University of Toronto
Department of Economics

Working Paper 342

Budget-Constrained Sequential Auctions with Incomplete Information

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October 22, 2008

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October 2008
Forthcoming in *Games and Economic Behavior*†

Abstract

I study a budget-constrained, private-valuation, sealed-bid sequential auction with two incompletely-informed, risk-neutral bidders in which the valuations and income may be non-monotonic functions of a bidder's type. Multiple equilibrium symmetric bidding functions may exist that differ in allocation, efficiency and revenue. The sequence of sale affects the competition for a good and therefore also affects revenue and the prices of each good in a systematic way that depends on the relationship among the valuations and incomes of bidders. The sequence of sale may affect prices and revenue even when the number of bidders is large relative to the number of goods. If a particular good, say α, is allocated to a strong bidder independent of the sequence of sale, then auction revenue and the price of good α are higher when good α is sold first.

Keywords: sequential auctions, budget constraints, efficiency, revenue, price, sequence.

JEL Classification: C7, C72, L1

1 Introduction

Much of the existing theoretical work on auctions concentrates on the allocation of a single good.[1] However, in actual auctions, several heterogeneous goods are often allocated sequentially. If there is no link among the goods then one may be able to apply the single-good analysis repeatedly. However, such a link may arise if budget constraints limit a bidder's ability to bid for later goods when earlier prices deplete the bidder's limited resources.

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† `http://www.elsevier.com/wps/find/journaldescription.cws_home/622836/description#description`

[1] For a survey of the literature, see Klemperer (1999).

Individual bidders whose valuations derive from consumption (rather than resale) may be budget-constrained. But the relevance of budget constraints extends well beyond this case. A theoretical literature argues generally that the existence of agency problems implies that firms are effectively budget-constrained in their investment decisions.[2] An empirical literature supports this idea.[3] In the context of auctions, even firms that are buying to re-sell may effectively be budget-constrained if the cost of borrowing increases with the amount borrowed[4] (a standard assumption in the finance literature) or if capital market imperfections result in budgets for projects being determined on a yearly basis, so that the firms allocate only a fixed amount of capital[5] for the completion of a project. Engelbrecht-Wiggans (1987) shows that budget constraints arise if a bidder is an agent of a principal.

When investments are relatively large then capital market imperfections can lessen the ability of even a large firm to borrow funds. The historic auction of radio spectrum by the FCC in the USA is a good example of an auction in which the investments are relatively large. Cramton (1994) finds it realistic to assume that all firms in PCS (personal communicating services) auctions face budget constraints.[6] As he explains, bidders must raise funds before the auction starts when they do not know exactly how much they will need. Given that fund-raising is time-consuming and costly, he argues that it is reasonable to assume that firms that come to such auctions are budget-constrained. In addition, only forty per cent of the narrow band PCS spectrum was for sale in the first spectrum auction held by the FCC, so that, though each spectrum auction was simultaneous, goods were allocated sequentially across auctions as well as simultaneously within an auction.

I find that the order of sale affects revenue in a private value budget-constrained sequential auction with imperfect information in which bids are continuous. The order of sale affects revenue and prices whether information is perfect and bids are discrete or whether information is imperfect and bids are continuous. The intuition derives from the fact that once good 1 is sold, there is an option to win good 2. The value of the option depends on demand for good 2 which in turn depends on the order of sale. Benoît and Krishna (1998) show that in a complete information common value auction of two goods and three budget-constrained bidders, selling the more highly valued good first always generates the highest revenue. Their result extends to two goods and n budget-constrained bidders since only the top three incomes are relevant. However, it is easy to generate budget-constrained sequential common value auctions in which

[2]See Lewis and Sappington (1989a, 1989b), Hart and Moore (1995) and Clementi and Hopenhayn (2003).

[3]See Fazzari and Athey (1987), Fazzari et. al. (1988), Whited (1992), Fazzari and Petersen (1993), Love (2003) and Clementi and Hopenhayn (2003) for empirical corroboration of budget constraints.

[4]See Cramton (1995) for a discussion of the budget constraints faced by firms making large investments in the nationwide narrowband PCS auction held in the United States in July 1994.

[5]See Hendricks and Porter (1992) for empirical evidence of capital constraints in land lease auctions.

[6]As do Burguet and McAfee (2005).

selling the most highly valued of three goods does not generate the highest revenue.[7]

The preceding paragraph illustrates that the relationship between the order of sale and revenue is unclear. The question that I address in this paper is whether any systematic rules govern the relationship among the prices of a good, the revenue, and the order of sale when the valuations are similar and when the income exceeds the valuation of each good. In Section 7, I provide a restricted set of auctions in which selling the more highly valued good first raises more revenue than selling it second.

I study a budget-constrained version of the benchmark model of a private-valuation sealed-bid sequential auction in which two risk-neutral bidders bid for two goods and information is incomplete.[8] When information is complete, revenue is affected, in a systematic way,[9] by the price-formation rule (i.e. the rule that specifies the price as a function of the bids). Assuming information is incomplete does not change this. In order to isolate the pure effect of the budget constraints on the prices of goods relative to their order of sale and the price-formation rule, I restrict to a world in which the expected revenue is constant within a class of price-formation rules that includes 1^{st} and 2^{nd} price rules (as would happen if there were only a single good and no effective budget constraints). In this world, I find that the auction revenue depends on the sequence of sale, that the price of a good depends on its position in the sequence of sale, and that this dependence has a natural interpretation.

In a budget-constrained sequential auction of two goods in which a bidder's type determines the value of the bidder's valuation and income functions, I restrict attention to symmetric bidding functions but do not assume monotonicity in a bidder's type. I find that multiple symmetric equilibrium bidding functions may exist that differ with respect to efficiency, revenue and allocation. Whether revenue is maximized or the allocation is efficient depends on the relationship between the bidding function and the valuation and income functions and not on the price rule.

Say that two real-valued functions f and g are *ordinally equivalent* on a common domain S if $f(x) > f(y)$ if and only if $g(x) > g(y)$ for any $x, y \in S$ (that is, they produce a common ranking of the domain elements). Ordinal equivalence can be useful in determining whether an equilibrium bidding function generates the highest revenue or an efficient outcome. An upper bound on the revenue generated is that generated by any bidding function that is ordinally equivalent to the income function (Theorem 2 and Corollary 3). An efficient allocation is generated if the bidding function is ordinally equivalent to the difference in valuation functions (Theorem 2 and

[7] Benoît and Krishna (1998).

[8] Benoît and Krishna (2000) consider budget constrained buyers with complete information. Che and Gale (1993) consider budget constrained buyers in one-good auctions. Pitchik and Schotter (1986), Pitchik and Schotter (1988) and Pitchik (1989) considers budget constrained buyers with incomplete information.

[9] See Benoît and Krishna (2000), Pitchik and Schotter (1988), and Pitchik and Schotter (1986).

Corollary 3). In particular, if the valuation and income functions are increasing in a bidder's type with one valuation function increasing more rapidly than the other, then, under first or second price rules, there exists an equilibrium bidding function that generates maximum revenue and an efficient allocation when the good whose valuation increases more rapidly in a bidder's type is sold first. Thus, even when there are only two goods, selling the highest valued good first need not generate the highest revenue. In particular, if one is auctioning the contents of a household, then selling a wall painting by an unknown artist (for whom bidders' tastes are highly variable) before the used ride-on lawn mower (whose value may be high but publicly known) maximizes revenue.

The price-formation rule may affect the price of a good even if it does not affect revenue (Theorem 7). For example, under a second price rule, a bidder might worry about not being awarded good 2 at a low price and so bid relatively high while, under a first price rule a bidder might worry about winning good 1 at a high price and so bid relatively low. In addition, auction revenue and allocation may differ across symmetric equilibrium bidding functions (Theorem 2, 8, and 10, and Corollary 3).

The law of one price does not hold for similar goods in a budget-constrained sequential auction (Theorems 4, 17, and 20, and Corollaries 5, 18, and 19). If the goods α and β are identically valued and the bidding function is ordinally equivalent to the income function, then the expected price is higher the later the good is sold. If the goods α and β are similarly valued (with a common mean) but the value of one good is even slightly more variable than the other and income is constant across types of bidders, then the expected price is higher the earlier the good is sold under a 2^{nd} price rule;[10] under a 1^{st} price rule, it is higher the later the good is sold.

In my model the goods may be heterogeneous. The revenue and the price of a good depend on the order of sale and the interaction among the valuations and income of bidder types. Each bidder obtains no more than one good *in equilibrium*, but the bidders are not constrained ex ante from obtaining both goods independent of the prices and bids; further, bidders know their valuations of both goods at the beginning of the auction. In addition, the valuations and income may be non-monotonic in type.

Previous models in the literature[11] use various assumptions to obtain the relationship between the price of a good and its order of sale so that the various results are hard to compare within the context of a single model. When I restrict to similar goods in my model, the effect of the order of sale on the price depends, in a systematic way, on the way that the order of sale affects the opportunity cost of winning good one.

[10] Ashenfelter (1989) and Ashenfelter and Genesove (1993) provide empirical evidence that ex ante identical goods fetch prices that depend on their position in the order of sale.

[11] Genesove (1993), Black and De Meza (1993), Bernhardt and Scoones (1993) and Gale and Hausch (1992).

Two previous models link revenue with the order of sale. Elmaghraby (2003) links revenue to the order of sale when a buyer auctions two heterogeneous jobs to capacity-constrained suppliers. Chakraborty et. al. (2006) obtain that the order of sale affects revenue when a seller may choose which of two goods to sell first based on her private information in the context of the standard auction model in Milgrom and Weber (1982).

Other work deals with the allocation of multiple goods to multiple bidders, but none of which I am aware specifically analyzes the allocation of multiple goods auctioned sequentially to a set of incompletely-informed, budget-constrained bidders with private valuations.[12] The aim of the paper is to understand how the relationship between auction revenue, allocation, prices and the order of sale depends on the relationship among the parameters of the model and the price-formation rule.

2 The Model

Two individuals, 1 and 2, bid for two heterogenous goods, α and β. The goods are sold sequentially; I refer to the first good sold as good 1. Each individual's privately known *type* is drawn independently from the publicly known distribution H, which is atomless, continuously differentiable, and increasing on its support $[0, 1]$. An individual of type t has income $I(t)$ and valuation $v_\gamma(t)$ for good γ, for $\gamma = \alpha$, β. An individual of type t is constrained to pay no more than $I(t)$ in the auction. Thus, if an individual of type t obtains good 1 at the price p_1, then $p_1 \leq I(t)$ and the individual can pay no more than $I(t) - p_1$ for good 2.

I assume that the functions $I : [0, 1] \longrightarrow [a, b]$ and $v_\gamma : [0, 1] \longrightarrow [a_\gamma, b_\gamma]$, $\gamma = \alpha$, β are continuously differentiable, with $a \geq 0$ and $a_\gamma \geq 0$ for $\gamma = \alpha$, β. I impose the mild condition that on no set of positive Lebesgue measure is it the case that $v_1' - v_2' + I' = 0$ and $v_1' - v_2' = 0$. So, for example, if v_1 and v_2 differ by a constant on some interval then I is not constant on this interval. Note that I do not impose monotonicity on v_1, v_2 or I.

Since I, v_α and v_β may not be monotonic, knowing the value $I(t)$ of one's opponent's income need not give any information about one's opponent's valuations $v_1(t)$ and $v_2(t)$. For example, no information is given when the function I is constant on $[0, 1]$. Note also that the income and valuations of one individual are not correlated with those of any other.

In order that the valuations be meaningful I assume that for each type, income is at least equal to each valuation; in order that the budget constraint be effective I assume that, for each type, income is at most the sum of the valuations. Precisely,

$$v_\gamma(t) \leq I(t) \leq v_\alpha(t) + v_\beta(t) \text{ for } \gamma = \alpha, \beta \text{ and } t \in [0, 1], \tag{1}$$

[12]See Benoît and Krishna (2000), Bernhardt and Scoones (1993), Bulow and Klemperer (2002), Engelbrecht-Wiggans and Weber (1979), Engelbrecht-Wiggans and Menezes (1993), Gale and Hausch (1992), Gale and Stegeman (2001), Krishna (1990,1993), McAfee and Vincent (1993), Menezes (1993), Ortega-Reichert (1968), Palfrey (1980), Pitchik and Schotter (1986,1987), von der Fehr (1994), Weber (1983) and Swinkels (1989).

where the first inequality is strict for $t \in (0,1)$.

This assumption has two implications. (1) The maximum amount that an individual of type t is willing and able to pay for both goods simultaneously is $I(t)$. (2) The maximum amount that an individual of type t is willing and able to pay for good 2, once good 1 is sold, is the minimum of $v_2(t)$ and any income remaining to the individual after payment for good 1 is made. I call this minimum the individual's *de facto valuation* of good 2. Later I define an individual's *de facto valuation* of good 1, which takes into account the fact that the higher is the price paid by the winner of good 1, the lower is this individual's de facto valuation of good 2.

