Transfer rate rules and core selections in NTU games

Vincent Iehlé *†

August 2004

Cahier de la MSE 2004.93, Série Bleue

Abstract

Different kinds of asymmetries between players can occur in core allocations, in that case the stability of the concept is questioned. One remedy consists in selecting robust core allocations. We review, in this note, results that all select core allocations in NTU games with different concepts of robustness. Within a unified approach, we deduce the existence of allocations in: the partnered core, the social stable core, the core intersected with average prekernel, the weak inner core. We use a recent contribution of Bonnisseau and Iehlé (2003) that states the existence of core allocations with a transfer rate rule equilibrium under a dependent balancedness assumption. It shall turn out to be manipulable tools for selecting the core. *Journal of Economic Literature* Classification Numbers: C60, C71.

Keywords: Cooperative games, dependent balancedness, core selections in NTU games.

Résumé

Dans cette note, on propose quelques applications directes d'un résultat d'existence de Bonnisseau et Iehlé (2003). Ces auteurs ont montré l'existence d'allocations du cœur dans les jeux NTU qui satisfont un équilibre de taux de transfert sous une condition de balancement dépendant. Il s'avère que la notion de balancement dépendant procure en fait un outil manipulable pour sélectioner le cœur. Pour illustrer ce fait, nous montrons que cette notion permet d'obtenir des résultats d'existence dans des modèles de cœur avec partenariat, cœur socialement stable, prekernel moyen intersecté avec le cœur et de cœur interne faible. *Journal of Economic Literature* Classification Numbers : C60, C71.

Mots-clés : Jeux coopératifs, balancement dépendant, sélections du cœur dans les jeux NTU.

*I thank Jean-Marc Bonnisseau for his constant guidance.

†CERMSEM-CNRS UMR 8095, Université Paris 1 Panthéon-Sorbonne, Maison des Sciences Economiques, 106-112 boulevard de l'Hôpital 75647 Paris Cedex 13, France. E-mail: iehle@univ-paris1.fr.

1 Introduction

The core of a cooperative game is the set of efficient payoffs for the grand coalition that cannot be improved by any coalition of players. One critic arising from the core concept in cooperative NTU games is that some core allocations may exhibit asymmetric dependencies (Reny and Wooders [13])[1] or inefficiencies if players agree for a transfer of utility (Qin [11, 12]). For example, at a core allocation, some players could contribute more than others. Then, the stability of the concept is questioned since the best contributors are likely to receive rewards for their participation. A remedy consists in selecting the core, that is, defining a criterion to find a utility vector that each member of the grand coalition finds acceptable. Hence, by selecting the core, we mean prescribing specific core allocations satisfying division schemes or stable matchings that are robust with respect to asymmetries or inefficiencies. We review, in this note, non-emptiness results with different concerns and frameworks that all selects specific core allocations in NTU games.

We propose a unified treatment based on a recent notion of balancedness with a transfer rate rule, generalizing the extant notions, and called dependent balancedness. The idea behind the notion is to consider a transfer rate rule depending on the payoffs to define a notion of balancedness whereas the usual transfer rate of the literature is supposed to be constant. It turns out that the class of dependent balancedness games is exactly the class of games with non empty cores [2].

Going beyond the non-emptiness of the core, it is also proved in Bonnisseau and Iehlé [3] that dependent balancedness is a sufficient condition to get the existence of core allocations with a transfer rate rule equilibrium. All the following selections of the core will coincide with a core allocation with a transfer rate rule equilibrium. For instance, the authors deduce from their existence result the non-emptiness of the partnered core of Reny and Wooders [13]. This specific core selection is the set of core allocations such that, for any pair players i, j, if the player i cannot achieve her core payoff without player j then player j cannot either achieve her core payoff without player j.

As further applications, three other results will illustrate the role of the transfer rate rule equilibrium, giving hints on its manipulation.

