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On the Equivalence of Bayesian and Dominant Strategy Implementation

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Abstract

We consider a standard social choice environment with linear utilities and independent, one-dimensional, private types. We prove that for any Bayesian incentive compatible mechanism there exists an equivalent dominant strategy incentive compatible mechanism that delivers the same interim expected utilities for all agents and the same ex ante expected social surplus. The short proof is based on an extension of an elegant result due to Gutmann et al. (*Annals of Probability*, 1991). We also show that the equivalence between Bayesian and dominant strategy implementation generally breaks down when the main assumptions underlying the social choice model are relaxed, or when the equivalence concept is strengthened to apply to interim expected allocations.

*The present study builds on the insights of two papers. Gershkov, Moldovanu and Shi (2011) uncovered the role of a theorem due to Gutmann et al. (1991) for the analysis of mechanism equivalence, and Goeree and Kushnir (2011) generalized the theorem to several functions, thus greatly widening its applicability.

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1. Introduction

In an inspiring recent contribution, Manelli and Vincent (2010) revisit Bayesian and dominant strategy implementation in the context of standard single-unit, private-value auctions. They prove that for any Bayesian incentive compatible (BIC) auction there exists an equivalent dominant strategy incentive compatible (DIC) auction that yields the same interim expected utilities for all agents. This equivalence result is surprising and valuable because dominant strategy implementation has important advantages over Bayesian implementation. In particular, dominant strategy implementation is robust to changes in agents' beliefs and does not rely on the assumptions of a common prior and equilibrium play.

The definition of equivalence in terms of interim expected utilities is a conceptual innovation of Manelli and Vincent (2010). Most of the earlier literature concerns the implementation of social choice functions (or correspondences) and defines two mechanisms to be equivalent if they provide the same ex post allocation.[1] Mookherjee and Reichelstein (1992) show that the latter condition for BIC-DIC equivalence generally fails unless the BIC allocation rule is itself monotonic in each coordinate. In contrast, Manelli and Vincent (2010) are not concerned with the implementation of a given allocation rule but rather construct, for any allocation rule that is Bayesian implementable, another allocation rule that is dominant strategy implementable and that delivers the same interim expected utilities.[2]

In this paper, we extend the BIC-DIC equivalence result to social choice environments with linear utilities and independent, one-dimensional, private types. Moreover, we present a novel and powerful proof method based on an elegant mathematical theorem due to Gutmann et al. (1991), which relates to some of the mathematical underpinnings of computed tomography.[3] The theorem states that for any bounded, non-negative function of several variables that generates monotone, one-dimensional marginals, there exists a non-negative function that respects the same bound, generates the same one-dimensional marginals, and is monotone in each coordinate.[4] The proof shows how the desired function can be found as a solution to a convex minimization problem.

[1]See, e.g., Gibbard (1973), Satterthwaite (1975), and Roberts (1979).

[2]A main focus of the mechanism design literature concerns the implementation of efficient mechanisms, e.g. Green and Laffont (1977), d'Aspremont and Gérard-Varet (1979), Laffont and Maskin (1979), and Williams (1999). In contrast, the BIC-DIC equivalence result of Manelli and Vincent (2010) applies to *every* BIC auction, not just efficient ones.

[3]Gutmann et al. (1991) build upon earlier contributions due to Lorenz (1949), Gale (1957), Ryser (1957), Kellerer (1961), and Strassen (1965). These authors study the existence of measures with given marginals in various discrete or continuous settings. Those insights are also relevant to the analysis of reduced form auctions, e.g., Border (1991).

[4]Simply generating the same monotone marginals via a monotone function is a trivial multiplication exercise. The difficulty behind the result stems from the constraint of keeping the same bound.

The original Gutmann et al. (1991) theorem pertains to a *single* function, which restricts its direct applicability to settings with two alternatives or to symmetric settings where all agents' utilities share the same functional form.[5] In order to analyze more general social choice environments we prove an extension of this theorem. The extension involves minimizing a quadratic functional of *several* functions satisfying certain boundary and marginal constraints. We use this minimization procedure to construct, for any BIC mechanism, an equivalent DIC mechanism.

Within the context of auction design the implications of BIC-DIC equivalence can be highlighted as follows. BIC-DIC equivalence implies that *any* auction, including any optimal auction (in terms of efficiency or revenue), can be implemented using a dominant strategy mechanism and nothing can be gained from designing more intricate auction formats with possibly more complex Bayes-Nash equilibria. This holds not only for single-unit auctions but also for multi-unit auctions with homogeneous or heterogeneous goods, combinatorial auctions, and the like, as long as bidders' private values are one-dimensional and independent, and utilities are linear.

We also delineate the limits of BIC-DIC equivalence. We first consider an alternative definition of equivalence that requires the same interim expected allocations. In the single-unit, private-value auction context studied by Manelli and Vincent (2010), this condition is equivalent to the existence of transfers that yield the same interim expected utilities for all agents. For the social choice environments studied in this paper, however, the two notions do not necessarily coincide. In particular, demanding the same interim allocations implies that there exist transfers such that agents' interim expected utilities are the same, but the converse is not necessarily true. Using a simple public goods example with three social alternatives we show that the condition that the interim allocations are the same cannot generally be met.

Next, using a series of simple examples we demonstrate that BIC-DIC equivalence generally fails when utilities are not linear or when types are not independent, one-dimensional, or private. In other words, once we relax the assumptions underlying our model, Bayesian implementation may have advantages over dominant strategy implementation. For example, we show that ex ante social surplus may be strictly higher under BIC implementation when values are interdependent. Likewise, with multi-dimensional values, BIC mechanisms may result in higher revenues than can be attained by any DIC mechanism.

The paper is organized as follows. Section 2 presents the social choice environment. We prove our main BIC-DIC equivalence result in Section 3 and delineate its limits in Section 4. Section 5 concludes. The Appendix contains proofs omitted in the main text.

[5]For instance, in a two-alternative social choice setting this single function can describe the probability with which one of the alternatives occurs while the other alternative occurs with complementary probability.

2. Model

We consider an environment with a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of risk-neutral agents and a finite set $\mathcal{K} = \{1, 2, \ldots, K\}$ of social alternatives. Agent i's utility in alternative k equals $u_i^k(x_i, t_i) = a_i^k x_i + c_i^k + t_i$ where x_i is agent i's private type, $a_i^k, c_i^k \in \mathbb{R}$ are constants with $a_i^k \geq 0$, and $t_i \in \mathbb{R}$ is a monetary transfer. Agent i's type x_i is distributed according to probability distribution λ_i with support X_i, where the type space $X_i \subseteq \mathbb{R}$ can be any (possibly discrete) subset of $\mathbb{R}$. Note that types are one-dimensional and independent. Let A denote the matrix with elements a_i^k where the player index i corresponds to the rows and the social alternative index k corresponds to the columns. Furthermore, let $X = \prod_{i \in \mathcal{I}} X_i$ and $\lambda = \prod_{i \in \mathcal{I}} \lambda_i$.

Our model fits many classical applications of mechanism design, including auctions (e.g. Myerson, 1981), public goods (e.g. Mailath and Postlewaite, 1990), bilateral trade (e.g. Myerson and Satterthwaite, 1983), and screening models (e.g. Mussa and Rosen, 1978). However, it is important to point out that even within the restricted class of linear environments, one-dimensional types generally cannot capture the full space of agents' possible preferences in arbitrary social choice environments.

Without loss of generality we consider only direct mechanisms characterized by $K + I$ functions, $\{q^k(\mathbf{x})\}_{k \in \mathcal{K}}$ and $\{t_i(\mathbf{x})\}_{i \in \mathcal{I}}$, where $\mathbf{x} = (x_1, \ldots, x_I) \in X$ is the profile of reports, $q^k(\mathbf{x}) \geq 0$ is the probability that alternative k is implemented with $\sum_{k \in \mathcal{K}} q^k(\mathbf{x}) = 1$, and $t_i(\mathbf{x})$ is the monetary transfer agent i receives. When agent i reports x_i' and all other agents report truthfully, the conditional expected probability (from agent i's point of view) that alternative k is chosen is $Q_i^k(x_i') = E_{\mathbf{x}_{-i}}(q^k(x_i', \mathbf{x}_{-i}))$ and the conditional expected transfer to agent i is $T_i(x_i') = E_{\mathbf{x}_{-i}}(t_i(x_i', \mathbf{x}_{-i}))$. For later use we define, for $i \in \mathcal{I}$ and $\mathbf{x} \in X$,

$$v_i(\mathbf{x}) \equiv \sum_{k \in \mathcal{K}} a_i^k q^k(\mathbf{x})$$

with marginals $V_i(x_i) = \sum_{k \in \mathcal{K}} a_i^k Q_i^k(x_i)$, and the modified transfers

$$\tau_i(\mathbf{x}) \;=\; t_i(\mathbf{x}) + \sum_{k \in \mathcal{K}} c_i^k q^k(\mathbf{x})$$

with marginals $\mathcal{T}_i(x_i) = E_{\mathbf{x}_{-i}}(\tau_i(x_i, \mathbf{x}_{-i})) = T_i(x_i) + \sum_k c_i^k Q_i^k(x_i)$. When agent i's type is x_i and she reports being of type x_i', her interim expected utility can then be written as

$$u_i(x_i') \;=\; V_i(x_i')x_i + \mathcal{T}_i(x_i').$$

A mechanism $(\tilde{q}, \tilde{t})$ is BIC if truthful reporting by all agents constitutes a Bayes-Nash equilibrium. A mechanism (q, t) is DIC if truthful reporting is a dominant strategy equilibrium. To relate BIC and DIC mechanisms we employ the following notion of equivalence.

