Directed Search for Equilibrium Wage-Tenure Contracts

Shouyong Shi*

Department of Economics, Univeristy of Toronto

150 St. George Street, Toronto, Ontario, Canada, M5S 3G7

(email: shouyong@chass.utoronto.ca)

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Abstract

I analyze the equilibrium in a labor market where firms offer wage-tenure contracts to direct the search of employed and unemployed workers. Each applicant observes all offers and there is no coordination among individuals. Workers' applications (as well as firms' recruiting decisions) are optimal. This optimality requires the equilibrium to be formulated differently from the that in the literature of undirected search. I provide such a formulation and show that the equilibrium exists. In the equilibrium, individuals explicitly tradeoff between an offer and the matching rate at that offer. This tradeoff yields a unique offer which is optimal for each worker to apply, and applicants are separated endogenously according to their current values. Despite such uniqueness and separation, there is a non-degenerate and continuous wage distribution of employed workers in the stationary equilibrium. The density of this distribution is increasing at low wages and decreasing at high wages. In all equilibrium contracts, wages increase with tenure, which results in quit rates to decrease with tenure. Moreover, the model makes novel predictions about individuals' job-to-job transition and comparative statics.

Keywords: Directed search, On-the-Job, Wage-tenure contracts.

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1. Introduction

Directed search is a matching process in which an individual can use his offer to affect his matching rate. The objective of this paper is to study the equilibrium in a labor market where firms offer wage-tenure contracts to direct workers' search. A wage-tenure contract is a time profile of wages which describes how a worker's wage will evolve with tenure. All firms post contracts before workers apply and each applicant observes all offers. Employed workers continue to search on the job for better contracts elsewhere. I characterize the market equilibrium and establish its existence. Then, I show that the equilibrium yields novel predictions about individual workers' job-to-job transition and aggregate outcomes.

To see why directed search is interesting to study, it is useful to contrast it with the large literature on undirected search developed from Diamond (1982), Mortensen (1982), and Pissarides (1990). There are two classes of models in this literature. In one class, as in the three pioneering works, prices (wages) are a result of bargaining after individuals are matched. In the other class, some individuals post prices but the searching individuals do not know who posted what prices (e.g., Burdett and Mortensen, 1998, and Burdett and Coles, 2003). In both classes of models, search is undirected because prices play no role, ex ante, to direct workers to particular matches.

Although undirected search captures important frictions, there are good reasons why it does not describe a search market adequately. First, some search is directed rather than completely random. For example, searching workers often have information about wages, from job advertisement, word of mouth, or referrals. This is particularly true for workers who search on the job. Second, it has been a long tradition in economics to treat prices as a useful mechanism to direct the allocation of resources, ex ante. By abandoning this role of prices, the literature of undirected search generates an array of inefficiencies in the market. The corrective policy depends on arbitrary details of matching and price determination processes (see Hosios, 1990). Directed search can eliminate most of these inefficiencies. Third, undirected search generates wage dispersion that is sensitive to the assumption on how many wages a worker knows before search. In these models, a searching worker knows either no wage beforehand as in the three pioneering works, or one wage (the worker's current wage) as in the on-the-job search model of Burdett and Mortensen (1998). If each searching worker knows two or more wages, instead, then wage dispersion disappears in these models. This sensitivity reduces the potency of undirected search as an explanation for wage dispersion. Directed search is immune to this sensitivity.

During the last fifteen years or so, a literature has grown to analyze directed search. Peters (1984, 1991) and Montgomery (1991) provide two of the earliest formulations. Examples of further explorations include Moen (1997), Acemoglu and Shimer (1999a,b), Julien, et al. (2000), Burdett, et al. (2001), Shi (2001), Galenianos and Kircher (2005), and Delacroix and Shi (2006). They have shown that an equilibrium with directed search and its efficiency properties are significantly different from those with undirected search.

This literature has not yet introduced wage-tenure contracts; instead, it has assumed that each firm posts a single fixed wage for the entire duration of the worker's employment with the firm. Moreover, only one model in this literature (i.e., Delacroix and Shi, 2006) has incorporated on-the-job search. Without wage-tenure contracts, the literature of directed search is unable to explain the empirical regularities that wages rise and quit rates fall with tenure (e.g., Farber, 1999). Without on-the-job search, a model cannot make predictions on job-to-job transitions which constitute a large part of the flow of workers in the data. There is an urgency to fill in these gaps between directed search theory and the data, given the appealing features of directed search discussed above.

