University of Toronto
Department of Economics

Working Paper 313

Forecasting Realized Volatility: A Bayesian Model Averaging Approach

By Chun Liu and John M Maheu

April 03, 2008

Forecasting Realized Volatility: A Bayesian Model Averaging Approach*

Chun Liu
School of Economics and Management
Tsinghua University

John M. Maheu
Dept. of Economics
University of Toronto

This version: March 2008

Abstract

How to measure and model volatility is an important issue in finance. Recent research uses high frequency intraday data to construct ex post measures of daily volatility. This paper uses a Bayesian model averaging approach to forecast realized volatility. Candidate models include autoregressive and heterogeneous autoregressive (HAR) specifications based on the logarithm of realized volatility, realized power variation, realized bipower variation, a jump and an asymmetric term. Applied to equity and exchange rate volatility over several forecast horizons, Bayesian model averaging provides very competitive density forecasts and modest improvements in point forecasts compared to benchmark models. We discuss the reasons for this, including the importance of using realized power variation as a predictor. Bayesian model averaging provides further improvements to density forecasts when we move away from linear models and average over specifications that allow for GARCH effects in the innovations to log-volatility.

*We are grateful to Olsen Financial Technologies GmbH, Zurich, Switzerland for making the high frequency FX data available. We thank the co-editor, Tim Bollerslev, and three anonymous referees for many helpful comments. We thank Tom McCurdy who contributed to the preparation of the data used in this paper, and the helpful comments from Doron Avramov, Chuan Goh, Raymond Kan, Mark Kamstra, Gael Martin, Alex Maynard, Tom McCurdy, Angelo Melino, Neil Shephard and participants of the Far Eastern Meetings of the Econometric Society, Beijing, and China International Conference in Finance, Xi'an. Maheu thanks the SSHRCC for financial support.

1 Introduction

How to measure and model volatility is an important issue in finance. Volatility is latent and not observed directly. Traditional approaches are based on parametric models such as GARCH or stochastic volatility models. In recent years, a new approach to modeling volatility dynamics has become very popular which uses improved measures of ex post volatility constructed from high frequency data. This new measure is called realized volatility (RV) and is discussed formally by Andersen, Bollerslev, Diebold and Labys (2001), Andersen, Bollerslev, Diebold and Ebens (2001) and Barndorff-Nielsen and Shephard (2002a,2002b).[1] RV is constructed from the sum of high frequency squared returns and is a consistent estimator of integrated volatility plus a jump component for a broad class of continuous time models. In contrast to traditional measures of volatility, such as squared returns, realized volatility is more efficient. Recent work has demonstrated the usefulness of this approach in finance. For example, Bollerslev and Zhou (2002) use realized volatility to simplify the estimation of stochastic volatility diffusions, while Fleming, Kirby and Ostdiek (2003) demonstrate that investors who use realized volatility improve portfolio decisions.

This paper investigates Bayesian model averaging for models of volatility and contributes to a growing literature that investigates time series models of realized volatility and their forecasting power. Recent contributions include Andersen, Bollerslev, Diebold and Labys (2003), Andersen, Bollerslev and Meddahi (2005), Andreou and Ghysels (2002), Koopman, Jungbacker and Hol (2005), Maheu and McCurdy (2002), and Martens, Dijk and Pooter (2004). These papers concentrate on pure time series specifications of RV, however, there may be benefits to model averaging and including additional volatility proxies.

Barndorff-Nielsen and Shephard (2004) have defined several new measures of volatility, and associated estimators. *Realized power variation* (RPV), is constructed from the sum of powers of the absolute value of high frequency returns. This is a consistent estimator of the integral of the spot volatility process raised to a positive power (integrated power variation). *Realized bipower variation*, which is defined as the sum of the products of intraday adjacent returns, is a consistent estimator of integrated volatility.

There are several reasons why RPV may improve the forecasting of volatility. Barndorff-Nielsen and Shephard (2004) show that power variation is robust to jumps. Jumps are generally large outliers that may have a strong effect on model estimates and forecasts. Second, the absolute value of returns displays stronger persistence than squared returns (Ding, Granger and Engle (1993)), and therefore may provide a better signal for volatility. Third, Forsberg and Ghysels (2007), Ghysels, Santa-Clara and Valkanov (2006) and Ghysels and Sinko (2006) demonstrate that absolute returns (power variation of order 1) enhance volatility forecasts. Forsberg and Ghysels (2007) argue that the gains are due to the higher predictability, smaller sampling error and a robustness to jumps.[2]

[1] Earlier use of realized volatility includes French, Schwert and Stambaugh (1987), Schwert (1989), and Hsieh (1991).

[2] Other papers that have used realized power variation include Ghysels et al. (2007). A range of different volatility estimators is discussed in Barndorff-Nielsen and Shephard (2005).

Building on this work, we show empirically for data from equity and foreign exchange markets that persistence is highest for realized power variation measures. The correlation between realized volatility and lags of realized power variation as a function of the order p, is maximized for $1.0 \leq p \leq 1.5$, and not $p = 2$, which corresponds to realized volatility. Compared with models using just realized volatility, daily squared returns or the intraday range, we find that power variation and bipower variation can provide improvements.

These observations motivate a wide range of useful specifications using realized volatility, power variation of several orders, bipower variation, a jump and an asymmetric term. We focus on the benefits of Bayesian model averaging (BMA) for forecasts of daily, weekly and biweekly average realized volatility. BMA is constructed from autoregressive type parameterizations and variants of the heterogeneous autoregressive (HAR-log) model of Corsi (2004) and Andersen et al. (2007) extended to include different regressors. Choosing one model ignores model uncertainty, understates the risk in forecasting and can lead to poor predictions (Hibon and Evgeniou (2004)).[3] BMA combines individual model forecasts based on their predictive record. Therefore, models with good predictions receive large weights in the Bayesian model average.

We compare models' density forecasts using the predictive likelihood. The predictive likelihood contains the out-of-sample prediction record of a model, making it the central quantity of interest for model evaluation (Geweke and Whiteman (2005)). The empirical results show BMA to be consistently ranked at the top among all benchmark models, including a simple equally weighted model average. Considering all data series and forecast horizons, the BMA is the dominate model. Although there are substantial gains in BMA based on density forecasts, point forecasts using the predictive mean show smaller improvements.

The importance of GARCH dynamics in time series models of log-realized-volatility has been documented by Bollerslev et al. (2007). We find that Bayesian model averaging provides further improvements to density forecasts when we move away from linear models and average over specifications that allow for GARCH effects. For example, it provides improvements relative to a benchmark HAR-log-GARCH model for daily density forecasts.

There are two main reasons why BMA delivers good performance. First, we show that no single specification dominates across markets and forecast horizons. For each market and forecast horizon there is considerable model uncertainty in all our applications. In other words, there is model risk associated with selecting any individual model. The ranking of individual models can change dramatically over data series and forecast horizons. Bayesian model averaging provides an optimal way to combine this information.[4] The second reason, is that based on the predictive likelihood, including RPV terms can dramatically improve forecasting power. Although specifications with RPV terms also display considerable model uncertainty, BMA gives them larger weights when they perform well.

[3]Recent examples of Bayesian model averaging in a macroeconomic context include Fernández, Ley, and Steel (2001), Jacobson and Karlsson (2004), Koop and Potter (2004), Pesaran and Zaffaroni(2005) and Wright (2003).

[4]Based on a logarithmic scoring rule, averaging over all the models provides superior predictive ability (Raftery et al. (1997)).

The relative forecast performance of the specifications that enter the model average is ordered as follows. As a group, models with RPV regressors deliver forecast improvements. Bipower variation delivers relatively smaller improvements over models with only realized volatility regressors. A realized jump term which is constructed from bipower variation is important in all model formulations.

This paper is organized as follows. Section 2 discusses the econometric issues for Bayesian estimation and forecasting. Section 3 reviews the theory behind the improved volatility measures: realized volatility, realized power variation and realized bipower variation. Section 4 details the data and the adjustment to RV and realized bipower variation in the presence of market microstructure noise. The selection of regressors is discussed in Section 5. Section 6 presents the different configurations that enter the model averaging while Section 7 discusses forecasting results as well as the role of realized power variation, and the performance of BMA when allowing for GARCH effects. The last section concludes. An appendix explains how to calculate the marginal likelihood, and describes the algorithm to estimate volatility models with GARCH innovations.

2 Econometric Issues

2.1 Bayesian Estimation and Gibbs Sampling

To conduct formal model comparisons and model averaging we use Bayesian estimation methods. All the models we consider take the form of a standard normal linear regression

$$y_t = X_{t-1}\beta + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2). \tag{1}$$

In the following let $Y_T = [y_1, ..., y_T]'$ be a vector of size T, and X a T x k matrix of regressors with row X_{t-1}. Inference focuses on the posterior density. By Bayes rule, the prior distribution $p(\beta, \sigma^2)$, given data and a likelihood function $p(Y_T|\beta, \sigma^2)$, is updated to the posterior distribution,[5]

$$p(\beta, \sigma^2|Y_T) = \frac{p(Y_T|\beta, \sigma^2)p(\beta, \sigma^2)}{\int\int p(Y_T|\beta, \sigma^2)p(\beta, \sigma^2)d\beta d\sigma^2}. \tag{2}$$

We specify independent conditionally conjugate priors for $\beta \sim N(b_0, B_0)$, and $\sigma^2 \sim IG\left(\frac{v_0}{2}, \frac{s_0}{2}\right)$, where $IG(\cdot, \cdot)$ denotes the inverse gamma distribution. Although the posterior is not a well known distribution we can obtain samples from the posterior based on a Gibbs sampling scheme. Specifically, the conditional distributions used in sampling are $\beta|Y_T, \sigma^2 \sim N(M, V)$, where $M = V\left(\sigma^{-2}X'Y_T + B_0^{-1}b_0\right)$, $V = \left(\sigma^{-2}X'X + B_0^{-1}\right)^{-1}$, and $\sigma^2|Y_T, \beta \sim IG\left(\frac{v}{2}, \frac{s}{2}\right)$ where $v = T + v_0$, $s = (Y_T - X\beta)'(Y_T - X\beta) + s_0$.

Good introductions to Gibbs sampling and Markov chain Monte Carlo (MCMC) methods can be found in Chib (2001) and Geweke (2005). Formally, Gibbs sampling involves the following steps. Select a starting value, $\beta^{(0)}$ and $\sigma^{2(0)}$, and number of iterations N, then iterate on

[5] To minimize notation we suppress the conditioning on X in the following derivations.

- sample $\beta^{(i)} \sim p(\beta|Y_T, \sigma^{2(i-1)})$.
- sample $\sigma^{2(i)} \sim p(\sigma^2|Y_T, \beta^{(i)})$.

