A NEW INTEGRAL FOR CAPACITIES

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April 10, 2005

Abstract. A new integral for capacities, different from the Choquet integral, is introduced and characterized. The main feature of the new integral is concavity, which might be interpreted as uncertainty aversion. The integral is then extended to fuzzy capacities, which assign subjective expected values to random variables (e.g., portfolios) and may assign subjective probability only to a partial set of events. An equivalence between minimum over sets of additive capacities (not necessarily probability distributions) and the integral w.r.t. fuzzy capacities is demonstrated. The extension to fuzzy capacities enables one to calculate the integral also when there is information only about a few events and not about all of them.

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1. Introduction

The Choquet integral with respect to a capacity (or a non-additive probability) has been extensively used in decision theory. Schmeidler (1989) was the first to use it for calculating expected utility. Gilboa (1987), Wakker (1989), and Sarin and Wakker (1992) contributed further to this literature. Dow and Werlang (1992, 1994) applied the Choquet integral to game theory and finance. Schmeidler (1986) and Groes et. al. (1998) provided various characterizations of the Choquet integral.

In this note we present a new integral with respect to capacities which differs from the Choquet integral. This integral is then axiomatically characterized in two ways.

Another prominent integral is the Sugeno (or fuzzy) integral (see, Sugeno 1974). It is expressed in maximum-minimum terms and it corresponds to the notion of the median rather than to that of the average. The Sugeno integral, as opposed to the Choquet integral and the one introduced here, does not coincide with the regular integral when the capacity is additive.[1]

The key property of the new integral is concavity: the integral of the sum of two functions is less than or equal to the sum of the integrals. In the context of decision under uncertainty this property might be interpreted as uncertainty aversion.

Three more axioms are needed to characterize the integral. The first requires that if the capacity is additive, then the integral coincides with the regular one. The second is a monotonicity axiom with respect to capacities. It states that if the capacity v assigns to every subset a value which is greater than or equal to that assigned by w, then the integral of any non-negative function with respect to v is greater than or equal to the integral taken with respect to w.

[1]For a futher discussion of this issue the reader is referred to Murofushi and Sugeno (1991).

The last axiom states that when integrating a function, say X, the integral does not depend on the values that the capacity takes on the subset where X vanishes. In other words, the integral of a function depends only on the values that the capacity ascribes to its support and its subsets.

The integral proposed here is a slight variation of the concavification of a cooperative game that appeared first in Azrieli and Lehrer (2004). In the last section we introduce an integral w.r.t. fuzzy capacities. Fuzzy capacities assign subjective expected values to some random variables (e.g. portfolios). In particular, a fuzzy capacity may assign subjective probabilities only to some events and to all. The new integral aggregates all available information and enables one to calculate an average value also when there is a partial information and the capacity does not provide the likelihood of every possible event.

The integral w.r.t fuzzy capacities is inspired by Azrieli and Lehrer (2005) who used extensively the operational technique and employed it to investigate cooperative population games.

It turns out that there is a strong relation between the minimum over additive capacities and the new integral. This phenomenon is demonstrated in Section 8. A full equivalence between the representation of an order over random variables as a minimum over additive capacities[2] and a representation by the integral w.r.t. fuzzy capacities is shown in Section 9.

2. The new integral

A *capacity* is a function v that assigns a non-negative real number to every subset of a finite set N and satisfies $v(\emptyset) = 0$. The capacity v is said to be defined over N. A capacity P is *additive* if for any two disjoint subsets $S, T \subseteq N$, $P(S) + P(T) = P(S \cup T)$.

Let $|N| = n$ and let v be a capacity defined over it.

[2]See Gilboa and Schmeidler (1989) for the case of probability distributions.

Definition 1. *(i) The* concavification *of v, denoted* $\mathbf{cav}v$*, is defined as the minimum of all concave and homogeneous functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that*[3] $f(\mathbb{1}_R) \geq v(R)$ *for every $R \subseteq N$.*
(ii) For any non-negative $X \in \mathbb{R}^n_+$, define

$$\int^{\text{cav}} X dv = \mathbf{cav}v(X).$$

Remark 1. *Since the minimum of a family of concave and homogeneous functions over $\mathbb{R}^n_+$ is concave and homogeneous, so is $\int^{\text{cav}} X dv$, as a function of X.*

Let v and w be two capacities. We say that $v \geq w$ if $v(S) \geq w(S)$ for every $S \subseteq N$.

Lemma 1. *(i) For every $X \in \mathbb{R}^N_+$,*

$$(1) \quad \int^{\text{cav}} X dv = \max \Big\{ \sum_{R \subseteq N} \alpha_R v(R); \ \sum_{R \subseteq N} \alpha_R \mathbb{1}_R = X, \ \alpha_R \geq 0 \Big\}.$$

(ii) $$\int^{\text{cav}} X dv = \min_{P \text{ is additive and } P \geq v} \int^{\text{cav}} X dP.$$

The proof of (i) is similar to that of Lemma 1 in Azrieli and Lehrer (2004) and the proof of (ii) is standard; therefore both are omitted.