First, we turn to the non-emptiness of a social stable core. To define such a concept, Herings et al. [5] introduce a power index for each players in the coalitions. And then, they prove the non-emptiness of the set of equipotent allocations in the core: the social stable core.

The second core selection is the average prekernel intersected with the core. The prekernel is the NTU extension of the usual notion of prekernel at stake in TU games. Though no interpretation is attached to the prekernel, it can be seen as a fair sharing allocation with respect to a surplus measure of the players. We improve the existence result originally given in Orshan et al. [8], by considering

[1] See also Bennett [2] and Bennett and Zame [1] for further developments on conflicts over gains from cooperation.

[2] See details in [3, 10].

the class of ∂-balanced games.

Lastly, we propose an existence result for a core allocation in the spirit of the original notion of inner core of Qin [11, 12]. x is in the inner core if it is feasible for the grand coalition, and there exists a transfer rate λ such that x is in the core of the λ-transfer game. The inner core is included in the core, hence it can be seen as a selection of the core. We exhibit a result for the non-emptiness of a weak inner core.

To deduce these results as corollaries of the abstract result of Bonnisseau and Iehlé [3], we construct explicitly a transfer rate rule for which the games are dependent balanced.

1.1 Game description and the general result

We will use the following notations[3]: $N = \{1, ..., n\}$ is the finite set of players; $\mathcal{N}$ is the set of the non-empty subsets of N, i.e. the coalitions of players; for each $S \in \mathcal{N}$, L_S is the $|S|$-dimensional subspace of $\mathbb{R}^N$ defined by $L_S = \{x \in \mathbb{R}^N \mid x_i = 0, \forall i \notin S\}$; L_{S+} (L_{S++}) is the non negative orthant (positive orthant) of L_S; for each $x \in \mathbb{R}^N$, x^S is the projection of x into L_S; $\mathbf{1}$ is the vector of $\mathbb{R}^N$ whose coordinates are equal to 1; $\mathbf{1}^\perp$ is the hyperplane $\{s \in \mathbb{R}^N \mid \sum_{i \in N} s_i = 0\}$; proj is the orthogonal projection mapping on $\mathbf{1}^\perp$; $\Sigma_S = \text{co}\{\mathbf{1}^{\{i\}} \mid i \in S\}$; $m^S = \frac{\mathbf{1}^S}{|S|}$; $\Sigma = \Sigma_N$ and $\Sigma_{++} = \Sigma \cap \mathbb{R}^N_{++}$.

A game $(V_S, S \in \mathcal{N})$ is a collection of subsets of $\mathbb{R}^N$ indexed by $\mathcal{N}$. $x \in \mathbb{R}^N$ is called a payoff; $V_S \subset \mathbb{R}^N$ is the set of feasible payoffs of the coalition S; $\mathcal{S}(x) = \{S \in \mathcal{N} \mid x \in \partial V_S\}$ is the set of coalitions, for which $x \in \mathbb{R}^N$ is an efficient payoff; $W := \cup_{S \in \mathcal{N}} V_S$ is the union of the payoffs sets.

We will assume in the remainder of the paper that the two following assumptions are satisfied.

(H1) (i) $V_{\{i\}}$, $i \in N$, and V_N are non-empty. (ii) For each $S \in \mathcal{N}$, V_S is closed, $V_S - \mathbb{R}^N_+ = V_S$, $V_S \neq \mathbb{R}^N$, and, for all $(x, x') \in (\mathbb{R}^N)^2$, if $x \in V_S$ and $x^S = x'^S$, then $x' \in V_S$.

(H2) There exists $m \in \mathbb{R}$ such that, for each $S \in \mathcal{N}$, for each $x \in V_S$, if $x \notin \text{int } V_{\{i\}}$ for all $i \in S$, then $x_j \leq m$ for all $j \in S$.

