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Joint Production Games with Mixed Sharing Rules

by

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Abstract We study Nash equilibria of joint production games under a mixed output sharing rule in which part of the output (the mixing parameter) is shared in proportion to inputs and the rest according to exogenously determined shares. This rule includes proportional sharing and equal sharing as special cases. We show that this game has a unique equilibrium and discuss comparative statics. When the game is large, players unanimously prefer the same value of the mixing parameter: the equilibrium value of the elasticity of production. For this value, equilibrium input and output are fully efficient. Our approach exploits the fact that payoffs in the joint production game are a function only of a player's input and the aggregate input and has independent interest as it readily extends to other "aggregative games".

Keywords Production externalities, non-cooperative games

J.E.L. Class C72, H42

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1 Introduction

Several procedures for distributing the output amongst the owners of a common property technology have been proposed. Under the average-return procedure each owner receives a share of the output equal to their share of the aggregate input. This widely studied sharing rule may be chosen on the basis of ethical principles or may simply be an inevitable consequence of free access to a jointly owned resource. In the latter case, with free entry, we are led to the familiar tragedy of the commons. More generally, if the technology exhibits decreasing returns to scale, there is over-production in equilibrium: a Pareto improvement can be achieved by reducing inputs. A polar opposite case is the equal shares procedure in which the output is divided equally amongst all the owners. Here, there is under-production when the technology exhibits increasing returns to scale. This sharing rule is readily generalised to exogenous sharing in which each owner receives an exogenously determined proportion of the output independent of her input. Once again, under-production is characteristic of this procedure. Serial cost sharing was introduced by Moulin and Shenker [20]. Under decreasing returns, the unique Nash equilibrium of this sharing rule can also be achieved as the unique equilibrium of an output sharing variant. Serial output sharing has a number of desirable properties. Notably, if owners have identical preferences or the technology has constant returns to scale, the level of output under the serial equilibrium is efficient.

In this article, we study a mixture of the average and exogenous sharing rules. More specifically, the output is split into two piles. Each owner receives a share of the first pile in proportion to her input and an exogenously determined share of the second pile. We refer to the proportion of total output in the first pile as the mixing parameter. Evidently, (by continuity) there will be a value of the mixing parameter for which the level of total output is efficient. When preferences satisfy suitable conditions, principally convexity and binormality, the mixed sharing rule has a unique Nash equilibrium. The choice of mixing parameter is most easily resolved in large games. When there are many players, even if their preferences fall into distinct types, all players will agree (to first order) on the most preferred value of the mixing parameter: the equilibrium value of the elasticity of production. Furthermore, with this value of the mixing parameter, the Nash equilibrium will be fully efficient. This is true of the equilibrium allocation, not just the equilibrium input and output. When all players have the same preferences, the elasticity of production is the only value of the mixing parameter for which the equilibrium allocation is efficient.

To establish these results, we adopt a novel method of analysis which exploits the fact that the choice of inputs under a mixed sharing rule is an example of an 'aggregative' game. Shubik comments that such games "clearly have much more structure than a game selected at random. How this structure influences the equilibrium points has not yet been explored in depth" [Shubik [27], p.325]. Several authors have suggested ways of exploiting the aggregative structure of a game to simplify its analysis. This paper offers an approach that builds on but significantly extends some of this earlier work to provide a simple and powerful method of analyzing such issues as the existence, uniqueness and comparative static and asymptotic properties of equilibrium[1]. Our methodology, which uses what we call 'share functions', provides a tool which reduces the study of fixed points of multi-dimensional mappings, entailed by conventional approaches using best-response functions, to the analysis of real-valued functions of the aggregate input. This simplification permits a straightforward and natural proof of existence and uniqueness, and allows us to characterize comparative statics and obtain results for large games. This methodology can also be applied to many other aggregative games; this is discussed briefly in the Conclusion.

Section 2 presents a simple numerical example of joint production under a mixed sharing rule and is designed to illustrate the use of share functions to obtain and characterize equilibria. Section 3 extends the example to the general case with some mild restrictions on preferences and the production function. The existence and properties of share functions are derived and used to demonstrate the existence of a unique Nash equilibrium. The next section shows how comparative statics may be derived from the properties of share functions. Section 5 applies share functions to an analysis of asymptotic properties of large games. This enables us to establish the optimality properties of setting the mixing parameter equal to the elasticity of production. In Section 6, we use share functions to extend our results to games in which the disutility of inputs depends also on aggregate input as well individual input. This includes cost sharing games with mixed sharing rules. Section 7 concludes.

[1] A seminal contribution is Selten [26], whose approach is later exploited by Szidarovszky and Yakowitz [28]. Novshek [22], [23] extends the approach to establish existence of equilibrium in Cournot models. Phlips [25] and Wolfstetter [32] provide expositions of Selten's approach. Okuguchi [24] and Cornes, Hartley and Sandler [11] have used a similar approach to analyze existence and uniqueness of equilibrium in the aggregative games that arise in models of oligopoly and public good provision. Corchon [6], [7] exploits the aggregative structure of such games using a slightly different approach.

2 A numerical example

Consider a jointly owned resource in the form of a fishing ground. Each of the n owners chooses how much input in the form of labor to devote to catching fish. The total catch, X, depends on the aggregate level L of labor applied according to the homogeneous production function $X = \sqrt{L}$. Suppose the output is shared according to a rule in which a proportion $\lambda(>0)$ of the total output is consumed by player i in proportion to the input that she supplies. The remaining output is shared equally amongst all the players:

$$\frac{x_i}{X} = \lambda \frac{\ell_i}{L} + (1-\lambda)\frac{1}{n},$$

where $\ell_i[x_i]$ is the input supplied [output consumed] by i. For example, $\lambda = 1$ generates the standard 'open access resource' problem in which each player's share of total output equals that player's share of the total input whereas λ close to zero characterizes the 'equal shares' rule for distributing output. An increase in the mixing parameter, λ, reflects a move towards a more egalitarian distribution.

Player i has a linear utility function: $u_i(x_i, l_i) = x_i - a_i l_i$. The parameter a_i reflects the player's disutility of labor. Alternatively, player i may be thought of as a profit-maximizer facing output and input prices of 1 and a_i respectively. When $\lambda = 1$, this example is formally equivalent to a Cournot oligopoly game with an isoelastic inverse demand function and constant unit costs, but equivalence breaks down for more general preferences. We have chosen linear preferences because they lead to particularly simple functions in our expository example.

Writing $L_{-i} \equiv L - \ell_i$, player i's payoff takes the form:

$$u_i(x_i, l_i) = \lambda \ell_i (\ell_i + L_{-i})^{-1/2} + (1-\lambda)\frac{(\ell_i + L_{-i})^{1/2}}{n} - a_i \ell_i.$$

This is a strictly concave function of ℓ_i for any L_{-i}, so a positive best response $\widehat{\ell}_i$ to L_{-i} must satisfy the stationarity condition:

$$\lambda\left(\widehat{\ell}_i + L_{-i}\right)^{-1/2} - \frac{\lambda \widehat{\ell}_i}{2\left(\widehat{\ell}_i + L_{-i}\right)^{3/2}} + \frac{(1-\lambda)}{2n\left(\widehat{\ell}_i + L_{-i}\right)^{1/2}} - a_i = 0. \quad (1)$$

Also, $\widehat{\ell}_i = 0$ if and only if the left hand side of (1) is non-positive. For $L > 0$, writing $s_i(L)$ for $\widehat{\ell}_i/L$, we can rearrange the first-order conditions as:

$$s_i(L) = \max\left\{2 + \frac{1-\lambda}{\lambda n} - \frac{2a_i}{\lambda}\sqrt{L}, 0\right\}. \quad (2)$$

In addition, an economically meaningful solution must satisfy the requirement that $L_{-i} \geqq 0$, or $\widehat{\ell}_i \leqq L$. This can only be consistent with (1) if $L \geqq \underline{L}_i$, where

$$\underline{L}_i = \left[\frac{1 + n\lambda - \lambda}{2na_i} \right]^2 .$$

We refer to $s_i(L)$ as the *share function* of player i and note that its domain is the set $[\underline{L}_i, \infty)$.

Knowledge of the share function of every player in the game allows us to determine the set of Nash equilibria. This follows from the natural consistency requirement that L^N is an equilibrium value of L if and only if the aggregate share function, defined as $S(L) \equiv \sum_j s_j(L)$, equals unity when evaluated at $L = L^N$. The strategy profile corresponding to L^N is the vector ℓ^N, where $\ell_i^N = s_i\left(L^N\right) L^N$.

Example 2.1 *Suppose* $\lambda = 1/2$ *and* $(a_1, a_2, a_3) = (1/30, 1/20, 1/15)$. *The individual and aggregate share functions are drawn in Figure 1. The unique Nash equilibrium corresponds to the point* N *at which* $L^N = 121$ *and the corresponding strategy profile is* $(13/15, 2/15, 0)\, L^N \cong (105, 16, 0)$ *(to the nearest integer values).*

For a second example, we leave the mixing parameter and the number of players unspecified.

Example 2.2 *Suppose* n *is even and* $a_i = 1$ *for half the players,* $a_i = 2$ *for the remainder. In this case, it is readily checked that the share function for players of the second type* ($a_i = 2$) *takes the value* 0 *at the lower bound* ($\underline{L}_i$) *of the domain of the share function of players of the first type. Hence, only players of the first type contribute positive input in equilibrium whereas the second type free rides on this input. The equilibrium value of* L *can be found by solving* $s_i(L) = 2/n$ *with* $a_i = 1$ *to find*

$$L = \left[\lambda \left(1 - \frac{3}{2n} \right) + \frac{1}{2n} \right]^2 \tag{3}$$

and $\ell_i = 2L/n$ *if* $a_i = 1$ *whereas* $\ell_i = 0$ *if* $a_i = 2$. *Note that when* $n = 2$, *the sole player with* $a_i = 1$ *supplies the whole input.*

Existence and uniqueness of Nash equilibrium follow from simple qualitative properties of the share functions. From (2) we see that $s_i(L)$ has a closed, semi-infinite domain of the form $[\underline{L}_i, \infty)$, it is continuous and strictly decreasing wherever positive, it equals zero for all large enough L and it equals 1 at $\underline{L}_i$. All these properties are inherited by the aggregate share function, except the last: $\sum_j s_j(L)$ either equals or exceeds 1 at $\underline{L}_i$. The deduction that there is a unique Nash equilibrium for any number of players is immediate.