I study a sealed-bid auction. First, good 1, which may be α or β, is brought up for sale. Each individual submits a bid for good 1; the bids are submitted simultaneously. The bid that an individual of type t is able to make is constrained to be nonnegative. The bidder who submits the higher bid obtains good 1. The price this bidder pays for the good depends on the price-formation rule. The price that an individual of type t is able to pay is at most $I(t)$. I assume that there is an explicity penalty that constrains the bids so that the winning price is at most the income of the winning bidder. (For example, if the bids and price-formation rule result in a price that the winner is unable to pay, then the winner must forfeit the good and in addition must pay a financial penalty that ensures compliance with this assumption.) I assume that the price-formation rule satisfies the following conditions. (Note that both first- and second-price rules satisfy these assumptions.)

(S1) Bidders are treated anonymously (the price of a good depends only on the collection of bids and not on the identity of which bidder made which bid).

(S2) The price is non-decreasing in the bids.

After good 1 is sold, the winner's budget is reduced by the price paid for good 1 and the winning bid is revealed. The individuals then simultaneously bid for good 2 ($\{\alpha, \beta\} = \{1, 2\}$); the price that an individual of type t is able to pay is constrained to be at most $I(t)$ minus any payment the individual made for good 1.

I restrict attention to symmetric equilibria—that is, equilibria in which the bid of an individual of type t at each stage depends on t and not on t's name. The fact that the loser in the first stage knows the winning bid and the equilibrium bidding function in the first stage allows the loser's beliefs about the winner's type to be updated. The inference problem complicates the model substantially without apparently affecting the key incentives that I wish to explore, so I make the simplifying assumption that once good 1 is allocated, the income of the winner of good 1 and the price paid for good 1, in addition to the winning bid, are public knowledge.

It is well-known that the price-formation rule can affect the revenue of an auction. I want to isolate the effect of the sequence of sale on the price of a good and therefore I restrict to parameters for which revenue is independent of the price-formation rule. In order to avoid looking at special cases, I assume that, though the income of a bidder of type t is large enough to pay either $v_\alpha(t)$ or $v_\beta(t)$ for either good, it is not so large that any bidder is allocated both goods in equilibrium. This property is satisfied when

$$\max_{s,t} \frac{v_1(t) - v_2(t) + I(s)}{2} \leq \min_t I(t) \leq \max_t I(t) \leq \min_{s,t} \frac{v_1(t) - v_2(t) + I(s)}{2} + \min_s v_2(s) \qquad (2)$$

for all $t, s \in [0, 1]$. This condition is explained in the next section.

Since the income, $I(t)$ say, of the winner of good 1 and its price, p_1 say, are known, the remaining income, $I(t) - p_1$, of the winner of good 1 is known. As shown in the next section condition (2) implies that this remaining income is less than the good 1 winner's valuation for good 2, so that $I(t) - p_1 < v_2(t)$. It implies also that, once a bidder of type t wins good 1 from a bidder of type s at a price p_1, it is public knowledge that

$$\min \{I(t) - p_1, v_2(t)\} = I(t) - p_1 < v_2(s)$$

That is, the good 1 winner's de facto valuation of good 2 and the fact that it is below the good 1 loser's valuation of good 2 are public knowledge

In the second stage, a single good is for sale. The auction in this stage differs from a standard auction in that the players' ability to pay are limited by their incomes. Given that neither bidder is allocated both goods in equilibrium and given that the de facto valuation of the good 1 winner is known and is less than the valuation of the good 1 loser, under first- and second-price-formation rules (with suitable tie-breaking rules if necessary), the second period auction has an equilibrium in which the bidder with the higher de facto valuation for good 2 obtains the good at a price equal to the lower de facto valuation. For example, in the standard equilibrium under a second-price rule, individuals bid their de facto valuations, and in the standard equilibrium under a first-price rule with a tie-breaking rule that favors the bidder with the higher de facto valuation, each individual bids the lower de facto valuation. I restrict attention to price-formation rules for which such an equilibrium exists. Further, I take this equilibrium to be the one that occurs. That is, I assume that

(S3) Good 2 is sold to the bidder with the higher de facto valuation of good 2 at a price equal to the lower de facto valuation of good 2.

Replacing the second stage of the game by the equilibrium payoffs in this equilibrium outcome we obtain a Bayesian game G which is the game that I study. The strategy set of a bidder of type t in G is $[0, \infty)$, the set of feasible bids on good 1. I consider a symmetric Nash equilibrium of G. That is, I look for a function $B : [0, 1] \to \Re$ that assigns a bid to each type with the property that (B, B) is a Nash equilibrium of G.

Suppose that, in equilibrium, a bidder of type t wins good 1 at the price p_1. Then the other bidder wins good 2 at the price[13] $p_2 = I(t) - p_1$ so that

$$\textit{Revenue} = \textit{income of the winner of good 1} \tag{R}$$

Since the expected revenue equals the income of the winner of good 1 and since the price-formation rule does not affect who wins good 1, the expected revenue does not depend on the price-formation rule. Since the expected revenue does not depend on the price-formation rule (as in an auction with a single good and no effective budget constraints), I am able to isolate the pure effect of a change in the order of sale or the price-formation rule on the prices of goods that occurs directly because of budget constraints. I note that the price-formation rule does not affect who wins good 1.

I consider budget-constrained sequential auctions for which the price-formation rule satisfies (S1), (S2) and (S3) and the valuation and income functions satisfy satisfy (1) and (2) so that (R) is true.

3 The Sequence of Sale Affects Revenue and Prices

I start by defining the maximum amount that a bidder of type t is willing and able to pay for good 1 when facing a bidder of type s. I call this maximum amount the bidder's *de facto valuation* of good 1 when facing a bidder of type s and denote it by $V(t, s)$. If a bidder of type t wins good 1 at a price of p_1 then the payoff to this bidder is $v_1(t) - p_1$. If good 1 is sold at the price p_1 to a bidder of type s, then the bidder of type t wins good 2 at the price $I(s) - p_1$, and thus obtains the payoff $v_2(t) - (I(s) - p_1)$. Hence the most that a bidder of type t is willing to pay for good 1 when facing a bidder of type s is the amount p_1 for which

$$v_1(t) - p_1 = v_2(t) - (I(s) - p_1) \text{ or } p_1 = \frac{v_1(t) - v_2(t) + I(s)}{2}$$

Thus, the de facto valuation for good 1 of a bidder of type t when facing a bidder of type s is

$$V(t, s) = \frac{v_1(t) - v_2(t) + I(s)}{2}$$

Suppose that f and g are real-valued functions on a domain S. I say that f is *ordinally equivalent* (denoted OE) to g on S whenever the level curves and upper contour sets of f are equal to those of g on S, that is, for all $x, y \in S$, $f(x) > f(y)$ if and only if $g(x) > g(y)$. I say that f is OE to g when f is OE to g on the entire domain. In particular, any strictly increasing function is OE to any other strictly increasing function. I say that f is *ordinally*

[13] This follows from condition (2).

reversed (denoted ORE) to g whenever there is one common set of level curves but the upper contour set of one function is the lower contour set of the other function. In particular, any strictly increasing function is ORE to any strictly decreasing function and any function f is ORE to $-f$.

In a standard private valuation auction for one good, a bidding function that is OE to the valuation function allocates the goods and money efficiently. In a budget-constrained auction, an equilibrium bidding function that is OE to $v_1 - v_2$ allocates the goods and income efficiently (Theorem 2).

As discussed in the previous section, condition (2) ensures that no bidder is allocated both goods in equilibrium. Since the price-formation rule does not affect who wins good 1, this condition can be used to show that revenue is independent of the price-formation rule. The argument is as follows. The assumption that $v_1(t) + v_2(t) \geq I(t)$ implies that the minimum *de facto valuation* for good 1 is less than or equal to the minimum valuation for good 1. Therefore, any bidder is willing to bid up the price for good 1 to the minimum *de facto valuation* for good 1. That is, the price of good 1 is at least equal to this minimum. The last inequality of (2) says that the difference between the maximum possible income of the winner of good 1 and the minimum *de facto valuation* for good 1 must be less than or equal to the minimum valuation of good 2. That is, the residual income of the winner of good 1 must be less than the minimum valuation of good 2. Thus, in equilibrium, each bidder who wins good 1 must lose good 2. Moreover, the income of the winner of good 1 is depleted to below the winner's valuation of good 2. It follows that the price of good 2 is the depleted income of the winner of good 1 so that (R) is satisfied and so a bidding function that is OE to the income function maximizes auction revenue.

In a standard auction of one good, a bidder is always willing and able to bid up to the bidder's valuation. Therefore the unique efficient allocation is that in which the good is allocated to the bidder with the highest valuation. An allocation of goods and income in a budget-constrained auction is efficient if there are no Pareto improving trades. In contrast to the standard auction, the willingness and ability of an individual to pay for a good depends not only on the bidder's valuation for the good but also on the bidder's remaining income. In any auction allocation of goods and income let t_θ denote the type allocated good θ and let R_θ denote the money allocated to type t_θ for $\theta \in (\alpha, \beta)$.

Definition 1 *The allocation* $((t_\alpha, R_\alpha), (t_\beta, R_\beta))$ *is efficient if*

$$v_\alpha(t_\beta) \leq v_\alpha(t_\alpha) \text{ or } R_\beta \leq v_\alpha(t_\alpha)$$
$$v_\beta(t_\alpha) \leq v_\beta(t_\beta) \text{ or } R_\alpha \leq v_\beta(t_\beta)$$

and

$$\begin{array}{ll} \textit{either} & v_\alpha(t_\alpha) - v_\beta(t_\alpha) \geq v_\alpha(t_\beta) - v_\beta(t_\beta) \\ \textit{or} & \begin{array}{l} R_\beta \leq v_\alpha(t_\alpha) - v_\beta(t_\alpha) \\ R_\alpha \leq v_\beta(t_\beta) - v_\alpha(t_\beta) \end{array} \end{array}$$

We next answer two questions when all individuals use a common bidding function. When is the resulting allocation efficient? When is the resulting revenue maximized? All proofs not in the text are in the Appendix. Let M be the maximum revenue generated over all equilibria of G under either order of sale.

Theorem 2 *In any symmetric equilibrium of a budget-constrained sequential auction, the expected revenue is independent of the price-formation rule.* M *is at most equal to the revenue generated by a common bidding function* $B(t)$ *that is OE to* $I(t)$*. Moreover, if the equilibrium bidding function* $B(t)$ *is OE to* $I(t)$*, the expected revenue is higher than that generated by any auction in which both goods are bundled and allocated simultaneously to one of the bidders. Whenever the common bidding function* $B(t)$ *is OE to* $v_1(t) - v_2(t)$*, the allocation of goods and money is efficient.*

Corollary 3 *If the bidding function* B *is OE to* $v_1(t) - v_2(t)$ *and* I*, then revenue equals* M *and the goods are allocated efficiently.*

The above theorem and corollary hold for any auction form under consideration for which the income constraints are binding. The proof here does not require the explicit calculation of equilibrium bidding functions. The bidders know their valuations ex ante and the bidders may bid on more than one good.

The next result implies that the law of one price need not hold for identical goods since $v_1 - v_2$ equals the zero function when the goods are identical.

Theorem 4 *If the equilibrium bidding function* $B(t)$ *is OE to* $v_1(t) - v_2(t) + I(t)$ *and* $I(t)$*, then the expected price of good* 1 *is less than that of good* 2 *whenever* $v_2(w_1) - v_1(w_1) \geq 0$*, where* w_1 *satisfies*

$$v_1(w_1) - v_2(w_1) + I(w_1) = E \max(v_1(t_1) - v_2(t_1) + I(t_1), v_1(t_2) - v_2(t_2) + I(t_2)),$$

where E *is the expectation operator.*

It follows immediately that whenever each bidder considers the two goods to be identical, the expected price of good 2 is higher than that of good 1 in such an equilibrium.

Corollary 5 *If $v_\alpha(t) = v_\beta(t)$ for all t and the equilibrium bidding function is OE to I, then the expected price of good 2 is higher than that of good 1.*

I now explore opportunities for arbitrage in this case.

Theorem 6 *If $v_\alpha(t) = v_\beta(t)$ for all t and the equilibrium bidding function is OE to I, then the expected price paid by a bidder in the auction is constant across bidders.*

Theorems 4 and 6 imply that when $v_\alpha(t) = v_\beta(t)$ for all t and the bidding function is OE to I, the expected price of good 1 is less than that of good 2 even though the expected price paid is constant across bidders. To see how this result is possible, let p^* be the common expected price that a bidder expects to pay in the auction. Since good 1 is allocated to the bidder with the higher bid and good 2 is allocated to the bidder with the lower bid, the bidder whose type is associated with the highest bid obtains good 1 for sure at a price of p^*, while the bidder whose type is associated with the lowest bid obtains good 2 for sure at a price of p^*. Since the price paid for a good is increasing in the bids, p^* is the highest price paid for good 1 and the lowest price paid for good 2. A bidder whose type is associated with a bid between the highest and lowest bids sometimes obtains good 1 for a price lower than p^* and sometimes obtains good 2 for a price higher than p^* but on average obtains a good for a price of p^*. Thus, the price that any single bidder expects to pay in the auction is constant across bidders even though the expected price of good 1 is lower than that of good 2. A violation of the law of one price does not imply opportunities for arbitrage.

4 The Expected Price of a Good Depends on the Price-Formation Rule

Theorem 2 shows that the revenue can be affected by the sequence of sale when income varies with type. It follows that, if income varies, then the sequence of sale can also affect the prices of the goods. In fact, as shown below, the price-formation rule may affect the prices even in the case that income is constant across types. In addition, the price-formation rule may affect the way in which the price of a good depends on the sequence of sale.