Zhang et. al. (2002) discussed expressions similar to that in the right hand side of eq. (1) with a further restriction that all the sets are required to be mutually disjoint. With this restriction the integral becomes analogous to Riemann integral.

Example 1: Let $N = \{1,2,3\}$, $v(N) = 1$, $v(12) = v(13) = \frac{3}{4}$, $v(23) = 1$ and $v(i) = 0$ for every $i \in N$. A function over N is a 3-dimensional vector. Consider $X = (1,1,1)$. As $(1,1,1) = \frac{1}{2}(1,1,0) + \frac{1}{2}(1,0,1) + \frac{1}{2}(0,1,1)$, $\int^{\text{cav}} X dv = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$. Notice that $\int^{\text{cav}} X dv > v(N)$. Now consider $X' = (0, \frac{6}{5}, \frac{6}{5})$. $\int^{\text{cav}} X' dv = \frac{6}{5}$. ∎

[3] $\mathbb{1}_R$ is the indicator of R: $\mathbb{1}_R = (\mathbb{1}^1_R, ..., \mathbb{1}^n_R)$, where $\mathbb{1}^i_R$ equals 1 if $i \in R$ and 0, otherwise.

3. Characterization

In this section we characterize the new integral. In what follows $\int X dv$ should be thought of as a function from pairs (X, v) to the reals. The goal is to find a set of plausible properties of such a function that characterizes it uniquely as the new integral.

The first property (including its title) is adopted from Groes et. al. (1998).

Accordance for Additive Measures - (AAM): if v is additive then $\int X dv$ is a regular integral.

The next axiom is the paramount property of the new integral.

Concavity - (CAV): For any v, X , Y and $\beta \in (0,1)$, $\int \beta X + (1-\beta) Y dv \geq \beta \int X dv + (1-\beta) \int Y dv$.

The following property is also shared by the Choquet integral.

Homogeneity - (HO): For any v, X and $\beta \geq 0$, $\int \beta X dv = \beta \int X dv$.

The next axiom states that if $v \geq w$, then the integral w.r.t. to v is greater than or equal to that w.r.t. w. Moreover, if $v \ngeq w$ then there is an indicator function whose integral w.r.t. w is greater than that w.r.t. v.

Strong monotonicity w.r.t. capacity - (SM): $v \geq w$ if and only if $\int \mathbb{1}_S dv \geq \int \mathbb{1}_S dw$ for every $S \subseteq N$.

Let S be a subset of N. The sub-capacity v_S is a capacity defined over S: $v_S(T) = v(T)$ for every $T \subseteq S$. The next axiom requires that the integral of an indicator function with respect to v is equal to the integral with respect to sub-capacity restricted to S. It suggests that the integral of a function depends on the values that v takes on the subset of N where the function is not vanishing.

The following axiom equates two integrals: one w.r.t. v over the domain N, and another w.r.t. v_S over a restricted domain, S.

Independent of irrelevant events - (IIE): For every S, $\int \mathbb{1}_S dv = \int \mathbb{1}_S dv_S$.

Proposition 1. (First Characterization) *The integral* $\int X dv$ *satisfies* (AAM), (CAV), (HO), (SM), *and* (IIE) *if and only if* $\int X dv = \int^{\text{cav}} X dv$ *for every non-negative* X.

Proof. The fact that $\int X dv$ satisfies (AAM), (CAV), (HO), (SM), and (IIE) is easy to check. As for the inverse direction, (SM) [4] implies that for every additive capacity P that satisfies $P \geq v$, $\int \mathbb{1}_S dP \geq \int \mathbb{1}_S dv$. Lemma 1 (ii) implies that $\int X dv$, as a function of X, is smaller than or equal to $\int^{\text{cav}} X dv$ (recall Definition 1). By (CAV) and (HO), $\int X dv$ is concave and homogeneous. It remains to show that $\int \mathbb{1}_S dv \geq v(S)$ for every $S \subseteq N$.

We proceed by induction on the size of S. For S such that $|S| = 1$, let $P(T) = v(T)$ if $T = S$ and $P(\emptyset) = 0$, otherwise. Thus, $v \geq P$ and by (SM), $\int \mathbb{1}_S dv \geq \int \mathbb{1}_S dP = v(S)$. Assume that $\int \mathbb{1}_S dv \geq v(S)$ for every $S \subseteq N$ with $|S| < \ell$ and we prove it for S of size ℓ.