Repeating these steps N times produces the draws $\left\{\theta^{(i)}\right\}_{i=1}^{N} = \{\beta^{(i)}, \sigma^{2(i)}\}_{i=1}^{N}$. To eliminate the effect of starting values, we drop the first N_0 draws and collect the next N. For a sufficiently large sample this Markov chain converges to draws from the stationary distribution which is the posterior distribution. A simulation consistent estimate of features of the posterior density can be obtained by sample averages. For example, the posterior mean of the function $g(\cdot)$ can be estimated as

$$E[g(\theta)|Y_T] \approx \frac{1}{N}\sum_{i=1}^{N} g\left(\theta^{(i)}\right)$$

which converges almost surely to $E\left[g(\theta)|Y_T\right]$ as N goes to infinity.

In this paper we compare forecasts of models based on the predictive mean. The predictive mean is computed as

$$E[y_{T+1}|Y_T] \approx \frac{1}{N}\sum_{i=1}^{N} X_T\beta^{(i)}. \tag{3}$$

As a new observation arrives the posterior is updated through a new round of Gibbs sampling and a forecast for y_{T+2} can be calculated.

2.2 Model Comparison

There is a long tradition in the Bayesian literature of comparing models based on predictive distributions (Box (1980), Gelfand and Dey (1994), and Gordon (1997)). In a similar fashion to the Bayes factor which is based on all the data, we can compare the performance of models on a specific out-of-sample period. Given the information set $Y_{s-1} = \{y_1, ..., y_{s-1}\}$, the *predictive likelihood* (Geweke (1995,2005)) for model M_k is defined for the data $y_s, ..., y_t$, $s < t$ as

$$p(y_s, ..., y_t|Y_{s-1}, M_k) = \int p(y_s, ..., y_t|\theta_k, Y_{s-1}, M_k)p(\theta_k|Y_{s-1}, M_k)d\theta_k, \tag{4}$$

where $p(y_s, ..., y_t|\theta_k, Y_{s-1}, M_k)$ is the conditional data density given Y_{s-1}. The predictive likelihood is the predictive density evaluated at the realized outcome $y_s, ..., y_t$. Note that integration is performed with respect to the posterior distribution based on the data Y_{s-1}. If $s = 1$, this is the *marginal likelihood* and the above equation changes to

$$p(y_1, ..., y_t|M_k) = \int p(y_1, ..., y_t|\theta_k, M_k)p(\theta_k|M_k)d\theta_k, \tag{5}$$

where $p(y_1, ..., y_t|\theta_k, M_k)$ is the likelihood and $p(\theta_k|M_k)$ the prior for model M_k.

The predictive likelihood contains the out-of-sample prediction record of a model, making it the central quantity of interest for model evaluation (Geweke and Whiteman (2005)). For example, (4) is simply the product of the individual predictive likelihoods,

$$p\left(y_s, ..., y_t | Y_{s-1}, M_k\right) \quad = \quad \prod_{j=s}^{t} p\left(y_j | Y_{j-1}, M_k\right), \tag{6}$$

where each of the terms $p\left(y_j | Y_{j-1}, M_k\right)$ has parameter uncertainty integrated out. The relative value of density forecasts can be compared using the realized data $y_s, ..., y_t$ with the predictive likelihoods for two or more models.

The Bayesian approach allows for the comparison and ranking of models by predictive Bayes factors. Suppose we have K different models denoted by M_k, $k = 1, \ldots, K$, then the predictive Bayes factor for the data $y_s, ..., y_t$ and models M_0 versus M_1 is

$$PBF_{01} = p(y_s, ..., y_t | Y_{s-1}, M_0) / p(y_s, ..., y_t | Y_{s-1}, M_1).$$

This summarizes the relative evidence for model M_0 versus M_1. An advantage of using Bayes factors for model comparison is that they automatically include Occam's razor effect in that they penalize highly parameterized models that do not deliver improved predictive content. For the advantages of the use of Bayes factors see Koop and Potter (1999). Kass and Raftery (1995) recommend considering twice the logarithm of the Bayes factor for model comparison, as it has the same scaling as the likelihood ratio statistic.[6] In this paper we report estimates of the predictive likelihood corresponding to an out-of-sample period in which point forecasts are also investigated.

2.3 Calculating the Predictive Likelihood

The previous results require the calculation of the predictive likelihood for each model. Following Geweke (1995), each of the individual terms of the right hand side of (6) can be estimated consistently from the Gibbs sampler output as

$$p\left(y_j | Y_{j-1}, M_k\right) \approx \frac{1}{N} \sum_{i=1}^{N} p\left(y_j | \theta_k^{(i)}, Y_{j-1}, M_k\right), \tag{7}$$

where $\theta_k^{(i)} = \{\beta_k^{(i)}, \sigma_k^{2(i)}\}$. $p(y_j | \theta_k^{(i)}, Y_{j-1}, M_k)$ in the context of (1) denotes the normal density with mean $X_{j-1}\beta_k^{(i)}$ and variance $\sigma_k^{2(i)}$, evaluated at y_j, and the Gibbs sampler draws are obtained based on the information set Y_{j-1}.

2.4 Bayesian Model Averaging

In a Bayesian context it is straightforward to entertain many models and combine their information and forecasts in a consistent fashion. There are several justifications for

[6]Kass and Raftery suggest a rule-of-thumb of support for M_0 based on $2 \log PBF_{01}$: 0 to 2 not worth more than a bare mention, 2 to 6 positive, 6 to 10 strong, and greater than 10 as very strong.

Bayesian model averaging. Min and Zellner (1993) show that the model average minimizes the expected predicted squared error when the models are exhaustive, while it is superior based on a logarithmic scoring rule (Raftery et al. (1997)). For an introduction to Bayesian model averaging see Hoeting et al. (1999) and Koop (2003). The probability of model M_k given the information set Y_T is[7],

$$p(M_k|Y_T) = \frac{p(Y_T|M_k)p(M_k)}{\sum_{i=1}^{K} p(Y_T|M_i)p(M_i)} \tag{8}$$

where K is the total number of models. In this equation, $p(M_k)$ is the prior model probability, and $p(Y_T|M_k)$ is the marginal likelihood. In the context of recursive out-of-sample forecasts, it is more convenient to work with a period-by-period update to model probabilities. Given Y_{T-1}, after observing a new observation y_T, we update as

$$p(M_k|y_T, Y_{T-1}) = \frac{p(y_T|Y_{T-1}, M_k)p(M_k|Y_{T-1})}{\sum_{i=1}^{K} p(y_T|Y_{T-1}, M_i)p(M_i|Y_{T-1})}. \tag{9}$$

$p(y_T|Y_{T-1}, M_k)$ is the predictive likelihood value for model M_k based on information Y_{T-1}, and can be estimated by (7). $p(M_k|Y_{T-1})$ is last period's model probability.

The predictive likelihood for BMA is an average of each of the individual model predictive likelihoods,

$$p(y_{T+1}|Y_T) = \sum_{i=1}^{K} p(y_{T+1}|Y_T, M_i)p(M_i|Y_T), \tag{10}$$

where each model's predictive density is estimated from (7). Similarly, the predictive mean of y_{T+1} is,

$$E[y_{T+1}|Y_T] = \sum_{i=1}^{K} E[y_{T+1}|Y_T, M_i]p(M_i|Y_T), \tag{11}$$

which is a weighted average, using the model probabilities, of model specific predictive means.

3 Realized Volatility, Power Variation and Bipower Variation

A good discussion of the class of special semi-martingales, which are stochastic processes consistent with arbitrage-free prices can be found in Andersen, Bollerslev, Diebold and Labys (2003). These processes allow for a wide range of dynamics including jumps in the mean and variance process as well as long memory.

For illustration, consider the following logarithmic price process:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + \kappa(t)dq(t), \ 0 \leq t \leq T, \tag{12}$$

[7] Note that (8) can be written as $p(M_k)/\sum_{i=1}^{K} BF_{ik}p(M_i)$, where $BF_{ik} \equiv p(Y_T|M_i)/p(Y_T|M_k)$ is the Bayes factor for model i versus model k.

where $\mu(t)$ is a continuous and locally bounded variation process, $\sigma(t)$ is the stochastic volatility process, $W(t)$ denotes a standard Brownian motion, $dq(t)$ is a counting process with $dq(t) = 1$ corresponding to a jump at time t and $dq(t) = 0$ corresponding to no jump, a jump intensity $\lambda(t)$, and $\kappa(t)$ refers to the size of a realized jump. The increment in *quadratic variation* from time t to $t+1$ is defined as

$$QV_{t+1} = \int_t^{t+1} \sigma^2(s)ds + \sum_{t<s\leq t+1, dq(s)=1} \kappa^2(s) \tag{13}$$

where the first component, called integrated volatility, is from the continuous component of (12), and the second term is the contribution from discrete jumps. Barndorff-Nielsen and Shephard (2004) consider *integrated power variation* of order p defined as

$$IPV_{t+1}(p) = \int_t^{t+1} \sigma^p(s)ds \tag{14}$$

where $0 < p \leq 2$. Clearly $IPV_{t+1}(2)$ is integrated volatility.

To consider estimation of these quantities, we normalize the daily time interval to unity and divide it into m periods. Each period has length $\Delta = 1/m$. Then define the Δ period return as $r_{t,j} = p(t + j\Delta) - p(t + (j-1)\Delta)$, $j = 1, ..., m$. Note that the daily return is $r_t = \sum_{j=1}^m r_{t,j}$. Barndorff-Nielsen and Shephard (2004) introduce the following estimator called *realized power variation* of order p defined as

$$RPV_{t+1}(p) = \mu_p^{-1} \Delta^{1-p/2} \sum_{j=1}^m |r_{t,j}|^p, \tag{15}$$

where $\mu_p = E\,|\mu|^p = 2^{p/2} \frac{\Gamma(\frac{1}{2}(p+1))}{\Gamma(\frac{1}{2})}$ for $p > 0$ where $\mu \sim N(0,1)$. Note that for the special case of $p = 2$ equation (15) becomes

$$RPV_{t+1}(2) = \sum_{j=1}^m {r_{t,j}}^2 \equiv RV_{t+1} \tag{16}$$

and we have the *realized volatility*, RV_{t+1}, estimator discussed in Andersen, Bollerslev, Diebold and Labys (2001), Barndorff-Nielsen and Shephard (2002b), and Meddahi (2002). To avoid confusion we refer to $RPV_{t+1}(p)$ for $p < 2$ as realized power variation, and to (16) as RV_{t+1}.

Another estimator considered in Barndorff-Nielsen and Shephard (2004) is *realized bipower variation* which is,

$$RBP_{t+1} \equiv \mu_1^{-2} \sum_{j=2}^m |r_{t,j-1}||r_{t,j}|, \tag{17}$$

where $\mu_1 = \sqrt{2/\pi}$.