The immediate challenge of this task is to formulate the equilibrium with contracts and prove its existence. To appreciate the challenge, it is useful to compare the task with the one in undirected search, which is accomplished by Burdett and Coles (2003, termed as BC henceforth). With undirected search, one does not need to formulate workers' application decisions, because workers are assumed to send their applications randomly to a pool of recruiting firms. With directed search, however, each worker's application must be optimal. In this decision, a worker makes the optimal tradeoff between an offer and the likelihood of obtaining the offer. Similarly, each firm understands that it can raise the offer to entice more workers to apply to the firm. To describe this tradeoff, I need two new objects in addition to the set of optimal contracts. One is the employment rate function, which describes how the rate at which an applicant gets a particular offer varies with the offer. The other is the hiring rate function, which describes how the rate at which a recruiting firm successfully hires a worker varies with the offer. These functions are equilibrium objects: They must be consistent with the aggregation of individuals' optimal choices, and the hiring rate must ensure that all equilibrium offers earn the same expected profit to a firm. A challenge is to show that these functions exist.

I formulate the equilibrium in an environment where all firm-worker pairs have the same productivity, and then establish the existence of the equilibrium. The equilibrium extends several realistic properties from the BC model of undirected search to directed search. First, wages increase and quit rates fall with tenure, because finding a higher offer becomes increasingly difficult as a worker's wage rises. Second, all equilibrium contracts are sections of a baseline contract. The baseline contract starts with the lowest equilibrium wage and then increases the wage with the worker's tenure in the firm. Any other equilibrium contract that starts at a different wage is identical to the remaining section of the baseline contract from that wage level onward. Third, wage-tenure contracts and on-the-job search generate wage dispersion among workers, even though all matches have the same productivity and all applicants observe all offers before they apply.

Beyond these similarities, the equilibrium with directed search has little resemblance to the one with undirected search. One main difference is the prediction about individuals' job-to-job transitions. With directed search, the employment rate is a decreasing function of the offer and the hiring rate is an increasing function of the offer. Thus, the tradeoff between an offer and the matching rate is non-trivial. Each worker chooses to apply to a unique offer and the applicants are separated endogenously. That is, a worker whose current state yields a higher value chooses to apply to a higher offer than another worker whose current state has a lower value. Such separation implies that wage mobility is limited endogenously by the worker's current wage. In contrast, undirected search models (e.g., BC, 2003, and Burdett and Mortensen, 1998) assume that any two workers have the same probability of receiving an offer that is higher than their current wages.

Another difference is the shape of the wage distribution of employed workers. There is a non-degenerate, continuous distribution of wages both in the current model and in the BC model. However, in the current model, the density function of the distribution of employed workers over wages is increasing at low wages and decreasing at high wages. This non-monotonic shape of the density function is an empirical regularity (see Kiefer and Neumann, 1993), but it is not the prediction of an undirected search model with homogeneous matches. Instead, an increasing density function of employed wages is *necessary* to support an equilibrium with undirected search and homogeneous matches. To eliminate this unrealistic prediction, the literature of undirected search has introduced sufficient heterogeneity across matches (e.g., van den Berg and Ridder, 1998). It is important to know that directed search can generate the non-monotonic wage density without such heterogeneity.

The third difference is comparative statics. An increase in unemployment benefits in the sense of the first-order stochastic dominance has no effect on the set of equilibrium contracts or individual workers' job-to-job transition rates, although it affects wage distributions of workers. If search were undirected, however, such an increase in unemployment benefits would increase the slope of the wage-tenure profile and increase the transition rate from low wages to high wages, as well as affecting wage distributions.

In general, the distributions of workers and offers have a much lesser role in determining equilibrium contracts and job-to-job transitions in the current model than in undirected search models. The reason is that, with directed search, the matching rate functions (rather than the distributions) play the critical role in determining the equilibrium. These functions form a fixed-point problem with optimal contracts and optimal applications. Only after solving this fixed-point problem can one deduce the implications on the distributions. The reverse is true with undirected search. There, in order to offer contracts optimally, the firms must know the distributions of workers and offers first.

The main extension of this paper to the literature of directed search is to incorporate wage-tenure contracts and on-the-job search, as discussed earlier. Here, a contrast must be made to Delacroix and Shi (2006), who examine directed search on the job by assuming that firms can offer only fixed wages over tenure. That paper shows that the equilibrium is a wage ladder. The discreteness of the set of equilibrium wages makes the characterization of the equilibrium quite messy in that model. Allowing for wage-tenure contracts not only captures empirical regularities, but also simplifies the characterization of the equilibrium – Any initial gap between two offers will eventually be filled in by the increasing wage profile. The analysis here is much more general than in Delacroix and Shi and, at the same time, preserves the feature of limited wage mobility.