If $v(S) \leq \int^{\text{cav}} \mathbb{1}_S dv$, then $\int^{\text{cav}} \mathbb{1}_S dv = \sum_{i=1}^{k} \alpha_i \int^{\text{cav}} \mathbb{1}_{R_i} dv$, where R_i is a proper subset of S and $\int^{\text{cav}} \mathbb{1}_{R_i} dv = v(R_i)$ for every $i = 1, ..., k$. Thus, $\int^{\text{cav}} \mathbb{1}_{R_i} dv = \sum_{i=1}^{k} \alpha_i v(R_i)$. By the induction hypothesis, $\int \mathbb{1}_{R_i} dv \geq v(R_i)$ for every $i = 1, ..., k$. And by (CAV), $\int \mathbb{1}_S dv \geq \sum_{i=1}^{k} \alpha_i v(R_i) = \int^{\text{cav}} \mathbb{1}_S dv \geq v(S)$, as desired.

We can therefore assume that $v(S) = \int^{\text{cav}} \mathbb{1}_S dv$ and it is strictly greater than any combination of the type $\sum_{i=1}^{k} \alpha_i \int^{\text{cav}} \mathbb{1}_{R_i} dv$, where R_i is a proper subset of S and $\alpha_i \geq 0$. The function $\int^{\text{cav}} X dv$ restricted to $\mathbb{R}_+^S$ (i.e., $\mathbb{R}_+^n$ restricted to the coordinates of S) is concave. Furthermore, the point $(\mathbb{1}_S, \int^{\text{cav}} \mathbb{1}_S dv)$ lies on an extreme ray of the graph of $\int^{\text{cav}} X dv$. The separation theorem ensures that there is an additive capacity P over S such that $P(S) = v(S)$ and $P(T) > v(T)$ for every

[4] In fact, at this point the 'only if' direction of (SM) suffices.

proper subset T of S. Let $\varepsilon > 0$ and consider the additive capacity P^ε defined by $P^\varepsilon(j) = \max(0, P(j) - \varepsilon)$ for every $j \in S$. When ε is sufficiently small, $P^\varepsilon(S) < v(S)$ and $P^\varepsilon(T) > v(T)$ for every proper subset T of S. Equivalently, $P_\varepsilon(S) < v_S(S)$ and $P^\varepsilon(T) > v_S(T)$ for every proper subset T of S.

We obtained that $P^\varepsilon \not\geq v_S$. By[5] (SM), there is $T \subseteq S$ such that $\int \mathbb{1}_T dv_S > \int \mathbb{1}_T dP^\varepsilon \geq P(T) - |T|\varepsilon$. I claim that $T = S$. Indeed, if T is a proper subset of S, then $P^\varepsilon_T > v_T$. This implies by (SM) that $\int \mathbb{1}_T dP^\varepsilon_T \geq \int \mathbb{1}_T dv_T = \int \mathbb{1}_T dv_S$ (the last inequality is by (IIE)), which is a contradiction. I conclude that $\int \mathbb{1}_S dv = \int \mathbb{1}_S dv_S > \int \mathbb{1}_S dP^\varepsilon \geq P(S) - |S|\varepsilon$. Since ε is arbitrary, $\int \mathbb{1}_S dv \geq P(S) = v(S)$, as desired and the proof is complete. ■

The following axiom is a relaxation of (SM).

Monotonicity w.r.t. capacity - (M): If $v \geq w$, then $\displaystyle\int \mathbb{1}_S dv \geq \int \mathbb{1}_S dw$ for every S.

Weak Indicator property - (WIP): For every S, $\displaystyle\int \mathbb{1}_S dv \geq v(S)$.

Schmeidler (1986) and Groes et. al. (1998) employ a strong version of (WIP), called the indicator property, which states that $\displaystyle\int \mathbb{1}_S dv = v(S)$.

The first part of the proof of Proposition 1 uses (AAM), (CAV), (HO), and only (M). The second part is devoted to show what (WIP) explicitly assumes. One therefore obtains,

Proposition 2. (Second Characterization) *The integral* $\displaystyle\int X dv$ *satisfies* (AAM)*,* (CAV)*,* (HO)*,* (M)*, and* (WIP) *if and only if* $\int X dv = \int^{\text{cav}} X dv$ *for every non-negative* X*.*

4. The new integral and Choquet integral

Let v be a capacity defined over N. The Choquet integral of X w.r.t v, denoted $\int^C X dv$, is defined by $\sum_{i=1}^n (X_{\sigma(i)} - X_{\sigma(i-1)}) v(R_i)$,

[5]At this point the 'if' part of (SM) is being used.

where σ is a permutation over N that satisfies $X_{\sigma(1)} \leq ... \leq X_{\sigma(n)}$ and $R_i = \{\sigma(i), ..., \sigma(n)\}$ ($X(\sigma(0)) = 0$, by convention).