As shown by the papers discussed above, as $m \to \infty$

$$RPV_{t+1}(p) \xrightarrow{p} IPV_{t+1}(p) = \int_t^{t+1} \sigma^p(s)ds \text{ for } p \in (0,2) \tag{18}$$

$$RV_{t+1} \xrightarrow{p} QV_{t+1} = \int_t^{t+1} \sigma^2(s)ds + \sum \kappa^2(s) \tag{19}$$

$$RBP_{t+1} \xrightarrow{p} IPV_{t+1}(2) = \int_t^{t+1} \sigma^2(s)ds. \tag{20}$$

Note that the asymptotics operate within a fixed time interval by sampling more frequently. RV converges to quadratic variation, and the latter measures the ex post variation of the process regardless of the model or information set. Therefore, realized volatility is the relevant quantity to focus on the modeling and forecasting of volatility. For further details on the relationship between RV and the second moments of returns see Andersen, Bollerslev, Diebold and Labys (2003), Barndorff-Nielsen and Shephard (2002a,2005) and Meddahi (2003).

From these results, it follows that the jump component in QV_{t+1} can be estimated by $RV_{t+1} - RBP_{t+1}$. $RPV_{t+1}(p)$ for $p \in (0,2)$ and RBP_{t+1} are robust to jumps. Forsberg and Ghysels (2007), Ghysels, Santa-Clara and Valkanov (2006) and Ghysels and Sinko (2006) have found that absolute returns (power variation of order 1) improve volatility forecasting using criterions such as adjusted R^2 and Mean Squared Error. They argue that improvements are due to the higher predictability, less sampling error and a robustness to jumps.

4 Data

We investigate model forecasts for equity and exchange rate volatility over several forecast horizons. For equity we consider the S&P 500 index by using the Spyder (Standard & Poor's Depository Receipts), which is an Exchange Traded Fund that represents ownership in the S&P 500 Index. The ticker symbol is SPY. Since this asset is actively traded, it avoids the stale price effect of the S&P 500 index. The Spyder price transaction data are obtained from the Trade and Quotes (TAQ) database. After removing errors from the transaction data[8], a 5 minute grid from 9:30 to 16:00 was constructed by finding the closest transaction price before or equal to each grid point time. The first observation of the day occurring just after 9:30 was used for the 9:30 grid time. From this grid, 5 minute intraday log returns are constructed. The intraday return data was used to construct daily returns (open to close prices), and the associated realized volatility, realized bipower variation and realized power variation of order 0.5, 1 and 1.5 following the previous section. An adjusted estimator of RV and RBP to correct for market microstructure dynamics is

[8]Data was collected with a TAQ correction indicator of 0 (regular trade) and when possible a 1 (trade later corrected). We also excluded any transaction with a sale condition of Z, which is a transaction reported on the tape out of time sequence, and with intervening trades between the trade time and the reported time on the tape. We also checked any price transaction change that was larger than 3%. A number of these were obvious errors and were removed.

discussed below. Given the structural break found in early February 1997 in Liu and Maheu (2008) our data begins in February 6, 1997 and goes to March 30, 2004.[9] We reserve the first 35 observations as startup values for the models. The final data has 1761 observations.

High frequency foreign exchange data on the JPY-USD and DEM-USD spot rates are from Olsen Financial Technologies. We adopt the official conversion rate between DEM and Euro after January 1, 1999 to obtain the DEM-USD rate. Bid and ask quotes are recorded on a five minute grid when available. To fill in the missing values on the grid we take the closest previous bid and ask. The spot rate is taken as the logarithmic middle price for each grid point over a 24 hour day. The end of a day is defined as 21:00:00 GMT and the start as 21:05:00 GMT. Weekends (21:05:00 GMT Friday - 21:00:00 GMT Sunday) and slow trading dates (December 24-26, 31 and January 1-2) and the moving holidays Good Friday, Easter Monday, Memorial Day, July Fourth, Labor Day, Thanksgiving and the day after were removed. A few slow trading days were also removed. From the remaining data, 5 minute returns where constructed, as well as the daily volatility measures and the daily return (close to close prices). The sample period for FX data is from February 3, 1986 to December 30, 2002. JPY-USD data has 4192 observations. The DEM-USD data has 4190 observations. Conditioning on the first 35 observations leaves us 4157 observations (JPY-USD) and 4155 observations (DEM-USD).

4.1 Adjusting for Market Microstructure

It is generally accepted that there are dynamic dependencies in high-frequency returns induced by market microstructure frictions, see Bandi and Russell (2006), Hansen and Lunde (2006a), Oomen (2005) and Zhang, Mykland and Ait-Sahalia (2005) among others. The raw RV constructed from (16) can be an inconsistent estimator. To reduce the effect of market microstructure noise[10], we employ a kernel-based estimator suggested by Hansen and Lunde (2006a) which utilizes autocovariances of intraday returns to construct realized volatility as,

$$RV_t^q = \sum_{i=1}^{m} r_{t,i}^2 + 2\sum_{w=1}^{q}\left(1 - \frac{w}{q+1}\right)\sum_{i=1}^{m-w} r_{t,i}r_{t,i+w} \tag{21}$$

where $r_{t,i}$ is the ith logarithmic return during day t, and q is a small non-negative integer. The theoretical results concerning this estimator is due to Barndorff-Nielsen et al. (2006a). This Bartlett-type weights ensure a positive estimate, and Barndorff-Nielsen et al. (2006b) show that it is almost identical to the subsample-based estimator of Zhang, Mykland and Ait-Sahalia (2005).

We list the summary statistics for daily squared returns, unadjusted RV and the adjusted RV for $q = 1, 2$ and 3 in Table 1. As a benchmark, the average daily squared

[9]The main effect of the break is on the variance of log-volatility. We also investigated breaks in the JPY-USD and DEM-USD realized volatility data discussed below and found no evidence of parameter change.

[10]An alternative is to sample the price process at a lower frequency to minimize market microstructure contamination. However, the asymptotics in Section 3 suggest a loss of information in lower sampling frequencies.

return can be treated as an unbiased estimator of the mean of latent volatility. However, it is very noisy which can be seen from its large variance. The RV row lists the statistics for the unadjusted RV. The average difference between the mean of daily squared returns and unadjusted RV is fairly large. This suggests significant market microstructure biases. The adjusted RV provides an improvement. In our work we use $q = 3$. The time series of adjusted $\log(RV_t)$ measures are shown in Figure 1.

Market microstructure also contaminates bipower variation. As in Andersen et al. (2007) and Huang and Tauchen (2005), using staggered returns will decrease the correlation in adjacent returns induced by the microstructure noise. Following their suggestion, we use an adjusted bipower variation as

$$\widehat{RBP}_{t+1} = \frac{\pi}{2}\frac{m}{m-2}\sum_{j=3}^{m} |r_{t,j-2}||r_{t,j}|. \tag{22}$$

In the following we refer to adjusted RV_t^q as RV_t, and $\widehat{RBP}_{t+1}$ as RBP_{t+1}. Of course market microstructure may also affect power variation measures, but it is much harder to correct for and empirically may be less important (Ghysels and Sinko (2006)).

5 Predictors of Realized Volatility

This section provides a brief discussion of the potential predictors that could be used to forecast realized volatility. Although daily squared returns are a natural measure of volatility, as shown by Andersen and Bollerslev (1998) they are extremely noisy. A popular proxy for volatility that exploits intraday information is the range estimator used in Brandt and Jones (2006). It is defined as $\text{range}_t = \log(P_{H,t}/P_{L,t})$, where $P_{H,t}$ and $P_{L,t}$ are the intraday high and low price levels on day t. According to Alizadeh, Brandt and Diebold (2002), the log-range has an approximately Gaussian distribution, and is more efficient than daily squared returns.

Besides lagged values of realized volatility, previous work by Forsberg and Ghysels (2007), Ghysels, Santa-Clara and Valkanov (2006) and Ghysels and Sinko (2006) has shown power variation of order 1 to be a good predictor. Other orders may be useful. Figure 2 displays the sample autocorrelation function for $\log(RV_t)$, $\log(RPV_t(.5))$, $\log(RPV_t(1))$, $\log(RPV_t(1.5))$ and $\log(RBP_t)$ for the JPY-USD. The ACF for realized volatility is below all the others. Each of the power variation measures is more persistent over a wide range of lags.

Figure 3 displays estimates of $\text{corr}(\log(RV_t), \log(RPV_{t-i}(p)))$ as a function of p for different lag lengths $i = 1, 5, 10, 20$. The order of RPV ranges from 0.01 to 2 with increments of 0.01. Recall that $RPV_t(2) = RV_t$. The correlation is maximized with a power variation order less than 2 in each case. The largest correlation for the JPY-USD data are: 0.6732 ($i = 1, p = 1.39$); 0.4906 ($i = 5, p = 1.31$); 0.3866 ($i = 15, p = 1.25$); and 0.2862 ($i = 20, p = 1.01$). On the other hand, the correlation with realized bipower (not shown in the figure) is always lower. For example, $\text{corr}(\log(RV_t), \log(RBP_{t-i}))$, is 0.6658 ($i = 1$), 0.4820 ($i = 5$), 0.3785 ($i = 10$), and 0.2745 ($i = 20$).

Finally, Table 2 gives out-of-sample predictive likelihoods[11] and several forecast loss functions for a linear model discussed in the next section. All models have the common regressand of $\log(RV_t)$ from JPY-USD or DEM-USD but differ by the regressors. Included are versions with realized volatility, RPV(1), RPV(1.5), RBP, the daily range and daily squared returns. The range provides a considerable improvement upon daily squared returns, while RV, RPV, and RBP provide further improvements. Based on the predictive likelihoods, for the JPY-USD market $RPV(1)$ has a marginally better performance than RV. On the other hand the version with RBP is the best in the DEM-USD market. Clearly, there is risk in selecting any one specification.

Based on this discussion we will confine model averaging to different specifications featuring realized volatility, realized power variation and realized bipower variation, and will not consider daily squared returns or the range. The specifications are discussed in the next section.

