}\right).$$

We now apply Prim's algorithm to $\left(S_0^p, C^{+S}\right)$ where $p < |S| + 1$. We first select an arc $(0, i)$ such that $c_{0i}^{+S} = \min_{j \in S^p} \left\{c_{0j}^{+S}\right\}$. We have already proved that $c_{0i}^{+S} = \min\left\{c_{i_{p-1}i}, c_{ii_p}\right\}$.

Since t satisfies $(A2)$,

$$\min_{j \in S^p} \left\{c_{0j}^{+S}\right\} = \min\left\{c_{i_{p-1}(i_{p-1}+1)}, c_{(i_p-1)i_p}\right\} = \min\left\{c_{0(i_{p-1}+1)}^{+S}, c_{(i_p-1)0}^{+S}\right\}.$$

Assume that the arc selected in $\left(S_0^p, C^{+S}\right)$ is $(0, i_{p-1}+1)$ (the case where the arc selected is $(i_p - 1, 0)$ is similar and we omit it). Notice that the arc $(0, i_{p-1}+1)$ in $\left(S_0^p, C^{+S}\right)$ corresponds to the arc $(i_{p-1}, i_{p-1}+1)$ in (N_0, C).

Using arguments similar to those used before, we can conclude that the second arc selected is the arc (i, j) such that

$$c_{ij}^{+S} = \min\left\{c_{(i_{p-1}+1)(i_{p-1}+2)}, c_{(i_p-1)i_p}\right\}.$$

If we continue with this procedure we obtain that

$$c\left(S_0^p, C^{+S}, t'_{S_0^p}\right) = c\left(N_0, C, t_{S^p \cup \{i_{p-1}, i_p\}} \setminus \{(i_{p'} - 1, i_{p'})\}\right)$$

where $(i_{p'}-1, i_{p'})$ satisfies that

$$c_{(i_{p'}-1)i_{p'}} = \max_{j=i_{p-1}+1,\dots,i_p} c_{(j-1)j}.$$

Since t satisfies $(A2)$, $c_{(i_{p'}-1)i_{p'}} = c_{i_{p-1}i_p}$. Thus,

$$c\left(S_0^p, C^{+S}, t'_{S_0^p}\right) = \sum_{j=i_{p-1}+1}^{i_p} c_{(j-1)j} - c_{i_{p-1}i_p}.$$

If $S^p = \emptyset$, $c\left(S_0^p, C^{+S}, t'_{S_0^p}\right) = 0$.

If $p = |S|+1$, using arguments similar to those used when $p < |S|+1$ we can prove that

$$c\left(S_0^p, C^{+S}, t'_{S_0^p}\right) = \sum_{j=i_p+1}^{n} c_{(j-1)j}.$$

Thus,

$$\begin{aligned} v_C^+(N \setminus S) &= \sum_{p=1}^{|S|}\left(\sum_{j=i_{p-1}+1}^{i_p} c_{(j-1)j} - c_{i_{p-1}i_p}\right) + \sum_{j=i_p+1}^{n} c_{(j-1)j} \\ &= \sum_{j=1}^{n} c_{(j-1)j} - \sum_{p=1}^{|S|} c_{i_{p-1}i_p}. \end{aligned}$$

Since t is an mt in (N_0, C), $\sum_{j=1}^{n} c_{(j-1)j} = m(N_0, C)$. In Bergantiños and Vidal-Puga (2005a, Proposition 3.3(a)) we proved that $\sum_{p=1}^{|S|} c_{i_{p-1}i_p} = v_C(S)$. Thus (a) holds.

(b) Let (N_0, C) and (N_0, C') be two tree-equivalent problems. Assume that $t = \{(i^0, i)\}_{i=1}^n$ is an mt in (N_0, C) and (N_0, C') satisfying that $c_{i^0 i} = c'_{i^0 i}$ for all $i = 1, ..., n$. For all $i \in N$, $i^0 \in N_0$ is the first node in the unique path from i to the source.

We proceed by induction on $|N|$. If $|N| = 1$ the result is trivial. Assume that the result holds when $|N| \leq \alpha - 1$. We now prove it when $|N| = \alpha$.

In order to simplify the notation, for all $i \in N$ we denote $N^{-i} = N \setminus \{i\}$.

We prove several claims.

Claim 1. Given a *mcstp* (N_0, C), $S \subset N$, and $j \in N \setminus S$,

$$\left(S_0, C^{+(N\setminus S)}\right) = \left(S_0, \left(C^{+j}\right)^{+\left(N^{-j}\setminus S\right)}\right).$$

Let $i, k \in S$ such that $i \neq 0$ and $k \neq 0$. Thus,

$$c_{ik}^{+(N\setminus S)} = c_{ik} = \left(c_{ik}^{+j}\right)^{+\left(N^{-j}\setminus S\right)}.$$

Given $i \in S$,

$$\begin{aligned} c_{0i}^{+(N\setminus S)} &= \min_{k\in(N\setminus S)_0}\{c_{ki}\} \\ &= \min\left\{\min_{k\in(N^{-j}\setminus S)}\{c_{ki}\}, \min\{c_{0i}, c_{ji}\}\right\} \\ &= \min\left\{\min_{k\in(N^{-j}\setminus S)}\left\{c_{ki}^{+j}\right\}, c_{0i}^{+j}\right\} \\ &= \min_{k\in(N^{-j}\setminus S)_0}\left\{c_{ki}^{+j}\right\} \\ &= \left(c_{0i}^{+j}\right)^{+\left(N^{-j}\setminus S\right)}. \end{aligned}$$