Note that,

$$X = \sum \alpha_i \mathbb{1}_{R(i)}, \tag{2}$$

where $\alpha_i = X_{\sigma(i)} - X_{\sigma(i-1)}$. Thus, X is a positive linear combination of indicator functions. Note that the sum in eq. (2) is of the kind allowed in eq. (1). This means that, for the calculation of the Choquet integral, X is expressed as a linear combination of indicator functions of a particular kind. In contrast, in the new integral all such combinations are allowed, and like in the definition of the Lebesgue integral (see next section), the one that achieves the maximum of the respective summation is chosen.

This implies, in particular, that always $\int^C X dv \leq \int^{\text{cav}} X dv$. Proposition 4 of Azrieli and Lehrer (2004) implies that $\int^C X dv = \int^{\text{cav}} X dv$ for every X if and only if v is convex (i.e., $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for every $S, T \subseteq N$).

Example 1 continued: Let X and X' be the functions considered in Example 1. $\int^C X dv = 1$ and $\int^C X' dv = \frac{6}{5}$. Thus, $\int^C X' dv = \frac{6}{5} > 1 = \int^C X dv$, while $\int^{\text{cav}} X' dv = \frac{6}{5} < \frac{5}{4} = \int^{\text{cav}} X dv$. Therefore, if the two types of integrals were to indicate which of the two random variables, X and X', is better, they would induce different preferences. ■

5. The new integral as an extension of Lebesgue integral

A function f is *simple* if it can be written as $f = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{R_i}$, where $\alpha_i \in \mathbb{R}$. For a simple function, the integral of f with respect to a measure μ is defined as $\sum_{i=1}^{k} \alpha_i \int \mathbb{1}_{R_i} \, d\mu = \sum_{i=1}^{k} \alpha_i \mu(R_i)$. And for a non-negative function f it is defined as

$$\int f \, d\mu := \sup \left\{ \int h \, d\mu; \; h \text{ is simple and } h \leq f \right\}.$$

Lemma 1 (i) implies that the definition of $\int^{\text{cav}} X dv$ is similar to this definition.

6. Properties

Properties of the new integral that are not mentioned explicitly in the axioms are listed in this section. In what follows X and X' are non-negative functions over N, or equivalently, points in $\mathbb{R}^n_+$.

(1) $\int^{\text{cav}} X dv$ is continuous in both, X and v.

(2) Monotonicity w.r.t. functions: if $X \geq X'$, then $\int^{\text{cav}} X dv \geq \int^{\text{cav}} X' dv$.

(3) Proposition 1 of Azrieli and Lehrer (2004) implies that for every $S \subseteq N$, $\int^{\text{cav}} \mathbb{1}_S dv \geq v(S)$. If $\int^{\text{cav}} \mathbb{1}_S dv > v(S)$, then there are scalars $\alpha_i > 0$ and $R_i \subseteq N$, $i = 1, ..., k$, such that $\int^{\text{cav}} \mathbb{1}_S dv = \sum_{i=1}^k \alpha_i v(R_i)$ and $\int^{\text{cav}} \mathbb{1}_{R_i} dv = v(R_i),\ i = 1, ..., k$.

(4) Proposition 1 of Azrieli and Lehrer (2004) implies that for any $R \subseteq N$, the core[6] of the sub-capacity v_R is not empty if and only if $\int^{\text{cav}} \mathbb{1}_R dv = v(R)$. Thus, $\int^{\text{cav}} \mathbb{1}_R dv = v(R)$ for every $R \subseteq N$ if and only if the capacity is totally balanced (i.e., the core of each of its sub-capacities is not empty).

(5) Let $S \subseteq N$. Define the capacity v^S as follows: $v^S(R) = v(R)$ if $R \neq S$ and $v^S(S) = \int^{\text{cav}} \mathbb{1}_S dv$. Then, $\int^{\text{cav}} X dv = \int^{\text{cav}} X dv^S$. Thus, increasing the value of the capacity from $v(S)$ to $\int^{\text{cav}} \mathbb{1}_S dv$ would not change the integral.

(6) Let v be a capacity. Define the capacity w as follows: $w(S) = \int^{\text{cav}} \mathbb{1}_S dv$ for every $S \subseteq N$. w is the totally balanced cover of v. Then, $\int^{\text{cav}} X dv = \int^{\text{cav}} X dw$ for every non-negative X.

(7) $\int^{\text{cav}} X dv$ is piecewise linear in X. That is, the set $\mathbb{R}^n_+$ can be divided into finitely many closed cones $F_1, ..., F_\ell$ such that $\int^{\text{cav}} X dv$ is linear in each one: for every $X, X' \in F_i$, $\int^{\text{cav}} X + X' dv = \int^{\text{cav}} X dv + \int^{\text{cav}} X' dv$.