We can also use share functions to obtain comparative static results. Suppose a_1 falls in Example 2.1. From (2), we see that the graph of the aggregate share function in Figure 1 moves to the right and its new value at the original equilibrium L exceeds unity. Consequently the equilibrium value of L increases. To examine the effect of this increase on player 2 in equilibrium, we can substitute the share function into the payoff function to obtain

$$u_2\left(\left\{\frac{s_2(L)}{2}+\frac{1}{6}\right\}\sqrt{L}, s_2(L)\,L\right)=\frac{4\sqrt{L}}{3}-\frac{13}{60}L+\frac{L^{3/2}}{100},$$

provided L does not reach $136\frac{1}{9}$. The latter is the value at which player 2's share function reaches the axis, above which she contributes no input. Differentiating this expression with respect to L gives

$$\frac{2}{3\sqrt{L}}-\frac{13}{60}+\frac{3\sqrt{L}}{200},$$

which is positive for all L exceeding the current equilibrium value of L. The nature of player 3's share function means that she remains inactive when L increases so her payoff is, we have

$$u_3\left(\frac{\sqrt{L}}{6},0\right)=\frac{\sqrt{L}}{6},$$

which is obviously increasing in L. Note that, this is also the formula for player 2's payoff if L exceeds $136\frac{1}{9}$ and we may conclude that both of player 1's rivals benefit from the fall in a_1.

Share functions are also useful tools for the derivation of results for large games. Consider Example 2.2 and suppose that n is large. Then (3) shows that L is approximately λ^2 and, using the same approximation, the payoff of a player i for which $a_i = 1$ is

$$\left\{\frac{2\lambda}{n}+\frac{1-\lambda}{n}\right\}\sqrt{L}-\frac{2}{n}L=\frac{\lambda-\lambda^2}{n}.$$

Similarly, we find that the payoff for the remaining players is

$$\frac{1-\lambda}{n}\sqrt{L}\cong\frac{\lambda-\lambda^2}{n}.$$

Note that all players (of both types) agree on their preferred value of the mixing parameter; payoffs are maximised at $\lambda = 1/2$. The fact that this is equal to the exponent of L in the production function is not coincidental as we shall see in Section 5 where we also show that similar conclusions apply even when payoffs are not linear.

3 The output sharing game: existence and uniqueness

In this section we generalise the example discussed in Section 2. Suppose that the total output X that is available for consumption by n players depends upon the aggregate level of labor, L, that is applied to the resource via a production function $F(L)$ exhibiting diminishing returns to labor. We consider an output-sharing rule that generalizes the sharing rule in Section 2. If player i supplies labor input ℓ_i, she receives the quantity x_i, where

$$x_i = \gamma_i(\ell_i, L) = \left\{\lambda\frac{\ell_i}{L} + (1-\lambda)\,\theta_i\right\} F(L). \tag{4}$$

Here λ is an exogenous mixing parameter satisfying $0 \leqq \lambda \leqq 1$ and the θ_i's are positive exogenous weights satisfying $\sum_{j\in I}\theta_j \leqq 1$. In Section 2, we assumed equal shares: $\theta_i = 1/n$ for all i, but the generalisation to arbitrary θ_i allows us, for example, to single out some more deserving players for receipt of more than equal share of that part of the total output which is distributed exogenously. Note also that we now allow the mixing parameter to take the value zero, which we label the "exogenous shares" case.

Player i's preferences are represented by a utility function $u_i(x_i, \ell_i)$. If $L > 0$, player i's payoff to strategy profile $(\ell_1, \ldots, \ell_n)$ is $u_i(x_i, \ell_i)$ where x_i is determined by (4). Otherwise it is $u_i(0,0)$. We make the following assumptions:

A.1(Preferences) Player i's utility function $u_i(x_i, \ell_i)$ is quasi-concave, locally non-satiable, increasing in x_i, decreasing in ℓ_i, continuous and continuously differentiable[2] for $x_i, \ell_i > 0$. Both x_i and ℓ_i are normal.

A.2(Technology) The production function $F(L)$ is increasing, strictly concave, continuous and continuously differentiable for $L > 0$, and $F(0) = 0$.

[2]Watts (1996) does not assume that utility functions are differentiable. However, we make this assumption purely for expository reasons. The proofs in the sequel go through *mutatis mutandis* in the nondifferentiable case if $\zeta_i(\sigma_i, L)$ is interpreted as the slope of a separating line to player i's upper preference set at $(x_i, l_i) = (\sigma_i F(L), \sigma_i L)$.

A.3(Boundedness) There exists a value of $L > 0$ such that $u_i(F(L), L) \leqq u_i(0,0)$.

The first two assumptions are standard. Our characterization of normality in A.1 follows Watts [30]. Suppose that the allocation (x_i', ℓ_i') is in player i's demand set when the budget set is $px_i - w\ell_i \leqq m_i'$. Both goods are normal if, for any $m_i'' > m_i'$, the demand set associated with the budget set $px_i - w\ell_i \leqq m_i''$ contains at least one point (x_i'', ℓ_i'') such that $x_i'' \geqq x_i'$ and $\ell_i'' \leqq \ell_i'$. Bearing in mind the fact that ℓ_i is a 'bad' this implies that, if preferences are strictly convex, all income expansion paths in (x_i, ℓ_i) space are downward-sloping. Assumption A.3 says that the indifference curve through the origin cannot lie entirely below the graph of the production function when ℓ_i is measured along the horizontal axis and x_i along the vertical axis.. This leaves two possibilities. The whole indifference curve may lie on or above the production function. If a player with such preferences had exclusive access to the lake she would choose not to fish. *A fortiori* this is the case if there are other players. Alternatively, the curves cross for some positive L and a monopoly owner of the resource would supply a positive but finite labor input. Sufficient conditions for A.3 are (a) $F'(L) \to 0$ as $L \to \infty$ or (b) the MRS of the indifference curve through the origin is unbounded. In particular, an upper bound imposed on the input of player i corresponds to vertical indifference curves at $\ell_i = \overline{\ell}_i$, so that (b) is satisfied.

It will prove convenient to write the first order conditions for best responses in terms of L and player i's share of total input at any allocation, $\sigma_i = \ell_i / L$. Because of the dependence of x_i on ℓ_i and L, both player i's payoff and also her marginal rate of substitution between x_i and ℓ_i can be written as functions of σ_i and L. In particular, we use $\zeta_i(\sigma_i, L)$ to denote the MRS evaluated at

$$x_i = \{\lambda \sigma_i + (1 - \lambda) \theta_i\} F(L) \tag{5}$$

and $\ell_i = \sigma_i L$. Note that an increase in either σ_i or L cannot lead to a decrease in either x_i or ℓ_i. Hence, by Assumption A1 and the subsequent discussion, the MRS cannot decrease.

Lemma 3.1 *Player i's marginal rate of substitution: $\zeta_i(\sigma_i, L)$ is a non-decreasing function of σ_i for fixed $L > 0$ and of L for fixed σ_i.*

Now consider the response of x_i to a change in ℓ_i when the input levels of all other players are taken as given. This response, which is i's marginal rate of transformation of input into consumption, can also be expressed as a function of σ_i and L. Holding all other players' input levels fixed and differentiating (5) with respect to ℓ_i, we obtain the following expression for i's marginal rate of transformation as a function of σ_i and L:

$$\frac{\partial x_i}{\partial \ell_i} = \{\lambda\sigma_i + (1-\lambda)\,\theta_i\}\,F'(L) + \lambda\,[1-\sigma_i]\,\frac{F(L)}{L} \equiv \tau_i(\sigma_i, L)\,. \qquad (6)$$

Our analysis will exploit the following properties of the MRT.

Lemma 3.2 *Player i's marginal rate of transformation: $\tau_i(\sigma_i, L)$ is a strictly decreasing function of L for fixed σ_i and of σ_i for fixed $L > 0$, provided $\lambda > 0$.*

Proof. For fixed σ_i, the conclusion follows immediately from Assumption A.2. This assumption also implies that $F'(L) < F(L)/L$. For fixed $L > 0$, increasing σ_i places more weight on $F'(L)$ and less on $F(L)/L$ and the value of the convex combination of marginal and average products must fall. ■

The first-order conditions for $\widehat{\ell}_i$ to be a best response to $L_{-i} = \sum_{j\in I, j\neq i} \ell_j$ can be written

$$\zeta_i(\widehat{\sigma}_i, L) \;\geqq\; \tau_i(\widehat{\sigma}_i, L) \qquad (7)$$

$$[\zeta_i(\widehat{\sigma}_i, L) - \tau_i(\widehat{\sigma}_i, L)]\,\widehat{\sigma}_i \;=\; 0 \qquad (8)$$

where $L = \widehat{\ell}_i + L_{-i}$ and $\widehat{\sigma}_i = \widehat{\ell}_i/L$. Note that

$$\frac{\partial^2}{\partial \ell_i^2}\gamma(\ell_i, \ell_i + L_{-i}) = \frac{2\lambda L_{-i}}{L^2}\left[F'(L) - \frac{F(L)}{L}\right] + \left\{\lambda\frac{\ell_i}{L} + (1-\lambda)\,\theta_i\right\}F''(L) < 0$$

for $\ell_i > 0$. This means that the sharing rule $\gamma_i(\ell_i, \ell_i + L_{-i})$ is a strictly concave function of ℓ_i. Since u_i is quasiconcave, conditions (7) and (8) are necessary and sufficient. Recall from Section 2, that the value of the share function of player i at L is equal to $\widehat{\sigma}_i$ where $\widehat{\ell}$ is a best response to $L_{-i} = L - \widehat{\ell}_i$. To establish that a share function is well-defined, we must demonstrate that these conditions have a unique solution in $\widehat{\sigma}_i$. We start by ruling out multiple solutions in the following proposition, the proof of which is an immediate consequence of Lemmas 3.1 and 3.2.