6 Models

We consider two families of linear models. The first is based on the heterogeneous autoregressive (HAR) model of realized volatility by Corsi (2004). Corsi (2004) shows that this model can approximate many of the features of volatility including long-memory. Specifically, we use the logarithmic version (HAR-log) similar to Andersen, Bollerslev and Diebold (2007). Our benchmark model is

$$\begin{aligned} \log(RV_{t,h}) &= \beta_0 + \beta_1 \log(RV_{t-1,1}) + \beta_2 \log(RV_{t-5,5}) + \beta_3 \log(RV_{t-22,22}) \\ &+ \beta_J J_{t-1} + u_{t,h},\ u_{t,h} \backsim NID(0,\sigma^2), \end{aligned} \tag{23}$$

where $RV_{t,h} = \frac{1}{h}\sum_{i=1}^{h} RV_{t+i-1}$ is the h-step ahead average realized volatility. This model postulates three factors that affect volatility: a daily ($h = 1$), weekly ($h = 5$) and monthly ($h = 22$) factor. The importance of jumps have been recognized by Andersen, Bollerslev, and Diebold (2007), Huang and Tauchen(2005), and Tauchen and Zhou (2005) among others. All of the models include a jump term defined as

$$J_t = \begin{cases} \log(RV_t - RBP_t + 1) & \text{when } RV_t - RBP_t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

where we add 1 to ensure $J_t \geq 0$. For the S&P 500, an asymmetric term is included in all specifications and is defined as

$$L_t = \begin{cases} \log(RV_t + 1) & \text{when daily return} < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

To consider other specifications define $RPV_{t,h}(p) = \frac{1}{h}\sum_{i=1}^{h} RPV_{t+i-1}(p)$ and $RBP_{t,h} = \frac{1}{h}\sum_{i=1}^{h} RBP_{t+i-1}$ as the corresponding average realized power and bipower variation, respectively. Note the special case $RV_{t,h} = RPV_{t,h}(2)$. A summary of the specifications is

[11] Using the notation of Section 2.2 the predictive likelihood is computed as $\prod_{j=s}^{t} p(y_j|Y_{j-1})$, $s < t$ over the out-of-sample period.

listed in Table 3. The first panel displays HAR-type configurations. Each row indicates the regressors included in a model. A 1 indicates a daily factor (e.g. $\log(RPV_{t-1,1}(1)))$ a 2 means a daily and weekly factor (e.g. $\log(RPV_{t-1,1}(1)), \log(RPV_{t-5,5}(1)))$ and a 3 means a daily, weekly and monthly factor (e.g. $\log(RPV_{t-1,1}(1)), \log(RPV_{t-5,5}(1)), \log(RPV_{t-22,22}(1)))$ using the respective regressor in that column.

Models 1–5 are HAR-log specifications in logarithms of either RV, RPV(.5), RPV(1), RPV(1.5) or RBP. Models 6–41 provide mixtures of volatility HAR terms. A typical model would have regressors of $\log(RV_{t-h,h})$ and $\log(RPV_{t-h,h}(p))$ or $\log(RBP_{t-h,h})$ for $h = 1, 5$ and 22 as well as a jump term J_{t-1}. For instance, specification 20 has regressors $X_{t-1} = [1 \ \log(RV_{t-1,1}) \ \log(RV_{t-5,5}) \ \log(RPV_{t-1,1}(0.5)) \ \log(RPV_{t-5,5}(0.5)) \ \log(RPV_{t-22,22}(0.5)) \ J_{t-1}]$. In the case of equity, L_{t-1} is included, while it is omitted in the FX applications.[12]

The next set of models are based on autoregressive type specifications. In the second panel of Table 3, models 42–56 are AR specifications in logarithms of either RV, RPV(.5), RPV(1), RPV(1.5) or RBP. Models 57–72 provide AR models of mixtures of volatility terms. For example, model 70 includes 10 lags of daily RV, and 5 lags of daily RPV(1), and has the form

$$\begin{aligned}\log(RV_{t,h}) &= \beta_0 + \beta_1 \log(RV_{t-1}) + \cdots + \beta_{10} \log(RV_{t-10}) + \beta_{11} \log(RPV_{t-1}(1)) \\ &+ \cdots + \beta_{15} \log(RPV_{t-5}(1)) + \beta_J J_{t-1} + u_{t,h}, \ u_{t,h} \backsim NID(0, \sigma^2). \end{aligned} \tag{26}$$

In total there are 72 different specifications that enter the model averages.

When $h > 1$, we ensure that our predictions of $\log(RV_{t,h})$ are true out-of-sample forecasts. For instance, for (23) if we used data till time t for estimation, the last regressand would be $\log(RV_{t-h+1,h})$, then the forecast is computed based on this information set for $E[\log(RV_{t+1,h})|RV_t, RV_{t-1}, ...]$. This is the predictive mean estimated following (3).

7 Results

We do Bayesian model comparison, and model averaging conditional on the following uniformative proper priors: $\beta \sim N(0, 100I)$, and $\sigma^2 \sim IG(0.001/2, 0.001/2)$. For the linear models the first 100 Gibbs draws were discarded and the next 5000 were collected for posterior inference. The output from the Gibbs sampler mixed well with a fast decaying autocorrelation function.

There are $K = 72$ specifications (Table 3) that enter the model averages. In performing Bayesian model averaging we follow Eklund and Karlsson (2007) and use predictive measures to combine individual models. We set the model probabilities to $P(M_k) = 1/K, k = 1, ..., K$ at observation 500.[13] Thereafter we update model probabilities according to Bayes rule. The training sample of 500 observations only affects BMA and Section 7.3 shows the results are robust to different sample sizes. The effect of the training sample is to put more weight on recent model performance. As Eklund and Karlsson (2007) show this

[12]Preliminary work showed no evidence of an asymmetric effect in FX data.

[13]For example, using (9) we set $P(M_k|Y_{500}) \equiv 1/K$ for $k = 1, ..., K$ and build up the model probabilities as new data arrives.

provides protection against in-sample overfitting and can improve forecast performance. To compute the predictive likelihood from the end of the training sample to the last in-sample observation we use the Chib (1995) method[14], see the Appendix for details, thereafter we update model probabilities period by period using (7) and (9).

The in-sample observations are 1000 for S&P 500, and 3000 for both JPY-USD and DEM-USD. The out-of-sample period extends from March 16, 2001 to March 30, 2004 (761 observations) for S&P 500, May 13, 1998 to December 30, 2002 (1157 observations) for JPY-USD, and May 15, 1998 to December 30, 2002 (1155 observations) for DEM-USD.

7.1 Bayesian Model Averaging

The predictive likelihood for BMA along with several benchmark alternatives is displayed in the left panel of Table 4.[15] Included are autoregressive models in $\log(RV_t)$, the HAR-log specification in (23), and a simple model average (SMA) which assumes equal weighting across all models through time.

Beginning with the S&P 500, BMA is very competitive. When $h = 1$, the log predictive likelihood is larger than all the benchmarks except for the AR(15) model, where BMA and AR(15) have very close values. The evidence for $h = 5$ and $h = 10$ is stronger. The $\log(PL)$ for the BMA is about 6 and 16 points larger than those from the best benchmarks. We also find the SMA is dominated by many of the benchmarks, and it has poor performance compared with BMA. The difference between these model averages is that BMA weights models based on predictive content while the SMA ignores it.

For JPY-USD market the Bayesian model average outperforms all the benchmarks for each time horizon. Compared to the HAR-log model used in Andersen, Bollerslev, and Diebold (2007) the log predictive Bayes factor in favor of BMA is 10.0, 6.9, and 12.6 for $h = 1, 5$, and 10, respectively. It also performs well for DEM-USD when $h = 1$. However for $h = 5$, and $h = 10$ the BMA is second to the AR(15) model.

In summary, in 6 out of 9 cases BMA delivers the best performance in terms of density forecasts, and when it is not the top model it is a close second.

7.2 Out-of-sample Point Forecasts

Although we focus on the predictive likelihood to measure predictive content, it is interesting to consider the out-of-sample point forecasts of average log volatility based on the predictive mean. Recent work by Hansen and Lunde (2006b) and Patton (2006) has emphasized the importance of using a robust criterion, such as mean squared error, to compare model forecasts against an imperfect volatility proxy like realized volatility. Therefore, the right panel of Table 4 reports the root mean squared forecast error (RMSE). The out-of-sample period corresponds exactly to the period used to calculate the predictive likelihood. Forecast performance is listed for the same set of models as in previous

[14] For instance, using the notation in Section 2, the log predictive likelihood for $y_{501}, ..., y_T$, where y_T is the last in-sample observation, can be decomposed as $\log(p(Y_T)) - \log(p(Y_{500}))$ and each term estimated by Chib (1995).

[15] The full set of results for individual models is available upon request from the authors.

section.

BMA performs well against the benchmarks. For S&P 500, BMA is better than all the benchmarks for $h = 5$, and 10, and second best when $h = 1$. For JPY-USD, BMA is the top performer. As with our previous results, BMA is weaker in the DEM-USD market. In this case the SMA and the AR(15) perform well.

BMA is competitive for all data series and forecast horizons, although any improvements it offers are modest.[16] In 5 of the 9 cases BMA has the lowest RMSE.

7.3 Training Sample

The above results are based on model combination using predictive measures. As previously mentioned, we set the model probabilities to $P(M_k) = 1/K, k = 1, ..., K$ at observation 500, thereafter model probabilities are updated according to Bayes rule. This training sample of 500 observations puts more weight on recent model performance and less on past model performance. To investigate the robustness of BMA to the size of the training sample, we calculate the results with different sample sizes as well as no training sample. Results are summarized in Table 5 for DEM-USD with similar results for the other data. Focusing on the predictive likelihood, we see that using more recent predictive measures has some benefit for $h = 10$.

7.4 The Role of Power Variation

In this section we investigate why BMA performs well. One reason is that it weights individual models based on past predictive content through the model probabilities. Over time model performance changes and BMA responds to it. Another possibility is that the specifications with power variation are better than existing models that only use RV.

To focus on this latter question we divide all the models that enter BMA into 3 groups according to their regressors. These are the "RV only group" which includes all models that have regressors constructed from only lag terms of $\log(RV_t)$. The "RPV group" includes all models in which at least 1 RPV regressor is used, and the "RBP group" is all models that have RBP regressors. Table 6 reports the predictive likelihood of the best models within each group for each of the forecast horizons $h = 1, 5$, and 10. The rank of the model among the $K = 72$ alternatives is also displayed.

Including RPV terms can improve forecasting power. For S&P 500 when $h = 1$, if we exclude RPV regressors, the best individual model has a log-predictive likelihood -527.0 with a rank of 2 out of the full 72 models. Among the models with RPV, the best one has $\log(PL)$ of -526.0 and it is also the best model overall. For $h = 5$, including RPV

[16] The statistical importance of the relative RMSE values could be assessed based on a posterior predictive assessment (Gelman et al. (1996)). Using the posterior estimates for the model average based on the full set of data the steps are: 1) a draw is taken from the model probabilities; 2) given this model, a draw is taken from the respective posterior for the parameters; 3) using this parameter and model, artificial log-realized-volatility data is generated. Each of the models and the BMA is estimated and out-of-sample forecasts are produced using the generated data. This produces a RMSE for each model. Repeating this many times provides a joint distribution of RMSE's for all models which can be used to assess the likelihood of observed results.

increases $\log(PL)$ from -310.1 for the RV group to -302.2 (rank from 13 to 1). For $h = 10$, the best RPV model achieves a -299.1 with rank 1, while the best RV model is -316.8 with rank 38.

The results from JPY-USD market provide very similar supportive evidence for the inclusion of RPV. In Panel B, the best models in the RPV group dominate those in the RV only group across h with much higher predictive likelihood values (-835.6 vs -844.6 for $h = 1$, -648.0 vs -659.4 for $h = 5$ and -647.9 vs -661.5 for $h = 10$) and ranking $(1, 1, 1$ compared with $40, 22$ and 22 for $h = 1, 5, 10)$. For DEM-USD data, when $h = 1$, the best model is from the RPV group. When $h = 5$ and 10, the top specification has only RV terms, however, the second best includes RPV.