Note that under Assumption H1, there exist continuous mappings p_N from $\mathbb{R}^N$ to ∂V_N, p_W from $\mathbb{R}^N$ to ∂W, λ_N and λ_W from $\mathbf{1}^\perp$ to $\mathbb{R}$ such that, for all $x \in \mathbb{R}^N$, $p_N(x) = \text{proj}(x) - \lambda_N(\text{proj}(x))\mathbf{1}$ and $p_W(x) = \text{proj}(x) - \lambda_W(\text{proj}(x))\mathbf{1}$[4]. Let us recall now the definitions of core and dependent balancedness, and, the main result obtained by Bonnisseau and Iehlé [3].

Definition 1 Let $(V_S, S \in \mathcal{N})$ be a game. A payoff x is in the core of the game if $x \in V_N \setminus \text{int } W$.

[3]For any set $Y \subset \mathbb{R}^N$, co(Y), ∂Y, int Y will denote respectively its convex hull, boundary, interior. For any set-valued mapping Γ, Gr Γ will denote its graph.

[4]See [3] for more details.

Definition 2 Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumption H1: (i) A transfer rate rule is a collection of set-valued mappings $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that for all $S \in \mathcal{N}$, φ_S is upper semi-continuous with non-empty compact and convex values from ∂V_S to Σ_S, and, ψ is upper semi-continuous with non-empty compact and convex values from ∂V_N to Σ. (ii) The game $(V_S, S \in \mathcal{N})$ is dependent balanced if there exists a transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that, for each $x \in \partial W$, if $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$, then $x \in V_N$.

Theorem 1 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1 and H2. If it is dependent balanced with respect to the transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$, there exists a core allocation x such that:* $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset$.

2 Applications: four selections of the core

Theorem 1 downsizes the core into specific core allocations with transfer rate rule equilibrium. In the following applications, the stake is to define indexes for contribution, power or transfer and to prescribe an allocation in the core of the game satisfying a division scheme with respect to these indexes. Thanks to Theorem 1, we unify different models where such a prescription is proposed. The following results are all deduced as corollaries, the proofs are given in Appendix.

2.1 The partnered core (Reny and Wooders [13])

To get a first application of this result, consider the following corollary of Theorem 1, due to Reny and Wooders [13] and already proved in Bonnisseau and Iehlé [3]. We recall before the notion of ∂-balancedness.

∂-balancedness The game is ∂-balanced if for all $x \in \partial W$ and any balanced family of coalitions $\mathcal{B} \subset \mathcal{N}$ such that $x \in \cap_{S\in\mathcal{B}} V_S$ then $x \in V_N$.

Corollary 1 *Let $(V_S, S \in \mathcal{N})$ be a ∂-balanced game satisfying Assumptions H1 and H2*[5]*. Suppose that for each pair of players i and j, there is a continuous mapping c_{ij}: $\partial W \to \mathbb{R}_+$ such that c_{ij} is zero on $V(S) \cap \partial W$ whenever $i \notin S$ and $j \in S$. Then there exists a core allocation x such that, for each $i \in N$:*

$$\eta_i(x) := \sum_{j\in N} (c_{ij}(x) - c_{ji}(x)) = 0.$$

The mappings c_{ij} can be interpreted as credit/debit mappings. Then, one can see $\eta_i(x)$ as the measure of the grand coalition's net indebtness to i or as i's net credit against the grand coalition. The previous result states the existence of a core allocation where the net credits of the players are all equal to 0. The result of Reny and Wooders[13] has been originally applied to a stable

[5]In Reny and Wooders [13], the result is stated for balanced games, it is slightly improved by considering ∂-balanced games.

matching problem. They state indeed that any balanced game has a non-empty partnered core, which is the set of core allocations such that, for any pair players i, j, if the player i cannot achieve her core payoff without player j then player j cannot either achieve her core payoff without player j. Formally, a payoff $x \in \partial W$ is said to be partnered if the family $\mathcal{S}(x)$ satisfies, for all $i, j \in N$, $\mathcal{S}_i(x) \subset \mathcal{S}_j(x) \Rightarrow \mathcal{S}_j(x) \subset \mathcal{S}_i(x)$, where $\mathcal{S}_i(x) = \{S \in \mathcal{S}(x) \mid i \in S\}$, and the partnered core is the set of all partnered core allocations.