Proposition 3.1 *If $\lambda > 0$ and $L > 0$, there is at most one $\widehat{\sigma}_i$ satisfying (7) and (8).*

The case $\lambda = 0$ may be accommodated by a slight strengthening of our assumptions. In this case, τ_i is a constant function of σ_i for any L, but the proposition would still hold provided ζ_i were *strictly* increasing in σ_i. This would follow from a slightly stricter interpretation of normality in which all income expansion paths were strictly downward sloping. We shall refer to this by describing player i's preferences as *strictly normal.* Observe, however, that strict normality would rule out the linear utility function considered in the example in Section 2.

Figure 2 summarizes the reasoning that establishes Proposition 3.1. Consider a particular value of L, say L^0 in the figure. Suppose that there is a share, σ_i^0, such that $\sigma_i^0 L^0$ is a best response to $(1 - \sigma_i^0)L^0$. If σ_i^0 is strictly positive, it is characterized by equality of $\zeta_i(.)$ and $\tau_i(.)$. By demonstrating that, for any such given value of L, $\zeta_i(.)$ is nondecreasing in σ_i and $\tau_i(.)$ is everywhere increasing in σ_i, we have established the uniqueness of such a share. For any $L > 0$ for which conditions (7) and (8) have a unique solution, we will write $s_i(L)$ for that solution. Note that Proposition 3.1 does not constrain the domain of s_i. However, the next result shows that the domain is a semi-infinite interval (if non-empty) as well as drawing attention to a useful property of share functions.

Proposition 3.2 *Suppose $\lambda > 0$, $L^1 > L^0 > 0$ and $s_i(L^0)$ exists. Then $s_i(L^1)$ exists and $s_i(L^1) \leqq s_i(L^0)$. This inequality is strict if $s_i(L^0) > 0$.*

Proof. From Lemmas 3.1 and 3.2 and the first-order conditions, we have

$$\zeta_i(s_i(L^0), L^1) \geqq \zeta_i(s_i(L^0), L^0) \geqq \tau_i(s_i(L^0), L^0) > \tau_i(s_i(L^0), L^1).$$

Hence, either (a) $\zeta_i(\sigma_i, L^1) > \tau_i(\sigma_i, L^1)$ for all $\sigma_i \in (0, 1)$, or (b) $\zeta_i(\widehat{\sigma}_i, L^1) = \tau_i(\widehat{\sigma}_i, L^1)$ for some $\widehat{\sigma}_i$. In case (a), $s_i(L^1) = 0 \leqq s_i(L^0)$ and the inequality is strict if $s_i(L^0) > 0$. In case (b), since ζ_i is non-decreasing in L and τ_i is strictly decreasing in L, we have $s_i(L^1) < s_i(L^0)$. Note that (b) can only occur if $s_i(L^0) > 0$. ■

(Again, if $\lambda = 0$, Proposition 3.2 is valid under an assumption of strictly normal preferences.) Figure 2 summarizes the argument. Briefly stated, an increase in the value of L from L^0 to L^1 shifts the graph of $\zeta_i(., L)$ up and that of $\tau_i(., L)$ down. Therefore, if player i's most preferred share is initially positive, it must fall.

Nothing in our argument to this point excludes the possibilities that the share function has empty domain, exhibits downward jumps or has a strictly positive limit as $L \to \infty$, any of which would pose difficulties for the existence of a Nash equilibrium. Fortunately, Proposition 3.3 rules out such pathological behavior:

Proposition 3.3 *If Assumption A.3 holds, either $s_i(L) = 0$ for all $L > 0$, or, for any $\sigma_i \in (0, 1)$, there is a value of L satisfying $s_i(L) = \sigma_i$.*

Proof. Figure 3 provides a justification of this proposition. The graph marked $X = F(L)$ represents the aggregate technology. Choose any $\sigma_i \in (0,1)$ and take a point on the graph of $X = F(L)$, such as A. The point a is the point on the ray OA with the property that $Oa/OA = \sigma_i$. As A moves along the graph of $F(L)$, the point a traces out the graph of the function $x_i = \sigma_i F(\ell_i/\sigma_i)$ for the chosen value of σ_i. Note that the slope of the graph through a equals that of the graph through A: $\partial x_i(\ell_i, \sigma_i)/\partial \ell_i = F'(L)$ evaluated at $\ell_i/\sigma_i = L$. It follows that both graphs share a common tangent at the origin.

If Assumption A.3 holds, either $F'(0) \leqq s_i(0,0)$ and we can deduce that $s_i(L) = 0$ for all $L > 0$ or [the case drawn in the figure] a point of intersection I exists, at some positive value of L, between the graph of $F(.)$ and player i's indifference curve through the origin. Hence, the graph of $x_i = \sigma_i F(\ell_i/\sigma_i)$ intersects this indifference curve at some point S in the positive orthant. Conditions (7) and (8) are satisfied with $\widehat{\sigma}_i = \sigma_i$ at some point on the graph between 0 and S. This follows from the observation that at S, player i's marginal rate of substitution exceeds both the marginal product and also the average product. It therefore exceeds their convex combination. At the origin, the opposite is true. Continuity ensures that equality holds for some intermediate value of L. ■

These propositions ensure that (a) $s_i(.)$ is continuous[3], (b) it is strictly decreasing where positive, (c) it approaches or equals 0 as $L \longrightarrow \infty$. Finally, either $s_i(L) = 1$ for some L or $s_i(L) = 0$ for all $L > 0$. If the latter holds for all i, then $L = 0$ is the only Nash equilibrium. Otherwise, $S(L) = \sum_{j=1}^{n} s_j(L) \geqq 1$ for some L and satisfies (a), (b) and (c). Consequently there is a unique $L^* > 0$ satisfying $S(L) = 1$. This establishes the next theorem.

Theorem 3.4 (Existence and Uniqueness) *Given assumptions A.1 - A.3, the surplus sharing game has a unique Nash equilibrium for $\lambda > 0$.*

As we have seen above, the case $\lambda = 0$ is easily admitted by requiring strict normality in Assumption A.1.

Figure 1, although derived for a particular numerical example, also serves to summarize our argument to this point. It shows the graphs of the individual share functions associated with a 3-player game. Each is continuous and strictly decreasing for positive values of the share, and the figure shows each reaching zero at some finite value of L. The aggregate share function, $S(L)$, inherits these properties. Consequently, there exists a unique value, L^N, at which $S(L^N) = 1$.

[3] A discontinuous, non-increasing function would have to exhibit downward jumps and these are ruled out by Proposition 3.3.

4 The output sharing game: comparative statics

Share functions can also be used to study comparative statics. To guarantee the existence and desirable properties of share functions, we assume, without explicit statement, that A.2 is satisfied and that A.1 and A.3 hold for all players. We also exclude players with a share function equal to zero for all $L > 0$. (When $\lambda = 0$, we additionally assume that preferences are strictly normal.) Any change in the game, for example, more or fewer players, modification of the strategy sets, or an alteration of the payoffs, has two effects on players. There will often be a direct effect (holding levels of input and output fixed) on the payoffs of some or all of the players. However, the altered payoffs will lead to new inputs and outputs leading to a secondary effect on payoffs. In particular, the aggregate share function will shift, altering the equilibrium L. Define $\omega_i(\sigma_i, L) \equiv u_i(x_i, \sigma_i L)$, where x_i is determined by the sharing rule (4). In equilibrium, player i's payoff is $\omega_i[s_i(L), L]$ where $s_i(L)$ is the player's share function. A change in other players' payoff functions or the advent of new players will change equilibrium L and the sign of the slope of ω_i can be used to sign the effect of such a change on player i's equilibrium payoff.

Recall that the domain of the share function for player i is $[\underline{L}_i, \infty)$ and that $s_i(\underline{L}_i) = 1$. Furthermore, either $s_i(.)$ is positive and strictly decreasing throughout its domain or it is strictly decreasing up to some finite value $\overline{L}_i$ [to which we refer as player i's *dropout value*] and takes the value zero for larger L. In the former case, we set $\overline{L}_i = \infty$. It is also convenient to write $\eta(L)$ for the elasticity of production, $LF'(L)/F(L)$.

Theorem 4.1 *Suppose $s_i(L)$ exists if and only if $L \geqq \underline{L}_i$ and is positive if and only if $\underline{L}_i \leqq L < \overline{L}_i$.*

(i) When $\overline{L}_i$ is finite, $\omega_i(s_i(L), L)$ is strictly decreasing for $L \geqq \overline{L}_i$ if $0 \leqq \lambda < 1$ and constant if $\lambda = 1$.

(ii) Let $\underline{L}_i < L < \overline{L}_i$. Then $\omega_i(s_i(L), L)$ is a strictly increasing [decreasing] function in a neighborhood of L if $\Delta_i(s_i(L), L) > 0$ [< 0], where

$$\Delta_i(\sigma, L) \equiv (1-\lambda)\,\eta(L)\,\theta_i - \lambda\,\{1-\eta(L)\}\,\sigma. \tag{9}$$

Proof. (i) If $L \geqq \overline{L}_i$, $s_i(L) = 0$, so

$$\omega_i(0, L) = u_i((1-\lambda)\,\theta_i F(L)\,, 0)$$

and (i) is immediate.