The continuous distribution of wages in the current model with homogeneous matches is another contribution to the literature of undirected search. In a similar setting, the literature of directed search generates only a finite number, or even a singleton, of equilibrium wages. In the current model, wage-tenure contracts provide a source of wage dispersion because they allow workers who got jobs earlier to earn more than workers who got jobs later, even if the two are employed under the same contract. The endogenous separation of applicants is another source of wage dispersion because workers who got jobs earlier apply to higher wages than other workers choose to.

To emphasize the differences between directed search and undirected search, I maintain four assumptions imposed by BC. First, workers are risk averse; second, the capital market is not perfect for workers to borrow against their future income. These assumptions are important for generating the wage-tenure relationship, as discussed by BC. Third, a firm does not respond to the worker's outside offers. One justification for this assumption might be that, since a worker can observe all firms' posted offers, he may be able to counterfeit other firms' offers. If it is difficult for a firm to verify such counterfeits, then it is optimal for the firm not to respond to outside offers. How reasonable this assumption is clearly varies across different types of labor markets. In checking the validity of the assumption, however, one must keep in mind that all workers are assumed to have the same productivity in the current model. In any case, the assumption is commonly imposed in the literature, and it enables me to compare the results clearly with those in BC. For a model of undirected search without this assumption, see Postel-Vinay and Robin (2002). Finally, I assume that the productivity of a firm-worker pair is public information and deterministic. For private information or learning about productivity, see Jovanovic (1979), Harris and Holmstrom (1982), and Moscarini (2005). These productivity differences between matches or over time are clearly important for wage dynamics and turnover, but abstracting from them enables me to have the clearest exploration of the role of search frictions.

2. The Model

Consider a labor market that lasts forever in continuous time. There is a unit measure of risk averse workers whose utility function is $u(w)$, where w is income. Workers do not have access to the financial market to borrow against their future wage income, and so the lower bound on wages is 0. All workers have the same productivity: when employed, each worker produces a flow of output, $y > 0$. When unemployed, a worker enjoys a flow of utility $u(b)$, which is derived from leisure and other benefits in unemployment. I will refer to b simply as the unemployment benefit.

It will become clear later that the analysis is simpler if b is distributed in an interval, rather than being concentrated on a discrete set. Accordingly, I assume that a worker who enters unemployment draws a value of b from the interval $[\underline{b}, \bar{b}]$ according to a continuous distribution H, where $0 < \underline{b} < \bar{b}$. Let the density function $h(b) = H'(b)$ be differentiable. To simplify the analysis further, I set $\bar{b} = \bar{w}$, where $\bar{w}$ is the highest wage specified later. Once a worker draws a value of b, the value will stay with him until he dies.

All workers face the process of death at a Poisson rate $\sigma \in (0, \infty)$. Dead workers are replaced with newborns who enter the labor market through unemployment and who draw unemployment benefits according to the distribution H. To simplify the algebra, assume that the rate of time preference is zero. However, the probability of death generates effective discounting on the future.

Assumption 1. *The utility function has the following properties:* $0 < u'(w) < \infty$ *and* $-\infty < u''(w) < 0$ *for all* $w \in (0, \infty)$; $u'(0) = \infty$; *and* $u(0) = -\infty$.

These properties are standard, except $u(0) = -\infty$. This additional property is required to ensure that wages are positive at all time in an optimal contract. As discussed extensively by BC, if $u(0) < \infty$, then the optimal contract may be a wage path which starts with zero wage for a finite duration and then increases continuously into positive wages. If, in addition, the workers are risk neutral, then the optimal wage path is a step function; i.e., the wage is zero initially, followed by a jump to a permanent level (see Stevens, 2004).

There are also a large number of identical firms that can enter the market. Entry is competitive: a firm can recruit by incurring a flow of vacancy cost $k > 0$. Each vacancy (firm) can recruit only one worker. Normalize the production cost to 0. Recruiting firms announce wage-tenure contracts to compete for workers. A contract offered at time s is a time path of wages, $W(s) = \{w(t)\}_{t=s}^{\infty}$, conditional on the continuation of the worker's employment with the firm. Although a worker can quit the firm at any time, the firm is assumed to commit to the contract. Thus, employment is permanent until the worker either quits the firm or dies.