Definition 1. *Two mechanisms* (q, t) *and* $(\tilde{q}, \tilde{t})$ *are equivalent if they deliver the same interim expected utilities for all agents and the same ex ante expected social surplus.*

The definition of equivalence in terms of interim expected utilities follows Manelli and Vincent (2010). In addition, we demand that the same ex ante expected social surplus is generated so that no money needs to be inserted to match agent's utilities.

3. BIC–DIC Equivalence

We first consider connected type spaces, i.e. $X_i = [\underline{x}_i, \overline{x}_i] \subseteq \mathbb{R}$. In this case a mechanism is BIC if and only if (i) for all $i \in \mathcal{I}$ and $x_i \in X_i$, $V_i(x_i)$ is non-decreasing in x_i and (ii) agents' interim expected utilities satisfy

$$u_i(x_i) \;=\; u_i(\underline{x}_i) + \int_{\underline{x}_i}^{x_i} V_i(s)ds,$$

see, for instance, Myerson (1981). Similarly a mechanism is DIC if and only if (i) for all $i \in \mathcal{I}$ and $\mathbf{x} \in X$, $v_i(x_i, \mathbf{x}_{-i})$ is non-decreasing in x_i and (ii) agents' utilities can be expressed as

$$u_i(x_i, \mathbf{x}_{-i}) \;=\; u_i(\underline{x}_i, \mathbf{x}_{-i}) + \int_{\underline{x}_i}^{x_i} v_i(s, \mathbf{x}_{-i})ds,$$

e.g., Laffont and Maskin (1980). Hence, with connected type spaces, agents' utilities are determined (up to a constant) by the allocation rule. This allows us to define equivalence in terms of the allocation rule only. Consider two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ and transfers such that $u_i(\underline{x}_i) = \tilde{u}_i(\underline{x}_i)$ for all $i \in \mathcal{I}$, then agents' interim expected utilities are the same under the two mechanisms if $V_i(x_i) = \tilde{V}_i(x_i)$ for all $i \in \mathcal{I}$, $x_i \in X_i$. Furthermore, the requirement that social surplus is the same is met when the ex ante probabilities of each alternative are the same for the two mechanisms, i.e. $E_{\mathbf{x}}(q^k(\mathbf{x})) = E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x}))$ for all $k \in \mathcal{K}$. To see this, recall that the expected social surplus is equal to the sum of (ex ante) expected utilities of the agents plus expected transfers. Since $u_i(x_i) = \tilde{u}_i(x_i)$ and $V_i(x_i) = \tilde{V}_i(x_i)$ imply that $\mathcal{T}_i(x_i) = \tilde{\mathcal{T}}_i(x_i)$, we have

$$E_{\mathbf{x}}(t_i(\mathbf{x})) \;=\; E_{\mathbf{x}}(\tilde{t}_i(\mathbf{x})) + \sum_{k \in \mathcal{K}} c_i^k \big(E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x})) - E_{\mathbf{x}}(q^k(\mathbf{x}))\big) \;=\; E_{\mathbf{x}}(\tilde{t}_i(\mathbf{x}))$$

Hence, the two mechanisms result in the same expected transfers and social surplus if the ex ante probabilities with which each alternative occurs are identical.

We now state and prove our main result. Define $\mathbf{v}(\mathbf{x}) = A \cdot \mathbf{q}(\mathbf{x})$ with elements $v_i(\mathbf{x}) = \sum_k a_i^k q^k(\mathbf{x})$ for $i \in \mathcal{I}$, and let $|| \cdot ||$ denote the usual Euclidean norm: $||\mathbf{v}(\mathbf{x})||^2 = \sum_{i \in \mathcal{I}} v_i(\mathbf{x})^2$. Throughout we identify functions that are equal almost everywhere with respect to λ.

Theorem 1. *Let X_i be connected for all $i \in \mathcal{I}$ and let $(\tilde{q}, \tilde{t})$ denote a BIC mechanism. An equivalent DIC mechanism is given by (q, t), where the allocation rule q solves*

$$\min_{\substack{\{q^k\}_{k\in\mathcal{K}} \\ q^k(\mathbf{x}) \geq 0 \quad \forall k,\mathbf{x} \\ \sum_k q^k(\mathbf{x}) = 1 \quad \forall \mathbf{x} \\ V_i(x_i) = \tilde{V}_i(x_i) \quad \forall i, x_i \\ E_{\mathbf{x}}(q^k(\mathbf{x})) = E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x})) \quad \forall k}} E_{\mathbf{x}}\left(||\mathbf{v}(\mathbf{x})||^2\right) \tag{1}$$

and the transfers are given by $t_i(\mathbf{x}) = \tau_i(\mathbf{x}) - \sum_{k\in\mathcal{K}} c_i^k q^k(\mathbf{x})$ with

$$\tau_i(x_i, \mathbf{x}_{-i}) = \tau_i(\underline{x}_i, \mathbf{x}_{-i}) + v_i(\underline{x}_i, \mathbf{x}_{-i})\underline{x}_i - v_i(x_i, \mathbf{x}_{-i})x_i + \int_{\underline{x}_i}^{x_i} v_i(s, \mathbf{x}_{-i})ds, \tag{2}$$

for $\mathbf{x} \in X$, $i \in \mathcal{I}$, where $\tau_i(\underline{x}_i, \mathbf{x}_{-i}) = (v_i(\underline{x}_i, \mathbf{x}_{-i})/\tilde{V}_i(\underline{x}_i))\tilde{\mathcal{T}}_i(\underline{x}_i)$.[6]

Proof. Existence of a solution to (1) is guaranteed because the constraints in (1) define a non-empty,[7] compact, and convex set and $E_{\mathbf{x}}\left(||\mathbf{v}(\mathbf{x})||^2\right)$ is a convex functional. The main difficulty of the proof is to establish that any solution $v_i(x_i, \mathbf{x}_{-i})$ to (1) is non-decreasing in x_i. We do so in three steps. First, we consider discrete and uniformly distributed types, then we extend to the continuous uniform types using a discrete approximation, and, finally, we generalize to arbitrary type distributions. The first step, which extends Theorem 6 of Gutmann et al. (1991) to allow for multiple functions and multiple constraints, is covered in the main text while the proofs for the more technical second and third steps can be found in the Appendix.