Let B be an equilibrium bidding function. Consider the direct revelation game $\overline{G}(B)$ in which each bidder's strategy set is the set $[0, 1]$ of types and the payoff of type t when s is announced is the payoff obtained by type t in G when type t bids $B(s)$. Since (B, B) is an equilibrium of G, the "truthful" strategy profile in which each type t chooses t is an equilibrium of $\overline{G}(B)$. I study equilibria of $\overline{G}(B)$ for any continuous bidding function B that is not constant on any interval. Riley and Samuelson (1981) study a standard auction of one good and so consider continuous bidding functions that increase in a bidder's type. They find that the equilibrium price (and therefore revenue) is independent of the price-formation rule. By contrast, in a budget-constrained sequential auction, even though the expected revenue is independent of the price-formation rule since the price-formation rule does not affect who wins good 1, the equilibrium price of good 1 varies with the rule.

Theorem 7 *The equilibrium price of good* 1 *may depend on the price-formation rule.*

Thus, even though the expected revenue and the winner of good 1 are independent of the price-formation rule, the expected prices vary with the rule. Che and Gale (1998) show that in a budget-constrained auction of one good, the expected price of the single good (which is equivalent to revenue in this case) is higher under the 1^{st} price rule than under the 2^{nd} price rule. In the auctions that we consider, revenue is constant across price rules but the price of each good varies across rules.

5 Character of Bidding Function

In this section, we analyze the relationship between the equilibrium bidding function and the functions $v_1 - v_2 + I$ and $v_1 - v_2$. We first show that if the equilibrium bidding function is non-monotonic then the function $v_1 - v_2$ is non-monotonic.

Theorem 8 *If there exist sets $\widehat{J}_1 \subset [0,1]$ and $\widehat{J}_2 \subset [0,1] \backslash \widehat{J}_1$ and a function $\widehat{x} : \widehat{J}_1 \longrightarrow \widehat{J}_2$ for which the equilibrium bidding function B satisfies $B(t) = B(\widehat{x}(t))$ for $t \in \widehat{J}_1$, then $v_1(t) - v_2(t) = v_1(\widehat{x}(t)) - v_2(\widehat{x}(t))$ for $t \in \widehat{J}_1$.*

Definition 9 *A continuous function $f : X \to \Re$ is S-monotonic on a set $T \subset X$ if*

$$\{z \in X : f(z) = f(x)\}$$

is a singleton for all $x \in T$.

Note that a function that is strictly monotonic over the entire domain X is S-monotonic but that strict monotonicity over a set $T \subset X$ is not enough to guarantee S-monotonicity on T. The level curve in X that passes through any element of the set T must be a singleton. While the function $f(x) = x(1-x)(2-x)$ is S-monotonic on $[A, \infty)$ for A large enough, f strictly increases but is not S-monotonic on $[2, \infty)$.

Whenever a bidder of type t faces a bidder of type s for which $B(t) > B(s)$, let $P(t, s)$ denote the price that the bidder of type t pays for good 1. Let D'_i denote the partial derivative operator with respect to i; D''_{ij}, the second partial derivative with respect to i and j.

Theorem 10 *Suppose that $v_1 - v_2 + I$ is differentiable and S-monotonic on $[0, 1]$. There exists an equilibrium bidding function that is OE to $v_1 - v_2 + I$ on $[0, 1]$ if $D''_{12}P(t, s) = 0$ for all (s, t) and* ***either*** *one of $D'_1P(t, s)$ and $D'_2P(t, s)$ equals zero for all (s, t)* ***or*** *$D'_2P(t, t) = \gamma D'_1P(t, t)$ for $\gamma > 0$.*

We note that when P is not a member of the class of price rules assumed by Theorem 10, then there may exist a monotonic B that is not OE to $v_1 - v_2 + I$.[14] Theorems 8 and 10 indicate the possibility of multiple bidding functions that differ with respect to allocation, revenue and efficiency.

Since the equilibrium bidding function need not be monotonic and since monotonicity affects the equations that are necessary in equilibrium, we can use Theorems 8 and 10 and Corollary 3 to determine the feasible ordinal equivalence classes of equilibrium functions. For example, if $I(t)$ and $v_\alpha(t)$ increase while $v_\beta(t)$ is constant, then the difference in values increases in type under the order (α, β). In this case, I is OE to $v_\alpha - v_\beta$. Under either a 1^{st} or 2^{nd} price rule, Theorem 10 implies that there exists an equilibrium bidding function that is OE to $v_\alpha - v_\beta$, I and $v_\alpha - v_\beta + I$. Corollary 3 then implies that revenue is maximized and that the allocation is efficient. Thus, one might expect that, in estate auctions, heavy equipment (for example, ride-on lawn mowers that have a fixed value) will be sold later than an item whose value may depend more heavily on taste (for example, used bedspreads).

6 1^{st} and 2^{nd} Price Rules

I restrict attention to 1^{st} and 2^{nd} price-formation rules in this section. In general, more than one equilibrium bidding function may exist and the equilibrium bidding functions may be neither S-monotonic nor result in an efficient allocation. I provide a sufficient condition for there to exist an S-monotonic equilibrium bidding function under either price-formation rule.

Theorem 11 *An S-monotonic equilibrium bidding function exists if $v_1 - v_2 + I$ is S-monotonic.*

I now provide a sufficient condition for the existence of an equilibrium bidding function whose allocation is efficient. This condition (that c, a function analogous to $v_1 - v_2 + I$, is monotonic on a set K whose construction depends on $v_1 - v_2$) bears some resemblance to the sufficient condition in Theorem 11 but requires elaboration to state.

I begin by constructing the set K when $v_1 - v_2$ is S-monotonically **increasing** whenever it is S-monotonic. (The derivative of any real-valued differentiable function f on $[0, 1]$ must have

[14] If $P(s, t) = P(t, s) = \sqrt{B(s)B(t)}$, Equation (4) is solved by $B(t) = (t + 1)^2$ for which $P(0, 0) = 1$ if either $v_1(t) - v_2(t) + I(t) = 3t^2 + 6t + \frac{1}{2} \geq \frac{1}{2}$ and $H(t) = t$ or if $v_1(t) - v_2(t) + I(t) = \frac{10}{3}t^2 + 8t - \frac{8}{3}t^{1/2} + 2 \geq 1.7802$ and $H(t) = t^{1/2}$.

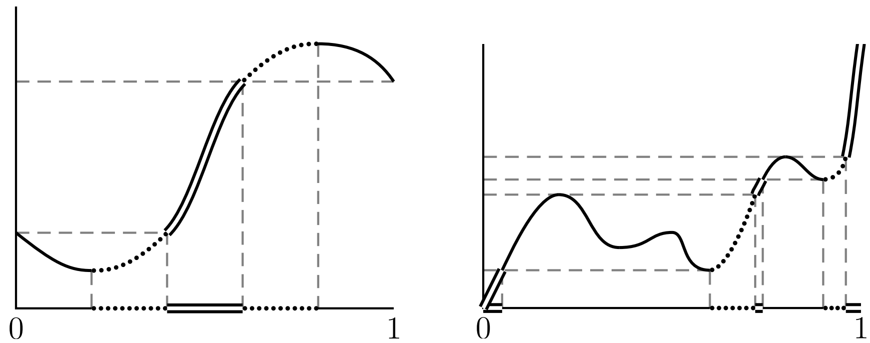

Figure 1: Two examples of the sets S and J. In each case, S is the union of the double line segments on the x-axis and J is the union of the dotted line segments on the x-axis.

a common sign over any part of its domain over which it is S-monotonic. Suppose instead that there exist two disconnected intervals S and T over which f is S-monotonic but the sign of f' over S differs from that over T. The Intermediate Value Theorem then implies that there exists $x \in [0,1] \backslash (S \cup T)$ and $y \in S \cup T$ for which $f(x) = f(y)$. This contradicts the assumption that f is S-monotonic on S and T.) Let $S \subset [0,1]$ be the smallest closed set containing the domain on which $v_1 - v_2$ is S-monotonically increasing. We construct $J \subset [0,1]$ to be an irreducible closed set on which $v_1 - v_2$ is monotonically increasing and

$$\{v_1(t) - v_2(t) : t \in S \cup J\} = \{v_1(t) - v_2(t) : t \in [0,1]\}$$

The set J is irreducible in the sense that discarding any of its interior members renders the above equality to be false. If $J \neq \emptyset$, then $S \cup J \subsetneqq [0,1]$. For each $x \in [0,1] \backslash (S \cup J)$ there exists $t \in J$ for which $v_1(x) - v_2(x) = v_1(t) - v_2(t)$. The restriction of $v_1 - v_2$ to the set $S \cup J$ is said to **cover** $v_1 - v_2$ on the entire set $[0,1]$ in that the set of image points is common.

Since there is a finite number of critical values of $v_1 - v_2$, each of the sets S and J is the union of a finite number of intervals. We construct J in an iterative fashion as follows[15]. Since $v_1 - v_2$ is a continuous function on $[0,1]$, its maximum exists by the Extreme Value Theorem. Let t_0 be the point closest to 1 for which the maximum of $v_1 - v_2$ occurs. In particular if the maximum occurs for some $t \in S$ then continuity and the Intermediate Value Theorem imply that it must occur for $t_0 = 1$ in which case let $t_{s(0)} = t_0 = 1$. In this case let $S_0 = [t_{s(1)}, t_{s(0)}] \subset S$ be the largest interval in S that contains $t_{s(0)}$. If $S_0 = [0,1]$, then $J = \emptyset$. If $S_0 \neq [0,1]$, then $J \neq \emptyset$ and we construct J_0 as follows. Let $t_{j(0)} = t_{s(1)}$ and let $[t_{j(1)}, t_{j(0)}]$ be the largest interval on which $v_\alpha - v_\beta$ increases and for which $(t_{j(1)}, t_{j(0)}) \cap S = \emptyset$. If instead, the maximum of $v_1 - v_2$ occurs for some $t_0 \notin S$, then $t_0 \leq 1$. In this case let $t_{j(0)} = t_0$ and we construct J_0 as follows. Let $J_0 = [t_{j(1)}, t_{j(0)}]$ be the largest interval on which $v_1 - v_2$ increases and for which $(t_{j(1)}, t_{j(0)}) \cap S = \emptyset$.

[15] For some graphical examples, see Figure 1.

We now provide the iterative step in the construction of the intervals of J. Suppose that we have obtained the intervals

$$S_0 = [t_{s(1)}, t_{s(0)}], S_1 = [t_{s(3)}, t_{s(2)}], ..., S_M = [t_{s(2M+1)}, t_{s(2M)}]$$

contained in S and the intervals

$$J_0 = [t_{j(1)}, t_{j(0)}], J_1 = [t_{j(3)}, t_{j(2)}], ..., J_N = [t_{j(2N+1)}, t_{j(2N)}]$$

contained in J as above and that $t_a \in (t_{k(2i+1)}, t_{k(2i)})$, $t_b \in (t_{k(2i-1)}, t_{k(2i-2)})$ implies $t_a < t_b$ for $k = s$ and $i = 1, ..., M$ or $k = j$ and $i = 1, ..., N$. If $t_{s(2M+1)} < t_{j(2N+1)}$ then let $t_{j(2N+2)} = t_{s(2M+1)}$ and let $J_{N+1} = [j_{(2N+3)}, j_{(2N+2)}]$ be the largest interval on which $v_1 - v_2$ increases and for which $(t_{j(2N+3)}, t_{j(2N+2)}) \cap S = \emptyset$. If $t_{j(2N+1)} < t_{s(2M+1)}$ then there exists $\varepsilon > 0$ for which either $(t_{j(2N+1)} - \varepsilon, t_{j(2N+1)}) \subset S$ or $(t_{j(2N+1)} - \varepsilon, t_{j(2N+1)}) \cap int(S) = \emptyset$. In the former case, let $t_{s(2M+2)} = t_{j(2N+1)}$ and let $[t_{s(2M+3)}, t_{s(2M+2)}]$ be the largest interval in S that contains $t_{s(2M+2)}$. In the latter case, let $t_{j(2N+2)} < t_{j(2N+1)}$ be the point closest to $t_{j(2N+1)}$ for which

$$v_1(t_{j(2N+2)}) - v_2(t_{j(2N+2)}) = v_1(t_{j(2N+1)}) - v_2(t_{j(2N+1)})$$

and define $J_{N+1} = [t_{j(2N+3)}, t_{j(2N+2)}]$ as the largest interval on which $v_1 - v_2$ increases and for which $(t_{j(2N+3)}, t_{j(2N+2)}) \cap S = \emptyset$.