2.2 Social stable core (Herings et al. [5])

In Herings et al. [5], the authors propose a generalization of NTU games. Firstly, they assume the possibility of internal organizations, that is, inside a given coalition, the members can choose among several possibilities of organization, it give rise to a multiplicity of possible payoffs sets for the given coalition [6]. Secondly, a power mapping that describes the power of agents inside each organization is introduced. Under a balancedness condition, it is shown that there exists an allocation lying in the core of the generalized NTU game and such that the agents are equally powerful.

For each coalition $S \in \mathcal{N}$, there is a finite number k_S of possible internal organizations. Denote $I^S = (I_1^S ... I_{k_S}^S)$ these organizations. Let $\mathcal{I}$ be the union over S of all internal organizations. For each $S \in \mathcal{N}$, each $I \in I^S$, define a payoff set $v_I \in \mathbb{R}^N$. Now, define the power of an agent within an internal organization by a power vector function p from $\mathcal{I}$ to $\mathbb{R}_+^N \setminus \{0\}$. For each $S \in \mathcal{N}$, each $I \in I^S$, $p(I) \in L_{S+} \setminus \{0\}$. A socially structured game is described by $(N, \mathcal{I}, v, p)$. In Herings et al. [5], the authors restate Assumptions H1 and H2 for this generalized game. We omit their statements, it is an easy matter to check that it amounts to consider that the game $(V_S, S \in \mathcal{N})$, where for each $S \in \mathcal{N}$, $V_S = \cup_{I \in I^S} v_I$, satisfies Assumption H1 and H2. Define the power cone of a payoff x as: $PC(x) = \{y \in \mathbb{R}^N \mid y = \sum_{I \in \mathcal{I}(x)} \lambda_I p(I), \lambda_I \geq 0, \text{ for all } I\}$, where $\mathcal{I}(x) = \{I \in \mathcal{I} \mid x \in \partial v_I\}$.

Definition 3 For a socially structured game, $(N, \mathcal{I}, v, p)$, a payoff vector $x \in \mathbb{R}^N$ is socially stable if:

$$\mathbf{1} \in PC(x).$$

A core allocation is a payoff vector $x \in \mathbb{R}^N$ such that $x \in v_I$ for some $I \in I^N$ and $x \notin \text{int } v_{\overline{I}}$ for all $\overline{I} \in \mathcal{I}$. A socially stable core is the set of socially stable core allocations.

(SSG) If a payoff vector x is socially stable then $x \in v_I$ for some $I \in I^N$[7].

[6]The reader can imagine that possibilities of a coalition are described by special pairwise links between its members that give rise to different networks, (e.g. see networks formation in Jackson [6]).

[7]This balancedness notion is sandwiched between the notion of b-balancedness of Billera and that of dependent balancedness.

We deduce the following result given in Herings et al. [5].

Corollary 2 *Let* $(N, \mathcal{I}, v, p)$ *be a socially structured game and suppose that* $(V_S, S \in \mathcal{N})$*, where for each* $S \in \mathcal{N}$*,* $V_S = \cup_{I \in I^S} v_I$*, satisfies Assumption H1 and H2. Under SSG, the socially stable core is non-empty.*

To prove the result, we will consider the transfer rate rule $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ where: $\psi = m^N$, and, for all $S \in \mathcal{N}$ and all $x \in \partial V_S$, $\varphi_S(x) = \text{co}\{\frac{p(I)}{\sum_{i \in S} p_i(I)} \mid I \in \mathcal{I}(x) \cap I^S\}$.

It is an easy matter to extend the result to parameterized games. It suffices to apply Theorem 3.1 of Bonnisseau and Iehlé [3]. Furthermore, the parameterization could allow us to define a sharper model of internal organization, continuously depending on the parameter set.