Lemma 1. *Suppose, for all $i \in \mathcal{I}$, X_i is a discrete set and λ_i is uniform distribution on X_i. Let $\{q^k\}_{k\in\mathcal{K}}$ be a solution to (1) then $v_i(\mathbf{x}) = \sum_k a_i^k q^k(x)$ is non-decreasing in x_i for all $i \in \mathcal{I}$, $\mathbf{x} \in X$.*

Proof. Suppose, in contradiction, that $v_j(x_j, \mathbf{x}_{-j}) > v_j(x'_j, \mathbf{x}_{-j})$ for some j, $x'_j > x_j$, and some $\mathbf{x}_{-j}$. Since $\{\tilde{q}^k\}_{k\in\mathcal{K}}$ is a BIC mechanism $E_{\mathbf{x}_{-j}}(v_j(x_j, \mathbf{x}_{-j})) = E_{\mathbf{x}_{-j}}(\tilde{v}_j(x_j, \mathbf{x}_{-j}))$ is non-decreasing in x_j. Hence, there exists $\mathbf{x}'_{-j}$ for which $v_j(x_j, \mathbf{x}'_{-j}) < v_j(x'_j, \mathbf{x}'_{-j})$. Let $\alpha \equiv \varepsilon/(v_j(x_j, \mathbf{x}_{-j}) - v_j(x'_j, \mathbf{x}_{-j}))$ and $\alpha' \equiv \varepsilon/(v_j(x'_j, \mathbf{x}'_{-j}) - v_j(x_j, \mathbf{x}'_{-j}))$. Then, for small enough $\varepsilon > 0$, we have $0 < \alpha < 1$ and $0 < \alpha' < 1$. Define the perturbations

$$\mathbf{q}'(x_j, \mathbf{x}_{-j}) = (1-\alpha)\mathbf{q}(x_j, \mathbf{x}_{-j}) + \alpha\mathbf{q}(x'_j, \mathbf{x}_{-j}), \quad \mathbf{q}'(x'_j, \mathbf{x}_{-j}) = (1-\alpha)\mathbf{q}(x'_j, \mathbf{x}_{-j}) + \alpha\mathbf{q}(x_j, \mathbf{x}_{-j})$$

$$\mathbf{q}'(x_j, \mathbf{x}'_{-j}) = (1-\alpha')\mathbf{q}(x_j, \mathbf{x}'_{-j}) + \alpha'\mathbf{q}(x'_j, \mathbf{x}'_{-j}), \quad \mathbf{q}'(x'_j, \mathbf{x}'_{-j}) = (1-\alpha')\mathbf{q}(x'_j, \mathbf{x}'_{-j}) + \alpha'\mathbf{q}(x_j, \mathbf{x}'_{-j})$$

and $\mathbf{q}'(\mathbf{x}) = \mathbf{q}(\mathbf{x})$ for other $\mathbf{x} \in X$. By construction ${q'}^k(\mathbf{x}) \geq 0$ and $\sum_{k\in\mathcal{K}} {q'}^k(\mathbf{x}) = 1$ for all $\mathbf{x} \in X$. Also $E_{\mathbf{x}}(\mathbf{q}'(\mathbf{x})) = E_{\mathbf{x}}(\mathbf{q}(\mathbf{x}))$ since $\mathbf{q}'(x_j, \mathbf{x}_{-j}) + \mathbf{q}'(x'_j, \mathbf{x}_{-j}) + \mathbf{q}'(x_j, \mathbf{x}'_{-j}) + \mathbf{q}'(x'_j, \mathbf{x}'_{-j}) =$

[6] Where 0/0 is interpreted as 1.

[7] Since $\tilde{q}$ satisfies the constraints.

$\mathbf{q}(x_j, \mathbf{x}_{-j}) + \mathbf{q}(x'_j, \mathbf{x}_{-j}) + \mathbf{q}(x_j, \mathbf{x}'_{-j}) + \mathbf{q}(x'_j, \mathbf{x}'_{-j})$. We next show that the perturbations $\mathbf{q}'$ also produce the same marginals as $\mathbf{q}$. Rewrite the above perturbations in terms of $\mathbf{v}'(\mathbf{x}) = A \cdot \mathbf{q}'(\mathbf{x})$:

$$\mathbf{v}'(x_j, \mathbf{x}_{-j}) = (1-\alpha)\mathbf{v}(x_j, \mathbf{x}_{-j}) + \alpha \mathbf{v}(x'_j, \mathbf{x}_{-j}), \quad \mathbf{v}'(x'_j, \mathbf{x}_{-j}) = (1-\alpha)\mathbf{v}(x'_j, \mathbf{x}_{-j}) + \alpha \mathbf{v}(x_j, \mathbf{x}_{-j})$$

$$\mathbf{v}'(x_j, \mathbf{x}'_{-j}) = (1-\alpha')\mathbf{v}(x_j, \mathbf{x}'_{-j}) + \alpha' \mathbf{v}(x'_j, \mathbf{x}'_{-j}), \quad \mathbf{v}'(x'_j, \mathbf{x}'_{-j}) = (1-\alpha')\mathbf{v}(x'_j, \mathbf{x}'_{-j}) + \alpha' \mathbf{v}(x_j, \mathbf{x}'_{-j})$$

and the equal-marginal condition as $E_{\mathbf{x}_{-i}}(v'_i(x_i, \mathbf{x}_{-i})) = E_{\mathbf{x}_{-i}}(v_i(x_i, \mathbf{x}_{-i}))$. For $i = j$, this condition follows from $\alpha(v_j(x_j, \mathbf{x}_{-j}) - v_j(x'_j, \mathbf{x}_{-j})) = \alpha'(v_j(x'_j, \mathbf{x}'_{-j}) - v_j(x_j, \mathbf{x}'_{-j}))$ when $x_i = x_j$ or $x_i = x'_j$, while for other values of x_i it follows trivially. For $i \neq j$, the condition follows since $\mathbf{v}'(x_j, \mathbf{x}_{-j}) + \mathbf{v}'(x'_j, \mathbf{x}_{-j}) = \mathbf{v}(x_j, \mathbf{x}_{-j}) + \mathbf{v}(x'_j, \mathbf{x}_{-j})$ and $\mathbf{v}'(x_j, \mathbf{x}'_{-j}) + \mathbf{v}'(x'_j, \mathbf{x}'_{-j}) = \mathbf{v}(x_j, \mathbf{x}'_{-j}) + \mathbf{v}(x'_j, \mathbf{x}'_{-j})$. Finally,

$$\begin{aligned} E_{\mathbf{x}}\left(||\mathbf{v}'(\mathbf{x})||^2 - ||\mathbf{v}(\mathbf{x})||^2\right) &= -\frac{2\alpha(1-\alpha)}{|X|}\,||\mathbf{v}(x_j, \mathbf{x}_{-j}) - \mathbf{v}(x'_j, \mathbf{x}_{-j})||^2 \\ &\quad -\frac{2\alpha'(1-\alpha')}{|X|}\,||\mathbf{v}(x'_j, \mathbf{x}'_{-j}) - \mathbf{v}(x_j, \mathbf{x}'_{-j})||^2 \end{aligned}$$

a contradiction since the right hand side is strictly negative and $\{q^k\}_{k\in\mathcal{K}}$ solves (1). *Q.E.D.*

Lemma 2. *Suppose, for all $i \in \mathcal{I}$, $X_i = [0,1]$ and λ_i is the uniform distribution on X_i. Let $\{q^k\}_{k\in\mathcal{K}}$ denote a solution to (1) then $v_i(\mathbf{x})$ is non-decreasing in x_i for all $i \in \mathcal{I}$, $\mathbf{x} \in X$.*

The proof can be found in the Appendix. The idea is to consider a partition of $[0,1]^{K|X|}$ and define a discrete approximation of the Bayesian mechanism $\{\tilde{q}^k\}_{k\in\mathcal{K}}$ by replacing the $\tilde{q}^k$ with their averages in each element of the partition. Lemma 1 ensures there exists an equivalent DIC mechanism $\{q^k\}_{k\in\mathcal{K}}$ for this discrete approximation. The q^k can be extended to piecewise constant functions over $[0,1]^{K|X|}$. The result follows by considering increasingly finer partitions of $[0,1]$.

Lemma 3. *Suppose, for all $i \in \mathcal{I}$, $X_i \subseteq \mathbb{R}$ and λ_i is some distribution on X_i. Let $\{q^k\}_{k\in\mathcal{K}}$ denote a solution to (1). Then $v_i(\mathbf{x})$ is non-decreasing in x_i for all $i \in \mathcal{I}$, $\mathbf{x} \in X$.*

The proof can be found in the Appendix. The intuition is to consider a transformation of variables and relate the uniform distribution covered by Lemma 2 to the case of a general distribution. In particular, if the random variable Z_i is uniformly distributed then $\lambda_i^{-1}(Z_i)$, with $\lambda_i^{-1}(z_i) = \inf\{x_i \in X_i | \lambda_i(x_i) \geq z_i\}$, is distributed according to λ_i.