We stop the iterative process of constructing the sets S and J as soon as there exist $M = \overline{M}$ and $N = \overline{N}$ such that

$$\{v_1(t) - v_2(t) : t \in [0, 1]\} = \left\{v_1(t) - v_2(t) : t \in \left(\cup_{i=0}^{\overline{M}} S_i\right) \cup \left(\cup_{i=0}^{\overline{N}} J_i\right)\right\}$$

so that

$$S = \cup_{i=0}^{\overline{M}} S_i \quad \text{and} \quad J = \cup_{i=0}^{\overline{N}} J_i$$

are well defined. Let $K_i = [t_{2i+1}, t_{2i}]$ for $i = 0, ..., \overline{M} + \overline{N}$ where $K_0, K_1, ..., K_{\overline{M+N}}$ represent the intervals

$$\left(\cup_{i=0}^{\overline{M}} S_i\right) \cup \left(\cup_{i=0}^{\overline{N}} J_i\right)$$

from right to left so that $t_i > t_{i+1}$ for $t_i \in int(K_i), t_{i+1} \in int(K_{i+1})$. Let $K = \cup_{i=0}^{\overline{M}+\overline{N}} K_i$.

In the case that $v_1 - v_2$ is S-monotonically decreasing whenever it is S-monotonic, the construction of the sets starts at t_0 closest to 0 at which $v_1 - v_2$ reaches its maximum and the construction of the sets progresses rightward with each new set being to the right of the already constructed set and the sets $K_0, ..., K_{\overline{M+N}}$ represent the intervals from left to right. This ends the construction of the set K.

Finally, we define the function c, the analogue of $v_1 - v_2 + I$ in Theorem 11. Let

$$c_i(t) = \frac{\sum_{x \in X(t)} (v_1(t) - v_2(t) + I(x)) H'(x) |x'(t)|}{\sum_{x \in X(t)} H'(x) |x'(t)|} \text{ for } i \in K_i,\ i = 1, ..., \overline{M} + \overline{N}$$

so that

$$c(t) = c_i(t) \text{ for } t \in K_i,\ i = 1, ..., \overline{M} + \overline{N}$$

Theorem 12 *If $v_1 - v_2$ has a finite number of critical points on $[0,1]$ and $c(t)$ is monotonic on K, then there exists an equilibrium bidding function whose allocation is efficient.*

Corollary 13 *If $v_1 - v_2$ has at most a finite number of critical points on $[0,1]$ and I is constant, then there exists an equilibrium bidding function whose allocation is efficient.*

The following corollary says that, if the value of one good is common to all types and the value of the other good is correlated with income, then revenue is maximized when the commonly valued good is sold last.

Corollary 14 *If I and v_α are S-monotonically increasing on $[0,1]$ and v_β is constant then, there exists a unique equilibrium bidding function under the order of sale (α, β). This unique equilibrium bidding function generates an efficient allocation and maximizes revenue among all equilibrium bidding functions under either order of sale.*

In general, there may exist multiple equilibrium bidding functions under both 1^{st} and 2^{nd} price rules. These functions may be neither S-monotonic nor efficient but the next result shows that for each equilibrium bidding function that exists under one rule, an order-equivalent bidding function exists under the other rule.

Theorem 15 *For each equilibrium bidding function B_y that exists under a y^{th}price rule, there exists, under a $3 - y^{th}$ price rule, an equilibrium bidding function B_{3-y} OE to B_y for $y \in \{1, 2\}$.*

The following two theorems compare outcomes under 1^{st} and 2^{nd} price rules for bidding functions that are OE. Theorem 16 generalizes a theoretical and experimental result for auctions with complete information in Pitchik and Schotter (1988).

Theorem 16 *If the equilibrium bidding function under a 1^{st} price rule is OE to the equilibrium bidding function under a 2^{nd} price rule, then the expected price of good 1 is higher under a 2^{nd} price rule than it is under a 1^{st} price rule.*

Intuitively, under a 1^{st} price rule, a bidder is worried about being allocated good 1 at a relatively high price and so makes a relatively conservative bid. Under a 2^{nd} price rule, a bidder is worried about not winning the good at a relatively low price.

Theorems 17 and 20 and their Corollaries compare the price of a good when it is sold first to the price of the good when it is sold second under 1^{st} and 2^{nd} price rules.

When income is an index of an individual's ability to use the good profitably, then the valuation for a good is correlated with income in which case it may be that $v_\alpha - v_\beta + I$ and $v_\beta - v_\alpha + I$ are OE to I. In this case, we may compare the expected price of a good under one sequence of sale with that under the other sequence of sale.

Theorem 17 *If $v_\alpha - v_\beta + I$, $v_\beta - v_\alpha + I$, and I are OE to the equilibrium bidding functions $B_{\alpha,\beta}$ under order α, β and $B_{\beta,\alpha}$ under order β, α, then the expected revenue is independent of the sequence of sale and the expected price of a good is higher, the later it is sold.*

The following two corollaries follow from Theorems 11 and 17.

Corollary 18 *If $v_\alpha - v_\beta + I$ and $v_\beta - v_\alpha + I$ are OE and S-monotonic, then, for each order of sale, there exists an equilibrium bidding function that is S-monotonic. Under any S-monotonic equilibrium bidding function, the expected revenue is independent of the sequence of sale and the expected price of a good is higher, the later it is sold.*

Corollary 19 *If $v_\alpha \equiv v_\beta$ and I is S-monotonic, then for each order of sale there exists an equilibrium bidding function that is S-monotonic. Among all outcomes associated with an S-monotonic equilibrium bidding function, the allocation is efficient, the expected revenue is independent of the sequence of sale, and the expected price of a good is higher, the later it is sold.*

If income varies widely relative to the value of either good then the expected price of the good is higher the later it is sold.

Under each price rule, the last theorem compares the expected price of a good under the two different sequences of sale. While it always true that $v_\alpha - v_\beta$ is ORE to $v_\beta - v_\alpha$ it may not be the case that $v_\alpha - v_\beta + I$ and $v_\beta - v_\alpha + I$ share any equivalence. We would like to compare the prices of goods as a function of the sequence but to do so we need to keep revenue constant in order to find the direct effect of the order of sale on the difference in price. In the case that $v_\alpha - v_\beta + I$ is ORE to $v_\beta - v_\alpha + I$, the expected revenue is independent of the sequence only if the income is independent of type. If income is constant across types then $v_\alpha - v_\beta + I$ is ORE to $v_\beta - v_\alpha + I$. In this case, the expected prices of the goods depend on the sequence of sale and on the price-formation rules as follows.[16]

Theorem 20 *If $I(t) = I$ is constant and $v_1 - v_2$ has a finite number of critical values, then for each order of sale and for each price rule, there exists a unique equilibrium bidding function. In*

[16] The last part of this result generalizes theoretical and experimental complete information results in Pitchik and Schotter (1988).

each case, the unique bidding function is OE to $v_1 - v_2 + I$ and allocates goods efficiently. Under each price rule, the bidding function under one order of sale is ORE to the bidding function under the other order of sale. In addition, (1) under a 1^{st} price rule the expected price of a good is higher the later it is sold, and (2) under a 2^{nd} price rule the expected price of a good is higher the earlier it is sold.

The reason the last part of this result differs from Theorems 4 and 17 is as follows. Because the variation in the differences in valuation for each good are essentially dwarfed by that of income in Theorems 4 and 17, income plays the following role in determining the allocation of each good. The individual with the higher income obtains good 1 under either sequence and under any price-formation rule. Thus, the results do not depend on the price-formation rule and the allocation varies with the sequence. However, in Theorem 20, the variation in the valuation of good α say dwarfs that of good β and income combined so that both the valuation and position (whether first or second) of good α play a role in determining the price of each good. In this case, under any price-formation rule and under any sequence, the individual who values good α more highly obtains good α.

Now suppose that the value of one good, say α, varies with a bidder's type while the income and the value of the other good, say β, is independent of the bidder's type. Further suppose that the values share a common mean. In this case, Theorem 20 implies that if the goods are similar, then the price of a good decreases with its position in the order of sale when the auction proceeds under a 2^{nd} price rule. Under a 2^{nd} price rule, a bidder is able to bid up the price of good 1 in order to obtain good 2 at a lower price than otherwise.

7 Robustness

In this section I show that the sequence of sale may affect revenue and/or prices when the assumptions of the model are relaxed. I provide three examples below.

I first consider a 2^{nd} price auction in the presence of $n > 2$ bidders who each have a common income that is less than the valuation of either good and who each highly value good α relative to good β and income. Specifically, $I(t) = I$ and $v_\alpha(t) > (n-1)(v_\beta(t) - I) + I > v_\beta(t) - I > I > 0$ for $t \in [0, 1]$. Since $I < v_\beta(t) < v_\alpha(t)$ for all $t \in [0, 1]$, no bidder obtains more than one good in equilibrium. I want to compare the price of each good if the order of sale is (α, β) to that when it is (β, α). If the order is (α, β) then each individual's equilibrium bidding function for good α is[17] $B(t) = I$ so that the expected equilibrium price of good α is I. So, when the goods are sold in the order (α, β), the price of good α is I; that of good β is I since $n > 2$. If instead,

[17] The payoff from a bid of I is $(v_\alpha(t) - I)/n + (v_\beta(t) - I)(n-1)/n(n-1)$. The payoff from bidding less than I results is $(v_\beta(t) - I)/(n-1)$.

the goods are sold in the order (β, α) then the expected price of β is less than I. The reason is as follows. Suppose that the expected price of β is I. In this case, at least two individuals bid I. Let $m \geq 2$ be the number of individuals who bid I. The payoff to each who bids I is $(v_\beta(t) - I)/m + [(m-1)/m](v_\alpha(t) - I)/(n-1)$ while the payoff to anyone who bids just less than I is $(v_\alpha(t) - I)/(n-1)$. By assumption $v_\alpha(t) - I > (n-1)(v_\beta(t) - I)$, so that it cannot be that $m \geq 2$ bidders bid I in equilibrium. Thus, the equilibrium price of β must be less than I. It follows that when the goods are sold in the order (β, α) the expected equilibrium price of β is less than I, as claimed. Thus, revenue is highest when α is sold first and the price of β increases with its position in the order of sale. The reason is that when β is sold second, there are always at least two individuals who are willing and able to pay I for β so that competition is intense; when β is sold first, competition is not as intense since each bidder wants to have income to bid on good α.

The above example illustrates that the revenue and prices of goods may depend on the sequence in which the goods are sold even when there are many bidders and only two goods. In the example, revenue is maximized when the more valuable good is sold first.

I now analyse the model when there are asymmetries among the bidders as well as changes to the assumptions on income relative to valuations so that a bidder may obtain both goods in equilibrium under a 2^{nd} price rule. Let's assume that the valuations of the two goods are fixed such that $v_\alpha(t) = A$, $v_\beta(t) = B < A$. Suppose that there are two pools of bidders and that bidder 1 has enough income to buy both goods whereas bidder 2 is relatively and absolutely poor. Specifically, let's assume that $I_1(t) \in [c_1, d_1]$ and that $I_2(t) = I$ where $A > d_1 > c_1 > 2I > B > I$. If the goods are auctioned in the order (α, β) then each bidder is willing to pay at least I for α so that bidder 1 obtains good α at a price of I in equilibrium. Once good α is allocated, each bidder is willing to pay at least I for good β. Since bidder 1 has more than double the income of bidder 2 and since good α is relatively highly valued, bidder 1 obtains both goods in equilibrium when the order of sale is (α, β). The equilibrium price of each good equals I and the equilibrium revenue is $2I$. However, when the goods are sold in the order (β, α), bidder 1's de facto valuation for good β is $B/2 < I$. Bidder 2 is willing to pay up to I for good β since otherwise, bidder 2 receives nothing. Thus, under the order $\beta\alpha$, bidder 2 obtains good β and bidder 1 obtains good α. The equilibrium price of β is $B/2$ and that of α is $I - B/2$. The equilibrium revenue is I. In summary, the revenue is higher when α (the more highly valued good) is sold first; the price of good α (the good that is allocated to the rich bidder independent of its order of sale) is higher when α is sold first; the price of good β (the good that is allocated to the stronger bidder only when it is sold second) is higher when β is sold second. That revenue is higher when α is sold first is consistent with the implications of Theorem 2 in which the revenue is higher when good 1 is sold to the richer bidder. In Theorem 20, the allocation of the goods is independent of the price rule and the price of a good is higher the earlier it is sold under a 2^{nd} price rule. In our example, α is allocated to the richer bidder independent of the order of sale so that competition for good α and its price are higher the earlier it is sold. That the price of good α is higher when sold first in our example is therefore consistent with Theorem 20. Under the assumptions of Theorem 17 the price of a good is higher the later it is sold when the competition for the good is higher the later it is sold. In our example, β is allocated to the rich bidder only when it is sold second so that competition for β is higher when it is sold second. That the price of good β is higher when sold second is consistent with Theorem 17.

Thus, even when one bidder is relatively strong (in the example, the bidder whose income is twice that of the other bidder is relatively strong), revenue and price may depend on the order of sale. Whether the price of a good increases or decreases with its position in the order of sale depends on how the order of sale affects the competition for the good. When good 1 goes to the stronger bidder independent of the order of sale, then there is no disadvantage in obtaining good 1 because, if there were, the stronger bidder would just mimic the weaker bidder. In this case, the price of a good must increase with its position in the order of sale. When a designated good goes to the stronger bidder, then the competition for the good is higher when it is sold first and so price decreases with its position in the order of sale.