[6] The core of v consists of all additive capacities P such that $P \geq v$ and $P(N) = v(N)$.

(8) Therefore, $\int^{\text{cav}} X dv$ is locally additive: Every X is included in an open cone, say U_X, such that for every $X' \in U_X$, $\int^{\text{cav}} X + X' dv = \int^{\text{cav}} X dv + \int^{\text{cav}} X' dv$.

7. First order stochastic dominance and concavity

Let (v, N) be a capacity, and X, X' be two non-negative functions over N. We say that X' (first order) *stochastically dominates* X w.r.t. v, denoted $X' \succeq^v X$, if for every number t, $v(X' \geq t) \geq v(X \geq t)$.

The Choquet integral is monotonic w.r.t. stochastic dominance. That is, if $X' \succeq^v X$, then $\int^C X' dv \geq \int^C X dv$. In Example 1 $X' \succeq^v X$ and nevertheless, $\int^{\text{cav}} X' dv < \int^{\text{cav}} X dv$. Thus, $\int^{\text{cav}}$ is not monotonic w.r.t. stochastic dominance. The question arises whether there is a reasonable integral which is monotonic w.r.t. stochastic dominance and concave (i.e., satisfies (CAV)) at the same time. The following example shows that there is no homogeneous (non-trivial) integral which possesses these two properties.

Example 2: Let $N = \{1, 2, 3\}$, $v(S) = 1$ if $|S| \geq 2$ and otherwise, $v(S) = 0$. If $|S| = 2$, then $\mathbb{1}_S \succeq^v \mathbb{1}_N$, and if $\int$ is monotonic w.r.t. stochastic dominance, then $\int \mathbb{1}_S dv \geq \int \mathbb{1}_N dv$. However, $\mathbb{1}_N = \sum_{S;\ |S|=2} \frac{1}{2} \mathbb{1}_S$, and if $\int$ is concave and homogeneous, then $\int \mathbb{1}_N dv \geq \sum_{S;\ |S|=2} \frac{1}{2} \int \mathbb{1}_S dv \geq \frac{3}{2} \int \mathbb{1}_N dv$. Therefore, in the presence of homogeneity, monotonicity w.r.t. stochastic dominance and concavity are not compatible, unless $\int \mathbb{1}_N dv \leq 0$. If, instead, $v(S) = 1 - \varepsilon$ ($\varepsilon > 0$) for every $S \subseteq N$ that contains two states, then v becomes monotonic w.r.t. stochastic dominance.

The set N can be thought of as a state space and the function $\frac{2}{3}\mathbb{1}_N$ can be thought of as a portfolio that ensures a payoff of $\frac{2}{3}$ at any state. However, $\frac{2}{3}\mathbb{1}_N$ can be decomposed as an average of three portfolios: $\frac{2}{3}\mathbb{1}_N = \sum_{S;\ |S|=2} \frac{1}{3}\mathbb{1}_S$. Thus, if each of the portfolios $\mathbb{1}_S$, $|S| = 2$ (i.e., a payoff of 1 is guaranteed if a state in S is realized) is selected with probability $\frac{1}{3}$, then, on average, a payoff of $\frac{2}{3}$ is guaranteed at any state. The idea behind concavity is that the value of $\frac{2}{3}\mathbb{1}_N$ should be at least the average of the values of the portfolios forming it. That is, $\int^{\text{cav}} \frac{2}{3}\mathbb{1}_N dv \geq \sum_{S;\ |S|=2} \frac{1}{3}\int^{\text{cav}} \mathbb{1}_S dv.$ ∎

8. Minimum over the core

The capacity v is *exact* (Schmeidler , 1972) if and only if for every $S \subseteq N$, there is P in the core of v such that $P(S) = v(S)$. It implies that $\int^{\text{cav}} \mathbb{1}_S dv = \min_{P \text{ in the core of } v} \int \mathbb{1}_S dP$ for every $S \subseteq N$. It turns out that a stronger statement is true:

Proposition 3. *v is exact if and only if*

$$\int^{cav} X dv = \min_{P \text{ in the core of } v} \int X dP$$

for every non-negative X.

Proof. The 'if' direction is immediate. Assume that v is exact and let X be a non-negative function over N. Let k be the minimal integer that satisfies $X = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{R_i}$, for $\alpha_i \geq 0$ and $R_i \subseteq N,\ i = 1, ...k$; and

$$\int^{\text{cav}} X dv = \sum_{i=1}^{k} \alpha_i v(R_i). \tag{3}$$

Due to minimality, $\alpha_i > 0,\ i = 1, ..., k$.