2.3 Average prekernel (Orshan et al. [8])

As another application, one can also prove the existence of an element lying in the core intersected with the average prekernel (also called bilateral consistent prekernel) as defined in Orshan and Zarzuelo [9], see also Serrano and Shimomura [14]. The average prekernel is the consistent extension of the usual prekernel at stake in TU games. Furthermore, the most interesting feature is that the following concept for multi-player games coincides with the Nash solution and intersects the core in a general class of games. We show how we can deduce this existence result under an assumption of balancedness.

Define additionally, for each coalition, the set of individually rational payoffs, $I_S = V_S \cap (\cap_{i \in S} (\text{int } V_{\{i\}})^c)$. Before introducing the average prekernel, we need two additional assumptions on the game, namely non-levelness (NL) and smoothness (SM).

- **(NL)** For each $S \in \mathcal{N}$, ∂V_S is non-leveled, that is: if $x, y \in \partial V_S$, $x \geq y$ and $y \in I_S$, then $x_i = y_i$.
- **(SM)** At each point $x \in \partial I_N$, there exists a unique vector $p(x)$ such that $\sum_{i \in N} p_i(x) = 1$. Moreover, for all $x \in \partial I_N$, $p(x) > 0$ and p is a continuous map.

Let us now define the individual excess functions, bilateral surplus functions and total loss functions as follows:
For each $x \in \mathbb{R}^N$, for each $S \in \mathcal{N}$, for $k \in S$, the individual excess of k with respect to S at x is :

$$e_k(S, x) = \begin{cases} \max\{y_k - x_k \mid (y_k, x_{-k}) \in V_S\} & \text{if } \{y_k \mid (y_k, x_{-k}) \in V_S\} \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

For every $k, \ell \in N$, $k \neq \ell$, define the surplus of k with respect to ℓ at x to be $s_{k\ell}(x) = \max\{e_k(S, x) \mid S \in \mathcal{N}, k \in S, l \notin S\}$.
For every $k \in N$ and $x \in \partial I_N$ denote $f_k(x) = \sum_{\ell \neq k} (p_k(x) s_{k\ell}(x) - p_\ell(x) s_{\ell k}(x))$ the total loss of player k at x. Let $f(x)$ be the vector $(f_1(x), ..., f_n(x))$.

Definition 4 The average prekernel of $(V_S, S \in \mathcal{N})$ is the set:

$$\{x \in \partial V_N \mid f(x) = 0\}.$$

In Orshan et al. [8], the authors have shown the non-emptiness of the core intersected with the average prekernel in ∂-separating games, here the result is improved by considering the larger class of ∂-balanced games[8].

Corollary 3 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1, H2, NL and SM. If it is a ∂-balanced game then there exists a core allocation that belongs to the average prekernel.*

We will deduce the result from Corollary 1, but it amounts to consider the transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ where: $\psi(x) = m^N - \tilde{f}(x)$, ($\tilde{f}(x) = f(x)$ up to a normalization), and for all $S \in \mathcal{N}$, $\varphi_S = m^S$.

2.4 Inner core (Qin [12])

The inner core is an alternative way of downsizing the core. The main results have been obtained by Qin [11, 12].

Let $(V_S, S \in \mathcal{N})$ be a game compactly generated. And let $\lambda \in \mathbb{R}^N_+$. Define a real valued set function v_λ on the set of all non-empty subsets of N by $v_\lambda(S) = \max\left\{\sum_{i\in S} \lambda_i \cdot y_i \mid y \in V_S\right\}$. Define the fictitious λ-transfer game $(V_S^\lambda, S \in \mathcal{N})$ associated with v_λ: for each $S \in \mathcal{N}$, the fictitious λ-transfer payoffs sets are: $V_S^\lambda = \left\{y \in \mathbb{R}^N \mid \sum_{i\in S} \lambda_i \cdot y_i \leq v_\lambda(S)\right\}$.