Lastly, I consider an example in which a bidder's type is two dimensional and each dimension is distributed independently. Suppose that the valuation for each good is common and distributed uniformly on $[6, 7]$ and income is independently distributed uniformly on $[7, 8]$. An increasing equilibrium bidding function (for which no bidder wins both goods) is

$$B(i) = \begin{cases} B_2(i) = \frac{i+4}{3} & \text{under a } 2^{nd} \text{ price rule} \\ B_1(i) = \frac{i}{3} + \frac{7}{6} & \text{under a } 1^{st} \text{ price rule} \end{cases}$$

and the distribution of bids is

$$G(b) = \begin{cases} G_2(b) = 3b - 11 \text{ on } \left[\frac{11}{3}, 4\right] & \text{under a } 2^{nd} \text{ price rule} \\ G_1(b) = 3b - \frac{21}{2} \text{ on } \left[\frac{7}{2}, \frac{23}{6}\right] & \text{under a } 1^{st} \text{ price rule} \end{cases}$$

In this case, the expected price of good 1 is

$$\begin{cases} 2\int_{\frac{11}{3}}^{4} b(1 - G_2(b))G_2'(b)db = \frac{34}{9} & \text{under a } 2^{nd} \text{ price rule} \\ 2\int_{\frac{7}{2}}^{\frac{23}{6}} bG_1(b)G_1'(b)db = \frac{67}{18} < \frac{68}{18} = \frac{34}{9} & \text{under a } 1^{st} \text{ price rule} \end{cases}$$

Since the expected revenue is

$$2\int_7^8 i(i-7)di = \frac{23}{3} \quad \text{under } 1^{st} \text{ and } 2^{nd} \text{ price rules}$$

we obtain that the expected price of good two is

$$\begin{cases} \frac{23}{3} - \frac{34}{9} = \frac{35}{9} & \text{under a } 2^{nd} \text{ price rule} \\ \frac{23}{3} - \frac{67}{18} = \frac{71}{18} > \frac{70}{18} = \frac{35}{9} & \text{under a } 1^{st} \text{ price rule} \end{cases}$$

Thus, even when the valuations are distributed independently from income, when the values are identical, the price of a good is affected by its order of sale. As in Theorem 17 and Corollary 19, there exists a bidding function that is OE to $v_1 - v_2 + I$ and I, and the expected price of good 2 is higher than that of good 1. As in Theorem 16, the expected price of good 1 is higher under a second price rule than under a first price rule. As in Theorem 7 the equilibrium price of good 1 depends on the price rule.

8 Conclusion

In the presence of budget constraints there may exist multiple symmetric equilibrium bidding functions and they may differ with respect to allocation, prices and revenue. Prices depend on the price-formation rule. In addition, even in the absence of arbitrage possibilities, identical goods may fetch different prices. The sequence of sale affects the expected revenue through the allocation of the goods. Whenever the winner of good 1 is the bidder with the higher income, expected revenue is maximized. Under 1^{st} and 2^{nd} price rules, whenever, independent of the sequence, the winner of good 1 is the bidder with the higher income, the expected price of a good is no lower the later it is sold. Intuitively, if good 1 is always sold to the stronger bidder, then there can be no disadvantage in winning good 1. This happens when the goods are similar enough and income is relatively variable. By contrast, if, independent of the sequence, the stronger bidder is allocated a designated good (which may be good 1 or good 2), the expected price of a good decreases with its position in the order of sale under a 2^{nd} price rule and increases under a 1^{st} price rule. Intuitively, under a 2^{nd} price rule, there is an incentive for the loser of good 1 to bid up its price, depleting the winner's income, in order to obtain good 2 at a lower price. Thus, under a 2^{nd} price rule, when the allocation of the goods is independent of the sequence, the expected price of a good declines with its position in the order of sale. Under a 1^{st} price rule, a higher bid of the loser does not affect the price of good 1 and may adversely affect the payoff of the loser so that bids are more conservative.

Basically, the price of a good is higher whenever competition for the good is higher. If bidders are drawn from populations that differ according to income, then goods that are always allocated to the richer bidder fetch a higher price when sold first. Goods that are sold to the richer bidder only when sold second fetch a higher price when sold second.

Other links between the goods can have the same effect as do budget-constraints. For example, if firms with limited plant capacities bid on projects let by the government, the results of letting any given contract will depend on the available capacity of firms in the industry. The results are not qualitatively different in this case.

9 Appendix

Proof of Theorem 2: By (R), expected revenue equals the expected income of the winner of good 1 so that revenue is highest if the winner of good one has the higher income. If $B(t)$ is OE to $I(t)$ under any price-formation rule then good 1 is allocated to the bidder with the higher income so that, by (R), the expected revenue is the expected value of the higher income so that, in the case of multiple equilibria, the expected revenue is highest if the equilibrium bidding function is OE to the income function. Thus, the expected revenue is independent of the price-formation rule since the winner of good 1 is unaffected by the price-formation rule.

We now compare revenue in the revenue-maximizing sequential auction to that in an auction in which both goods are sold simultaneously. When both goods are sold simultaneously to one of the bidders, rather than sequentially, an individual t's de facto valuation of holding both goods is $I(t)$ by assumption (1). Thus, the auction in which both goods are sold simultaneously is equivalent to an auction in which one good is sold whose value to individual t is $I(t)$. The result follows since no auction of a single good can yield an expected equilibrium revenue equal to the expected value of the highest valuation (Riley and Samuelson [1981]).

In the case that the bidding function is OE to $v_1 - v_2$, the individual who obtains good 1 (say type t_1) has the higher $v_1 - v_2$ while the individual who obtains good 2 (say type t_2) has the lower. By assumption, in equilibrium, the remaining income R_1 of the individual t_1 is less than $v_2(t_2)$, the valuation for good 2 of individual t_2. Thus, if the bidding function B is OE to $v_1 - v_2$, then the allocation is efficient. In the case of multiple equilibria, the allocation is efficient only if the bidding function is OE to the difference in the value functions. ∎

Proof of Theorem 4: The de facto valuation of good 1 for a bidder of type t who faces a bidder of type s is $V(t,s) = (v_1(t) - v_2(t) + I(s))/2$. Since B is OE to $v_1 - v_2 + I$ and I, the average price that a bidder of type t expects to pay for good 1 is strictly less than bidder t's critical value $v(t) = (v_1(t) - v_2(t) + I(t))/2$. In this case the expected price of good 1 must be strictly less than the critical value of the expected winner. Thus, the expected price of good 1 must be strictly less than $(v_1(w_1) - v_2(w_1) + I(w_1))/2$. However, by (R), the expected revenue is $I(w_1)$. Thus, the expected price of good 2 must be strictly greater than $(v_2(w_1) - v_1(w_1) + I(w_1))/2$ which proves the result. ∎

Proof of Theorem 6: If the other bidder uses the equilibrium strategy $B(t)$, let $\Delta(x)$ be the probability that a bidder of type t who pretends to be type x wins good 2 and let $\widetilde{P}(x)$ be the *expected* payment made by such a bidder. The *expected* payoff $\Pi(t,x)$ of such a bidder equals the *expected* benefit minus the *expected* payment

$$\Pi(t,x) = v_1(t)(1 - \Delta(x)) + v_2(t)(\Delta(x)) - \widetilde{P}(x)$$

In equilibrium, $D_2'\Pi(t,x)$ equals 0 when $x=t$ so that

$$-(v_1(t)-v_2(t))\Delta'(t)=\widetilde{P}'(t) \tag{3}$$

Since $v_1(t)=v_2(t)$, (3) implies $\widetilde{P}'(t)=0$ for all t as required. ∎

In order to prove the next Theorem, let $P^*(x)$ denote the *expected* price that an individual (who claims to be of type x) pays for good 1 and $P_*(x)$ denote the *expected* price that an individual (who claims to be of type x) pays for good 2. Thus, $\widetilde{P}(t,x)=P^*(x)+P_*(x)$. Let $P(x,z)$ denote the price that an individual who claims to be of type x pays for good 1 if the other individual claims to be of type z.

Let $\sigma_1(t,x)$ be the set of opponents who lose good 1. Let $\sigma_2(t,x)$ denote the set of opponents who lose only good 2. By assumption, no bidder wins both goods in equilibrium so that, in equilibrium, $\sigma_{12}(t,t)=\varnothing$ (which implies that $\sigma_1(t,t)+\sigma_2(t,t)=1$) and $v_2(s)>I(s)-P(s,t)$ for $s\in\sigma_2(t,t)$. Thus, if $s\in\sigma_1(t,x)$, then individual t obtains good 1 at the price $P(x,s)$; if instead $s\in\sigma_2(t,x)$ then individual t loses good 1 and obtains good 2 at the price $I(s)-P(s,x)$. In equilibrium, the expected payoff must be maximized when $x=t$. If $s(x)$ is an upper end point of an interval in $\sigma_1(t,x)$ that varies with x then $s(x)$ is a lower end point of an interval in $\sigma_2(t,x)$ so that the derivative (evaluated at $x=t$) of $\int_{\sigma_1(t,x)}H'(s)ds$ with respect to x equals the the negative of the derivative (evaluated at $x=t$) of $\int_{\sigma_2(t,x)}H'(s)ds$ with respect to x. Thus, in equilibrium,

$$\widetilde{P}(t)=P^*(t)+P_*(t)=\int_{\sigma_1(t,t)}P(t,s)H'(s)ds+\int_{\sigma_2(t,t)}\left(I(s)-P(s,t)\right)H'(s)ds$$

Proof of Theorem 7: In equilibrium, $P(s,t)=P(t,t)$ for all endpoints s of intervals in $\sigma_2(t,t)$ that depend on t. Since, in equilibrium,

$$\widetilde{P}'(t)=\int_{\sigma_1(t,t)}D_1'P(t,s)H'(s)ds-\int_{\sigma_2(t,t)}D_2'P(s,t)H'(s)ds-2P(t,t)\Delta'(t)+\frac{d}{dt}\int_{\sigma_2(t,t)}I(s)H'(s)ds$$

(3) implies that $2P(t,t)\Delta'(t)=$

$$(v_1(t)-v_2(t))\Delta'(t)+\frac{d}{dt}\int_{\sigma_2(t,t)}I(s)H'(s)ds-\int_{\sigma_2(t,t)}D_2'P(s,t)H'(s)ds+\int_{\sigma_1(t,t)}D_1'P(t,s)H'(s)ds \tag{4}$$

So, since

$$P^{*\prime}(t)=\int_{\sigma_1(t,t)}D_1'P(t,s)H'(s)ds-P(t,t)\Delta'(t)$$

we can substitute for $P(t,t)\Delta'(t)$ from (4) to obtain

$$P^{*\prime}(t)=\tfrac{1}{2}\Big(-(v_1(t)-v_2(t))\Delta'(t)-\delta'(t)+\int_{\sigma_2(t,t)}D_2P(s,t)H'(s)ds+\int_{\sigma_1(t,t)}D_1P(t,s)H'(s)ds\Big) \tag{5}$$

The result follows since the right-hand side of (5) depends on the price-formation rule.[18] ∎

Proof of Theorem 8: Equation (4) implies that, for $t \in \widehat{J}_1$

$$\begin{aligned} &\int_{\sigma_2(t,t)} D_2'P(s,t)H'(s)ds - \int_{\sigma_1(t,t)} D_1'P(t,s)H'(s)ds + 2P(t,t)\frac{d}{dt}\int_{\sigma_2(t,t)} H'(s)ds \\ = \;& (v_1(t) - v_2(t))\frac{d}{dt}\int_{\sigma_2(t,t)} H'(s)ds + \frac{d}{dt}\int_{\sigma_2(t,t)} I(s)H'(s)ds \end{aligned} \tag{6}$$

and for $t \in \widehat{J}_2$

$$\begin{aligned} &\int_{\sigma_2(t,t)} D_2'P(s,t)H'(s)ds - \int_{\sigma_1(t,t)} D_1'P(t,s)H'(s)ds + 2P(t,t)\frac{d}{dt}\int_{\sigma_2(t,t)} H'(s)ds \\ = \;& (v_1(t) - v_2(t))\frac{d}{dt}\int_{\sigma_2(t,t)} H'(s)ds + \frac{d}{dt}\int_{\sigma_2(t,t)} I(s)H'(s)ds \end{aligned} \tag{7}$$

If instead of t varying in $\widehat{J}_2$, we have t varying in $\widehat{J}_1$ and $\widehat{x}(t)$ varying in $\widehat{J}_2$, we can rewrite (7) for $t \in \widehat{J}_1$, as

$$\begin{aligned} &\int_{\sigma_2(\widehat{x}(t),\widehat{x}(t))} D_2'P(s,\widehat{x}(t))H'(s)ds - \int_{\sigma_1(\widehat{x}(t),\widehat{x}(t))} D_1'P(\widehat{x}(t),s)H'(s)ds \\ &\qquad + 2P(\widehat{x}(t),\widehat{x}(t))\frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} H'(s)ds \\ &= (v_1(\widehat{x}(t)) - v_2(\widehat{x}(t)))\frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} H'(s)ds + \frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} I(s)H'(s)ds \end{aligned} \tag{8}$$