Denote by $[(\mathbb{1}_{R_i}, v(R_i)), (\mathbb{1}_N, v(N))]$ the interval connecting the points $(\mathbb{1}_{R_i}, v(R_i))$ and $(\mathbb{1}_N, v(N))$ (both in $\mathbb{R}^{n+1}$). Since v is exact, this interval is on the graph of $\int^{\text{cav}} X dv$ (as a function of X). Let $D = \mathrm{conv} \cup_{\mathrm{i=1}}^{\mathrm{k}} [(\mathbb{1}_{\mathrm{R_i}}, \mathrm{v(R_i)}), (\mathbb{1}_{\mathrm{N}}, \mathrm{v(N)})]$.

We claim that D is on the graph of $\int^{\text{cav}} X dv$ D. Otherwise, D is not on the graph of $\int^{\text{cav}} X dv$, which means that there is Y such that Y can be written as a positive combination of $\mathbb{1}_{R_i},\ i = 1, ..., k$ and

$$\int^{\text{cav}} Y dv > \sum_{i=1}^{k} \beta_i v(R_i) \tag{4}$$

for every $\beta_i \geq 0,\ i = 1, ..., k$ that satisfy $Y = \sum_{i=1}^{k} \beta_i \mathbb{1}_{R_i}$.

Fix a representation of Y: $Y = \sum_{i=1}^{k} \beta_i 1\!\!1_{R_i}$, $\beta_i \geq 0$, $i = 1, ..., k$. For $\delta > 0$ sufficiently small $\delta\beta_i < \alpha_i$, $i = 1, ..., k$. Thus, $X = \sum_{i=1}^{k} \alpha_i 1\!\!1_{R_i} = \sum_{i=1}^{k} (\alpha_i - \delta\beta_i) 1\!\!1_{R_i} + \delta\beta_i 1\!\!1_{R_i} = \sum_{i=1}^{k} (\alpha_i - \delta\beta_i) 1\!\!1_{R_i} + \delta Y$. Homogeneity and monotonicity imply that $\int^{\text{cav}} X dv \geq \sum_{i=1}^{k} (\alpha_i - \delta\beta_i) v(R_i) + \delta \int^{\text{cav}} Y dv = \sum_{i=1}^{k} \alpha_i v(R_i) + \delta\big(\int^{\text{cav}} Y dv - \sum_{i=1}^{k} \beta_i v(R_i)\big) = \int^{\text{cav}} X dv + \delta\big(\int^{\text{cav}} Y dv - \sum_{i=1}^{k} \beta_i v(R_i)\big)$. The last equality is due to eq. (3). We therefore obtained, $\int^{\text{cav}} X dv \geq \int^{\text{cav}} X dv + \delta\big(\int^{\text{cav}} Y dv - \sum_{i=1}^{k} \beta_i v(R_i)\big) > \int^{\text{cav}} X dv$, where the last inequality is due to eq. (4). This is a contradiction and we therefore conclude that D is on the graph of $\int^{\text{cav}} X dv$.

Since D is convex, and by the separation theorem, there is an additive P such that $P(R_i) = \int^{\text{cav}} 1\!\!1_{R_i} dv = v(R_i)$ (the latter equality is due to exactness), $i = 1, ..., k$. In particular, $\int^{\text{cav}} X dP = \sum_{i=1}^{k} \alpha_i P(R_i) = \sum_{i=1}^{k} \alpha_i v(R_i)$. Thus, P is in the core of v, and $\displaystyle\int^{\text{cav}} X dv \geq \min_{P \text{ in the core of } v} \int X dP$. The inverse inequality is implied by Lemma 1 (ii), and therefore, the desired equality. ■

The analogous statement of Proposition 3 for the Choquet integral is due to Schmeidler (1986). He showed that v is convex if and only if

$$\int^{C} X dv = \min_{P \text{ in the core of } v} \int X dP$$

for every non-negative X.

9. An integral w.r.t. a fuzzy capacity

9.1. **Fuzzy capacity.** Let $I = [0,1]^n$ be the unit square. For every $a \in I$ let $|a|$ be the sum of its coordinates. Any subset of N can be identified with its indicator, which is an extreme point of I. Thus, a capacity is a function v that assigns to each extreme point of I a non-negative number and $v(0, ..., 0) = 0$. The notion of capacity is extended here as follows:

Definition 2. *(1) The pair* (v, A) *is a* fuzzy capacity *if* $(1, ..., 1) \in A \subseteq I$, $v : A \to \mathbb{R}_+$ *is continuous, and there is a positive* K *such that*

$v(a) \leq K|a|$ for every $a \in A$.
(2) (P, A) is an additive fuzzy capacity *if there are non-negative constants, $p_1, ..., p_n$, such that for every $a = (a_1, ..., a_n) \in A$, $P(a) = \sum_{i=1}^{n} a_i p_i$.*

While a capacity v assigns values (subjective probabilities) to events, a fuzzy capacity assigns values (subjective expected value) to random variables. The data-base of an agent might enable her to evaluate the expected values of some random variables (e.g., portfolios) and not of others. Furthermore, it might enable her to assess the probability of some events, but not of all of them. The set of variables about which the agent has firm assessments is represented by A. Note that A might contain only points of the form $\mathbb{1}_S$, where $S \subseteq N$. In this case v evaluates only the probability of events, and not necessarily all of them.