In many cases models with RBP improve upon those with only RV. They increase log-predictive likelihood for S&P 500 when $h = 10$, for JPY-USD across all forecast horizons, and for DEM-USD when $h = 1$. However, the improvement is not as large as models with RPV terms.

In summary, as a group, specifications with RPV regressors deliver forecast improvements. Bipower variation delivers relatively smaller improvements over models with only realized volatility regressors. However, specifications with RPV or RBP terms also display considerable model uncertainty, but BMA gives them larger weights when they perform well.

7.5 Model Risk

The message of this paper is that one should model average to reduce risk. Models that perform well in one market and forecast horizon generally do not in other cases. In fact they often perform poorly. Table 7 displays the differences in top models over markets and forecast horizons. For instance, the top JPY-USD $h = 5$ model, which is labeled "HAR: RV=3, PV(1)=2", achieves $\log(PL) = -648.0$. However, a rather different AR(15) in realized volatility is the best model in DEM-USD, $h = 5$. In fact, the previous model does relatively poorly with only $\log(PL) = -478.40$, about 6 points worse than the AR(15). This example also illustrates the need to have a wide range of different model specifications, and not just the apparent top specifications that include realized power variation.

7.6 BMA with GARCH Effects

Bollerslev et al. (2007) find evidence of GARCH dynamics in time-series models of log-volatility. To investigate the importance of this for our results we consider the same set of models but include a GARCH(1,1) specification for each model. The new class of models extends those of Section 2.1 to

$$y_t = X_{t-1}\beta + \sqrt{h_t}u_t, \quad u_t \sim N(0,1) \tag{27}$$

$$h_t = \omega + a\,(y_{t-1} - X_{t-2}\beta)^2 + bh_{t-1}. \tag{28}$$

Gibbs sampling is not readily available for this model.[17] Instead, we adopt the random walk Metropolis-Hastings algorithm following Vrontos et al. (2000). The details of estimation for this model and the predictive likelihood computation are presented in the appendix.

Table 8 compares the out-of-sample log predictive likelihood for models with GARCH errors for $h = 1$. All models that enter the model average have GARCH(1,1), including the benchmark specifications. Compared with Table 4, the GARCH alternatives dominate their homoskedastic counterparts. For example, the HAR-log-GARCH model improves upon the HAR-log with increases in the $\log(PL)$ of 7.3 for S&P 500, 33.5 for JPY-USD, and 34.8 for DEM-USD.

Consistent with our previous results, BMA provides overall good performance in extracting predictive content from the underlying models. For JPY and DEM data, it is the top specification while it is a close second for the S&P 500. In summary, averaging over a better class of models, BMA remains a useful approach to reduce model risk and provide consistently good density forecasts.

8 Conclusion

This paper advocates a Bayesian model averaging approach to forecasting volatility. Recent research provides a range of potential regressors. Model averaging reduces the risk compared to selecting any one particular model. Bayesian model averaging, ranked by any of the criteria studied in this paper, is the top performer, or very close to it. This occurs over 3 different markets of realized volatility and 3 different forecast horizons. Density forecasts show the most improvement while point forecasts show only modest gains over existing benchmark models.

We find that Bayesian model averaging provides further improvements to density forecasts when we move away from linear models and average over specifications that allow for GARCH effects. Other models that may be useful to average over include specifications with nonlinear terms and fat-tailed innovations.

9 Appendix

9.1 Marginal Likelihood

In this appendix we review the estimation of the marginal likelihood following Chib (1995). Denote the parameters $\theta = \{\beta, \sigma^2\}$. A rearrangement of Bayes rule gives the marginal likelihood, $ML(Y_T)$, as

$$\log ML\,(Y_T) = \log p\left(Y_T|\beta^*, \sigma^{2*}\right) + \log p\left(\beta^*, \sigma^{2*}\right) - \log p\left(\beta^*, \sigma^{2*}|Y_T\right) \tag{29}$$

where $p\left(Y_T|\beta^*, \sigma^{2*}\right)$ is the likelihood function, $p\left(\beta^*, \sigma^{2*}\right)$ is the prior and $p\left(\beta^*, \sigma^{2*}|Y_T\right)$ is the posterior ordinate, each evaluated at β^*, σ^{2*} which we set to the posterior mean. The

[17] Bauwens and Lubrano (1998) use a Griddy Gibbs sampler for GARCH models. This involves a numerical inversion of the conditional posterior densities.

likelihood and prior are available and to compute the posterior ordinate note

$$p\left(\beta^*, \sigma^{2*}|Y_T\right) = p\left(\beta^*|Y_T\right) p\left(\sigma^{2*}|Y_T, \beta^*\right). \tag{30}$$

The first term at the right hand side is,

$$p\left(\beta^*|Y_T\right) = \int p\left(\beta^*|Y_T, \sigma^2\right) p\left(\sigma^2|Y_T\right) d\sigma^2 \tag{31}$$

and can be estimated as $p\widehat{\left(\beta^*|Y_T\right)} = \frac{1}{N}\sum_{i=1}^{N} p\left(\beta^*|Y_T, \sigma^{2(i)}\right)$, where the draws $\{\sigma^{2(i)}\}_{i=1}^{N}$ are available directly from our Gibbs estimation step, and the conditional density $p\left(\sigma^{2*}|Y_T, \beta^*\right)$ is inverse-gamma as in Section 2.1 given β^*.

9.2 Estimation of Models with GARCH

We set all priors in the regression equation as before, they are independent normal $N(0, 100)$. The GARCH parameters have independent normal $N(0, 100)$ truncated to $\omega > 0, a \geq 0, b \geq 0$, and $a + b < 1$. These priors are uninformative.

Denote all the parameters by $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_L\}$. Since the conditional distributions for some of the model parameters are unknown, Gibbs sampling is not available. Instead we use a random walk Metropolis-Hastings algorithm. If we denote all the parameters except for γ_l as $\Gamma_{-l} = \{\gamma_1, \cdots, \gamma_{l-1}, \gamma_{l+1}, \cdots, \gamma_L\}$, we sample a new γ_l given Γ_{-l} fixed. With Γ as the previous value of the chain we iterate on the following steps:

Step 1: Propose a new Γ' according to $\Gamma'_{-l} = \Gamma_{-l}$, with element l determined as

$$\gamma'_l = \gamma_l + e_l, \;\; e_l \sim N(0, \xi_l^2). \tag{32}$$

Step 2: Accept Γ' with probability

$$\min\left\{\frac{p(Y_T|\Gamma')p(\Gamma')}{p(Y_T|\Gamma)p(\Gamma)}, 1\right\}$$

and otherwise reject. $p(\Gamma)$ is the prior, and

$$\log p\left(Y_T|\Gamma\right) = \sum_{t=1}^{T}\left[-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(h_t) - \frac{(y_t - X_{t-1}\beta)^2}{2h_t}\right] \tag{33}$$

where $h_t = \omega + a(y_t - X_{t-1}\beta)^2 + bh_{t-1}$.[18] Each ξ_l^2 is selected to give an acceptance frequency between 0.3–0.5. Running Step 1-2 above for all the parameters $l = 1, \cdots, L$, we obtain a new draw Γ which is one iteration. We perform 200,000 iterations and use the last 100,000 for posterior inference.

For the marginal likelihood we use the method of Gelfand and Dey (1994) adapted by Geweke (2005) (Section 8.2.4). This estimate is based on $\frac{1}{N}\sum_{i=1}^{N} g(\Gamma^{(i)})/[p(Y_T|\Gamma^{(i)})p(\Gamma^{(i)})] \to p(Y_T)^{-1}$ as $N \to \infty$, where $p(Y_T|\Gamma)$ is the likelihood, and $g(\Gamma^{(i)})$ is a truncated multivariate Normal. Note that the prior, likelihood and $g(\Gamma)$ must contain all integrating constants. Finally, to avoid underflow/overflow we use logarithms in this calculation.

[18] To start up the conditional variance we set $h_0 = \omega/(1 - a - b)$.

References

[1] Alizadeh S, Brandt MW and Diebold FX. 2002. Range-Based Estimation of Stochastic Volatility Models. *Journal of Finance* 57: 1047-1091.

[2] Andersen T and Bollerslev T. 1998. Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts. *International Economic Review* 39: 885-905.

[3] Andersen TG, Bollerslev T and Diebold FX. 2007. Roughing It Up: Including Jump Components in the Measurement, Modeling and Forecasting of Return Volatility. *Review of Economics and Statistics*, 89, 701-720.

[4] Andersen TG, Bollerslev T, Diebold FX and Ebens H. 2001. The distribution of realized stock return volatility. *Journal of Financial Economics* 61: 43-76.

[5] Andersen TG, Bollerslev T, Diebold FX and Labys P. 2003. Modeling and forecasting realized volatility. *Econometrica* 71(2): 579-625.

[6] Andersen TG, Bollerslev T, Diebold FX and Labys P. 2001. The distribution of exchange rate volatility. *Journal of the American Statistical Association* 96: 42-55.

[7] Andersen TG, Bollerslev T and Meddahi N. 2005. Correcting the errors: volatility forecast evaluation using high-frequency data and realized volatilities. *Econometrica* 73(1): 279-296.

[8] Andreou E and Ghysels E. 2002. Rolling-sample volatility estimators: some new theoretical, simulation, and empirical results. *Journal of Business and Economic Statistics* 20(3): 363-76.

[9] Bandi FM and Russell JR. 2006. Separating Microstructure Noise from Volatility. *Journal of Financial Economics*: forthcoming.

[10] Barndorff-Nielsen OE, Hansen P, Lunde A and Shephard N. 2006a. Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise. OFRC Working Papers Series, Oxford Financial Research Centre.

[11] Barndorff-Nielsen OE, Hansen P, Lunde A and Shephard N. 2006b. Subsampling realised kernels. OFRC Working Papers Series, Oxford Financial Research Centre.

[12] Barndorff-Nielsen OE and Shephard N. 2005. Variation, jumps, market frictions and high frequency data in financial econometrics. Working paper, Nuffield College, University of Oxford.

[13] Barndorff-Nielsen OE and Shephard N. 2004. Power and bipower variation with stochastic volatility and jumps. *Journal of Financial Econometrics* 2(1): 1-37.

[14] Barndorff-Nielsen OE and Shephard N. 2002a. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society*, Series B, 64: 253-280.

[15] Barndorff-Nielsen OE and Shephard N. 2002b. Estimating quadratic variation using realised variance. *Journal of Applied Econometrics* 17: 457-477.

[16] Bauwens L. and Lubrano M. 1998 Bayesian Inference on GARCH Models Using the Gibbs Sampler. *The Econometrics Journal* 1(1): 23-46.