An allocation x is in the inner core of the game $(V_S, S \in \mathcal{N})$ if $x \in V_N$ and there exists at least one $\lambda \in \Sigma$ such that x is in the core of $(V_S^\lambda, S \in \mathcal{N})$. Note that the inner core is included in the core of $(V_S, S \in \mathcal{N})$. The requirements imposed in the definition of the inner core are very strong. However, Qin [12] proposes a class of balanced games for which the inner core is non-empty[9]. In the following, we relax the definition and deduce from Theorem 1 the non-emptiness of a weak inner core.

The interpretation of this weak concept of inner core differs from the initial inner core. We consider a group of players who agree for transfer rate rules within each coalition. Then a global transfer rate rule is prescribed, this rule must belong to the set of admissible transfer rates, defined below. At this prescribed rate λ, the players can transfer utility among themselves. The core allocation x is in the weak inner core if x is an efficient point in the fictitious λ-transfer payoff set of the grand coalition.

Formally, the transfer set induced by the transfer rate mappings is defined as follows: for each $x \in \partial W$, $TS(x) = \text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\}$. Then, $(\lambda, x) \in \text{Gr}\, TS$ means that λ is an admissible transfer rate at the point x. λ defines a fictitious transfer game and one is led to the following definition.

[8] A ∂-separating game is ∂-balanced. See [8].
[9] The non-emptiness of the inner core is proved in games that cover the class of compactly generated and balanced-with-slacks games.

Definition 5 A pair $(\lambda \in \Sigma, x \in \partial W) \in \text{Gr } TS$ is said to be internally stable if:

$$(\lambda, x) \in \text{Gr } TS \text{ and } \lambda \cdot x \geq v_\lambda(N).$$

An allocation x is in the weak inner core of the game $(V_S, S \in \mathcal{N})$ if x belongs to the core of the game and there exists at least one $\lambda \in \Sigma$ such that (λ, x) is internally stable.

Suppose the players can transfer utility at a prescribed rate λ. The pair $(\lambda, x) \in \text{Gr } TS$ is not internally stable if each player can get a strictly better payoff in the fictitious λ-transfer payoff set V_N^λ. Denote N_{V_N} the normal cone of convex analysis of the set V_N.

Corollary 4 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1 and H2. Suppose also that V_N is a convex set. If it is dependent balanced with respect to the transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, N_{V_N} \cap \Sigma)$, then the weak inner core is non-empty.*

We omit the proof of the last result which is a direct application of Theorem 1. To apply Theorem 1, it suffices to notice that the transfer rate mapping $\psi = N_{V_N} \cap \Sigma$ satisfies the conditions of Definition 2(i).

3 A concluding remark

As a related topic, we want to mention that a recent stream of research defines the notion of extended core, see Gomez[4] and Keiding and Pankratova [7]. The problematic is the following: in the case of an empty core, which feasible allocations should be considered as potential candidates for guaranteeing the stability? Briefly, the construction of the extended core consists in blowing up the feasible payoff set of the grand coalition to get at least an allocation in the core of the extended game. Then, two tools based on mechanisms, that either downsize/select the core or blow up the payoffs are now available. They provide a rich articulation around the core concept.

Appendix

Proof of Corollary 2. Let define the induced coalitional game $(V_S, S \in \mathcal{N})$ where for each $S \in \mathcal{N}$, $V_S = \cup_{I\in I^S} v_I$. Let us normalize the power mappings by setting $\overline{p}(I) = \frac{p(I)}{\sum_{i\in S} p_i(I)}$. Now, define the transfer rates rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$. For each $x \in \partial V_N$, $\psi(x) = m^N$ and for all $S \in \mathcal{N}$ and all $x \in \partial V_S$, $\varphi_S(x) = \text{co}\{\overline{p}(I) \mid I \in \mathcal{I}(x) \cap I^S\}$. It is now routine to check that the assumptions of Theorem 1 are all fulfilled (Assumption SSG is actually a special case of dependent balancedness).