Claim 2. Assume that t^* is an *mt* in (N_0, C) and $j \in N$. Let $g = \{(i_{p-1}, i_p)\}_{p=1}^{r}$ be the unique path in t^* from $0\ (= i_0)$ to $j\ (= i_r)$. Let q such that $c_{i_{q-1}i_q} = \max_{p=1,\dots,r}\left\{c_{i_{p-1}i_p}\right\}$. Given $A_j^* = \{(j,k) : (j,k) \in t^* \setminus \{(i_{q-1}, i_q)\}\}$,

$$t_j^* = \left(t^* \setminus A_j^*\right) \cup \left\{(0,k) : (j,k) \in A_j^*\right\} \setminus \{i_{q-1}, i_q\} \tag{1}$$

is an *mt* in $\left(N_0^{-j}, C^{+j}\right)$.

First, we note that each arc $(0,k)$ in (1) for $\left(N_0^{+j}, C^{+j}\right)$ corresponds to the arc (j,k) for (N_0, C) (notice that j becomes a source itself when connected). Hence,

$$c\left(N_0, C, t^*\right) = c\left(N_0^{-j}, C^{+j}, t_j^*\right) + c_{i_{q-1}i_q}.$$

Suppose that t_j^* is not an *mt* in $\left(N_0^{-j}, C^{+j}\right)$. There exists a tree t' in $\left(N_0^{-j}, C^{+j}\right)$ such that

$$c\left(N_0^{-j}, C^{+j}, t'\right) < c\left(N_0^{-j}, C^{+j}, t_j^*\right).$$

Let S_j denote the set of agents in N^{-j} "connected to the source in t' through agent j". We now define S_j formally. For each $i \in N^{-j}$, let $\left\{ (0, l_1^i), (l_1^i, l_2^i), ..., (l_{s-1}^i, i) \right\}$ be the unique path in t' from the source to i. We define

$$S_j = \left\{ i \in N^{-j} : c_{0l_1^i}^{+j} = c_{jl_1^i} \right\}.$$

We can find $(i_{t-1}, i_t) \in g \subset t^*$ such that $i_{t-1} \in N_0^{-j} \setminus S_j$ and $i_t \in S_j \cup \{j\}$. Let $A'_j = \left\{ (0, l) \in t' : c_{0l}^{+j} = c_{jl} \right\}$. Thus,

$$t'_j = \left(t' \setminus A'_j \right) \cup \left\{ (j, l) : (0, l) \in A'_j \right\} \cup \{ (i_{t-1}, i_t) \}$$

is a tree in (N_0, C).

Since

$$\begin{aligned} c\left(N_0, C, t'_j\right) &= c\left(N_0^{-j}, C^{+j}, t'\right) + c_{i_{t-1}i_t}, \\ c\left(N_0, C, t^*\right) &= c\left(N_0^{-j}, C^{+j}, t_j^*\right) + c_{i_{q-1}i_q}, \text{ and} \\ c_{i_{t-1}i_t} &\leq c_{i_{q-1}i_q} \end{aligned}$$

we deduce that

$$c\left(N_0, C, t'_j\right) < c\left(N_0, C, t^*\right)$$

which is a contradiction because t^* is an mt of (N_0, C).

Claim 3. For all $j \in N$, $\left(N_0^{-j}, C^{+j}\right)$ and $\left(N_0^{-j}, C'^{+j}\right)$ are tree-equivalent.

Given $j \in N$, let t_j be the mt in $\left(N_0^{-j}, C^{+j}\right)$ obtained from the mt t in (N_0, C) as in the statement of Claim 2. Similarly, let t'_j be the mt in $\left(N_0^{-j}, C'^{+j}\right)$ obtained from the mt t in (N_0, C') as in the statement of Claim 2.

It is not difficult to see that $t_j = t'_j$. Moreover, for all $(i, k) \in t_j$, $c_{ik} = c'_{ik}$. Thus, $\left(N_0^{-j}, C^{+j}\right)$ and $\left(N_0^{-j}, C'^{+j}\right)$ are tree-equivalent.

Claim 4. v_C^+ coincides with $v_{C'}^+$.

We prove that $v_C^+(S) = v_{C'}^+(S)$ for all $S \subset N$. If $S = N$,

$$v_C^+(N) = m(N_0, C) = m(N_0, C') = v_{C'}^+(N).$$

Assume that $S \neq N$. Take $j \in N \setminus S$. Under Claim 1, $v_C^+(S) = v_{C^{+j}}^+(S)$ and $v_{C'}^+(S) = v_{C'^{+j}}^+(S)$.

Under Claim 3, $\left(N_0^{-j}, C^{+j}\right)$ and $\left(N_0^{-j}, C'^{+j}\right)$ are tree-equivalent. Under the induction hypothesis, $v_{C^{+j}}^+(S) = v_{C'^{+j}}^+(S)$. Thus, $v_C^+(S) = v_{C'}^+(S)$.

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