[17] Bollerslev T, Kretschmer U, Pigorsch C and Tauchen G. 2007. A Discrete-Time Model for Daily S&P 500 Returns and Realized Variations: Jumps and Leverage Effects. forthcoming *Journal of Econometrics*.

[18] Bollerslev T and Zhou H. 2002. Estimating stochastic volatility diffusion using conditional moments of integrated volatility. *Journal of Econometrics* 109(1): 33-65.

[19] Box GEP. 1980. Sampling and Bayes' inference in scientific modelling and robustness. *Journal of the Royal Statistical Society* A, 143: 383-430.

[20] Brandt MW and Jones CS. 2006. Volatility Forecasting with Range-Based EGARCH Models. *Journal of Business and Economic Statistics* 24: 470-486.

[21] Chib S. 1995. Marginal likelihood from the Gibbs output. *Journal of the American Statistical Association* 90(432): 1313–1321.

[22] Chib S. 2001. Markov Chain Monte Carlo Methods: Computation and Inference. In *Handbook of Econometrics*: volume 5, Heckman JJ and Leamer E, eds. North Holland, Amsterdam, 3569-3649.

[23] Corsi F. 2004. A simple long memory model of realized volatility. Working paper, University of Southern Switzerland.

[24] Ding Z, Granger CWJ and Engle RF. 1993. A long memory property of stock market returns and a new model. *Journal of Empirical Finance* 1: 83-106.

[25] Eklund J and Karlsson S. 2007. Forecast Combination and Model Averaging Using Predictive Measures. *Econometric Reviews* 26(2-4): 329 - 363.

[26] Fernandez C, Ley E and Steel MJF. 2001. Model Uncertainty in Cross-Country Growth Regressions. *Journal of Applied Econometrics* 16(5): 563-576.

[27] Fleming J, Kirby C and Ostdiek B. 2003. The economic value of volatility timing using 'realized' volatility. *Journal of Financial Economics* 67: 473-509.

[28] Forsberg L and Ghysels E. 2007. Why Do Absolute Returns Predict Volatility So Well? *Journal of Financial Econometrics* 5: 31-67.

[29] French K, Schwert GW and Stambaugh RF. 1987. Expected Stock Returns and Volatility. *Journal of Financial Economics* 19: 3-29.

[30] Gelfand AE and Dey D. 1994. Bayesian model choice: asymptotic and exact calculations. *Journal Royal Statistical Society* B, 56: 501-514.

[31] Gelman A, Meng X and Stern H. 1996. Posterior Predictive Assessment of Model Fitness via Realized Discrepancies. *Statistica Sinica*. 6(4) 733-807.

[32] Geweke J. 2005. *Contemporary Bayesian Econometrics and Statistics*. John Wiley & Sons Ltd.

[33] Geweke J. 1995. Bayesian comparison of Econometric Models. Working paper, Research Department, Federal Reserve Bank of Minneapolis.

[34] Geweke J and Whiteman C. 2005. Bayesian Forecasting. Forthcoming in *Handbook of Economic Forecasting*, Graham E, Granger C and Timmermann A, eds. Elsevier.

[35] Ghysels E, Santa-Clara P and Valkanov R. 2006. Predicting volatility: How to get most out of returns data sampled at different frequencies. *Journal of Econometrics,* 131: 59-95.

[36] Ghysels E and Sinko A. 2006. Comment on Hansen and Lunde JBES paper. *Journal of Business and Economic Statistics* 24(2): 192-194.

[37] Ghysels E, Sinko A and Valkanov R. 2007. MIDAS Regressions: Further Results and New Directions. *Econometric Reviews* 26(1): 53 - 90.

[38] Gordon S. 1997. Stochastic trends, deterministic trends, and business cycle turning points. *Journal of Applied Econometrics* 12: 411-434.

[39] Hansen PR and Lunde A. 2006a. Realized Variance and Market Microstructure Noise. *Journal of Business and Economic Statistics* 24(2) 127-161.

[40] Hansen, PR and Lunde A. 2006b. Consistent ranking of volatility models. *Journal of Econometrics* 131: 97-121.

[41] Hibon M and Evgeniou T. 2004. To combine or not combine: selecting among forecasts and their combinations. *International Journal of Forecasting* 21(1): 15-24.

[42] Hoeting JA, Madigan D, Raftery A and Volinsky CT. 1999. Bayesian Model Averaging: A Tutorial. *Statistical Science* 14(4): 382-417.

[43] Hsieh D. 1991. Chaos and Nonlinear Dyanmics: Application to Financial Markets. *Journal of Finance* 46: 1839-1877.

[44] Huang X and Tauchen G. 2005. The Relative Contribution of Jumps to Total Price Variance. *Journal of Financial Econometrics* 3: 456-499.

[45] Jacobson T and Karlsson S. 2004. Finding good predictors for Inflation: A Bayesian Model Averaging Approach. forthcoming, *Journal of Forecasting.*

[46] Kass RE and Raftery AE. 1995. Bayes factors and model uncertainty. *Journal of the American Statistical Association* 90: 773-795.

[47] Koop G. 2003. *Bayesian Econometrics*. John Wiley & Sons Ltd.

[48] Koop G and Potter S. 1999. Bayes factors and nonlinearity: Evidence from economic time series. *Journal of Econometrics* 88: 251-282.

[49] Koop G and Potter S. 2004. Forecasting in dynamic factor models using Bayesian model averaging. *The Econometrics Journal* 7(2): 550-565.

[50] Koopman SJ, Jungbacker B and Hol E. 2005. Forecasting daily variability of the S&P 100 stock index using historical, realised and implied volatility measurements. *Journal of Empirical Finance* 12(3): 445-475.

[51] Liu C and Maheu J. 2008. Are there structural break in Realized Volatility? *forthcoming Journal of Financial Econometrics*.

[52] Maheu J and McCurdy TH 2002. Nonlinear Features of Realized FX Volatility. *Review of Economics and Statistics* 84(4): 668-681.

[53] Martens M and van Dijk D.J.C. and de Pooter M. 2004. Modeling and forecasting S&P 500 volatility: Long memory, structural breaks and nonlinearity. Discussion Paper, Tinbergen Institute.

[54] Meddahi N. 2002. A Theoretical Comparison Between Integrated and Realized Volatility. *Journal of Applied Econometrics* 17: 479-508.

[55] Meddahi N. 2003. ARMA Representation of Integrated and Realized Variances. *The Econometrics Journal* 6: 334-355.

[56] Min C and Zellner A. 1993. Bayesian and non-Bayesian methods for combining models and forecasts with applications to forecasting international growth rates. *Journal of Econometrics* 56(1-2): 89-118.

[57] Oomen RCA. 2005. Properties of bias-corrected realized variance under alternative sampling schemes. *Journal of Financial Econometrics* 3(4): 555-577.

[58] Patton A. 2006. Volatility Forecast Comparison using Imperfect Volatility Proxies. Working paper 175, Quantitative Finance Research Centre, University of Technology Sydney.

[59] Pesaran MH and Zaffaroni P. 2005. Model Averaging and Value-at-Risk Based Evaluation of Large Multi-Asset Volatility Models for Risk Management. CEPR Discussion Papers 5279.

[60] Raftery A.E. and Madigan D. and Hoeting J.A. 1997. Bayesian Model Averaging for Linear Regression Models. *Journal of the American Statistical Association* 92(437) 179-191.

[61] Schwert GW. 1989. Why Does Stock Market Volatility Change Over Time? *Journal of Finance* 44: 1115-1154.

[62] Tauchen G and Zhou H. 2005. Identifying Realized Jumps on Financial Markets. Working Paper, Duke University.

[63] Vrontos I. D. and Dellaportas P. and Politis D. N. 2000. Full Bayesian Inference for GARCH and EGARCH Models. *Journal of Business and Economics Statistics* 18(2), 187-198.

[64] Wright JH. 2003. Forecasting U.S. inflation by Bayesian Model Averaging. International Finance Discussion Paper, The Federal Reserve Board.

[65] Zhang L and Mykland P and Ait-Sahalia Y. 2005. A tale of two time scales: determining integrated volatility with noisy high-frequency data. *Journal of American Statistical Association* 100(472), 1394-1411.

Table 1: Summary Statistics for Measures of Volatility

S&P 500				
Measures	Mean	Variance	Min	Max
r^2	1.3136	11.8238	0.0000	67.8305
RV	1.6332	3.5078	0.0766	33.2162
RV^1	1.4429	3.2324	0.0714	30.7842
RV^2	1.3536	2.8899	0.0635	25.2236
RV^3	1.3076	2.8392	0.0572	26.3249
JPY-USD				
	Mean	Variance	Min	Max
r^2	0.5253	2.0089	0.0000	52.9709
RV	0.6160	0.5963	0.0316	32.5053
RV^1	0.5801	0.7541	0.0244	39.7883
RV^2	0.5622	0.8262	0.0212	42.9220
RV^3	0.5491	0.8473	0.0184	43.6814
DEM-USD				
	Mean	Variance	Min	Max
r^2	0.4744	0.8366	0.0000	12.4406
RV	0.5645	0.3157	0.0347	15.3277
RV^1	0.5298	0.3234	0.0341	14.9701
RV^2	0.5122	0.3270	0.0297	14.9052
RV^3	0.4987	0.3244	0.0280	14.8933

S&P 500 data is from February 6, 1997 to March 30, 2004 (1796 observations). JPY-USD data, February 3, 1986 to December 30, 2002 (4192 observations). DEM-USD data, February 3, 1986 to December 30, 2002 (4190 observations). r^2 is the daily squared return, RV is unadjusted realized volatility, RV^1, RV^2 and RV^3 are adjusted RV^q for q=1,2 and 3 following equation (21).

Table 2: Model Comparison Using Different Volatility Predictors for FX Volatility

JPY-USD

	Regressors				
	RV	$RPV(1)$	RBP	Range	Squared Return
$\log(PL)$	-845.32	-845.12	-856.87	-863.47	-997.31
RMSE	0.4918	0.4864	0.4949	0.5025	0.5620
MAE	0.3659	0.3601	0.3684	0.3851	0.4285
R^2	0.5419	0.5594	0.5498	0.5193	0.4141

DEM-USD

	Regressors				
	RV	$RPV(1.5)$	RBP	Range	Squared Return
$\log(PL)$	-754.71	-752.04	-748.06	-765.03	-872.84
RMSE	0.4467	0.4450	0.4437	0.4521	0.4937
MAE	0.3346	0.3374	0.3311	0.3889	0.3729
R^2	0.3954	0.3988	0.3997	0.3996	0.3085

This table compares the out-of-sample forecasting power of different regressors using JPY-USD and DEM-USD data. The out-of-sample period is May 13, 1998 to December 30, 2002 (1157 observations) for JPY-USD, and May 15, 1998 to December 30, 2002 (1155 observations) for DEM-USD. The common model is a HAR-log

$$\log(RV_t) = \beta_0+\beta_1 \log(V_{t-1,1})+\beta_2 \log(V_{t-5,5})+\beta_3 \log(V_{t-22,22})+u_t, \quad u_t \sim N(0,\sigma^2),$$

where $V_{t,h} = \frac{1}{h}\sum_{i=1}^{h} W_{t+i-1}$. For the JPY-USD column 2 sets $W_{t-1} = RV_{t-1}$, column 3 $W_{t-1} = RPV_{t-1}(1)$, column 4 $W_{t-1} = RBP_{t-1}$, column 5 $W_{t-1} = \text{range}_{t-1} = \log(P_{H,t-1}/P_{L,t-1})$, and column 6 $W_{t-1} = r_{t-1}^2$, the daily squared return. It is identical for the DEM-USD except that column 3 sets $W_{t-1} = RPV_{t-1}(1.5)$. $P_{H,t-1}$ and $P_{L,t-1}$ are the intraday high and low price levels on day $t-1$. For the out-of-sample period we report the log predictive likelihood (PL), root mean square error (RMSE) and mean absolute error (MAE) for the predictive mean, and the R^2 from a forecast regression of realized volatility on a constant and the predictive mean.