Then there exists a core allocation (for the game $(V_S, S \in \mathcal{N})$) x such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset$. It can be rewritten as: there exist, for all

$S \in \mathcal{S}(x)$, $\rho_S \in \mathbb{R}_+$, $b_S \in \varphi_S(x)$, such that $\sum_{S\in\mathcal{S}(x)} \rho_S = 1$ and $\sum_{S\in\mathcal{S}(x)} \rho_S b_S = m^N$. Furthermore, $b_S \in \varphi_S(x)$ is equivalent to: for each $I \in I^S$, there exist $\nu_I^S \in \mathbb{R}_+$ such that $\sum_{I\in\mathcal{I}(x)\cap I^S} \nu_I^S = 1$ and $b_S = \sum_{I\in\mathcal{I}(x)\cap I^S} \nu_I^S \overline{p}(I)$. If we consider for each $S \in \mathcal{N}$ and each $I \in I^S$, $\lambda_I = \frac{|N|\rho_S \nu_I^S}{\sum_{i\in S} p_i(I)}$, then one gets the result. Indeed, firstly we remark that:
$\sum_{S\in\mathcal{S}(x)} \rho_S \sum_{I\in\mathcal{I}(x)\cap I^S} \nu_I^S \overline{p}(I) = \sum_{I\in\mathcal{I}(x)} \sum_{\{S|I\in I^S\}} \rho_S \nu_I^S \overline{p}(I) = \sum_{I\in\mathcal{I}(x)} \frac{\lambda_I p(I)}{|N|} = m^N$, that is $\sum_{I\in\mathcal{I}(x)} \lambda_I p(I) = \mathbf{1}$. Secondly it is an easy matter to check that x is in the core of the game $(V_S, S \in \mathcal{N})$ if and only if x is in the core of the socially structured game $(N, \mathcal{I}, v, p)$ in the sense of Definition 2, as was to be proved. □

Proof of Corollary 3. First, we remark that, without any loss of generality, one can extend Assumption SM on the whole boundary of the set V_N since the core solution lies on the set of individually rational payoffs.

Lemma 3.1 *If the game satisfies the non levelness assumption (NL), the mappings $s_{k\ell}$ are non positive and continuous on ∂W.*

Proof of Lemma 3.1. First, remark that for each $x \in \partial W$, one has $x \in I_S$, so non-levelness applies. The non positivity is straightforward from Assumption H1. Furthermore, the mappings $s_{k\ell}$, $k, \ell \in N$, are well defined on ∂W (consider the coalition $T := \{k\}$). We now show the continuity of the mappings $s_{k\ell}$ which derives from Assumption NL. Let $k, \ell \in N$, $x \in \partial W$ and denote S^* the set of coalitions (satisfying $k \in S^*$ and $\ell \notin S^*$) maximizing $e_k(., x)$, i.e. $S^* := argmax\{e_k(S, x) \mid S \in \mathcal{N}, k \in S, \ell \notin S\}$; let x^ν be a sequence in ∂W converging toward x and denote, for each ν, S^ν the set of coalitions (satisfying $k \in S^\nu$ and $\ell \notin S^\nu$) maximizing $e_k(., x^\nu)$, i.e. $S^\nu := argmax\{e_k(S, x^\nu) \mid S \in \mathcal{N}, k \in S, \ell \notin S\}$. We first remark that, since S^* is a finite set there exists a real m such that for all $\nu \geq m$, $S^\nu \subseteq S^*$. Consider now some $T \in \mathcal{N}$ such that $T \in \mathcal{S}^\nu$ for each ν big enough (taking a subsequence if necessary), then from the definition of the mappings e_k, there exist two real numbers y_k and y_k^ν respectively solutions of $e_k(S, x)$ and $e_k(S^\nu, x^\nu)$ and satisfying $(y_k, x_{-k}) \in \partial V_T$ and $(y_k^\nu, x_{-k}^\nu) \in \partial V_T$. Now suppose we do not have the convergence, that is, there exists $\epsilon > 0$ such that $|y_k - y_k^\nu| > \epsilon$ for all ν sufficiently high. Then, taking the limit components by components, this contradicts assumption NL. Indeed, it implies that $(\lim_{\nu\to\infty} y_k^\nu, x_{-k})$ and (y_k, x_{-k}) belong to ∂V_T, but $|y_k - \lim_{\nu\to\infty} y_k^\nu| > \epsilon$, which is impossible. □