Finally, we establish that the modified payments $\tau_i(x_i, \mathbf{x}_{-i})$ in (2) are such that the interim expected utilities $u_i(x_i)$ in the DIC mechanism (q,t) are the same as the interim expected utilities $\tilde{u}_i(x_i)$ in the BIC mechanism $(\tilde{q}, \tilde{t})$. Taking expectations over $\mathbf{x}_{-i}$ in (2) yields

$$\begin{aligned}
\mathcal{T}_i(x_i) &= \tilde{\mathcal{T}}_i(\underline{x}_i) + V_i(\underline{x}_i)\underline{x}_i - V_i(x_i)x_i + \int_{\underline{x}_i}^{x_i} V_i(s)ds \\
&= \tilde{\mathcal{T}}_i(\underline{x}_i) + \tilde{V}_i(\underline{x}_i)\underline{x}_i - \tilde{V}_i(x_i)x_i + \int_{\underline{x}_i}^{x_i} \tilde{V}_i(s)ds \\
&= \tilde{u}_i(x_i) - \tilde{V}_i(x_i)x_i \;=\; \tilde{\mathcal{T}}_i(x_i)
\end{aligned}$$

and, hence, $u_i(x_i) = V_i(x_i)x_i + \mathcal{T}_i(x_i) = \tilde{V}_i(x_i)x_i + \tilde{\mathcal{T}}_i(x_i) = \tilde{u}_i(x_i)$. Furthermore, the constraint that $E_{\mathbf{x}}(q^k(\mathbf{x})) = E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x}))$ for all $k \in \mathcal{K}$ ensures that the expected transfers are the same under the BIC and DIC mechanisms, and, hence, so is expected social surplus. *Q.E.D.*

Remark 1. Note that the constructed equivalent DIC mechanism satisfies ex post individual rationality if and only if the original BIC mechanism satisfies interim individual rationality.

Remark 2. Theorem 1 can be adapted to include other objectives to construct different equivalent DIC mechanisms. For example, we can replace the squared norm in the minimization problem (1) by $\sum_{i\in\mathcal{I}} E_{\mathbf{x}}\big(\mathcal{C}_i(v_i(\mathbf{x}))\big)$ where the $\mathcal{C}_i(\cdot)$ can be *arbitrary* strictly convex functions.

Remark 3. The constraint $E_{\mathbf{x}}(q^k(\mathbf{x})) = E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x}))$ ensures that the expected transfers and social surplus are the same. This constraint is also important when there are additional costs or benefits of implementing various alternatives or when the designer is not risk neutral.

Lemma 3 above applies to *any* distribution, not just continuous ones. We used the assumption of continuous type spaces only to invoke payoff equivalence, which allowed us to define the DIC transfers as in (2). We next prove BIC-DIC equivalence for discrete type spaces. For each $i \in \mathcal{I}$ let $X_i = \{x_i^1, \ldots, x_i^{N_i}\}$, where $x_i^n > x_i^{n-1}$ for $n = 2, \ldots, N_i$. A mechanism $(\tilde{q}, \tilde{t})$ is BIC if and only if (i) for all $i \in \mathcal{I}$ and $x_i \in X_i$, $\tilde{V}_i(x_i)$ is non-decreasing in x_i and (ii) the transfers satisfy

$$(\tilde{V}_i(x_i^n) - \tilde{V}_i(x_i^{n-1}))x_i^{n-1} \;\le\; \tilde{\mathcal{T}}_i(x_i^{n-1}) - \tilde{\mathcal{T}}_i(x_i^n) \;\le\; (\tilde{V}_i(x_i^n) - \tilde{V}_i(x_i^{n-1}))x_i^n \tag{3}$$

for $n = 2, \ldots, N_i$. Similarly, a mechanism (q, t) is DIC if and only if (i) for all $i \in \mathcal{I}$ and $\mathbf{x} \in X$, $v_i(x_i, \mathbf{x}_{-i})$ is non-decreasing in x_i and (ii) the transfers satisfy

$$(v_i(x_i^n, \mathbf{x}_{-i}) - v_i(x_i^{n-1}, \mathbf{x}_{-i}))x_i^{n-1} \;\le\; \tau_i(x_i^{n-1}, \mathbf{x}_{-i}) - \tau_i(x_i^n, \mathbf{x}_{-i}) \;\le\; (v_i(x_i^n, \mathbf{x}_{-i}) - v_i(x_i^{n-1}, \mathbf{x}_{-i}))x_i^n \tag{4}$$

For $n = 2, \ldots, N_i$ let

$$\alpha_i^n \;\equiv\; \frac{\tilde{\mathcal{T}}_i(x_i^{n-1}) - \tilde{\mathcal{T}}_i(x_i^n)}{\tilde{V}_i(x_i^n) - \tilde{V}_i(x_i^{n-1})}$$

when $\tilde{V}_i(x_i^n) \neq \tilde{V}_i(x_i^{n-1})$ and $\alpha_i^n = x_i^n$ otherwise.

Theorem 2. *Let X_i be discrete for all $i \in \mathcal{I}$ and let $(\tilde{q}, \tilde{t})$ denote a BIC mechanism. An equivalent DIC mechanism is given by (q, t), where the allocation rule q solves (1) and the transfers are given by $t_i(\mathbf{x}) = \tau_i(\mathbf{x}) - \sum_{k \in \mathcal{K}} c_i^k q^k(\mathbf{x})$ with*

$$\tau_i(x_i^n, \mathbf{x}_{-i}) \;=\; \tau_i(x_i^1, \mathbf{x}_{-i}) - \sum_{m=2}^{n} (v_i(x_i^m, \mathbf{x}_{-i}) - v_i(x_i^{m-1}, \mathbf{x}_{-i}))\alpha_i^m \tag{5}$$

for $n = 2, \ldots, N_i$, $i \in \mathcal{I}$, where $\tau_i(x_i^1, \mathbf{x}_{-i}) = (v_i(x_i^1, \mathbf{x}_{-i})/\tilde{V}_i(x_i^1))\tilde{\mathcal{T}}_i(x_i^1)$.

Remark 4. Payoff equivalence does not apply to the discrete type case, which allows for a wider range of transfers and, generally, two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ can be equivalent even when their marginals $V_i(x_i)$ and $\tilde{V}_i(x_i)$ are not the same. Theorem 2 focuses on equivalent DIC mechanisms that have the same marginals and the same expected transfers.

We end this section by comparing our approach to that of Manelli and Vincent (2010). Importantly, our analysis is not restricted to the single-unit auction case and includes multi-unit auctions for homogeneous and heterogeneous goods, combinatorial auctions, and the like.[8] Moreover, our BIC-DIC equivalence result goes well beyond the auction context, see Section 4.1 where we apply it to a public goods provision problem.

But even for single-unit auctions, our approach differs in several respects. First, Manelli and Vincent (2010) restrict attention to continuous distributions with connected supports. The discrete case covered by our Theorem 2 thus provides an important extension of their results. Second, Manelli and Vincent (2010) assume that $c_i^k = 0$, which means that keeping the same interim expected utility for all agents implies the same expected social surplus. In our setting, the latter is ensured by the additional constraint $E_{\mathbf{x}}(q^k(\mathbf{x})) = E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x}))$ for all $k \in \mathcal{K}$. Finally, Manelli and Vincent (2010) first prove BIC-DIC equivalence for the case with symmetric bidders (their Theorem 1), then introduce asymmetries between bidders (Theorem 2), and, finally, allow for the seller to have her own private value for the object (Theorem 3).

These different cases are all covered by the minimization approach in (1). To see this, consider a setup with $I + 1$ agents (I bidders plus one seller) and $K = I + 1$ alternatives. If the seller has no private value for the object we simply set $a_i^i = 1$ for $i = 1, \ldots, I$ and $a_i^k = 0$ otherwise (and $c_i^k = 0$). By including the seller as the $(I + 1)$-th agent, the possibility that the object does not sell is included. In fact, the constraint $\sum_{k \in \mathcal{K}} q^k(\mathbf{x}) = 1$ in (1) becomes

$$\sum_{k=1}^{I} q^k(\mathbf{x}) \;=\; 1 - q^{I+1}(\mathbf{x}),$$

which combined with $E_{\mathbf{x}}(q^k(\mathbf{x})) = E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x}))$ for all $k \in \mathcal{K}$ implies that if the seller does not sell with some probability in the original BIC mechanism then she does not sell with the

[8] Assuming types are one-dimensional, independent, and private.