Table 3: Model Specifications

HAR-type						AR-type					
Model	RV	RPV(.5)	RPV(1)	RPV(1.5)	RBP	Model	RV	RPV(.5)	RPV(1)	RPV(1.5)	RBP
1	3	0	0	0	0	42	5	0	0	0	0
2	0	3	0	0	0	43	0	5	0	0	0
3	0	0	3	0	0	44	0	0	5	0	0
4	0	0	0	3	0	45	0	0	0	5	0
5	0	0	0	0	3	46	0	0	0	0	5
6	1	1	0	0	0	47	10	0	0	0	0
7	1	2	0	0	0	48	0	10	0	0	0
8	1	3	0	0	0	49	0	0	10	0	0
9	1	0	1	0	0	50	0	0	0	10	0
10	1	0	2	0	0	51	0	0	0	0	10
11	1	0	3	0	0	52	15	0	0	0	0
12	1	0	0	1	0	53	0	15	0	0	0
13	1	0	0	2	0	54	0	0	15	0	0
14	1	0	0	3	0	55	0	0	0	15	0
15	1	0	0	0	1	56	0	0	0	0	15
16	1	0	0	0	2	57	5	1	0	0	0
17	1	0	0	0	3	58	5	0	1	0	0
18	2	1	0	0	0	59	5	0	0	1	0
19	2	2	0	0	0	60	5	0	0	0	1
20	2	3	0	0	0	61	5	5	0	0	0
21	2	0	1	0	0	62	5	0	5	0	0
22	2	0	2	0	0	63	5	0	0	5	0
23	2	0	3	0	0	64	5	0	0	0	5
24	2	0	0	1	0	65	10	1	0	0	0
25	2	0	0	2	0	66	10	0	1	0	0
26	2	0	0	3	0	67	10	0	0	1	0
27	2	0	0	0	1	68	10	0	0	0	1
28	2	0	0	0	2	69	10	5	0	0	0
29	2	0	0	0	3	70	10	0	5	0	0
30	3	1	0	0	0	71	10	0	0	5	0
31	3	2	0	0	0	72	10	0	0	0	5
32	3	3	0	0	0						
33	3	0	1	0	0						
34	3	0	2	0	0						
35	3	0	3	0	0						
36	3	0	0	1	0						
37	3	0	0	2	0						
38	3	0	0	3	0						
39	3	0	0	0	1						
40	3	0	0	0	2						
41	3	0	0	0	3						

The first panel displays HAR-type model configurations. A 1 indicates a daily factor (e.g. $\log(RPV_{t-1,1}(1))$) a 2 means a daily and weekly factor (e.g. $\log(RPV_{t-1,1}(1)), \log(RPV_{t-5,5}(1))$) and a 3 means a daily, weekly and monthly factor (e.g. $\log(RPV_{t-1,1}(1)), \log(RPV_{t-5,5}(1)), \log(RPV_{t-22,22}(1))$) using the respective regressor in that column. The second panel is for AR-type models. Each row lists the number of lagged regressors from the respective column. All specifications include a jump term and the S&P 500 has a leverage term.

Table 4: Out-of-Sample Forecasts, $\log(RV_{t,h})$

A: S&P 500

	log(PL)			RMSE		
Model	$h = 1$	$h = 5$	$h = 10$	$h = 1$	$h = 5$	$h = 10$
AR(5)	-535.9	-332.0	-341.2	0.4865	0.3652	0.3757
AR(10)	-529.1	-316.0	-320.2	0.4819	0.3564	0.3635
AR(15)	-527.0	-310.1	-316.8	0.4803	0.3523	0.3610
HAR-log	-534.4	-320.2	-331.7	0.4855	0.3595	0.3715
SMA	-537.0	-319.0	-317.5	0.4862	0.3545	0.3597
BMA	-527.7	-304.4	-300.7	0.4808	0.3518	0.3545

B: JPY-USD

	log(PL)			RMSE		
Model	$h = 1$	$h = 5$	$h = 10$	$h = 1$	$h = 5$	$h = 10$
AR(5)	-847.6	-675.4	-693.2	0.4955	0.4293	0.4429
AR(10)	-844.6	-666.4	-675.2	0.4943	0.4257	0.4355
AR(15)	-845.6	-661.0	-664.3	0.4947	0.4232	0.4311
HAR-log	-846.6	-659.4	-661.5	0.4950	0.4224	0.4303
SMA	-838.1	-663.8	-671.5	0.4904	0.4220	0.4324
BMA	-836.6	-652.5	-648.9	0.4899	0.4198	0.4258

C: DEM-USD

	log(PL)			RMSE		
Model	$h = 1$	$h = 5$	$h = 10$	$h = 1$	$h = 5$	$h = 10$
AR(5)	-749.7	-493.3	-422.0	0.4456	0.3565	0.3345
AR(10)	-745.6	-476.3	-400.0	0.4447	0.3514	0.3275
AR(15)	-743.4	-472.1	-394.5	0.4436	0.3497	0.3253
HAR-log	-748.9	-483.6	-410.2	0.4455	0.3535	0.3307
SMA	-750.6	-494.5	-422.5	0.4415	0.3514	0.3287
BMA	-742.1	-475.0	-397.7	0.4423	0.3514	0.3279

This table reports the out-of-sample log predictive likelihood ($\log(PL)$), and the out-of-sample root mean square forecast error (RMSE) for the predictive mean. The results are for Bayesian Model Averaging (BMA), a simple equally-weighted model average (SMA), a HAR-log model (23) and several AR benchmarks using $\log(RV_t)$. The out-of-sample period is from March 16, 2001 to March 30, 2004 (761 observations) for S&P 500, May 13, 1998 to December 30, 2002 (1157 observations) for JPY-USD, and May 15, 1998 to December 30, 2002 (1155 observations), for DEM-USD.

Table 5: Robustness to Training Sample Size, DEM-USD

	h=1		h=5		h=10	
Obs	$\log(PL)$	RMSE	$\log(PL)$	RMSE	$\log(PL)$	RMSE
0	-743.1	0.4426	-476.9	0.3514	-407.5	0.3308
200	-741.5	0.4419	-475.3	0.3515	-397.7	0.3279
500	-742.1	0.4423	-475.0	0.3514	-397.7	0.3279
1000	-742.4	0.4424	-474.3	0.3511	-397.4	0.3282

The first column is the number of observations in the training sample. A 0 denotes no training sample. Other columns show the results for BMA, including log predictive likelihood (PL) and root mean squared forecast error (RMSE).

Table 6: The Best Models with Different Regressors

A: S&P 500

	$h=1$		$h=5$		$h=10$	
Regressors	$\log(PL)$	Rank	$\log(PL)$	Rank	$\log(PL)$	Rank
RV only	-527.0	2	-310.1	13	-316.8	38
With RPV	-526.0	1	-302.2	1	-299.1	1
With RBP	-527.9	4	-310.2	14	-310.5	18

B: JPY-USD

	$h=1$		$h=5$		$h=10$	
Regressors	$\log(PL)$	Rank	$\log(PL)$	Rank	$\log(PL)$	Rank
RV only	-844.6	40	-659.4	22	-661.5	22
With RPV	-835.6	1	-648.0	1	-647.9	1
With RBP	-840.1	17	-653.0	11	-654.4	13

C: DEM-USD

	$h=1$		$h=5$		$h=10$	
Regressors	$\log(PL)$	Rank	$\log(PL)$	Rank	$\log(PL)$	Rank
RV only	-743.4	18	-472.1	1	-394.5	1
With RPV	-704.3	1	-475.3	2	-397.7	2
With RBP	-741.0	4	-476.1	4	-400.2	5

This table reports the best model among a specific set of regressors. “RV only” is the best model with regressors constructed from only lags of RV terms. “With RPV” row reports the best model which includes RPV in the regressors and “with RBP” is the best model with RBP terms. The candidate models are summarized in Table 3.

Table 7: Model Recommendations: Top Models by Category

A: S&P 500

h=1		h=5		h=10	
Model Specification	log(PL)	Model Specification	log(PL)	Model Specification	log(PL)
AR: RV=10, PV(.5)=1	-526.0	HAR: RV=2, PV(.5)=3	-302.2	AR: PV(.5)=15	-299.1
AR: RV=15	-527.0	HAR: RV=3, PV(.5)=3	-303.6	AR: PV(.5)=10	-299.6
AR: RV=10, PV(1)=1	-527.0	AR: RV=10, PV(.5)=5	-304.5	HAR: RV=3, PV(.5)=3	-300.6
AR: RV=10, RBP=1	-527.9	HAR: RV=3, PV(.5)=2	-305.9	HAR: RV=1, PV(.5)=3	-300.8
AR: RV=10, PV(1.5)=1	528.5	AR: RV=10, PV(1)=5	-306.8	AR: RV=10, PV(.5)=5	-300.9

B: JPY-USD

h=1		h=5		h=10	
Model Specification	log(PL)	Model Specification	log(PL)	Model Specification	log(PL)
AR: RV=10, PV(1)=5	−835.6	HAR: RV=3, PV(1)=2	−648.0	HAR: RV=3, PV(1)=2	−647.9
AR: RV=10, PV(1)=1	−836.0	HAR: RV=3, PV(1)=3	−649.4	HAR: RV=3, PV(1)=3	−648.6
HAR: RV=3, PV(1)=2	−836.5	HAR: RV=2, PV(1)=3	−649.6	HAR: RV=1, PV(1.5)=3	−649.8
HAR: RV=2, PV(1)=3	−836.5	HAR: RV=3, PV(1.5)=2	−650.1	HAR: RV=3, PV(1.5)=2	−649.8
HAR: RV=3, PV(1)=3	−836.6	HAR: RV=2, PV(1.5)=3	−650.3	HAR: RV=2, PV(1.5)=3	−650.1

C: DEM-USD

h=1		h=5		h=10	
Model Specification	log(PL)	Model Specification	log(PL)	Model Specification	log(PL)
HAR: RV=2, PV(1)=3	−740.3	AR: RV=15	−472.2	AR: RV=15	−394.5
HAR: RV=2, PV(1.5)=3	−740.3	AR: RV=10, PV(.5)=5	−475.3	AR: RV=10, PV(.5)=5	−397.7
HAR: RV=3, PV(1)=2	−741.0	AR: RV=10, PV(.5)=1	−476.0	AR: RV=10, PV(.5)=1	−398.7
HAR: RV=2, RBP=3	−741.0	HAR: RV=1, RBP=3	−476.1	AR: RV=10	−400.0
HAR: RV=3, PV(1.5)=2	−741.1	AR: RV=10	−476.3	AR: RV=10, RBP=1	−400.2

This table lists the top 5 models for different horizons and across markets. The odd columns are the best models according to predictive likelihood and even columns present the log predictive likelihood (log(PL)).