If we then multiply both side of (8) above by $\widehat{x}'(t)$ we obtain that, for $t \in \widehat{J}_1$,

$$\begin{aligned} &\int_{\sigma_2(\widehat{x}(t),\widehat{x}(t))} D_2'P(s,\widehat{x}(t))\widehat{x}'(t)H'(s)ds - \int_{\sigma_1(\widehat{x}(t),\widehat{x}(t))} D_1'P(\widehat{x}(t),s)\widehat{x}'(t)H'(s)ds \\ &\qquad + 2P(\widehat{x}(t),\widehat{x}(t))\widehat{x}'(t)\frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} H'(s)ds \\ &= (v_1(\widehat{x}(t)) - v_2(\widehat{x}(t)))\widehat{x}'(t)\frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} H'(s)ds + \widehat{x}'(t)\frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} I(s)H'(s)ds \end{aligned} \tag{9}$$

However, by definition of $\widehat{x}(t)$, $\sigma_2(t,t) = \sigma_2(\widehat{x}(t),\widehat{x}(t))$, $\sigma_1(t,t) = \sigma_1(\widehat{x}(t),\widehat{x}(t))$, $P(s,t) = P(s,\widehat{x}(t))$, $D_2'P(s,t) = D_2'P(s,\widehat{x}(t))\widehat{x}'(t)$, $P(t,s) = P(\widehat{x}(t),s)$, $D_1'P(t,s) = D_2'P(\widehat{x}(t),s)\widehat{x}'(t)$, $P(t,t) = P(\widehat{x}(t),\widehat{x}(t))$, and

$$\begin{aligned} \frac{d}{dt}\int_{\sigma_2(t,t)} H'(s)ds &= \widehat{x}'(t)\frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} H'(s)ds \\ \frac{d}{dt}\int_{\sigma_2(t,t)} I(s)H'(s)ds &= \widehat{x}'(t)\frac{d}{dt}_{|t=\widehat{x}(t)}\int_{\sigma_2(t,t)} I(s)H'(s)ds \end{aligned}$$

[18] The analog of equation (5) in the standard one good auction is $P^{*\prime}(t) = v(t)H'(t)$ which is independent of the price formation rules.

so equation (9), for $t \in \widehat{J}_1$, is equivalent to

$$\int_{\sigma_2(t,t)} D_2' P(s,t) H'(s) ds - \int_{\sigma_1(t,t)} D_1' P(t,s) H'(s) ds + 2P(t,t) \frac{d}{dt} \int_{\sigma_2(t,t)} H'(s) ds \tag{10}$$
$$= (v_1(\widehat{x}(t)) - v_2(\widehat{x}(t))) \frac{d}{dt} \int_{\sigma_2(t,t)} H'(s) ds + \frac{d}{dt} \int_{\sigma_2(t,t)} I(s) H'(s) ds$$

Thus, (6) and (10) must hold for $t \in \widehat{J}_1$ which implies the result. ∎

Proof of Theorem 10: We look for an equilibrium bidding function B that is monotonically increasing on $[0,1]$ so that when setting up equation (4), $\sigma_2(t,t) = (t,1)$, $\sigma_1(t,t) = (0,t)$ and $\Delta(t) = 1 - H(t)$. Let D denote the total derivative so that $D_1' P(t,t) +$ $D_2' P(t,t) = DP(t,t)$.

If $B(t)$ solves (4), then, by construction, bidder t gains most by bidding according to $B(t)$ rather than according to $B(s)$ for any $s \neq t$. I first show that there is no incentive for any bidder to bid outside the range of bids. Continuity of the payoff function and the fact that the payoff function decreases as the distance between the out-of-equilibrium bid and the set of equilibrium bids increases imply that no bidder can gain by deviating, proving the result. It remains to show that there exists a solution to (4) that is monotonically increasing for which no bidder obtains both goods in equilibrium. If the bidding function satisfies $P(0,0) \geq (v_1(0) - v_2(0) + I(0))/2$, condition (2) would be sufficient to imply that no bidder obtains both goods in equilibrium.

Let D denote the total derivative. In the case that one of $D_1' P(t,s)$ or $D_2' P(s,t)$ equals zero, then, since $D_1' P(t,t) +$ $D_2' P(t,t) = DP(t,t)$ and $D_{12}''(t,t) = D_{21}''(t,t) = 0$, (4) implies that there exists a monotonically increasing B that is an equilibrium bidding function if there exists a monotonically increasing bidding function for which

$$P(0,0) \geq \min_{s,t \in [0,1]} \frac{v_1(t) - v_2(t) + I(s)}{2}$$

and **either**

$$DP(t,t)(1 - H(t)) - 2P(t,t)H'(t) = -(v_1(t) - v_2(t) + I(t))H'(t)$$

or

$$DP(t,t)H(t) + 2P(t,t)H'(t) = (v_1(t) - v_2(t) + I(t))H'(t)$$

so that **either**

$$P(t,t)(1 - H(t))^2 = \int_1^t (v_1(s) - v_2(s) + I(s)) \frac{d(1 - H(s))^2}{2}$$

implies

$$P(t,t) = \frac{v_1(t) - v_2(t) + I(t)}{2} - \frac{\int_1^t (1 - H(s))^2 (v_1'(s) - v_2'(s) + I'(s)) ds}{2(1 - H(t))^2}$$

or

$$P(t,t)(H(t))^2 = \int_0^t (v_1(s) - v_2(s) + I(s)) \frac{d((H(s))^2)}{2} ds$$

implies

$$P(t,t) = \frac{(v_1(t) - v_2(t) + I(t))}{2} - \frac{\int_0^t (H(s))^2 (v_1'(s) - v_2'(s) + I'(s)) ds}{2(H(t))^2}$$

Since $P(t,t)$ is a strictly increasing function of $B(t)$ and $v_1'(s) - v_2'(s) + I'(s) > 0$, there exists a bidding function that solves (4) for which **either**

$$DP(t,t) = \frac{2H'(t) \int_t^1 (1 - H(s))^2 (v_1'(s) - v_2'(s) + I'(s)) ds}{2(1 - H(t))^3} > 0 \text{ on } (0,1)$$

and

$$P(0,0) \geq \min_{s,t \in [0,1]} \frac{v_1(t) - v_2(t) + I(s)}{2}$$

or

$$DP(t,t) = \frac{2H'(t) \int_0^t (H(s))^2 (v_1'(s) - v_2'(s) + I'(s)) ds}{2(H(t))^3} > 0 \text{ on } (0,1)$$

and

$$P(0,0) \geq \min_{s,t \in [0,1]} \frac{v_1(t) - v_2(t) + I(s)}{2}$$

Thus, when $v_1' - v_2' + I' > 0$ on $(0,1)$, $D_{12}''(t,t) = D_{21}''(t,t) = 0$ and one of $D_1'P(t,s)$ or $D_2'P(s,t)$ equals zero, $DP > 0$ for $t \in (0,1)$. Thus, since $P(t,t)$ is a strictly increasing function of $B(t)$, $v_1'(s) - v_2'(s) + I'(s) > 0$ on $[0,1]$ implies that there exists a monotonically increasing equilibrium bidding function.

In the case that $D_{12}''(t,t) = D_{21}''(t,t) = 0$ and $D_2'P(t,t) = \gamma D_1'P(t,t)$ for $\gamma > 0$, and $D_1'P(t,t) + D_2'P(t,t) = DP(t,t)$, (4) implies that

$$DP(t,t)\left(\frac{\gamma}{1+\gamma} - H(t)\right) - 2P(t,t)H'(t) = -(v_1(t) - v_2(t) + I(t))H'(t)$$

so that

$$P(t,t)\left(\frac{\gamma}{1+\gamma} - H(t)\right)^2 = \int_{H^{-1}\left(\frac{\gamma}{1+\gamma}\right)}^t (v_1(s) - v_2(s) + I(s)) \frac{d\left(\frac{\gamma}{1+\gamma} - H(s)\right)^2}{2}$$

implies

$$P(t,t) = \frac{(v_1(t) - v_2(t) + I(t))}{2} - \frac{\int_{H^{-1}\left(\frac{\gamma}{1+\gamma}\right)}^t \left(\frac{\gamma}{1+\gamma} - H(s)\right)^2 (v_1'(s) - v_2'(s) + I'(s)) ds}{2\left(\frac{\gamma}{1+\gamma} - H(t)\right)^2}$$

Since $P(t,t)$ is a strictly increasing function of $B(t)$, $v_1'(s) - v_2'(s) + I'(s) > 0$ this implies that there exists a bidding function that solves (4) for which

$$DP(t,t) = -\frac{\int_{H^{-1}\left(\frac{\gamma}{1+\gamma}\right)}^{t} \frac{(1-H(s))^2}{2}(v_1'(s) - v_2'(s) + I'(s))ds}{\left(\frac{\gamma}{1+\gamma} - H(t)\right)^3} > 0 \text{ on } (0,1)$$

and

$$P(0,0) \geq \min_{s,t \in [0,1]} \frac{v_1(t) - v_2(t) + I(s)}{2}$$

so that again $v_1'(s) - v_2'(s) + I'(s) > 0$ on $[0,1]$ implies that there exists an equilibrium bidding function B that is monotonically increasing on $[0,1]$. Analogous arguments show that the results remain when $v_1(s) - v_2(s) + I(s)$ is S-decreasing on $[0,1]$. ∎

Proof of Theorem 11: Under first price rules, $D_2P(t,s) = 0$ and under second price rules, $D_1P(t,s) = 0$. Thus, $D_{12}P(t,s) = 0$ and the result follows from Theorem *10*. ∎

Proof of Theorem 12: We show that there exists an equilibrium bidding function that is OE to $v_1 - v_2$. If all play according to a bidding function B that is OE to $v_\alpha - v_\beta$, let

$$\Delta_i(t) = \int_{\sigma_2(t,t)} H'(s)ds$$

denote the probability that player $t \in K_i \subset S \cup J$ obtains good 2. Let $X(t)$ represent the set of points in $[0,1]$ for which $v_1(x) - v_2(x) = v_1(t) - v_2(t)$. If $t \in int(S)$, $X(t) = \{t\}$ where int denotes the interior. If $t \in int(J)$, then $X(t) \supsetneqq \{t\}$. In this case, as t varies so does each point in $X(t)$. We abuse notation and let $x \in X(t)$ stand for a point and also for a function that varies with t whenever t is not a critical point of $v_1 - v_2$. In this case, the sign of $x'(t)$ equals that of $v_1'(x) - v_2'(x)$ for $x \neq t$. Let $|z|$ denotes the absolute value of z. Since $v_1 - v_2$ increases at $t \in int(S) \cup int(J)$ we obtain that, for $k = s$ and $i = 0, ..., \overline{M} + \overline{N}$,

$$\Delta_i'(t) = -\sum_{x \in X(t)} H'(x)\left|x'(t)\right| \leq 0$$

Since $v_1 - v_2$ is maximized at t_0, $\Delta_0(t_0) = 0$. We note that $\Delta_i(t) > 0$ at all other right-hand endpoints of K_i, $i = 1, ..., \overline{M} + \overline{N}$.

We provide the proof explicitly under a second price rule in the case that $v_1 - v_2$ is S-monotonically increasing whenever it is S-monotonic. We begin construction of B with a differential equation on K_0. In this case, since $v_1(x) - v_2(x) = v_1(t) - v_2(t)$ for $x \in X(t)$, the ODE that characterizes B on K_0 is

$$B'(t)\Delta_0(t) + 2B(t)\Delta_0'(t) = c(t)\Delta_0'(t) \tag{11}$$

with initial condition $B(t_0) = c(t_0)/2$.

Thus the solution to ODE (11) and its initial condition is

$$B(t)\Delta_0^2(t) = \int_{t_0}^{t} c_0(s)\Delta_0'(s)\Delta_0(s)ds$$

for $t \in [t_1, t_0] = K_0$. The proof that $B'(t) > 0$ for $t \in int\,(K_0)$ follows the exact reasoning offered in the proof of Theorem 10 with c_0 in the current proof taking on the role of $v_1 - v_2 + I$ in the previous proof so that the sign of $B' > 0$ as required.

We now derive B for the iterative step. Suppose that B has been derived and that $B'(t) > 0$ for $t \in \cup_{i=0}^{i=k} K_r$ as above. Suppose that, for $t \in K_k = [t_{2k+1}, t_{2k}]$,

$$B(t)\Delta_k^2(t) = \int_{t_{2k}}^{t} c_k(s)\Delta_k'(s)\Delta_k(s)ds + B(t_{2k})\Delta_k^2(t_{2k}) \quad \text{and}$$

$$B(t_{2k})\Delta_k^2(t_{2k}) = -\sum_{r=0}^{r=k-1} \int_{t_{2r}}^{t_{2r+1}} c_r(s)\Delta_r'(s)\Delta_r(s)ds$$

where $\Delta_r(t_{2r}) = \Delta_{r-1}(t_{2r-1})$ and $c_r(t_{2r+1}) = c_{r+1}(t_{2r+2})$ on K_r for $r = 0, ..., k$. We now derive B for $t \in K_{k+1}$ and show that $B' > 0$ on K_{k+1}.