(ii) By definition, for active players,

$$\omega_i(s_i(L), L) = \max_{\ell_i \geqq 0} u_i\left(\gamma_i\left(\ell_i, L - Ls_i(L) + \ell_i\right), \ell_i\right), \tag{10}$$

where the maximum is achieved at $\ell_i = Ls_i(L) > 0$. The first-order conditions for an interior solution together with our assumptions on differentiability imply that the share function is differentiable at L. Applying the envelope theorem,

$$\frac{\partial \omega_i(s_i(L), L)}{\partial L} = \frac{\partial u_i\left(\gamma_i\left(.\right), \ell_i\right)}{\partial x_i} \frac{\partial \gamma_i\left(\ell_i, L\right)}{\partial L} \left\{1 - s_i(L) - Ls_i'(L)\right\}.$$

Since $\partial u_i / \partial x_i > 0$, $s_i(L) < 1$ and $s_i'(L) \leqq 0$, we have

$$\text{sign} \frac{\partial \omega_i(s_i(L), L)}{\partial L} = \text{sign} \frac{\partial \gamma_i\left(\ell_i, L\right)}{\partial L}.$$

The proof is completed by noting that $F\Delta_i / L = \partial \gamma_i / \partial L$. ■

From (i), inactive players never lose from an increase in L and strictly gain if $\lambda > 0$. To determine the effect on active players we must sign Δ_i and this can be done unambiguously for extreme values of λ. In particular, if $\lambda = 0$, then $\Delta_i(\sigma, L) = \theta_i \eta(L) > 0$ for all L. By contrast, if $\lambda = 1$, then $\Delta_i(\sigma, L) = -\sigma\{1 - \eta\left(L\right)\} < 0$ for all L. More generally, if $\lambda < 1$, then Δ_i has the same sign as

$$\frac{\eta(L)\theta_i}{1 - \eta(L)} - \frac{\lambda \sigma}{1 - \lambda}.$$

Hence, if $\eta - \lambda$ and $\theta_i - \sigma$ share the same sign, that sign is also shared by Δ_i. The following corollary summarises these observations.

Corollary 4.2 *Player i's equilibrium payoff increases if*

(i) $\lambda = 0$ *(exogenous shares), and* L *increases,*

(ii) $\lambda = 1$ *(proportional shares), player* i *is active and* L *decreases,*

(iii) $0 < \lambda < 1$, $\eta(L_0) > \lambda$, $s_i\left(L_0\right) < \theta_i$ *and* L *increases slightly from* L_0,

(iv) $0 < \lambda < 1$, $\eta(L_0) < \lambda$, $s_i\left(L_0\right) > \theta_i$, *player* i *is active and* L *decreases slightly from* L_0,

(v) $0 \leqq \lambda < 1$, *player* i *is inactive and* L *increases.*

By continuity, the conclusion for $\lambda = 0$ [$\lambda = 1$] continues to hold when $0 < \lambda < 1$ provided λ is small [large] enough. Parts (iii) and (iv) continue to hold even if the one of the inequalities is weak. Also, the restriction to small changes in L in these parts reflects the fact that $\eta(L)$ is not necessarily a monotonic function of L. However, if the production function has constant elasticity: $F(L) \propto L^\alpha$ and the mixing parameter is equal to the elasticity: $\lambda = \alpha$, then an active player benefits from an increase in L above L_0 if she is a *net beneficiary* at L_0: $s_i(L_0) < \theta_i$ since s_i is strictly decreasing. Similarly, *net contributors at* L_0 *(*players for which $s_i(L_0) > \theta_i$*)*, benefit from a decrease in L.

To exploit these results, we must examine how aggregate input changes in response to a modification of the game. Perhaps the easiest case to consider is a change in the number of players. If a game has at least three players and one of the active players leaves the game, the aggregate share function will shift left, strictly at the current equilibrium. Hence, L will fall and, under exogenous shares, all players will be worse off. Under proportional shares, active players will gain; inactive players are unaffected. If $F(L) \propto L^\lambda$, net beneficiaries lose and net contributors gain from exit of an active player. All these conclusions are predicated on the assumption that the weights in the sharing rule: θ_i do not change. But this means that some of the output is wasted in the reduced game. Under an alternative rule such as equal shares, $\theta_i = 1/n$, the direct effect on payoffs of a decrease in n is positive since θ_i rises. These two effects work in opposite directions under exogenous shares and determining the overall sign requires more detailed information on payoffs. This also applies to intermediate values of λ although, if elasticity is constant and equal to the mixing parameter, both effects have the same sign for net contributors who always benefit from exit.

Now suppose that each player is subjected to an exogenously set upper limit on the individual input level. This simply involves adding to the open access resource game a set of constraints of the form $\ell_i \leqq \ell_i^{\max}$ and has the effect of changing player i's share function from s_i to s_i^*, where

$$s_i^*(L) = \min\{s_i(L), \ell_i^{\max}/L\}.$$

We will say that the player i's quota $\ell_i^{\max}$ *strictly binds* if $\ell_i^{\max} < Ls_i(L)$ at the equilibrium value of L. If no quota binds strictly, the equilibrium is unchanged. Otherwise, imposing quotas shifts the aggregate share function down, strictly in equilibrium, leading to a strict decrease in equilibrium aggregate labor input. We may deduce that, under exogenous shares, all players are worse off. Note that this includes those players for whom the quota binds, the direct effect of a quota is always adverse and therefore reinforces the indirect effect. Abolition of quotas is Pareto improving. This is not the case with a proportional sharing rule, for active players benefit from the imposition of a quota which does not bind them but strictly binds at least one of their competitors. We cannot even rule out the latter benefiting, for the direct and indirect effects are in opposite directions. For intermediate values of the mixing parameter, typically some players will benefit, others lose. For example, if elasticity is constant and equal to the mixing parameter, net contributors are made better off whereas net beneficiaries are made worse off if a rival is subject to a strictly binding quota. The payoff of net beneficiaries suffering a strictly binding quota also falls.

Next, we consider technological changes in players' inherent productivities. This may be modelled by supposing that each unit of nominal input by player j generates e_j units of effective input, where e_j is an exogenous parameter, so that $\ell_j = e_j h_j$ for $j = 1, \ldots, n$. The variable h_j may be interpreted as the actual number of hours applied by player j to the productive activity. We assume that player i's preferences over output and hours satisfy Assumption A.1 and, slightly abusing notation, we write u_i for the corresponding utility function. This also allows us write the payoff and marginal rate of substitution as functions of share, aggregate input and the i's technological factor: $\omega_i(\sigma_i, L, e_i) \equiv u_i(x_i, \ell_i/e_i)$ and $\zeta_i(\sigma_i, L, e_i) \equiv f_i(x_i, \ell_i/e_i)$, where x_i satisfies the sharing rule (4) and $\ell_i = \sigma_i L$. In this formulation, the natural generalisation of A.3 is that there should exist an $L > 0$ satisfying $u_i(e_i F(L), L) \leqq u_i(0, 0)$. It is possible for this to be true, say, for $e_i = 1$ (the original A.3) but not for some $e_i > 1$, so throughout the remainder of the section, we shall assume that this generalisation of A.3 is true for all players and all relevant e_i.

Consider a technological shock which exogenously increases the inherent productivity, or ability, of a subset I' of players, while leaving that of the other players unaffected. Specifically, suppose e_i increases from e_i^1 to e_i^2 for all $i \in I'$. The effect on share functions can be seen by holding σ_i and L fixed. Then, player i enjoys the same value of x_i, and applies a lower level of h_i. Consequently, $\omega_i(\sigma_i, L, e_i)$ increases and $\zeta_i(\sigma_i, L, e_i)$ falls. If $\zeta_i(.)$ now falls short of $\tau_i(.)$, player i can further enhance her payoff by increasing her share σ_i. This means that $s_i(L, e_i^2) > s_i(L, e_i^1)$ - the increase in i's ability implies an upward shift in the graph of her share function in Figure 1, and therefore also in that of the aggregate share function. Provided that player i is active at the new equilibrium, the equilibrium level of L therefore increases. Under exogenous shares, this adds to the direct effect for players in I' and results in a Pareto improvement provided at least one of the players in I' is active. When output is shared proportionally, the change in payoffs for players in I' is ambiguous, direct and indirect effects working in opposite directions, whilst active players not in I' are made worse off. When elasticity is constant and equal to the mixing parameter, net beneficiaries benefit from the increases in productivity whether or not they are members of I' whereas the payoff of net contributors who are not in I' falls.

Finally, we investigate changes in the mixing parameter. If this increases there is a direct effect on all players: for fixed L, net contributors gain and net beneficiaries lose. To determine the secondary effect, we see from (6) that an increase in λ of, say, $\delta\lambda$ changes the MRT of player i for fixed σ_i and L by

$$\delta\tau_i(\sigma_i, L) = \delta\lambda\,[\sigma_i - \theta_i]\,F'(L) + \delta\lambda\,[1 - \sigma_i]\,\frac{F(L)}{L} > 0,$$

exploiting the fact that the average product exceeds the marginal product. Since the MRT as a function of σ_i shifts upwards and the MRS is unchanged, the intersection moves to the right. Thus share functions move to the right and equilibrium L increases. If $\lambda = 0$, a move to positive λ benefits net contributors; the effect on net beneficiaries depends on whether the loss of their share of the output is outweighed by the increase in aggregate output. Similarly, if $\lambda = 1$, a reduction in λ leads to a rise in the payoff of net beneficiaries[4]. For intermediate values of λ, no firm conclusions can be drawn, even if the production function has constant elasticity, say α. For, if $\lambda \leqq \alpha$, net beneficiaries gain from an increase in L by Part (iii) of the corollary but the direct effect is a loss for such players and the overall result is ambiguous. A similar conclusion holds when $\lambda \geqq \alpha$. Even inactive players, who benefit from an increase in L, lose from the direct effects of an increase in λ since inactive players are always net beneficiaries. Nevertheless, it turns out that when there are many players, approximately, payoffs of all players increase with λ up to α and decrease thereafter. In the next section, we justify this claim as part of an application of share functions to large games.