The mappings $s_{k\ell}$ are non positive on the boundary of the game ∂W and continuous from Lemma 3.1. Let $x \in \partial W \cap V_S$ for some $S \in \mathcal{N}$ with $j \in S$ and $i \notin S$, since: $argmax\{y_j - x_j \mid (y_j, x^{S\setminus\{j\}}) \in V_S\} = \{x_j\}$. We deduce that $s_{ji}(x) = 0$. Let $c_{ij}(x) := -p_j(p_W(x))s_{ji}(x)$. Obviously, from the assumption SM which guarantees the positivity and continuity of the mapping p, we deduce that, for each pair of players i and j, the mapping c_{ij}: $\partial W \to \mathbb{R}_+$ is continuous and satisfies: c_{ij} is zero on $V(S) \cap \partial W$ whenever $i \notin S$ and $j \in S$.

We deduce from Corollary 1 that there exists a core allocation x such that for each $i \in N$, $\sum_{j\in N}(c_{ij}(x) - c_{ji}(x)) = 0$.

Equivalently, remarking that $p_W(x) = x$ on ∂W, $\sum_{j \in N} p_i(x)s_{ij}(x) - p_j(x)s_{ji}(x) = f_i(x) = 0$. Hence, x is a core allocation that belongs to the average prekernel, as was to be proved. □

References

[1] E. Bennett, *Multilateral bargaining problems*, Games and Economic Behavior **19** (1997), 151–179.

[2] E. Bennett and W.R. Zame, *Bargaining in cooperative games*, International Journal of Game Theory **17** (1988), 279–300.

[3] J.-M. Bonnisseau and V. Iehlé, *Payoffs-dependent balancedness and cores*, Cahier de la MSE 2003.45, Université Paris 1, 2003.

[4] J.C. Gomez, *An extension of the core solution concept*, Working Paper 04.01., University of Copenhagen, 2003.

[5] P.J.J. Herings, G. van der Laan, and D. Talman, *Socially structured games and their applications*, CentER Discussion paper 2003-40, Tilburg University, 2003.

[6] M.O. Jackson, *A survey of models of network formation: stability and efficiency*, Networks, Clubs and Coalitions (G. Demange and M. Wooders, eds.), Cambridge University Press, Cambridge, 2003.

[7] H. Keiding and Y. Pankratova, *The extended core of a cooperative NTU game*, Research project, University of Copenhagen, 2004.

[8] G. Orshan, F. Valenciano, and J.M. Zarzuelo, *The bilateral consistent prekernel, the core and NTU bankruptcy problems*, Mathematics of Operations Research **28** (2003), 268–282.

[9] G. Orshan and J.M. Zarzuelo, *The bilateral consistent prekernel for NTU games*, Games and Economic Behavior **32** (2000), 67–84.

[10] A. Predtetchinski and P.J.J. Herings, *A necessary and sufficient condition for non-emptiness of the core of a non-transferable utility game*, Journal of Economic Theory **116** (2004), 84–92.

[11] C.-Z. Qin, *The inner core and the strictly inhibitive set*, Journal of Economic Theory **59** (1993), 96–106.

[12] ———, *The inner core of an N-person game*, Games and Economic Behavior **6** (1994), 431–444.

[13] P.J. Reny and M.H. Wooders, *The partnered core of a game without side payments*, Journal of Economic Theory **70** (1996), 298–311.

[14] R. Serrano and K.-I. Shimomura, *A comparison of the average prekernel and the prekernel*, working paper 01-30, Brown University, 2001.