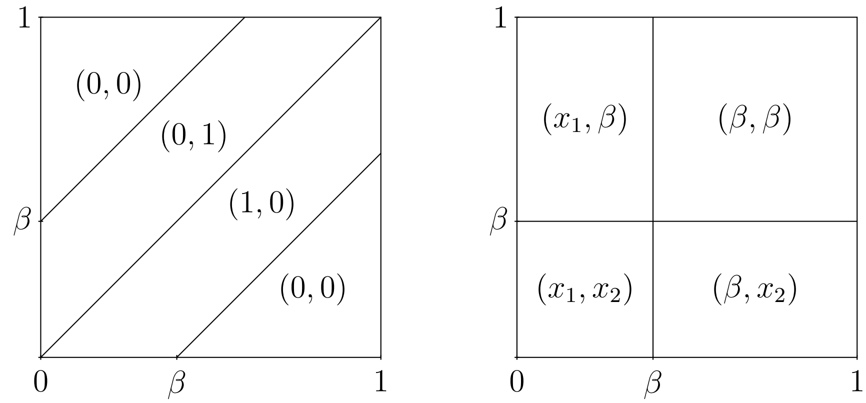

Figure 1: *BIC allocation rule (left) and DIC allocation rule (right) for $\beta \leq 1/2$. Here (q_1, q_2) represent the probabilities that bidders $(1, 2)$ win the object.*

same probability in the equivalent DIC mechanism. Furthermore, by including the seller as the $(I+1)$-th agent, the minimization approach in (1) implies that the constructed DIC mechanism generates the same expected revenue for the seller, since expected revenue is equal to minus the sum of bidders' expected transfers. To summarize, the constructed DIC mechanism is efficiency and revenue equivalent to the original BIC mechanism.

Moreover, if the original BIC mechanism is symmetric, an equivalent symmetric DIC mechanism can be found by including symmetry as a constraint in (1).[9] Alternatively, without this additional constraint, one could symmetrize any solution to (1) by permuting the agents and taking an average over all permutations.[10] Finally, the minimization approach in (1) also applies when the seller's private value is distributed over some range. In this case, we simply treat the seller like the bidders and set $a_i^i = 1$ for $i = 1, \ldots, I+1$ and $a_i^k = 0$ otherwise.

To illustrate, consider a single-unit private value auction with $I = 2$ bidders whose values, labeled x_1 and x_2, are independently and uniformly distributed on $[0, 1]$. Suppose the seller does not allocate the object if the difference between bidders' values is too high,[11] i.e. when $|x_1 - x_2| > \beta$ where, for simplicity, we assume that $\beta \leq 1/2$. In all other cases, the seller allocates the object efficiently, see the left panel of Figure 1. The allocation rule is not monotone and, hence, cannot be implemented in dominant strategies (Mookherjee and Reichelstein, 1992).

Denote the probability that bidder $k = 1, 2$ gets the object by $\tilde{q}^k$ and the probability that the seller keeps the object by $\tilde{q}^3$. So there are $K = 3$ social alternatives, $a_1^1 = a_2^2 = 1$ and $a_i^k = 0$

[9] Note that the resulting constraint set is again non-empty, compact, and convex.

[10] Permuting the agents honors the constraints in (1) if the original BIC mechanism is symmetric.

[11] Suppose the x_i for $i = 1, 2$ represent cost reductions from an innovation. A market regulator may prohibit the introduction of the innovation when the cost reductions are too asymmetric to avoid the advantaged firm being able to push the rival out of the market and gain monopoly power.

otherwise (and $c_i^k = 0$). For $i \neq j \in \{1, 2\}$ the allocation rule can be stated as

$$\tilde{q}^i(\mathbf{x}) \;=\; \begin{cases} 1 & \text{if } \; x_j \leq x_i \leq x_j + \beta \\ 0 & \text{otherwise} \end{cases}$$

while $\tilde{q}^3(\mathbf{x}) = 1 - \tilde{q}^1(\mathbf{x}) - \tilde{q}^2(\mathbf{x})$. This allocation rule has non-decreasing marginals

$$\int_0^1 \tilde{q}^i(\mathbf{x}) dx_j \;=\; \min(x_i, \beta)$$

for $i \neq j \in \{1, 2\}$, and is thus Bayesian implementable. For $\beta \leq 1/2$ the allocation rule

$$q^i(\mathbf{x}) \;=\; \min(x_i, \beta)$$

for $i = 1, 2$ and $q^3(\mathbf{x}) = 1 - \min(x_1, \beta) - \min(x_2, \beta)$ is a solution to minimization problem (1). This solution is shown in the right panel of Figure 1. Since the q^i are everywhere non-decreasing in x_i for $i = 1, 2$, they are dominant strategy implementable: supplemented with appropriate payments, they define an equivalent DIC mechanism.

4. The Limits of BIC–DIC Equivalence

In this section we present a series of examples, based on environments with two agents and discrete types, which delineate the limits of BIC-DIC equivalence. We start with a discussion of a stronger equivalence notion while maintaining the main assumptions of the social choice model: linear utilities, and independent, one-dimensional, private types. Subsequently we return to the equivalence notion of Definition 1 while relaxing these assumptions. In each case, we show how BIC-DIC equivalence fails.

4.1. Equivalence Based on Interim Expected Allocations

We now discuss a stronger notion of equivalence based solely on properties of the social choice function and does not involve agents' utilities. This notion becomes relevant when, for instance, the designer is not utilitarian or when preferences of agents outside the mechanism play a role.[12]

Definition 2. *Two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ are equivalent if they deliver the same interim expected allocation probabilities, i.e. $Q_i^k(x_i) = \tilde{Q}_i^k(x_i)$ for all $i \in \mathcal{I}$, $x_i \in X_i$, and $k \in \mathcal{K}$.*

[12] Consider, for example, a dynamic setting where a public decision affects both current and future generations. The distribution of values for future agents may be unknown and may depend on current realizations. Thus, current private information enters the "proxy" utility functions used for future agents, and a designer need not be indifferent between two mechanisms that are equivalent from the point of view of the current agents.

With continuous types, Definitions 1 and 2 are equivalent in settings with only two social alternatives or in the single-unit auction setting studied by Manelli and Vincent (2010).[13] More generally, however, requiring the same interim expected allocations is more stringent than Definition 1 and we next show that it fails in a simple public goods setting.

Suppose there are $K = 3$ alternatives, e.g. building a tunnel or a bridge or neither, and $I = 2$ symmetric agents, each with two equally likely and independent types $x^1 < x^2$. The utility, net of any transfers, of an agent with type x^j, for $j = 1, 2$, is $x^j + c^1$ in alternative 1, $ax^j + c^2$ with $0 < a \leq 1$ in alternative 2, and c^3 (independent of the agent's type) in alternative 3. The utility parameters are summarized by the matrices

$$A = \begin{pmatrix} 1 & a & 0 \\ 1 & a & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} c^1 & c^2 & c^3 \\ c^1 & c^2 & c^3 \end{pmatrix},$$

where rows correspond to agents and columns to social alternatives. To economize on notation we also represent the allocation rule with two-by-two matrices, where the rows correspond to agent 1's type and the columns to agent 2's type. Consider the following symmetric allocation rule

$$\tilde{q}^1 = as \begin{pmatrix} 1 & 1 \\ 1 & 13 \end{pmatrix}, \qquad \tilde{q}^2 = s \begin{pmatrix} 9 & 1 \\ 1 & 1 \end{pmatrix},$$

and $\tilde{q}^3 = 1 - \tilde{q}^1 - \tilde{q}^2$ where s is some small number, say $s = 1/20$. Note that $\tilde{q}^1 + a\tilde{q}^2$ is not increasing in each coordinate but its marginals $(6as, 8as)$ are. In other words, the allocation rule is BIC but not DIC. The symmetric allocation rules that are equivalent according to Definition 2 are summarized by[14]

$$\hat{q}^1 = as \begin{pmatrix} 2-\alpha & \alpha \\ \alpha & 14-\alpha \end{pmatrix}, \qquad \hat{q}^2 = s \begin{pmatrix} 10-\beta & \beta \\ \beta & 2-\beta \end{pmatrix},$$

for $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$. Note that $\hat{q}^1 + a\hat{q}^2$ is DIC only if $6 \leq \alpha + \beta \leq 8$, a contradiction. Of course, it is straightforward to solve the minimization problem in (1) to find equivalent DIC allocation rules in the sense of Definition 1:

$$q^1 = as \begin{pmatrix} 3 & 6 \\ 6 & 1 \end{pmatrix}, \qquad q^2 = s \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix},$$

so that $q^1 + aq^2$ is increasing in each coordinate.

[13]Since $\sum_{k\in\mathcal{K}} a_i^k Q_i^k(x_i) = \sum_{k\in\mathcal{K}} a_i^k \tilde{Q}_i^k(x_i)$ reduces to $Q_i^k(x_i) = \tilde{Q}_i^k(x_i)$ for all $k \in \mathcal{K}$ when there are only $K = 2$ alternatives or when $a_i^k = 0$ unless $i = k$ as in the single-unit auction case. In addition, Definition 2 implies the ex ante probabilities of each alternative are the same, i.e. $E_{\mathbf{x}}(q^k(\mathbf{x})) = E_{\mathbf{x}}(\tilde{q}^k(\mathbf{x}))$ for all $k \in \mathcal{K}$.