5 Large games

In previous sections, we have treated the number of players as exogenous. However, the presence of positive payoffs will attract potential entrants and, if costs of entry are small, we would expect to see many players. Results for large games are often sharper than those for smaller games. There are two reasons for this. Firstly, strategic effects are weakened in such games[5]

[4]For active players, this follows from part (ii) of Corollary 4.2. If i is inactive, $\Delta_i(0, L) = 0$ when $\lambda = 1$ and the direct effect is positive.

[5]Though not necessarily eliminated. See Cornes and Hartley [????] for analysis of this point for rent-seeking contests.

and, secondly, input and output are small, permitting the use of a linear approximation to the utility function. We shall analyse output sharing games as the number of players tends to infinity and, once again, share functions will provide the vital analytical tool, particularly when players are heterogeneous.

We make four further mild assumptions throughout this section in addition to A.1 - A.3. Firstly, we add to A.1 the assumption that the marginal rate of substitution at the origin is strictly positive for all players. Since, indifference curves are upward sloping by A.1, we are only ruling out the possibility that the slope falls to zero as the curve through the origin reaches it. This assumption allows us to approximate preferences in the neighbourhood of the origin by linear preferences which continue to satisfy A.1. Secondly, we add to A.2 the requirement that $F(L)/L \longrightarrow 0$ as $L \longrightarrow \infty$. Note that this restriction is satisfied if the production function is bounded, though we do not need to impose such a severe restriction. Concavity of F implies that the marginal product is less than the average product and therefore also approaches zero for large L. This assumption means that individual shares of outputs become small as the number of players becomes large. Finally, we assume that the graph of F has infinite slope at the origin exceeds the marginal rate of substitution at the origin. This assumption may appear to be more severely restrictive, though it does include the important special case of constant elasticity. However, our results are readily modified to cope with bounded marginal product and we briefly discuss these extensions at the end of the section. Finally, we assume that $\lambda > 0$.

We will analyse the limit as n, the number of players, approaches infinity. The analysis is simplest when all players have identical payoffs and an equal value of the exogenous sharing parameter θ_i. To ensure that we do not share out more than the total output, we must reduce θ_i as n increases to satisfy $n\theta_i \leqq 1$. A simple way of achieving this is to assume that, for all n, a proportion $\mu \leqq 1$ of the output set aside for exogenous sharing is distributed equally amongst the players: $\theta_i = \mu/n$. Theorem 3.4 shows that there is a unique equilibrium value of aggregate labor for each n, which we denote L^n, which satisfies $s_i(L^n) = 1/n$. At this value MRS and MRT must be equal:

$$f_i\left(\left\{\frac{\lambda}{n} + \frac{(1-\lambda)\mu}{n}\right\} F(L^n), \frac{L^n}{n}\right) = \left\{\frac{\lambda}{n} + \frac{(1-\lambda)\mu}{n}\right\} F'(L^n) + \lambda\left(1 - \frac{1}{n}\right)\frac{F(L^n)}{L^n}, \qquad (11)$$

where $f_i(x, \ell)$ is the MRS of player i evaluated at (x, ℓ). Under Assumption A.1, $f_i(x, \ell) \geqq f_i(0, 0)$ for all $(x, \ell) \geqq (0, 0)$. Under Assumption A.2,

$F' < F/L$ and we can deduce that $f_i(0,0) \leqq F(L^n)/L^n$. The additional assumptions made in this section (specifically $f_i(0,0) > 0$ and bounded average product) allow us to conclude that the sequence $\{L^n\}$ is confined to a bounded interval. From Lemma 8.1 in the Appendix, we may deduce that this sequence has a limit $\widetilde{L}$. (To apply the lemma, take $J = 2$, g_1 the difference between the LHS and RHS in (11) and $g_2 = -g_1$). Letting $n \longrightarrow \infty$ and $L^n \longrightarrow \widetilde{L}$ in (11), we have

$$f_i(0,0) = \frac{\lambda F\left(\widetilde{L}\right)}{\widetilde{L}}. \tag{12}$$

Note that the right hand side of (12), is decreasing in $\widetilde{L}$, exceeds $f_i(0,0)$ for $\widetilde{L}$ close enough to zero (since it approaches infinity) and is less than $f_i(0,0)$ for large enough $\widetilde{L}$ (since the LHS is positive and the RHS approaches zero for large $\widetilde{L}$). It follows that (12) has a unique solution for any $\lambda \in (0,1]$. Note also that the solution depends on the mixing parameter but not on the exogenous proportion, μ. Summarizing, we have the following result.

Lemma 5.1 *If all players are identical and $n \longrightarrow \infty$, then $L^n \longrightarrow \widetilde{L}$, the unique solution (12).*

There is a useful alternative interpretation of $\widetilde{L}$. Firstly, we show that, for any n, the share function reaches the axis. Recall that we refer to this as a players dropout value. We will write $\overline{L}^n$ for this value. Note, from Figure 2, that $\overline{L}^n$ satisfies $\zeta_i\left(0,\overline{L}^n\right) = \tau_i\left(0,\overline{L}^n\right)$ and we justify the assertion that $\overline{L}^n$ exists by proving that this equation has a solution. Observe that

$$\zeta_i(0,L) = f_i\left(\frac{(1-\lambda)\mu}{n}F(L),0\right)$$

and

$$\tau_i(0,L) = \frac{(1-\lambda)\mu}{n}F'(L) + \lambda\frac{F(L)}{L}.$$

Hence, as $L \longrightarrow 0$,

$$\tau_i(0,L) \longrightarrow \left[\frac{(1-\lambda)\mu}{n} + \lambda\right]F'(0)$$

using A.2. By our third additional assumption, this limit is infinite, so exceeds $\zeta_i(0,0)$. By A.1 and our first additonal assumption $\zeta_i(0,L) \geqslant \zeta_i(0,0) > 0$ for all $L > 0$ which means that $\tau_i(0,L) < \zeta_i(0,L)$ for large enough L. Since $\tau_i(0,L)$ and $\zeta_i(0,L)$ are both continuous, we may deduce that there is some value of L for which they are equal.

Our main claim is that $\overline{L}^n \longrightarrow \widetilde{L}$ as $n \longrightarrow \infty$. Similar arguments to those used for L^n shows that

$$0 < f_i(0,0) \leqslant \zeta_i\left(0,\overline{L}^n\right) = \tau_i\left(0,\overline{L}^n\right) \leqslant \frac{F\left(\overline{L}^n\right)}{\overline{L}^n}$$

and we may deduce that the sequence $\left\{\overline{L}^n\right\}$ is bounded. This allows us to apply Lemma 8.1 to deduce that it has a limit. Letting $n \longrightarrow \infty$ in $\zeta_i\left(0,\overline{L}^n\right) = \tau_i\left(0,\overline{L}^n\right)$ shows that the limit satisfies (12) and is therefore equal to $\widetilde{L}$.

We have assumed that the average product falls to zero, so it is not surprising that individual payoffs approach the reservation value, $u_i(0,0)$, in the limit. However, the behavior of aggregate payoff is less obvious. Nevertheless, we can use Lemma 5.1 to show that this generally has a finite limit. The aggregate excess payoff (over the reservation value) is

$$\begin{aligned}
n&\left[u_i\left(\left\{\frac{\lambda}{n}+\frac{(1-\lambda)\mu}{n}\right\}F(L^n),\frac{1}{n}L^n\right)-u_i(0,0)\right]\\
&\longrightarrow \frac{\partial u_i(0,0)}{\partial x_i}\left[\{\lambda+(1-\lambda)\mu\}F\left(\widetilde{L}\right)-\widetilde{L}f_i(0,0)\right]\\
&=\frac{\partial u_i(0,0)}{\partial x_i}(1-\lambda)\mu F\left(\widetilde{L}\right)\text{ as } n\longrightarrow\infty, \qquad (13)
\end{aligned}$$

where we have used (12) to obtain the final line. Under proportional sharing ($\lambda = 1$) the limit is zero: as the number of players increases, the surplus is fully competed away. When the mixing parameter is positive, the aggregate benefit of that portion of the output that is shared proportionally vanishes; what remains in the limit is the exogenously shared part which has a positive limit even in a large game.