Since $v_1(t_{2k+2}) - v_2(t_{2k+2}) = v_1(t_{2k+1}) - v_2(t_{2k+1})$, and since B is OE to $v_\alpha - v_\beta$, the ODE (11) over K_{k+1} is accompanied by the initial condition $B(t_{2k+2}) = B(t_{2k+1})$where

$$B(t_{2k+1})\Delta_k^2(t_{2k+1}) = \sum_{r=0}^{r=k} \int_{t_{2r}}^{t_{2r+1}} c_r(s)\Delta_r'(s)\Delta_r(s)ds$$

where the equality $\Delta_r(t_{2r}) = \Delta_{r-1}(t_{2r-1})$ holds for $r = 0, ..., k+1$ so that

$$B(t_{2k+2})\Delta_{k+1}^2(t_{2k+2}) = B(t_{2k+1})\Delta_k^2(t_{2k+1})$$

Since $\Delta_0(t_0) = 0$, the solution to ODE (11) with initial condition is

$$B(t)\Delta_{k+1}^2(t) = \int_{t_{2k+2}}^{t} c_{k+1}(s)\Delta_{k+1}'(s)\Delta_{k+1}(s)ds + \sum_{r=0}^{r=k} \int_{t_{2r}}^{t_{2r+1}} c_r(s)\Delta_r'(s)\Delta_r(s)ds$$

for $t \in K_{k+1}$ and its derivative is

$$\begin{aligned} B'(t) = {} & \frac{c_{k+1}\Delta_{k+1}'(t)\Delta_{k+1}(t)}{\Delta_{k+1}^2(t)} \\ & -\frac{2\Delta_{k+1}'(t)}{\Delta_{k+1}^3(t)} \left(\int_{t_{2k+2}}^{t} c_{k+1}(s)\Delta_{k+1}'(s)\Delta_{k+1}(s)ds + \sum_{r=0}^{r=k} \int_{t_{2r}}^{t_{2r+1}} c_r(s)\Delta_r'(s)\Delta_r(s)ds \right) \end{aligned}$$

for $t \in K_{k+1}$. Using integration by parts and that facts that c increases, $\Delta_r(t_{2r}) = \Delta_{r-1}(t_{2r-1})$ on K_r for $r = 0, ..., k$ and $\Delta_0(t_0) = 0$ we obtain that

$$B'(t) = -\frac{2\Delta_{k+1}'(t)}{\Delta_{k+1}^3(t)} \left(-\int_{t_{2k+2}}^{t} \frac{c_{k+1}'(s)\Delta_{k+1}(s)}{2} ds \right) - \frac{2\Delta_{k+1}'(t)}{\Delta_{k+1}^3(t)} \sum_{r=0}^{r=k} \left(-\int_{t_{2r}}^{t_{2r+1}} \frac{c_r'(s)\Delta_r(s)}{2} ds \right) > 0$$

since $c_r' > 0$ for $r = 0, ..., k+1$, $\Delta_{k+1}' < 0$, $t_{2r+1} < t_{2r}$, $t \in [t_{2k+3}, t_{2k+2}]$.

By construction, for $x \in X(t) \subset [0,1] \backslash (S \cup J)$, there exists $t \in J$ for which $v_1(x) - v_2(x) = v_1(t) - v_2(t)$ so that we may extend B letting $B(x) = B(t)$. In this case, $B'(x(t))x'(t) = B'(t)$ for $x \in X(t)$ as required.

Thus far, we have obtained the unique bidding function that is OE to $v_1 - v_2$ and that satisfies the appropriate ODE (and initial condition) required to guarantee that, for $t \in [t_{i+1}, t_i]$ bidding $B(t)$ is best among bids $B(s)$ for $t \in [t_{i+1}, t_i]$. It remains to show that, for any bidder $t \in [t_{i+1}, t_i]$ say, bidding any bid $b = B(s)$ for $s \in K_r$ for $r \neq i$ or $b \notin \{B(s) : s \in S \cup J\}$ is weakly dominated by bidding $B(t)$.

First note that if $s \in S \cup J$, then bidding $B(s)$ results in

$$\pi(t,s) = \int_{\sigma_2(t,s)} (v_\alpha(t) - B(u))\, H'(u)du + \int_{\sigma_1(t,s)} (v_\beta(t) - I(u) + B(s))\, H'(u)du$$

If all types bid according to B, then $\partial\pi(s,s)/\partial s = 0$. Now, suppose that $s \in J \cup S$, then $B(s) < B(t)$ implies $\partial\pi(t,s)/\partial s > \partial\pi(s,s)/\partial s$ since $v_\alpha(s) - v_\beta(s) < v_\alpha(t) - v_\beta(t)$ whenever $B(s) < B(t)$ and $B(s) > B(t)$ implies that $\partial\pi(t,s)/\partial s < \partial\pi(s,s)/\partial s$ since $v_\alpha(s) - v_\beta(s) > v_\alpha(t) - v_\beta(t)$ whenever $B(s) > B(t)$. Thus, bidder t prefers to bid $B(t)$ rather than $B(s)$ for any $s \in S \cup J$, $s \neq t$. Continuity of the payoff guarantees that bidder t prefers to bid $B(t)$ rather than $b < \min_{s \in S \cup J} B(s)$ or $b > \max B(s)$ since the payoff function increases in bids $b < \min_{s \in S \cup J} B(s)$ and decreases in bids $b > \max B(s)$. One can prove the theorem under a 1^{st} price rule either analogously or by appealing to Theorem 15. ∎

Proof of Corollary 13: $c(t)$ is monotonic on K so that Theorem 12 implies the result. ∎

Proof of Corollary 14: Efficiency, existence, uniqueness and revenue maximization follows from the proof of Theorems 12, 11, 8 and 2. ∎

Proof of Theorem 15: As the reasoning is entirely analogous in the two cases, we prove the result starting with an equilibrium bidding function that exists under a 2^{nd} price rule. Begin by letting

$$B_1(t) = B_2(t) - \frac{\int_{(c,t) \cap \widehat{K}} B_2'(t) \nabla(s) ds}{(\nabla(t))^2}$$

as stated at the end of the proof of Lemma 22. It is immediate from reversing the Proof of Lemma 22 that B_1 is an equilibrium bidding function under a 1^{st} price rule. Since the sign of ∇' equals that of B_2' and by definition of ∇, it is also immediate that B_1 is OE to B_2. ∎

The following lemmas are used to prove Theorems 16, 17, and 20.

Lemma 21 *Let $W : [0,1] \longrightarrow [0,1]$ be onto and continuous. Let $\widehat{K}$ be the union of intervals over which the cover of W is strictly monotonic on $\widehat{K}$. Let $W^{-1}(0) = C$, $W^{-1}(1) = D$ where*

$C<D$ or $D<C$ are the endpoints of $\widehat{K}$. Let

$$\begin{aligned} F(t) &= W(t)\ln W(t) - \frac{(2W(t)-1)^2}{4} \\ G(t) &= 2(1-W(t))^2\ln(1-W(t)) - \frac{(2W(t)-1)^2}{2} \end{aligned}$$

Then

$$F(t) \leq -\frac{1}{4},\ G(t) \geq -\frac{1}{2}$$

If L is OE to W, then

$$\begin{aligned} \int_{(C,D)\cap J} L'(t)F(t)dt &\leq -\frac{1}{4}\int_{(C,D)\cap J} L'(t)dt \\ \int_{(C,D)\cap J} L'(t)G(t)dt &\geq -\frac{1}{2}\int_{(C,D)\cap J} L'(t)dt \end{aligned}$$

If L is ORE to W and $W'>0$, then

$$\int_{(C,D)\cap J} L'(t)F(t)dt \geq -\frac{1}{4}\int_{(C,D)\cap J} L'(t)dt$$

Proof. We first note that $F(C) = -1/4 = F(D)$, $G(C) = -1/2 = G(D)$.

$$\begin{gathered} F'(t) = W'(t)\left(\ln W(t) + 2 - 2W(t)\right) = W'(t)M(t) \\ G'(t) = 2W'(t)\left(-2(1-W(t)\ln(1-W(t)) - W(t)\right) = 2W'(t)K(t) \end{gathered}$$

where

$$\begin{aligned} M'(t) &= W'(t)\left(\frac{1}{W(t)} - 2\right) \\ K'(t) &= W'(t)\left(2\ln(1-W(t)) + 1\right) \end{aligned}$$

Since $W'>0$ if and only if $C<D$, as t increases along $(C,D)\cap\widehat{K}$, $F(t)$ decreases then increases while $G(t)$ increases then decreases which proves the first pair of results.

Let L be OE to W. In this case, $W'>0$ implies $L'>0$, $C<D$ which implies

$$\begin{aligned} \int_{(C,D)\cap\widehat{K}} L'(t)F(t)dt &\leq -\frac{1}{4}\int_{(C,D)\cap\widehat{K}} L'(t)dt \\ \int_{(C,D)\cap\widehat{K}} L'(t)G(t)dt &\geq -\frac{1}{2}\int_{(C,D)\cap\widehat{K}} L'(t)dt \end{aligned}$$

and $W'<0$ implies $L'<0$, $D<C$ which implies

$$\begin{aligned} \int_{(C,D)\cap\widehat{K}} L'(t)F(t)dt &= -\int_{(D,C)\cap\widehat{K}} L'(t)F(t)dt \leq \frac{1}{4}\int_{(D,C)\cap\widehat{K}} L'(t)dt = -\frac{1}{4}\int_{(C,D)\cap\widehat{K}} L'(t)dt \\ \int_{(C,D)\cap\widehat{K}} L'(t)G(t)dt &= -\int_{(D,C)\cap\widehat{K}} L'(t)G(t)dt \geq \frac{1}{2}\int_{(D,C)\cap\widehat{K}} L'(t)dt = -\frac{1}{2}\int_{(C,D)\cap\widehat{K}} L'(t)dt \end{aligned}$$

Let L be ORE to W and $W' > 0$. $C < D$ and $L' < 0$ implies the last result. ∎

Given any bidding function B and its associated probability $\nabla(t) = 1 - \Delta(t)$, there is a monotonic cover of B. Denote by $\widehat{K}$ the closure of union of intervals over which the cover is strictly monotonic and let $c = \nabla^{-1}(0)$, $D = \nabla^{-1}(1)$ denote the end-points of $\widehat{K}$. Either the cover is strictly increasing over $\widehat{K}$ and $C < D$ or the cover is strictly decreasing over $\widehat{K}$ and $D < C$. There may be gaps in $\widehat{K}$ but, by nature of a cover, the values of the bidding function form a continuous range as t ranges over $\widehat{K}$. Below, the subscripts indicate the price rule.

Lemma 22 *If the equilibrium bidding function under a 1^{st} price rule is OE to that under a 2^{nd} price rule, then*

$$B_1(t) = B_2(t) - \frac{\int_{(C,t)\cap\widehat{K}} B_2'(t)\nabla(s)ds}{(\nabla(t))^2}$$

Proof. Since B_1 and B_2 are OE there are OE monotonic covers of B_1 and B_2 with a common associated $\widehat{K}$ and ∇. Since $\nabla(t) = 1 - \Delta(t)$, (4) implies

$$\begin{aligned} B_2'(t)(1-\nabla(t)) - 2B_2(t)\nabla'(t) &= -(v_1(t) - v_2(t))\nabla'(t) + \delta'(t) \\ B_1'(t)\nabla(t) + 2B_1(t)\nabla'(t) &= (v_1(t) - v_2(t))\nabla'(t) - \delta'(t) \end{aligned}$$

so that

$$\left(B_1'(t) - B_2'(t)\right)\nabla(t) + 2\left(B_1(t) - B_2(t)\right)\nabla'(t) = -B_2'(t)$$

which implies that, for $t \in \widehat{K}$,

$$B_1(t) = B_2(t) - \frac{\int_{(C,t)\cap\widehat{K}} B_2'(s)\nabla(s)ds}{(\nabla(t))^2} \quad ∎$$

The next two Lemmas restrict to 1^{st} (denoted by $a = 0$) and 2^{nd} price rules (denoted by $a = 1$). Let the subscripts on B denote the order of sale.