[14]It is easy to see that an equivalent dominant strategy mechanism must be symmetric.

4.2. Relaxing the Conditions of Theorems 1 and 2

In this subsection we demonstrate that BIC-DIC equivalence generally does not hold when we relax the assumption of linear utilities or when types are not one-dimensional, private, and independent. Recall from Section 3 that the constructed DIC mechanism is efficiency and revenue equivalent to the original BIC mechanism, which will prove useful in understanding the design of the counter-examples. Denote the seller's expected revenue by R and expected social surplus by W. Relaxing constraints in a revenue-maximization problem can only increase the achieved revenue level, so

$$\max_{\text{BIC, IR}} R \quad \geq \quad \max_{\text{DIC, IR}} R \quad \geq \quad \max_{\text{equivalent DIC, IR}} R \tag{6}$$

where IR, DIC, and BIC represent the interim individual rationality, dominant strategy incentive compatibility, and Bayesian incentive compatibility constraints respectively. For BIC-DIC equivalence to hold, these conditions have to be met with equality.[15] Conversely, if one of the conditions does not hold with equality, e.g. if the optimal DIC mechanism yields strictly less revenue than the optimal BIC mechanism, then BIC-DIC equivalence fails. A similar logic applies to social surplus. Importantly, in (6) we impose the same interim individual rationality constraints for all three cases so that any differences between the DIC and BIC mechanisms are not due to differences in participation constraints.

Interdependent Values

As noted by Manelli and Vincent (2010), Cremer and McLean (1988, Appendix A) construct an example with correlated types for which a BIC mechanism extracts all surplus from the buyers, while full-surplus extraction is not possible with a DIC mechanism. We therefore focus here on a setting with interdependent values but with independent types.

In this environment it is more natural to employ the notion of ex post incentive compatibility (EPIC), which requires that, for each type profile, agents prefer to report their types truthfully when others do. This characterization is akin to the definition of DIC for private values settings for which the two notions coincide (Bergemann and Morris, 2005). Unlike DIC, however, EPIC does not depend on agents' beliefs when there are value interdependencies.

Consider a discrete version of an example due to Maskin (1992). There are two bidders, labeled $i = 1, 2$, who compete for a single object. There are $K = 3$ possible alternatives corresponding to the cases where bidder 1 wins the object ($k = 1$), bidder 2 wins the object

[15] It is important to point out that our BIC-DIC equivalence result in Section 3 is *not* constrained to revenue-maximizing BIC mechanisms. Here we limit attention to surplus-maximizing and revenue-maximizing BIC mechanisms only to derive conditions under which BIC-DIC equivalence fails.

($k = 2$), or the seller keeps the object ($k = 3$). Bidder i's value for the object is $x_i + 2x_j$, where $i \neq j \in \{1, 2\}$ and the signal x_i is equally likely to be $x^1 = 1$ or $x^2 = 10$. Because of the higher weight on the other's signal, the first-best symmetric allocation rule is to assign the object to the lowest-signal bidder (with ties broken randomly)

$$q^1 = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and $q^2 = (q^1)^T$, i.e. the transpose of q^1, so that $q^3 = 1 - q^1 - q^2 = 0$, i.e. the object is always assigned. (As before, the rows of the q^k correspond to bidder 1's type and the columns to bidder 2's type.) The expected social surplus generated by the first-best allocation rule is $W = 150/8$.

Maskin (1992) used a continuous version of this example to show that the first-best allocation rule is not Bayesian implementable. Here this follows simply because the marginals are decreasing in a bidder's signal. It is a simple linear programming problem to find the surplus-maximizing allocation rule that respects Bayesian incentive compatibility:

$$q^1 = \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \tag{7}$$

and $q^2 = (q^1)^T$, yielding a total surplus of $W = 135/8$. Note that this "second-best" allocation rule does not always assign the object ($q^3_{11} = 1$) and that the marginal probability of winning is constant. Importantly, the allocation rule is not monotone, so the second-best solution is not ex post incentive compatible.[16]

For this example, the EPIC mechanism that maximizes surplus is given by

$$q^1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and $q^2 = (q^1)^T$, yielding a total surplus of $W = 132/8$. In other words, there exists no EPIC mechanism that generates the same total surplus as the second-best solution in (7).

This non-equivalence result does not hinge on the assumptions of discrete types or the fact that single crossing is violated.[17] Suppose, for instance, that signals are continuous and uniformly distributed and that bidder i's value is $x_i + \alpha x_j$ for $i \neq j \in \{1, 2\}$ and $0 \leq \alpha \leq 1$.

[16]Hernando-Veciana and Michelucci previously demonstrated these properties for a continuous version of Maskin's (1992) example where the signals x_i are uniformly distributed on $[0, 1]$. They also provide a general characterization of second-best efficient mechanisms and show that, with two bidders, the second-best solution can be implemented via an English auction (see Hernando-Veciana and Michelucci, 2011).

[17]Singe crossing is violated because in the agent's value the weight on the other's signal is twice as large as the weight on the agent's own signal.

Consider the following continuous extension of the second-best BIC allocation rule in (7)

$$\tilde{q}^1(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_1 < \frac{1}{2},\ x_2 < \frac{1}{2} \\ \frac{3}{4} & \text{if} \quad x_1 < \frac{1}{2},\ x_2 \geq \frac{1}{2} \\ \frac{1}{4} & \text{if} \quad x_1 \geq \frac{1}{2},\ x_2 < \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad x_1 \geq \frac{1}{2},\ x_2 \geq \frac{1}{2} \end{cases}$$

and $\tilde{q}^2(x_1, x_2) = \tilde{q}^1(x_2, x_1)$. It is readily verified that the marginals are constant, i.e. $\tilde{Q}^1(x_1) = \tilde{Q}^2(x_2) = \frac{3}{8}$. Since any EPIC allocation rule $q^1(x_1, x_2)$ has to be non-decreasing in x_1 for all x_2, the only way to match this constant marginal is if $q^1(x_1, x_2)$ is independent of x_1 (and, likewise, $q^2(x_1, x_2)$ is independent of x_2). Among the feasible EPIC allocation rules that match the constant marginals of $\frac{3}{8}$, the one that maximizes social surplus is given by

$$q^1(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_2 < \frac{1}{4} \\ \frac{1}{2} & \text{if} \quad x_2 \geq \frac{1}{4} \end{cases}$$

and $q^2(x_1, x_2) = q^1(x_2, x_1)$.

Even though the EPIC rule produces the same marginals as the BIC allocation rule and, hence, the same interim expected utilities for the bidders, it does not result in the same social surplus

$$W = \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \int_0^1 \int_0^1 (x_i + \alpha x_j) q^i(x_1, x_2) dx_1 dx_2$$

A straightforward computation shows that the social surplus under BIC and EPIC is given by $W = \frac{3}{8} + \frac{1}{2}\alpha$ and $W = \frac{3}{8} + \frac{15}{32}\alpha$ respectively. So with value interdependencies ($\alpha > 0$), the designer would have to insert money to implement an equivalent EPIC mechanism.

More generally, consider an environment with linear value interdependencies: agent i's value from alternative k equals $a_i^k x_i + \sum_{j \neq i} a_{ij}^k x_j$ for some non-negative a_{ij}^k (see also Jehiel and Moldovanu, 2001). Theorems 1 and 2 can be used to construct for any BIC allocation rule an EPIC rule that produces the same marginals and, hence, the same interim expected utilities for all agents. However, with interdependent values, social surplus is not determined by marginals alone and the constructed EPIC mechanism may generate less social surplus.

Multi-Dimensional Signals

There are two reasons why an equivalence result for multi-dimensional signals is not to be expected. First, monotonicity is not sufficient for implementation, and it must be complemented by an "integrability" condition, reflecting the various directions in which incentive constraints may bind (see, e.g., Rochet, 1987; Jehiel et al., 1999). Second, Gutmann et al. (1991) show that their result fails for higher dimensional marginals, which corresponds here to conditional expected probabilities given a multi-dimensional type. We explore here the first reason.