To analyze the case when players differ, we start by considering the m-fold replication of a basic game with T distinct types of player. We write $\mathcal{G}^m$ for this game and use a subscript enclosed in parentheses to refer to players of type t. So we set $\theta_i = \mu_{(t)}/m$ if player i is of type t, where the $\mu_{(t)}$ are positive type-weights which satisfy $\sum_{t=1}^{T}\mu_{(t)} \leqq 1$. For each type there will be a level of labor input which satisfies (12). We write $\widetilde{L}_{(t)}$ for this value and note that it is the unique solution of $f_{(t)}(0,0) = \lambda F\left(\widetilde{L}_{(t)}\right)/\widetilde{L}_{(t)}$. Writing $L^m_{(t)}$ for the dropout value for type t in $\mathcal{G}^m$, we also have $L^m_{(t)} \longrightarrow \widetilde{L}_{(t)}$ as $m \longrightarrow \infty$. It is also convenient to define a 'limiting' share function for players of type t by letting $m \longrightarrow \infty$ in the first order conditions (7) and (8). We can write $\widetilde{s}_{(t)}$ for this function and note that these conditions can be written:

$$f_{(t)}\left(\lambda\widetilde{s}_{(t)}(L)F(L), L\widetilde{s}_{(t)}(L)\right) \geqslant \lambda\widetilde{s}_{(t)}(L)F'(L) + \lambda\left[1-\widetilde{s}_{(t)}(L)\right]\frac{F(L)}{L} \tag{14}$$

with equality if $\widetilde{s}_{(t)}(L) > 0$. Lemmas 3.1 and 3.2 still apply to the LHS and RHS of (14) respectively and allow us to conclude that $\widetilde{s}_{(t)}$ has the properties of a share function set out in Section 3 and, furthermore, has dropout value $\widetilde{L}_{(t)}$. We can also show that $\widetilde{s}_{(t)}$ is the pointwise limit of type-t share functions in $\mathcal{G}^m$:

Lemma 5.2 *If $L < \widetilde{L}_{(t)}$ and $s^m_{(t)}$ is the share function of players of type t in $\mathcal{G}^m$, then $s^m_{(t)}(L) \longrightarrow \widetilde{s}_{(t)}(L)$ as $m \longrightarrow \infty$.*

Proof. The proof is another application of Lemma 8.1. For any $\sigma \in [0,1]$, define

$$\begin{aligned}\nabla^m(\sigma) &= f_{(t)}\left(\left\{\lambda\sigma + (1-\lambda)\frac{\mu_{(t)}}{m}\right\}F(L), \sigma L\right) \\ &\quad -\lambda(1-\sigma)\frac{F(L)}{L} - \left\{\lambda\sigma + (1-\lambda)\frac{\mu_{(t)}}{m}\right\}F'(L).\end{aligned}$$

The first order conditions for $s^m_{(t)}$ to be the share function for players of type t in $\mathcal{G}^m$ are $\nabla^m\left(s^m_{(t)}(L)\right) \geqq 0$ and $s^m_{(t)}(L)\nabla^m\left(s^m_{(t)}(L)\right) = 0$ for all $L > 0$. Note that ∇^m approaches the left hand side of (14)) as $m \longrightarrow \infty$ and this has a unique solution $\widetilde{s}_{(t)}(L)$. Since the sequence $\left\{s^m_{(t)}(L)\right\}_{m=1}^{\infty}$ is also bounded between 0 and 1, all the requirements of the Lemma 8.1 are satisfied. (Replace n with m and x with σ in the Lemma. Then take $J = 3$, $g_1(\sigma) = -\nabla^m(\sigma)$, $g_2(\sigma) = \sigma\nabla^m(\sigma)$ and $g_3 = -g_2$.) The desired limit follows. ■

We can always label the types so that $\widetilde{L}_{(t)} \leqslant \widetilde{L}_{(T)}$ for all $t \neq T$. Consider such a type t such that $\widetilde{L}_{(t)} < \widetilde{L}_{(T)}$ and define $L^* = \left[\widetilde{L}_{(t)} + \widetilde{L}_{(T)}\right]/2$. Note that $\widetilde{s}_{(T)}(L^*) > 0$ which means that there is a positive integer $\overline{m}_1$ such that $m\widetilde{s}_{(T)}(L^*) \geqq 2$ for $m > \overline{m}_1$. The preceding lemma implies that there is a positive integer $\overline{m}_2$ such that $s^m_{(T)}(L^*) > \widetilde{s}_{(T)}(L^*)/2$ for all $m > \overline{m}_2$. Furthermore, convergence of dropout values implies that there is a positive integer $\overline{m}_3$ such that $L^m_{(t)} < L^*$ for all $m > \overline{m}_3$. Hence, if $m > \max\{\overline{m}_1, \overline{m}_2, \overline{m}_3\}$, we may conclude that $ms^m_{(T)}\left(L^m_{(t)}\right) > 1$, exploiting the fact that share functions are non-increasing. Hence, the equilibrium L in $\mathcal{G}^m$ exceeds $L^m_{(t)}$, the dropout value of players of type t. If aggregate input is greater than the dropout value of a type, no player of that type supplies positive labor input in $\mathcal{G}^m$; only players of type T participate in equilibrium.

Proposition 5.1 *If $\widetilde{L}_{(t)} < \widetilde{L}_{(T)}$ players of type t supply no labor input in $\mathcal{G}^m$ for all large enough m.*

This proposition shows that only players of types maximising $\widetilde{L}_{(t)}$ participate in the limit. For ease of exposition, we will assume throughout the rest of this section that there is a unique type maximising $\widetilde{L}_{(t)}$ and we choose labels so that $\widetilde{L}_{(t)} < \widetilde{L}_{(T)}$ for all $t \neq T$. Note that, in view of equation (12) this is equivalent to assuming $f_{(T)}(0,0) < f_{(t)}(0,0)$ for $t \neq T$. (Nevertheless, all substantial results continue to hold *mutatis mutandis* without this additional restriction.) With this assumption, the proposition shows that a large replicated game is essentially symmetric in that only type T actively participates in equilibrium and aggregate labor input approaches $\widetilde{L}_{(T)}$. Hence, we can use the formula (13) to derive the aggregate excess payoff for players of type T. Other types supply no labor once m is large enough, so the aggregate excess payoff is given by

$$m\left[u_{(t)}\left(\frac{(1-\lambda)\,\mu_{(t)}}{m}F\left(L_{(t)}^{m}\right),0\right)-u_{(t)}\left(0,0\right)\right].$$

This has the same limit as type T when $m \longrightarrow \infty$ and we summarise our conclusions in the following corollary.

Corollary 5.2 *The equilibrium payoff to players of type t in $\mathcal{G}^m$ for large m is given by*

$$u_{(t)}\left(0,0\right)+\frac{1}{m}\frac{\partial u_{(t)}\left(0,0\right)}{\partial x_i}\mu_{(t)}\phi+o\left(\frac{1}{m}\right),$$

where

$$\phi=\left(1-\lambda\right)F\left(\widetilde{L}_{(T)}\right) \tag{15}$$

and $\widetilde{L}_{(T)}$ is the solution of (12) for players of type T.

In order to implement the sharing rule, we have assumed that either individual inputs can be observed or players do not misreport their inputs. From Corollary 5.2, a player's first-order payoff depends on individual input only through aggregate input. Hence, if aggregate input is observable, players have no incentive to misrepresent individual input levels in large games even collusively.

The corollary also has an interesting consequence for the choice of mixing parameter in large games for, to first order in $1/m$, every player will agree on their most preferred value of λ, namely the value that maximises ϕ regarded as a function of λ. Differentiating (12) with respect to λ and rearranging the result leads to the expression:

$$\frac{d\widetilde{L}_{(T)}}{d\lambda} = \frac{\widetilde{L}_{(T)}}{\lambda\left[1-\eta\left(\widetilde{L}_{(T)}\right)\right]} > 0.$$

Differentiating (15), using the previous result and rearranging, we find:

$$\frac{d\phi}{d\lambda} = F\left(\widetilde{L}_{(T)}\right)\left\{\frac{\eta\left(\widetilde{L}_{(T)}\right)[1-\lambda]}{\lambda\left[1-\eta\left(\widetilde{L}_{(T)}\right)\right]} - 1\right\}.$$

We may deduce that $\lambda = \eta\left(\widetilde{L}_{(T)}\right)$ is the unique stationary point of ϕ. Furthermore, since $d\phi/d\lambda$ is positive [negative] if $\lambda <[>] \eta\left(\widetilde{L}_{(T)}\right)$, it is the maximand of ϕ. The unanimously preferred value of the mixing parameter in a large game is the equilibrium value of the elasticity of production. When this is constant, i.e. the production function is proportional to L^{α}, where $0 < \alpha < 1$, then all players prefer α to any other value of the mixing parameter.

If we suppose that the value of λ is chosen by some group decision process on the part of the players, a voting scheme consistent with unanimity will choose $\lambda = \eta\left(\widetilde{L}_{(T)}\right)$. More generally, if we suppose that the mixing parameter is subject to alteration through a process that respects unanimity we are led to dynamics of the form $\dot{\lambda} = \varphi(\lambda)$ where φ is positive [negative] if a small increase in λ results in a rise [fall] in payoffs of *all* players. Such a process has $\lambda = \eta\left(\widetilde{L}_{(T)}\right)$ as its unique stable solution.

Setting the mixing parameter to equilibrium elasticity is also efficient. It is well known that exogenous sharing ($\lambda = 0$) results in too little output whereas proportional sharing ($\lambda = 1$) leads to over-exploitation (see, for example, Moulin [19]). With $\lambda = \eta\left(\widetilde{L}_{(T)}\right)$, we will see that the unique Nash equilibrium of a large output sharing game is Pareto efficient to first order provided no output is wasted: $\sum_{t=1}^{T} \mu_{(t)} = 1$.