Lemma 23 *If $v_a - v_\beta + I$, $v_\beta - v_\alpha + I$ and I are each OE to the equilibrium bidding function under order α, β and under order β, α then, for $t \in \widehat{K}$*

$$B_{\alpha,\beta}(t) + B_{\beta,\alpha}(t) = I(t) - a\frac{\int_{(D,t)\cap\widehat{K}}(1-\nabla(s))^2 I'(s)ds}{(1-\nabla(t))^2} - (1-a)\frac{\int_{(C,t)\cap\widehat{K}} \nabla^2(s)I'(s)ds}{\nabla^2(t)}$$

Proof. Since $B_{\alpha,\beta}$ and $B_{\beta.\alpha}$ are OE, the associated $\widehat{J}$ and Δ are common. Since $\nabla(t) = 1 - \Delta(t)$, (4) and the fact that I is OE to both bidding functions imply

$$\begin{aligned} B_{\alpha,\beta}'(t)\left(a(1-\nabla(t)) - (1-a)\nabla(t)\right) - 2B_{\alpha,\beta}(t)\nabla'(t) &= -(v_\alpha(t) - v_\beta(t) + I(t))\nabla'(t) \\ B_{\beta,\alpha}'(t)\left(a(1-\nabla(t)) - (1-a)\nabla(t)\right) - 2B_{\beta,\alpha}(t)\nabla'(t) &= -(v_\beta(t) - v_\alpha(t) + I(t))\nabla'(t) \end{aligned}$$

$$\left(B_{\alpha,\beta}'(t) + B_{\beta,\alpha}'(t)\right)\left(a(1-\nabla(t)) - (1-a)\nabla(t)\right) - 2\left(B_{\alpha,\beta}(t) + B_{\beta,\alpha}(t)\right)\nabla'(t) = -2I(t)\nabla'(t) \quad ∎$$

Lemma 24 *If $v_a - v_\beta + I$ is ORE to $v_\beta - v_\alpha + I$ and each bidding function is OE to its associated critical value $v(t)$ then, if $I(t) = I$*

$$B_{\alpha,\beta}(t) + B_{\beta,\alpha}(t) = I + \frac{\int_{(C,t)\cap\widehat{K}} \left(aB'_{\alpha,\beta}(s) + (1-a)B'_{\beta,\alpha}(s)\right) \nabla_{\alpha,\beta}(s)ds}{\nabla^2_{\alpha,\beta}(t)} \quad \textit{for } t \in \widehat{K}$$

Proof. Since $\nabla(t) = 1 - \Delta(t)$, (4) implies that

$$\begin{aligned} B'_{\alpha,\beta}(t)\left(a(1-\nabla_{\alpha,\beta}(t)) - (1-a)\nabla_{\alpha,\beta}(t)\right) - 2B_{\alpha,\beta}(t)\nabla'_{\alpha,\beta}(t) &= -(v_\alpha(t) - v_\beta(t) + I)\nabla'_{\alpha,\beta}(t) \\ B'_{\beta,\alpha}(t)\left(a(1-\nabla_{\beta,\alpha}(t)) - (1-a)\nabla_{\beta,\alpha}(t)\right) - 2B_{\beta,\alpha}(t)\nabla'_{\beta,\alpha}(t) &= -(v_\beta(t) - v_\alpha(t) + I)\nabla'_{\beta,\alpha}(t) \end{aligned}$$

By assumption, $\nabla_{\alpha,\beta}(t) + \nabla_{\beta,\alpha}(t) = 1$ so that after adding the two equations, we obtain

$$\begin{aligned} &\left(a(B'_{\alpha,\beta}(t) + B'_{\beta,\alpha}(t)) + (1-a)(B'_{\alpha,\beta}(t) + B'_{\beta,\alpha}(t))\right)\nabla_{\alpha,\beta}(t) + 2(B_{\alpha,\beta}(t) + B_{\beta,\alpha}(t))\nabla'_{\alpha,\beta}(t) \\ &\qquad = 2I\nabla'_{\alpha,\beta}(t) + aB'_{\alpha,\beta}(t) + (1-a)B'_{\beta,\alpha}(t) \end{aligned}$$

which implies the result where $\nabla_{\alpha,\beta}(C) = 0$, $\nabla_{\alpha,\beta}(D) = 1$, $C, D \in \widehat{K}$ where $\nabla_{\alpha,\beta}$ is OE to $B_{\alpha,\beta}$ on $\widehat{K}$ which is ORE to $B_{\beta,\alpha}$ on $\widehat{K}$. ∎

Proof of Theorem 16: By assumption, there is a common ∇ and $\widehat{J}$. Since the expected price of good 1 under a 2^{nd} price rule is

$$\exp p_1^2 = 2\int_{(c,d)\cap\widehat{J}} B_2(t)(1 - \nabla(t)\nabla'(t)dt$$

and the expected price of good 1 under a 1^{st} price rule is

$$\exp p_1^1 = 2\int_{(c,d)\cap\widehat{J}} B_1(t)\nabla(t)\nabla'(t)dt$$

the expected difference in the price of good 1 under 1^{st} and 2^{nd} price rules is

$$\exp p_1^1 - \exp p_1^2 = 2\int_{(c,d)\cap\widehat{J}} B_1(t)\nabla(t)\nabla'(t)dt - 2\int_{(c,d)\cap\widehat{J}} B_2(t)(1 - \nabla(t)\nabla'(t)dt$$

After using Lemma 22 and grouping terms we obtain that $\exp p_1^1 - \exp p_1^2$ equals

$$\int_{(c,d)\cap\widehat{J}} B_2(t)d\frac{(2\nabla(t)-1)^2}{2} - 2\int_{(c,d)\cap\widehat{J}}\int_{(c,t)\cap\widehat{J}} B'_2(s)\nabla(s)dsd\ln(\nabla(t))$$

After integrating by parts (using $\nabla(c) = 0$, $\nabla(d) = 1$), we obtain $\exp p_1^1 - \exp p_1^2$ equals

$$2\left(\frac{B_2(d)}{4} - \frac{B_2(c)}{4}\right) + 2\int_{(c,d)\cap J} B'_2(t)\left(\nabla(t)\ln\nabla(t) - \frac{(2\nabla(t)-1)^2}{4}\right)dt$$

By Lemma 21 using $\nabla(t) = W(t)$ is OE to $L = B_2$ we obtain

$$\exp p_1^1 - \exp p_1^2 \leq 2\left(\frac{B_2(d)}{4} - \frac{B_2(c)}{4}\right) - \frac{2}{4}\int_{(c,d)\cap\widehat{J}} B_2'(t)dt = 0 \quad \blacksquare$$

Proof of Theorem 17: Since$I$ is OE to the equilibrium bidding functions $B_{\alpha,\beta}$ and $B_{\beta,\alpha}$ the expected income of the winner of good one is independent of the sequence of sale so that the expected revenue is independent of the sequence of sale. Since the sum of the expected prices of the goods equals $\exp\max\{I(t_1), I(t_2)\}$, the difference $\exp_\alpha^{\beta,\alpha} - \exp p_\alpha^{\alpha,\beta}$ in the expected price of good α under sequence (β,α) and that under sequence (α,β) is

$$2\int_{(c,d)\cap\widehat{J}} I(t)\nabla(t)\nabla'(t)dt - 2a\int_{(c,d)\cap\widehat{J}}(B_{\beta,\alpha}(t) + B_{\alpha,\beta}(t))\nabla'(t)\left(1-\nabla(t)\right)dt$$
$$- 2(1-a)\int_{(c,d)\cap\widehat{J}}(B_{\beta,\alpha}(t) + B_{\alpha,\beta}(t))\nabla'(t)\nabla(t)dt$$

Lemma 23 implies that $\exp_\alpha^{\beta,\alpha} - \exp p_\alpha^{\alpha,\beta} =$

$$a\int_{(c,d)\cap\widehat{J}} I(t)d\left(\frac{(2\nabla(t)-1)^2}{2}\right) - 2a\int_{(c,d)\cap\widehat{J}}\left(\int_{(d,t)\cap\widehat{J}}(1-\nabla(s))^2 I'(s)ds\right)d\left(\ln\left(1-\nabla(t)\right)\right)$$
$$+ 2(1-a)\int_{(c,d)\cap\widehat{J}}\int_{(c,t)\cap\widehat{J}}\nabla^2(s)I'(s)dsd\left(\ln\nabla(t)\right)$$

Since $\nabla(c) = 0$, $\nabla(d) = 1$ integration by parts implies $\exp_\alpha^{\beta,\alpha} - \exp p_\alpha^{\alpha,\beta} =$

$$a\frac{I(d)}{2} - a\frac{I(c)}{2} + a\int_{(c,d)\cap\widehat{J}} I'(t)\left(2\left(\ln\left(1-\nabla(t)\right)\right)\left(1-\nabla(t)\right)^2 - \frac{(2\nabla(t)-1)^2}{2}\right)dt$$
$$- 2(1-a)\int_{(c,d)\cap\widehat{J}}\ln\nabla(t)\nabla^2(t)I'(t)dt$$

By Lemma 21 using $\nabla = W, I = L$, we obtain that $\exp_\alpha^{\beta,\alpha} - \exp p_\alpha^{\alpha,\beta} \geq$

$$a\frac{I(d)}{2} - a\frac{I(c)}{2} - \frac{a}{2}\int_{(c,d)\cap\widehat{J}} I'(t)ds - 2(1-a)\int_{(c,d)\cap\widehat{J}}\ln\nabla(t)\nabla^2(t)I'(t)dt$$
$$= -2(1-a)\int_{(c,d)\cap\widehat{J}}\ln\nabla(t)\nabla^2(t)I'(t)dt \geq 0 \quad \blacksquare$$

Proof of Corollary 18: Theorem 11 therefore implies existence of an S-monotonic equilibrium bidding function. Theorem 17 then implies the result. ∎

Proof of Corollary 19: Existence follows from Theorem 11. Efficiency follows from the fact that $v_\alpha = v_\beta$. The remainder of the theorem follows from Theorem (17). ∎

Proof of Theorem 20: Uniqueness follows from Theorems (8) and (10). Existence follows from Theorem (12) since $v_1 - v_2$ is OE to $v_1 - v_2 + I$. I now prove points (1) and (2). WLOG, I focus on α and assume $B_{\alpha,\beta}$ strictly increases on $\widehat{J}$.

The expected price of good α under order β, α minus that under order α, β is

$$\begin{aligned} & \exp p_{\alpha}{}^{\beta,\alpha} - \exp p_{\alpha}^{\alpha,\beta} \\ = \; & I - 2a\left(\int_{(d,c)\cap\widehat{J}} B_{\beta,\alpha}(t)(1-\nabla_{\beta,\alpha}(t))\nabla'_{\beta,\alpha}(t)dt + \int_{(c,d)\cap\widehat{J}} B_{\alpha,\beta}(t)(1-\nabla_{\alpha,\beta}(t))\nabla'_{\alpha,\beta}(t)dt\right) \\ & -2(1-a)\left(\int_{(d,c)\cap\widehat{J}} B_{\beta,\alpha}(t)\nabla_{\beta,\alpha}(t)\nabla'_{\beta,\alpha}(t)dt + \int_{(c,d)\cap\widehat{J}} B_{\alpha,\beta}(t)\nabla_{\alpha,\beta}(t)\nabla'_{\alpha,\beta}(t)dt\right) \end{aligned}$$

Lemma 24, $\nabla_{\alpha,\beta}(t) + \nabla_{\beta,\alpha}(t) = 1$, $I(t) = I$ implies $\exp p_{\alpha}{}^{\beta,\alpha} - \exp p_{\alpha}^{\alpha,\beta} =$

$$\begin{aligned} & 2a\left(\int_{(c,d)\cap\widehat{J}} B_{\alpha,\beta}(t)d\left(\frac{(2\nabla_{\alpha,\beta}(t)-1)^2}{4}\right) - \int_{(c,d)\cap\widehat{J}}\int_{(c,t)\cap\widehat{J}} B'_{\alpha,\beta}(s)\nabla_{\alpha,\beta}(s)dsd\ln\nabla_{\alpha,\beta}(t))\right) \\ & +2(1-a)\left(\int_{(c,d)\cap\widehat{J}} B_{\beta,\alpha}(t)d\left(\frac{(2\nabla_{\alpha,\beta}(t)-1)^2}{4}\right) - \int_{(c,d)\cap\widehat{J}}\int_{(c,t)\cap\widehat{J}} B'_{\beta,\alpha}(s)\nabla_{\alpha,\beta}(s)dsd\ln\nabla_{\alpha,\beta}(t))\right) \end{aligned}$$

Since $\nabla_{\alpha,\beta}(c) = 0$, $\nabla_{\alpha,\beta}(d) = 1$, integrating by parts implies $\exp p_{\alpha}{}^{\beta,\alpha} - \exp p_{\alpha}^{\alpha,\beta} =$

$$\begin{aligned} & 2a\left(\frac{B_{\alpha,\beta}(d)}{4} - \frac{B_{\alpha,\beta}(c)}{4} + \int_{(c,d)\cap\widehat{J}} B'_{\alpha,\beta}(t)\left(\ln\nabla_{\alpha,\beta}(t)\nabla_{\alpha,\beta}(t) - \frac{(2\nabla_{\alpha,\beta}(t)-1)^2}{4}\right)dt\right) \\ & +2(1-a)\left(\frac{B_{\beta,\alpha}(d)}{4} - \frac{B_{\beta,\alpha}(c)}{4} + \int_{(c,d)\cap\widehat{J}} B'_{\beta,\alpha}(t)\left(\ln\nabla_{\alpha,\beta}(t)\nabla_{\alpha,\beta}(t) - \frac{(2\nabla_{\alpha,\beta}(t)-1)^2}{4}\right)dt\right) \end{aligned}$$

By Lemma 21, since $B_{\alpha,\beta}$ is OE to $\nabla_{\alpha,\beta}$ and $B_{\beta,\alpha}$ is ORE to $\nabla_{\alpha,\beta}$, under a 2^{nd} price rule,

$$\exp p_{\alpha}{}^{\beta,\alpha} - \exp p^{\alpha,\beta} \leq 2\left(\frac{B_{\alpha,\beta}(d)}{4} - \frac{B_{\alpha,\beta}(c)}{4} - \frac{1}{4}\int_{(c,d)\cap\widehat{J}} B'_{\alpha,\beta}(t)dt\right) = 0$$

and, under 1^{st} price rule,

$$\exp p_{\alpha}{}^{\beta,\alpha} - \exp p^{\alpha,\beta} \geq 2\left(\frac{B_{\beta,\alpha}(d)}{4} - \frac{B_{\beta,\alpha}(c)}{4} - \frac{1}{4}\int_{(c,d)\cap J} B'_{\beta,\alpha}(t)dt\right) = 0 \quad \blacksquare$$

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