Consider a two-unit auction with $I = 2$ ex ante symmetric bidders whose types are equally likely to be $x^1 = (1,1)$, $x^2 = (2,1)$, or $x^3 = (5,3)$, where the first (second) number represents the marginal value for the first (second) unit. Note that marginal values are non-increasing for all three types, i.e. goods are substitutes. For simplicity we assume that both units sell so that there are only $K = 3$ possible alternatives: bidder 1 wins both units ($k = 1$), both bidders win a unit ($k = 2$), and bidder 2 wins both units ($k = 3$). It is a standard linear-programming exercise to find the *unique* symmetric BIC allocation rule that maximizes seller revenue:

$$\tilde{q}^1 = \begin{pmatrix} \frac{1}{2} & \frac{11}{20} & 0 \\ \frac{9}{20} & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

with $\tilde{q}^3 = (\tilde{q}^1)^T$ and $\tilde{q}^2 = 1 - \tilde{q}^1 - \tilde{q}^3$. The highest interim transfers that support this allocation rule as part of a BIC mechanism are given by $(\tilde{T}^1, \tilde{T}^2, \tilde{T}^3) = (-\frac{21}{30}, -\frac{23}{30}, -\frac{147}{30})$ for both bidders, resulting in seller revenues of $R = \frac{191}{45}$.

The allocation rule is not DIC, however. To see this, suppose the rival bidder's type is x^1. Then the condition for a bidder of type x^1 not to report being of type x^2 is $t^{21} - t^{11} \leq \frac{1}{10}$, where the superscripts correspond to the bidder's type and the other's type respectively. Similarly, the condition for a bidder of type x^2 not to report x^1 is $t^{21} - t^{11} \geq \frac{3}{20}$, a contradiction.[18] The unique symmetric allocation rule that maximizes revenue under the DIC constraints is given by

$$q^1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

and $q^3 = (q^1)^T$ and $q^2 = 1 - q^1 - q^3$. The transfers that support this allocation rule as part of a DIC mechanism are

$$t = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ -5 & -5 & -5 \end{pmatrix},$$

where rows correspond to the bidder's own type and columns to the other bidder's type. The resulting seller revenue is $R = \frac{38}{9}$. In other words, the optimal DIC mechanism produces strictly less revenues than the optimal BIC mechanism.

[18] In other words, when the opponent's type is x^1 the allocation rule violates one of Rochet's (1987) cycle conditions for dominant strategy implementability. However, the allocation rule does satisfy the "averaged" cycle conditions (where the average is taken over the opponent's type) that are necessary and sufficient for Bayesian implementation, see Müller, Perea, and Wolf (2007).

Non-Linear Utilities

We can reinterpret the multi-dimensional type example of the previous subsection in terms of non-linear utilities. A bidder's utility when her type is x^j and the alternative is k, for $j, k = 1, 2, 3$, is summarized by the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 8 & 5 & 0 \end{pmatrix}.$$

Obviously, only a non-linear model can fit all the payoffs in the matrix. Consider the one-dimensional types, $y^1 = 1$, $y^2 = 2$, and $y^3 = 5$, and, for both bidders, the non-linear utility functions $g^k(y)$ for $k = 1, 2, 3$, with $g^1(y) = \frac{1}{6}(y)^2 + \frac{1}{2}y + \frac{4}{3}$, $g^2(y) = y$, and $g^3(y) = 0$. It is readily verified that this non-linear model reproduces the utilities in the above matrix. Hence, bidders' interim expected utilities and their incentives to deviate are identical to those in the multi-dimensional example, and again there is an optimal BIC mechanism that produces strictly higher revenues than is possible under DIC implementation.

5. Discussion

This paper establishes a link between dominant strategy and Bayesian implementation in social choice environments. When utilities are linear and types are one-dimensional, independent, and private, we prove that for any social choice rule that is Bayesian implementable there exists a (possibly different) social choice rule that yields the same interim expected utilities for all agents, the same social surplus, and is implementable in dominant strategies. While Bayesian implementation relies on the assumptions of common prior beliefs and equilibrium play, dominant strategy implementation is robust to changes in agents' beliefs and allows agents to optimize without having to take into account others' behavior.

This paper also delineates the boundaries for BIC-DIC equivalence. When types are correlated, Cremer and McLean (1988) provide an example where a BIC mechanism yields strictly higher seller revenue than is attainable by any DIC mechanism. The examples in Section 4.2 show that BIC implementation may result in more social surplus or more revenue when values are interdependent, types are multi-dimensional, or utilities non-linear.

In general, the equivalence of Bayesian and dominant strategy implementation thus requires linear utilities and one-dimensional, independent, and private types. When these conditions are met, Bayesian implementation provides no more flexibility than dominant strategy implementation.

A. Appendix

Proof of Lemma 2. The intuition behind the proof is to relate the solution to that of Lemma 1 by taking a discrete approximation. For $i \in \mathcal{I}$, $n \geq 1$, $l_i = 1, \ldots, 2^n$, define the sets $S_i(n, l_i) = [(l_i - 1)2^{-n}, l_i 2^{-n})$, which yield a partition of $[0,1)$ into 2^n disjoint intervals of equal length. Let $\mathcal{F}_i^n$ denote the set consisting of all possible unions of the $S_i(n, l_i)$. Note that $\mathcal{F}_i^n \subset \mathcal{F}_i^{n+1}$. Also let $\mathbf{l} = (l_1, ..., l_I)$ and $S(n, \mathbf{l}) = \prod_{i \in \mathcal{I}} S_i(n, l_i)$, which defines a partition of $[0,1)^I$ into disjoint half-open cubes of volume 2^{-nI}. Let $\{\tilde{q}^k\}_{k \in \mathcal{K}}$ define a BIC mechanism and consider, for each $i \in \mathcal{I}$, the averages

$$\tilde{q}^k(n, \mathbf{l}) \;=\; 2^{nI} \int_{S(n,\mathbf{l})} \tilde{q}^k(\mathbf{x}) d\mathbf{x} \tag{A.1}$$

$$E_{\mathbf{l}_{-i}} \tilde{v}_i(n, \mathbf{l}) \;=\; 2^n \int_{S_i(n,l_i)} E_{\mathbf{x}_{-i}} \tilde{v}_i(\mathbf{x}) dx_i \tag{A.2}$$

Since $\tilde{q}^k(\mathbf{x}) \geq 0$ and $\sum_k \tilde{q}^k(\mathbf{x}) = 1$ we have $\tilde{q}^k(n, \mathbf{l}) \geq 0$ and $\sum_k \tilde{q}^k(n, \mathbf{l}) = 1$. By construction $\sum_{\mathbf{l}_{-i}} \tilde{v}_i(n, \mathbf{l}) = 2^{n(I-1)} E_{\mathbf{l}_{-i}} \tilde{v}_i(n, \mathbf{l})$, which is non-decreasing in l_i by (A.2).

Lemma 1 applied to the case where, for each $i \in \mathcal{I}$, $X_i = \{1, \ldots, 2^n\}$ and λ_i is the discrete uniform distribution on X_i, implies there exist $\{q^k(n, \mathbf{l})\}_{k \in \mathcal{K}}$ with $q^k(n, \mathbf{l}) \geq 0$ and $\sum_k q^k(n, \mathbf{l}) = 1$ such that $\sum_{\mathbf{l_{-i}}} v_i(n, \mathbf{l}) = \sum_{\mathbf{l_{-i}}} \tilde{v}_i(n, \mathbf{l})$, $\sum_{\mathbf{l}} q^k(n, \mathbf{l}) = \sum_{\mathbf{l}} \tilde{q}^k(n, \mathbf{l})$, and $v_i(n, \mathbf{l})$ is non-decreasing in l_i for all $\mathbf{l}$.

For each $i \in \mathcal{I}$, $n \geq 1$ define $q^k(n, \mathbf{x}) = q^k(n, \mathbf{l})$ for all $\mathbf{x} \in S(n, \mathbf{l})$. Then $q^k(n, \mathbf{x}) \geq 0$, $\sum_k q^k(n, \mathbf{x}) = 1$, and for each $i \in \mathcal{I}$, $v_i(n, \mathbf{x})$ is non-decreasing in x_i for all $\mathbf{x}$. Furthermore

$$\begin{aligned} \textstyle\int_{S_i(n,l_i)} E_{\mathbf{x}_{-i}} \tilde{v}_i(\mathbf{x}) dx_i \;&=\; 2^{-n} E_{\mathbf{l}_{-i}} \tilde{v}_i(n, \mathbf{l}) \;=\; 2^{-nI} \textstyle\sum_{\mathbf{l}_{-i}} \tilde{v}_i(n, \mathbf{l}) \;=\; 2^{-nI} \textstyle\sum_{\mathbf{l}_{-i}} v_i(n, \mathbf{l}) \\ &=\; \textstyle\sum_{l_{-i}} \int_{S(n,\mathbf{l})} v_i(n, \mathbf{x}) d\mathbf{x} \;=\; \int_{S_i(n,l_i) \times [0,1]^{I-1}} v_i(n, \mathbf{x}) d\mathbf{x} \end{aligned}$$