First, observe that, since players of the same type have the same share function, all such players enjoy the same input and output in equilibrium;

we call such allocations *type-symmetric*. Quasiconcavity of utility functions implies that, if a type-symmetric allocation is dominated by some other allocation, it is dominated by the type-symmetric allocation found by giving all players the average input and output for their type. This means we only need consider re-allocations between types. Writing $L_{(t)}$ for the aggregate input and $X_{(t)}$ the aggregate output of players of type t, we can take the type-t payoff to be the limiting aggregate excess, $\psi_{(t)}$, where

$$\begin{aligned}\psi_{(t)} &= \lim_{m\longrightarrow\infty} m\left[u_{(t)}\left(\frac{X_{(t)}}{m},\frac{L_{(t)}}{m}\right)-u_{(t)}(0,0)\right]\\ &= \frac{\partial u_{(t)}(0,0)}{\partial x_i}\left[X_{(t)}-L_{(t)}f_{(t)}(0,0)\right].\end{aligned}$$

We can establish efficiency by showing that the limiting equilibrium allocation maximises a positive weighted combination of the $\psi_{(t)}$. Writing $\left\{\alpha_{(t)}\right\}_{t=1}^{T}$ for the weights, note that, if we set $\alpha_{(t)}=\left[\partial u_{(t)}(0,0)/\partial x_i\right]^{-1}$ for each t, we have

$$\Psi\equiv\sum_{\tau=1}^{T}\alpha_{(\tau)}\psi_{(\tau)}\left(L_{(1)},\ldots,L_{(T)}\right)=F(L)-\sum_{\tau=1}^{T}L_{(\tau)}f_{(\tau)}(0,0).$$

Here, we have written $L=\sum_{t=1}^{T}L_{(t)}$ and used the feasibility requirement $\sum_{t=1}^{T}X_{(t)}=F(L)$. Since Ψ is a concave function of $\left(L_{(1)},\ldots,L_{(T)}\right)$, it is maximised at the equilibrium allocation, $L_{(t)}=0$ for $t\neq T$ and $L_{(T)}=\widetilde{L}_{(T)}$, if

$$\frac{\partial\Psi}{\partial L_{(t)}}=F'\left(\widetilde{L}_{(T)}\right)-f_{(t)}(0,0)\leq 0,$$

holds with equality for $t=T$. Note that equation (12) is valid for players of type T. If $\lambda=\eta\left(\widetilde{L}_{(T)}\right)$, then $F'\left(\widetilde{L}_{(T)}\right)=f_{(T)}(0,0)<f_{(t)}(0,0)$ for $t\neq T$, so these conditions are satisfied. This establishes Pareto efficiency.

Note that these results are independent of the exogenous weights $\left\{\mu_{(t)}\right\}_{t=1}^{T}$. However Corollary 5.2 shows that the choice of weights does affect the payoffs of particular types. More specifically, to first order in a large game, there is a unique efficient level of input: $\widetilde{L}_{(T)}$ provided by players of type T with no input from other types. Any distribution of the output $F\left(\widetilde{L}_{(T)}\right)$ results in an efficient allocation and this constitutes the set of all such allocations. Any efficient allocation which gives every player a non-negative payoff (individual rationality) is an equilibrium outcome for some choice of exogenous weights: $\{\mu_{(t)}\}_{t=1}^{T}$ provided $\lambda = \eta\left(\widetilde{L}_{(T)}\right)$. Changing the weights allows us the freedom to redistribute at will without compromising efficiency.

When there is a single type, we can additionally assert that, in large games, $\eta\left(\widetilde{L}_{(T)}\right)$ is the unique value of the mixing parameter for which the equilibrium is Pareto efficient. To see this, we simply note that, to first order in $1/m$, all players have payoffs proportional to $F(L) - Lf_{(T)}(0,0)$ which is uniquely maximised at the solution of $F'(L) = f_{(T)}(0,0)$. This is solved at $L = \widetilde{L}_{(T)}$ only if $\lambda = \eta\left(\widetilde{L}_{(T)}\right)$.

It is interesting to contrast these results with those for serial surplus sharing. With the appropriate choice of mixing parameter, the equilibrium solution with a mixed sharing rule is fully efficient. Under serial surplus sharing, under the assumptions above, equilibria typically involve overproduction. However, if preferences are linear, equilibrium input is efficient although individual allocations are inefficient. Full efficiency requires restrictions on preferences (the same for all players) or on the technology (linear production function). Note that in the latter case, the efficient value of the mixing parameter is one: the average return procedure. On the other hand, the results for serial sharing are valid for any number of players whereas our conclusions for mixed sharing rules are asymptotic ones, only valid to first order in large games.

For ease of exposition, we have focused on the case of a replicated game in which there are the same (large) number of players of each type but this is not really necessary. The conclusion, in Proposition 5.1, that only type T participates in a large game requires only that there are enough players of this type. Thus aggregate input approaches $\widetilde{L}_{(T)}$ provided the number of players of type T tends to infinity. Similarly, the asymptotic form of payoffs in Corollary 5.2 remain valid provided that the number of players of each type is large, but not necessarily equal. Our conclusions that, approximately, $\eta\left(\widetilde{L}_{(T)}\right)$ is the unanimously preferred value of the mixing parameter and is Pareto efficient do not depend on having equal numbers of each type.

Finally, we consider the effect of relaxing the assumption that $F'(L) \longrightarrow \infty$ as $L \longrightarrow 0$. Suppose $f_i(0,0) < F'(0) < \infty$. It is convenient to impose the former inequality, otherwise player i never participates against any opponents. If $\overline{\lambda} = f_i(0,0)/F'(0)$, then (12) has a solution $\widetilde{L}$ if and only if $\lambda > \overline{\lambda}$. For such λ, Lemma 5.1 and the remaining results continue to hold. For $\lambda < \overline{\lambda}$, it is necessary to consider boundary solutions and we will have $L^n = 0$ for large enough n. Once again, Lemma 5.1 continues to hold, provided we set $\widetilde{L} = 0$. It can also be shown that, for large enough n, the share function is zero for all $L > 0$. If we adopt the convention that the dropout value is zero in such a case, we again have convergence of dropout values to $\widetilde{L}$. Indeed, with these interpretations, the results for asymmetric games extend to finite $F'(0)$. Note that the preferred value of λ will always exceed $\overline{\lambda}$, so our comments on preferred values of the mixing parameter remain valid.

6 Cost and surplus sharing

We have assumed that the disutility of supplying labor input ℓ_i is purely a function of ℓ_i alone. In this section, we extend the analysis to games in which this disutility also depends on aggregate (labor) input. For example, this includes games in which the sum of all players' inputs exerts a negative externality such as pollution, and the costs of amelioration are shared amongst the players according to some rule. More particularly, consider the following sharing rule for player i

$$y_i = \gamma_i^*(\ell_i, L) = \left\{\lambda^* \frac{\ell_i}{L} + (1-\lambda^*)\,\theta_i^*\right\} F^*(L), \tag{16}$$

where $\lambda^* \in (0,1)$ and θ_i^* are positive weights satisfying $\sum_{j\in I}\theta_j^* = 1$ and take player i's payoff to be $u_i(\gamma_i(\ell_i, L), \gamma_i^*(\ell_i, L))$. We shall continue to assume that preferences satisfy A.1 in Section 3 (with y_i replacing ℓ_i). The other two assumptions need slight modification to cope with the generalized model, as follows.

A.2* The functions F and F^* are increasing, continuous and continuously differentiable for $L > 0$ and $F(0) = F^*(0) = 0$. Either (i) F is strictly concave and F^* is convex, or (ii) F is concave and F^* is strictly convex.

A.3* There exists a value of $L > 0$ such that $u_i(F(L), F^*(L)) \leqq u_i(0,0)$.

The surplus sharing model discussed in previous sections has $\lambda^* = 1$ and $F^*(L) = L$. (The value of θ_i^* is irrelevant.) Pure cost sharing, as discussed recently by Watts [31] is the special case $\lambda = \lambda^* = 1$ and $F(L) = L$.

The analysis proceeds along the same lines as the two preceding sections and so we only sketch the results. Defining $\zeta_i(\sigma_i, L)$ to be the MRS evaluated at x_i given by (5) and $y_i = \{\lambda^*\sigma_i + (1-\lambda^*)\,\theta_i^*\} F^*(L)$, Lemma 3.1 still holds. The formula for the MRT becomes

$$\tau_i(\sigma_i, L) \equiv \frac{\{\lambda\sigma_i + (1-\lambda)\,\theta_i\} F'(L) + \lambda[1-\sigma_i] F(L)/L}{\{\lambda^*\sigma_i + (1-\lambda^*)\,\theta_i^*\} F^{*\prime}(L) + \lambda^*[1-\sigma_i] F^*(L)/L}.$$

If Assumption A.2* (i) holds, $F' <[\leqq] F/L$ and both are strictly decreasing and $F^{*\prime} \geqq F^*/L$ and both are weakly increasing. For fixed σ_i, we may conclude that the MRT is strictly decreasing in L. For fixed $L > 0$, under (i) an increase in σ_i implies a strict decrease in the numerator and a weak increase in the denominator leading to a fall in τ_i. The same conclusions may be drawn under A.2* (ii). Either way, Lemma 3.2 continues to hold when A.2 is replaced by A.2*. Propositions 3.1 and 3.2 follow for the extended model using the same arguments as in Section 3. The proof of Proposition 3.3 can also be modified for the extended model. We omit the details, simply noting that the graph of F must replaced by the locus of points of the form $(F(L), F^*(L))$ for $L > 0$ in the argument; it has the same key properties under A.3* as the locus of $(F(L), L)$ under A.3. Existence and uniqueness follows.

Theorem 6.1 *Given assumptions A.1, A.2* and A.3* (i) [A.3* (ii)], there is a unique Nash equilibrium if* $\lambda > 0$, $\lambda^* > 0$.

Once again, the qualifications on the mixing parameters are not required if normality holds strictly in A.1.

All the comparative statics results in Section 4 rest on Theorem 4.1. Part of this theorem can be generalized as follows, defining $\omega_i(\sigma_i, L)$ to be $u_i(x_i, y_i)$ where x_i is given by (4) and y_i is given by (16).