The integral aggregates all available information, including individual assessments of the likelihood of events and expected values of variables, into a comprehensive picture. Upon observing the comprehensive picture the agent might re-evaluate the likelihood of events or the expected values she assigns to random variables and change her mind.

We say that $(x_1, ..., x_n) \geq (y_1, ..., y_n)$ if $x_i \geq y_i$, $i = 1, ..., n$. A function f over $\mathbb{R}^n_+$ is said to be *monotonic* if for every $X, Y \in \mathbb{R}^n_+$, $X \geq Y$ implies $f(X) \geq f(Y)$.

Similar to the definition in Section 2 we define the *concavification* of (v, A), denoted $\mathbf{cav}v$, as the minimum of all concave, monotonic and homogeneous functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that $f(a) \geq v(a)$ for every $a \in A$. The minimum of all concave, monotonic and homogeneous functions is well defined and possesses the same properties. The integral w.r.t. (v, A) is defined as

$$\int^{\text{cav}} X dv = \mathbf{cav}v(X)$$

for every non-negative X. Similarly to Lemma 1 one obtains,

$$(5) \qquad \int^{\text{cav}} X dv = \max\Big\{\sum_{i=1}^{k} \alpha_i v(a_i)\Big\},$$

where the maximum is taken over all $a_i \in A, \alpha_i \le 0,\ i = 1, ..., k$ that satisfy $\sum_{i=1}^{k} \alpha_i a_i \le X$. Denote by cone$A$ the convex cone generated by A. That is, cone$A = \{\sum \alpha_i a_i;\ a_i \in A \text{ and } \alpha_i \ge 0\}$. Note that in eq. (5) $\sum_{i=1}^{k} \alpha_i a_i$ is allowed to be less than or equal, and not necessarily equal, to X as in Lemma 1. Inequality is allowed since cone$A$ might be a strict subset of $I\!R^n_+$. Note also that if (P, A) is additive, then $\int^{\text{cav}} X dP = P(X)$ for every $X \in$ cone$A$.

Example 3: Let $N = \{1, 2\}$. Thus, $I = [0, 1] \times [0, 1]$. Define the fuzzy capacity (v, A) as follows: $A = \{(1, 1), (\frac{1}{2}, \frac{1}{3})\}$, $v(1, 1) = 1$ and $v(\frac{1}{2}, \frac{1}{4}) = \frac{1}{3}$. Consider $X = (1, \frac{3}{4})$. $X = \frac{1}{2}(1, 1) + (\frac{1}{2}, \frac{1}{4})$ and this representation of X attains the maximum of the right hand side of eq. (5). Thus, $\int^{\text{cav}} X dv = \frac{1}{2} \cdot 1 + \frac{1}{3} = \frac{5}{6}$. Now let $Y = (2, 3)$. $Y = (2, 3) \ge 2(1, 1)$ and this attains the maximum of the right hand side of eq. (5). Therefore, $\int^{\text{cav}} Y dv = 2$. ■

The *core* of (v, A) (see also[7] Aubin (1979) and Azrieli and Lehrer (2005)) consists of all the additive fuzzy capacities P such that $P(1, ..., 1) = v(1, ..., 1)$ and for every $a \in A$, $P(a) \ge v(a)$. The fuzzy capacity (v, A) is *exact* if for every $a \in A$ there is P in the core of v such that $P(a) = v(a)$.

9.2. **Minimum over additive capacities and the integral.** Let $\mathcal{P}$ be a compact set of additive capacities. Denote the fuzzy capacity $(v_{\mathcal{P}}, I)$ as follows: $v_{\mathcal{P}}(a) = \min_{P \in \mathcal{P}} \int^{\text{cav}} a dP = P(a)$ for every $a \in I$.

Remark 2. *For any compact set of additive capacities, $\mathcal{P}$, denote by* conv$\mathcal{P}$ *the convex hull of $\mathcal{P}$. For any $a \in A$, the value $v_{\text{conv}\mathcal{P}}(a)$ is attained at an extreme point of* conv$\mathcal{P}$*, which is in $\mathcal{P}$. Therefore, $v_{\mathcal{P}} = v_{\text{conv}\mathcal{P}}$.*

[7]Both referred to the special case where $A = I$.

The following example illustrates the main idea demonstrated in this section.