Thus $v_i(n, \mathbf{x}) - E_{\mathbf{x_{-i}}}(\tilde{v}_i(\mathbf{x}))$ integrates to 0 over every set $S_i \times [0,1]^{I-1}$ with $S_i \in \mathcal{F}_i^n$. Similarly $q^k(n, \mathbf{x}) - \tilde{q}^k(\mathbf{x})$ integrates to 0 over every set $[0,1]^I$. Consider any (weak*) convergent subsequence from the sequence $\{q^k(n, \mathbf{x})\}_{k \in \mathcal{K}}$ for $n \geq 1$, with limit $\{q^k(\mathbf{x})\}_{k \in \mathcal{K}}$. Then $\{q^k(\mathbf{x})\}_{k \in \mathcal{K}}$ defines a DIC mechanism that is equivalent to $\{\tilde{q}^k(\mathbf{x})\}_{k \in \mathcal{K}}$. *Q.E.D.*

Proof of Lemma 3. The intuition behind the proof is to relate the unique solution to (1) to that of the uniform case of Lemma 2. Recall that if the random variable Z_i is uniformly distributed then $\lambda_i^{-1}(Z_i)$ is distributed according to λ_i.[19] Hence, consider for all $i \in \mathcal{I}$ and $\mathbf{z} \in [0,1]^I$, the functions $\tilde{q}'^k(\mathbf{z}) = \tilde{q}^k(\lambda_1^{-1}(z_1), \ldots, \lambda_I^{-1}(z_I))$. Since

$$E_{\mathbf{z}_{-i}}(\tilde{v}'_i(\mathbf{z})) \;=\; E_{\mathbf{x}_{-i}}(\tilde{v}_i(\lambda_i^{-1}(z_i), \mathbf{x}_{-i}))$$

the mechanism defined by $\{\tilde{q}'^k\}_{k \in \mathcal{K}}$ is BIC and by Lemma 2 there exists an equivalent DIC mechanism $\{q'^k\}_{k \in \mathcal{K}}$ where $q'^k : [0,1]^I \to [0,1]$. In particular, q' minimizes $E_{\mathbf{z}}(||\mathbf{v}(\mathbf{z})||^2)$ and

[19] Where $\lambda_i^{-1}(z_i) = \inf\{x_i \in X_i | \lambda_i(x_i) \geq z_i\}$.

satisfies the constraints $q'^k(\mathbf{z}) \geq 0$, $\sum_k q'^k(\mathbf{z}) = 1$, and $E_{\mathbf{z}_{-i}}(v_i'(\mathbf{z})) = E_{\mathbf{x}_{-i}}(\tilde{v}_i(\lambda_i^{-1}(z_i), \mathbf{x}_{-i}))$ for all $i \in \mathcal{I}$. Now define $\{q^k\}_{k\in\mathcal{K}}$ with $q^k : X \to [0,1]$ where $q^k(\mathbf{x}) = q'^k(\lambda_1(x_1), \ldots, \lambda_I(x_I))$. Then $\{q^k\}_{k\in\mathcal{K}}$ solves (1) since $E_{\mathbf{x}}(||\mathbf{v}(\mathbf{x})||^2) = E_{\mathbf{z}}(||\mathbf{v}'^k(\mathbf{z})||^2)$ and $q^k(\mathbf{x}) \geq 0$, $\sum_k q^k(\mathbf{x}) = 1$, and $E_{\mathbf{x}_{-i}}(v_i(\mathbf{x})) = E_{\mathbf{z}_{-i}}(v_i'(\lambda_i(x_i), \mathbf{z}_{-i})) = E_{\mathbf{x}_{-i}}(\tilde{v}_i(\mathbf{x}))$ for all $i \in \mathcal{I}$ and $x_i \in X_i$. Furthermore, $v_i(\mathbf{x}) = \sum_k a_i^k q^k(\mathbf{x}) = \sum_k a_i^k q'^k(\lambda_1(x_1), \ldots, \lambda_I(x_I))$ is non-decreasing in x_i for all $k \in \mathcal{K}$, $x \in X$ since $\{q'\}_{k\in\mathcal{K}}$ is a DIC mechanism, λ is non-decreasing, and $a_i^k \geq 0$. *Q.E.D.*

Proof of Theorem 2. We first show the necessary conditions (3) and (4) are also sufficient. Consider (3) which ensures that deviating to an adjacent type, e.g. from x_i^{n-1} to x_i^n, is not profitable. Now consider types $x_i^p < x_i^q < x_i^r$. We show that if it is not profitable for type x_i^p to deviate to type x_i^q and it is not profitable for type x_i^q to deviate to type x_i^r then it is not profitable for type x_i^p to deviate to type x_i^r. The assumptions imply

$$\tilde{V}_i(x_i^p)x_i^p + \tilde{\mathcal{T}}_i(x_i^p) \;\geq\; \tilde{V}_i(x_i^q)x_i^p + \tilde{\mathcal{T}}_i(x_i^q), \quad \tilde{V}_i(x_i^q)l_i^q + \tilde{\mathcal{T}}_i(x_i^q) \;\geq\; \tilde{V}_i(x_i^r)x_i^q + \tilde{\mathcal{T}}_i(x_i^r)$$

and, hence,

$$\tilde{V}_i(x_i^p)x_i^p + \tilde{\mathcal{T}}_i(x_i^p) \;\geq\; \tilde{V}_i(x_i^r)x_i^p + \tilde{\mathcal{T}}_i(x_i^r) + (\tilde{V}_i(x_i^r) - \tilde{V}_i(x_i^q))(x_i^q - x_i^p) \;\geq\; \tilde{V}_i(x_i^r)x_i^p + \tilde{\mathcal{T}}_i(x_i^r)$$

since $\tilde{V}_i(x_i)$ is non-decreasing and $x_i^q > x_i^p$. Similarly, if it is not profitable for type x_i^r to deviate to type x_i^q and it is not profitable for type x_i^q to deviate to type x_i^p then it is not profitable for type x_i^r to deviate to type x_i^p. The same logic applies to the DIC constraints in (4).[20]

Next, consider the transfers defined by (5). Note that the BIC constraints (3) imply that $x_i^{n-1} \leq \alpha_i^n \leq x_i^n$ for $n = 2, \ldots, N_i$, which, in turn, implies that the difference in DIC transfers

$$\tau_i(x_i^{n-1}, \mathbf{x}_{-i}) - \tau_i(x_i^n, \mathbf{x}_{-i}) \;=\; (v_i(x_i^n, \mathbf{x}_{-i}) - v_i(x_i^{n-1}, \mathbf{x}_{-i}))\alpha_i^n$$

satisfies the bounds in (4). Let $\{q^k\}_{k\in\mathcal{K}}$ denote a solution to minimization problem in (1). Lemma 1 ensures that the associated $v_i(\mathbf{x})$ is non-decreasing in x_i for all $i \in \mathcal{I}$, $\mathbf{x} \in X$, and by construction $V_i(x_i) = E_{x_{-i}}(v_i(x_i, \mathbf{x}_{-i})) = \tilde{V}_i(x_i)$. Taking expectations over $\mathbf{x}_{-i}$ in (5) yields

$$\begin{aligned}\mathcal{T}_i(x_i^n) \;&=\; \tilde{\mathcal{T}}_i(x_i^1) - \sum_{m=2}^{n}(V_i(x_i^m) - V_i(x_i^{m-1}))\alpha_i^m \\ &=\; \tilde{\mathcal{T}}_i(x_i^1) + \sum_{m=2}^{n}(\tilde{\mathcal{T}}_i(x_i^m) - \tilde{\mathcal{T}}_i(x_i^{m-1})) \;=\; \tilde{\mathcal{T}}_i(x_i^n)\end{aligned}$$

for $n = 1, \ldots, N_i$. Hence, $u_i(x_i) = V_i(x_i)x_i + \mathcal{T}_i(x_i) = \tilde{V}_i(x_i)x_i + \tilde{\mathcal{T}}_i(x_i) = \tilde{u}_i(x_i)$, i.e. the DIC mechanism (q, t) yields the same interim expected utilities as the BIC mechanism $(\tilde{q}; \tilde{t})$.

The expected social surplus is the same because $\mathcal{T}_i(x_i) = \tilde{\mathcal{T}}_i(x_i)$ for all $x_i \in X_i$ and the ex ante expected probability with which each alternative occurs is the same under the BIC and DIC mechanisms. *Q.E.D.*

[20]Importantly, this derivation does not apply to multi-dimensional types, see Section 4.2.

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