Theorem 6.2 *Suppose that,* $s_i(L)$ *exists and is positive. Then* $\omega_i(s_i(L), L)$ *is a strictly decreasing function in a neighborhood of* L *if* $\Delta_i(s_i(L), L) < 0$*, where* Δ_i *is defined in (9).*

By a minor modification of the proof of the second part of Theorem 4.1, we find

$$\frac{\partial \omega_i(s_i(L), L)}{\partial L} = \left\{ \frac{\partial u_i}{\partial x_i} \Delta_i(s_i(L), L) + \frac{\partial u_i}{\partial y_i} \frac{\partial \gamma_i^*}{\partial L} \right\} \{1 - s_i(L) - L s_i'(L)\}.$$

Assumption A.1 implies that $\partial u_i / \partial x_i > 0$ and $\partial u_i / \partial y_i < 0$, and

$$\frac{\partial \gamma_i^*(\sigma_i, L)}{\partial L} = \lambda^* \sigma_i \left\{ F^{*\prime}(L) - \frac{F^*(L)}{L} \right\} + (1 - \lambda^*) \theta_i^* F^{*\prime}(L) \geq 0, \qquad (17)$$

by Assumption A.2*. Since $s_i(L) < 1$ and $s_i'(L) \leqq 0$, the second term in braces is positive and part (ii) of the theorem follows. Note that, if either $\lambda^* < 1$ or F^* is strictly concave, the inequality in (17) is strict, and the requirement in the theorem can be weakened to $\Delta_i(s_i(L), L) \leq 0$. This is important because under pure cost sharing, $\Delta_i = 0$ for all arguments and F^* is strictly concave under A.2*. We may deduce the following corollary which includes pure cost sharing as a special case.

Corollary 6.3 *Suppose* $\lambda = 1$ *and* $F(L)$. *Equilibrium payoffs of currently active players decrease if* L *increases.*

If $\Delta > 0$, the sign of $\partial \omega_i / \partial L$ is ambiguous unless $\lambda^* = 1$ and F^* is linear which implies equality in (17). This is the case of pure surplus sharing which is covered by Theorem 4.1.

Note that the comparative statics of pure cost sharing do not depend on the mixing parameter by contrast with the case of pure surplus sharing. We may apply the corollary in a similar manner to Section 4 to a number of comparative statics propositions. For example, under pure cost sharing or, more generally, a mixed game in which the surplus is shared proportionally, players are made worse off by extra players joining the game or by an increase in ability of another player and better off by the imposition of a binding quota on a competitor.

7 Conclusion

We have shown that, under conventional assumptions on preferences and technology, the joint production game, with a mixed rule for sharing the surplus or costs (or both simultaneously), has a unique Nash equilibrium. We have also established results in comparative statics and for large games. Throughout, we have used the method of share functions but this approach is much more widely applicable. Indeed, it is potentially applicable to any model that has the aggregative structure in which the payoff of every player can be expressed as a function of that player's own strategy choice and the sum of the strategies chosen by all players in the game. Examples include pure and impure public good provision (Cornes and Sandler [13], [14], [15]), Bergstrom, Blume and Varian [3]), Cournot oligopoly with undifferentiated products (Friedman [17]), rent-seeking contests (Nitzan [21]) and pollution models (Tulkens [29], Barrett [1], Hoel [18], Chander and Tulkens [4]). Cournot oligopoly is a special case of the model discussed in this paper. (Let preferences be quasilinear in x_i and interpret ℓ_i as the quantity produced by owner/firm i and $F(L)/L$ as the demand function.) Public goods are studied using this approach in [11] and the use of share functions in rent-seeking has been studied by Cornes and Hartley when contestants are risk neutral [9] as well as when they are risk averse [10].

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8 Appendix

Lemma 8.1 *Suppose that the sequence* $\{x^n\}_{n=1}^{\infty}$ *is bounded and satisfies*

$$g_j(x^n, 1/n) \leq 0 \text{ for } j = 1, \ldots, J, \tag{18}$$

where each g_j *is a continuous function (of both arguments). Suppose further that* $\widetilde{x}$ *is the unique solution of* $g_j(x, 0) \leq 0$ *for* $j = 1, \ldots, J$. *Then* $x^n \longrightarrow \widetilde{x}$ *as* $n \longrightarrow \infty$.

Proof. By contradiction. If the conclusion were false, we could choose an open neighborhood $\mathcal{N}$ of $\widetilde{x}$ such that infinitely many members of the sequence $\{x^n\}_{n=1}^{\infty}$ fall outside $\mathcal{N}$. Since the sequence $\{x^n\}_{n=1}^{\infty}$ is bounded, it would contain a convergent subsequence falling outside $\mathcal{N}$, with limit x^{∞}, say. Openness of $\mathcal{N}$ means $x^{\infty} \notin \mathcal{N}$ and taking limits in (18) on the subsequence, using the continuity of the g_j we deduce that $g_j(x^{\infty}, 0) \leq 0$ for $j = 1, \ldots, J$. But this means $x^{\infty} = \widetilde{x}$ contradicting $\widetilde{x} \in \mathcal{N}$. ■

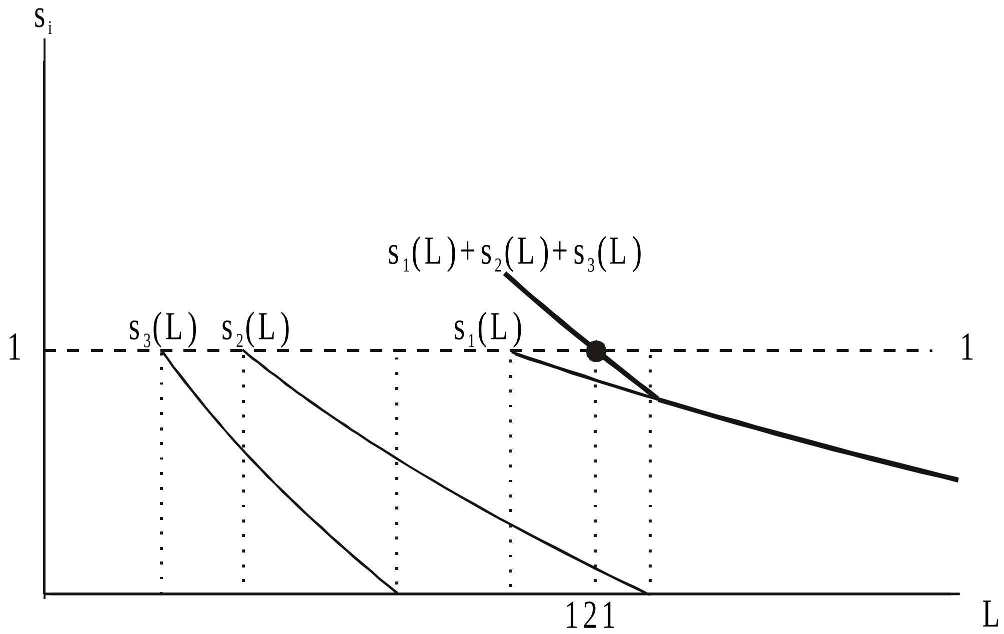


Figure 1:

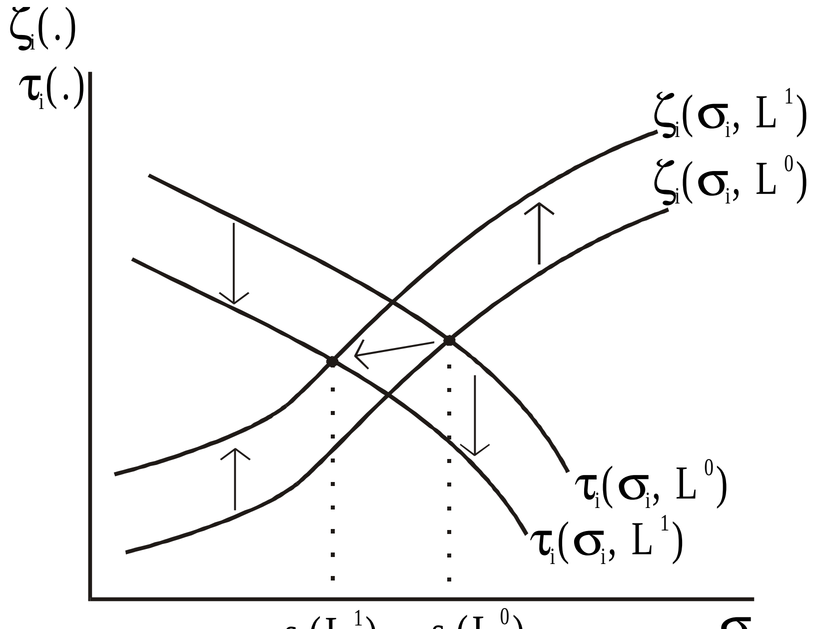
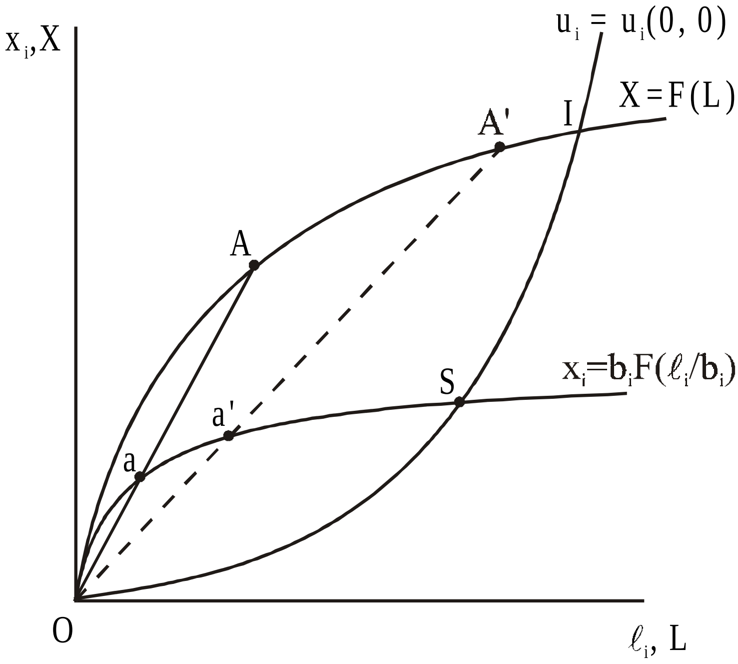


Figure 2:



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