Example 4: Let $N = \{1, 2, 3\}$ and consider the set $\mathcal{P}$ which consists of the probability distributions $P_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $P_2 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and $P_3 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$. Denote by w the capacity $v_{\mathcal{P}}$ restricted to $A = \{\mathbb{1}_S;\ S \subseteq N\ \}$. Thus,[8] $w(N) = 1$ and $w(S) = |S|\frac{1}{4}$ for $|S| \leq 2$. In this case for every non-negative X, $\min_{P\in\mathcal{P}} E_P(X) = \int^{\text{cav}} X dw$.

Now consider $P_4 = (\frac{2}{16}, \frac{7}{16}, \frac{7}{16})$ and $\mathcal{P}' = \{P_1, P_2, P_3, P_4\}$. Denote by u the capacity $v_{\mathcal{P}'}$ restricted to A. Thus, $u(N) = 1$, $u(S) = \frac{1}{2}$ if $|S| = 2$, $u(1) = \frac{1}{8}$, and $u(2) = u(3) = \frac{1}{4}$. In order to show that $\min_{P\in\mathcal{P}'} E_P(X) \neq \int^{\text{cav}} X du$ for some non-negative X, consider $X = (\frac{3}{5}, \frac{2}{5}, 0)$. On one hand, $\min_{P\in\mathcal{P}'} E_P(X) = \frac{1}{4}$ and on the other, $\int^{\text{cav}} X du = \frac{1}{5}u(1,0,0) + \frac{2}{5}u(1,1,0) = \frac{1}{5}\frac{1}{8} + \frac{2}{5}\frac{1}{2} = \frac{9}{40} < \frac{1}{4}$. In other words, in order to get equality between $\int^{\text{cav}} X dv_{\mathcal{P}'}$ and $\min_{P\in\mathcal{P}} E_P(X)$, one cannot restrict oneself to A.

We enlarge A: let $A' = A \cup \{(\frac{3}{5}, \frac{2}{5}, 0), (\frac{3}{5}, 0, \frac{2}{5})\}$. Define the fuzzy capacity (w', A') as follows: it coincides with u on A, and $w'(\frac{3}{5}, \frac{2}{5}, 0) = w'(\frac{3}{5}, 0, \frac{2}{5}) = \frac{1}{4}$. We obtained that for every non-negative X, $\min_{P\in\mathcal{P}'} E_P(X) = \int^{\text{cav}} X dw'$. For instance, let $X = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$. $\min_{P\in\mathcal{P}'} E_P(X) = E_{P_4}(X) = \frac{2}{16}\frac{3}{5} + \frac{7}{16}\frac{1}{5} + \frac{7}{16}\frac{1}{5} = \frac{1}{4}$ and $\int^{\text{cav}} X dw' = \frac{1}{2}w'(\frac{3}{5}, \frac{2}{5}, 0) + \frac{1}{2}w'(\frac{3}{5}, 0, \frac{2}{5}) = \frac{1}{4}$.

The information embedded in $\mathcal{P}'$ cannot be compressed into a capacity defined only over the extreme points of I (i.e., to subsets on N). The values of w' over the points $(\frac{3}{5}, \frac{2}{5}, 0)$ and $(\frac{3}{5}, 0, \frac{2}{5})$ are necessary. On the other hand, the values of w' on A' are sufficient to provide all the information needed to obtain $\min_{P\in\mathcal{P}} E_P(X)$ through the integral. ■

Lemma 2. *If for any* $P, P' \in \mathcal{P}$, $P(1, ..., 1) = P'(1, ..., 1)$, *then* $v_{\mathcal{P}}$ *is exact.*

Gilboa and Schmeidler (1989) characterized those preference orders over acts (which are translated to non-negative functions) that can be represented by a minimum over a compact and convex set of probability distributions. It turns out that the representations as a minimum over

[8]In this example we identify a subset of N with its indicator.

additive capacities (not necessarily probability distributions) and as an integral w.r.t. a fuzzy capacity are equivalent. Formally,

Proposition 4.

(1) *Let $\mathcal{P}$ be a compact set of additive capacities. Then,*

$$\min_{P\in\mathcal{P}} P(X) = \int^{cav} X dv_{\mathcal{P}}$$

for every non-negative X. Furthermore, if $\mathcal{P}$ is either finite or a polygon, then there is a fuzzy capacity (v, A) with A being finite such that $\min_{P\in\mathcal{P}} P(X) = \int^{cav} X dv$.

(2) *For every fuzzy capacity (v, A) there is a compact and convex set of additive capacities (not necessarily probability distributions), $\mathcal{P}$, such that*

$$\int^{cav} X dv = \min_{P\in\mathcal{P}} P(X).$$

Moreover, if (v, A) is exact, then $P(1, ..., 1) = P'(1, ..., 1)$ for every $P, P' \in \mathcal{P}$.

The proof[9] is rather standard and is therefore omitted.

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