HAR-log type models are denoted with prefix "HAR" and AR type models are denoted with prefix "AR". Given a HAR type model then a 1 indicates a daily term, 2 a daily and weekly term, and 3 a daily, weekly and monthly term for that volatility measure. If AR type model the number of lags of each regressor is listed. For example, "HAR: RV=3, PV(.5)=3" means the regressors of this HAR type model include all the daily, weekly and monthly components of RV and PV of order 0.5. "AR: RV=10,PV(0.5)=5" means 10 lags of log(RV_t) and 5 lags of log($RPV_t(0.5)$). All models include a jump component and all S&P 500 models include an asymmetric term.

Table 8: Predictive Likelihoods for GARCH Models, $h = 1$

Model	S&P 500	JPY-USD	DEM-USD
AR(5)	-530.0	-815.7	-717.3
AR(10)	-523.3	-814.9	-715.7
AR(15)	-521.1	-815.4	-712.9
HAR-log	-527.1	-813.1	-714.1
BMA	-522.8	-808.9	-706.4

This table reports the out-of-sample log predictive likelihood ($\log(PL)$) for models with GARCH. The results are for Bayesian Model Averaging (BMA), a HAR-log model (23) and several AR benchmarks using $\log(RV_t)$. All models include GARCH effects. The out-of-sample period is from March 16, 2001 to March 30, 2004 (761 observations) for S&P 500, May 13, 1998 to December 30, 2002 (1157 observations) for JPY-USD, and May 15, 1998 to December 30, 2002 (1155 observations), for DEM-USD.

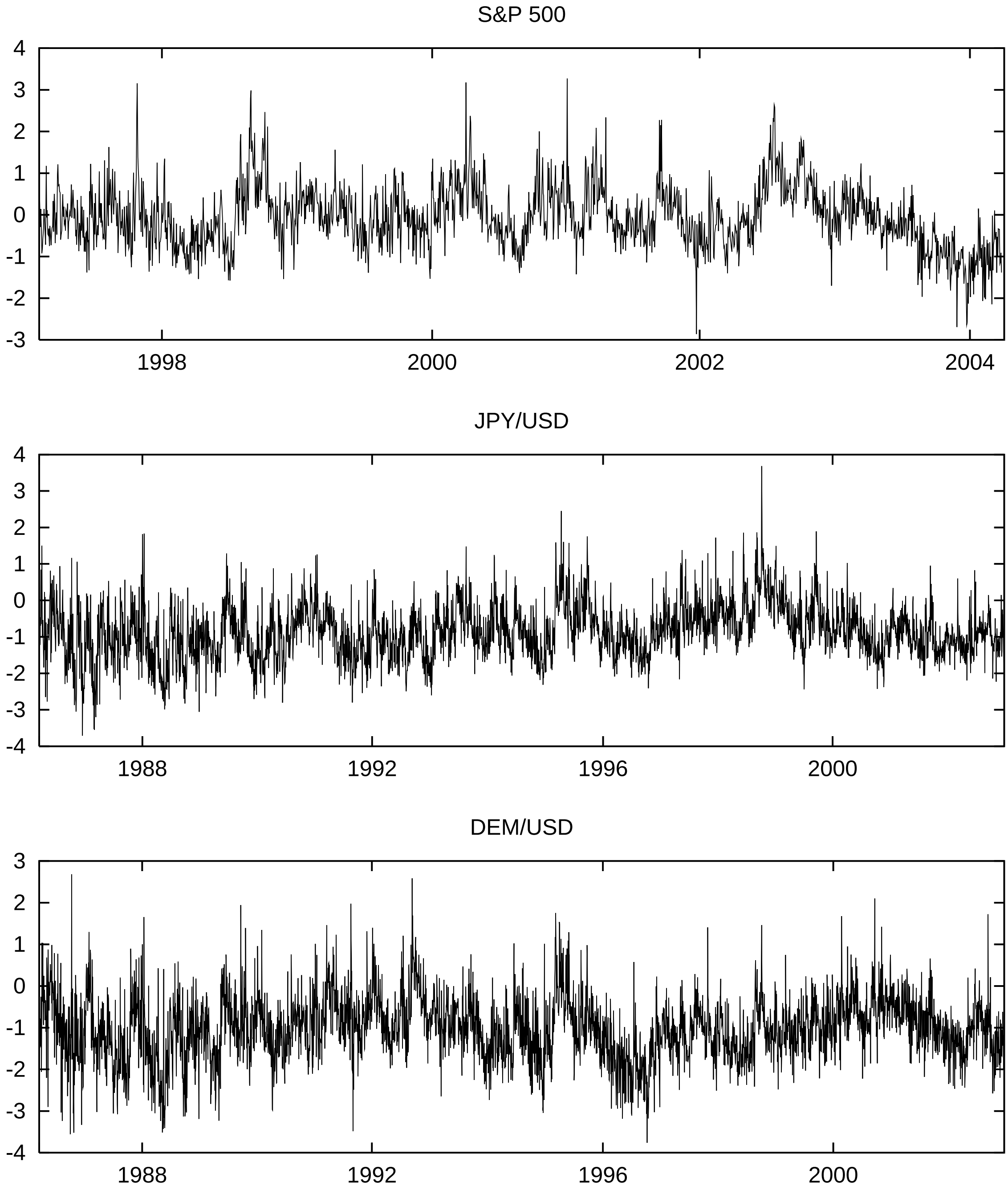


Figure 1: Time Series of Daily Adjusted Log-Realized Volatility

Figure 2: Autocorrelation Functions of Volatility Measures for JPY-USD

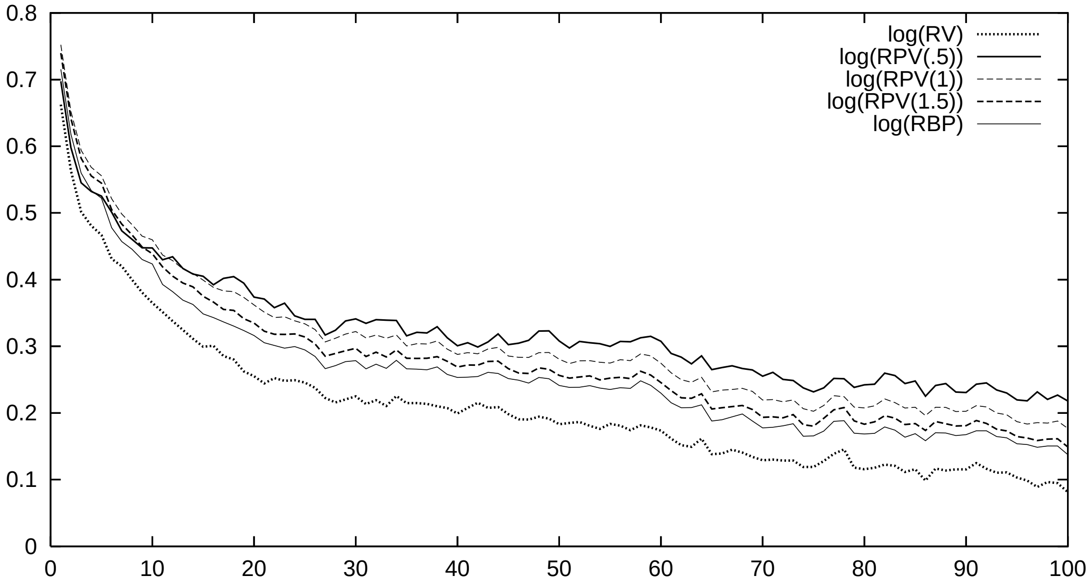
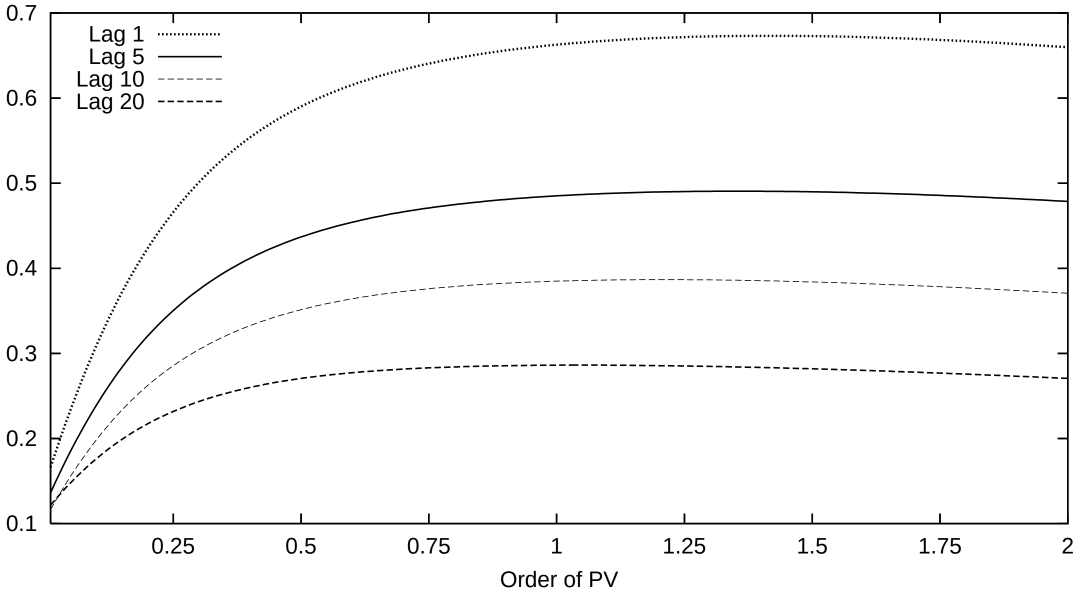


Figure 3: Correlation Function between volatility measures for JPY-USD



This figure displays estimates of $\text{corr}(\log(RV_t), \log(RPV_{t-i}(p)))$ as a function of p for different lag lengths $i = 1, 5, 10, 20$.