Let $V(t, s)$ be the remaining value of the contract to the worker whose tenure in the firm is $(t - s)$. This value is the expected utility to the worker in the lifetime generated by the remaining wage path in the contract from t onward, given the worker's optimal quitting strategy in the future. I will refer to an offer by its value to the worker at the time of the offer, $V(s, s)$, because this is all that matters to an applicant. All offers are bounded in $\left[\underline{V}, \bar{V}\right]$, where

$$\bar{V} = u(\bar{w})/\sigma, \quad \underline{V} = u(\underline{b})/\sigma$$

$\bar{w}$ is the highest wage which will be given by Lemma 3.3. The upper bound $\bar{V}$ is the lifetime utility of a worker who is employed at the highest wage permanently until death. The lower bound $\underline{V}$ is the lifetime utility of a worker who has the lowest unemployment benefit forever until death. However, because an unemployed worker has the opportunity of finding employment, all equilibrium offers are likely to be strictly higher than $\underline{V}$. I say that a result holds for all V if it holds for all $V \in [\underline{V}, \bar{V}]$.

Both unemployed and employed workers can search for jobs. At any instant, an unemployed worker receives an opportunity to apply to a job with probability λ_0, and an employed worker receives the opportunity with probability λ_1.[1] I allow for the possibility $\lambda_0 = \lambda_1 = 1$ by letting λ_0, $\lambda_1 \in (0, 1]$. A worker who receives the application opportunity observes all firms' offers instantly without any cost and then chooses the offer to which he applies. As in most search models, each worker can apply to only one offer.[2]

[1]Note that the λ'es are not Poisson rates, but rather the probabilities of receiving a job application opportunity at any instant. As such, they are bounded above by one.

[2]Let me clarify two assumptions here. One is that an applicant observes all offers. This assumption is not necessary, because the essential results in directed search are the same if each applicant is assumed to observe two offers that are randomly drawn from the offer distribution (see Acemoglu and Shimer, 1999b). The second assumption is that each applicant can apply to only one offer at a time. (For a directed search

There is no coordination among firms' recruiting decisions or workers' applications. When there are two or more firms making an offer to which a worker wants to apply, the worker randomly selects one to apply. Similarly, a firm may receive more than one applicant, in which case the firm randomly selects one to employ. If the selected worker is employed elsewhere, the worker must quit that job before accepting the offer. I assume that firms do not match the worker's outside offers, as discussed in the introduction. A job is destroyed when either the worker accepts another firm's offer or the worker dies.

Because workers observe the offers before they apply to the jobs, the offers can direct the search. That is, both workers and firms make an explicit tradeoff between an offer and the matching rate at that offer. When offering a value V, a firm will succeed in hiring a worker at a rate $q(V)$ according to the Poisson process. By changing the offer, the firm knows that its *hiring rate* will change according to $q(.)$. Similarly, when applying to V, a worker will obtain the job at a rate $p(V)$. By applying to a different offer, a worker understands that his *employment rate* will change according to $p(.)$. Note that p and q are Poisson rates instead of probabilities, and so they can exceed one.

More importantly, the functions $q(.)$ and $p(.)$ are equilibrium objects, since they must satisfy two equilibrium requirements. First, they must be consistent with aggregation. That is, as firms and workers make their choices under these functions, the resulting matching rates must indeed be given by these functions. Second, the hiring rate function must ensure that the expected profit of recruiting be the same for all equilibrium offers. Delaying the second requirement to section 4, I formulate the first requirement below.

Aggregate consistency imposes a link between the two functions, $q(.)$ and $p(.)$. To see this, let $M(x, 1)$ be a linearly homogenous matching function that determines the measure of matches between a measure x of workers and a unit measure of firms. Given the two functions of the matching rates, individuals' decisions result in a tightness $\theta(V)$ for each offer V, which is the ratio of applicants for V to recruiting firms at V. Then, $q(V) = M(\theta(V), 1)$ and $p(V) = M(\theta(V), 1)/\theta(V)$. Using these relationships to eliminate θ, I can express $p(V) = P(q(V))$. This relationship between the two matching rates is what aggregate consistency requires.

The function $P(q)$ embodies all essential properties of the matching function. From now on, I will take $P(q)$ as a primitive of the model and refer to it as the *matching function*.[3] To

model with multiple applications, see Galenianos and Kircher, 2005). In continuous time, this assumption is not as restrictive as it may sound. Although a worker in reality may be able to send out multiple applications, the probability with which two or more of his applications will be received by different firms at the same instant can be very small.

[3]Some directed search models have gone one step further to derive the matching function endogenously

specify the properties of the matching function, let $q(V) \in [\underline{q}, \bar{q}]$ for all V, with $0 < \underline{q} < \bar{q}$, where $\bar{q}$ is given by the matching function and $\underline{q}$ will be defined later by (5.4).

Assumption 2. *The matching function $P(q)$ has the following features: (i) $P(q)$ is continuous for all $q \in [\underline{q}, \bar{q}]$ and, for all q in the interior of $(\underline{q}, \bar{q})$, the derivatives $P'(q)$ and $P''(q)$ exist and are finite; (ii) $\bar{q} < \infty$ and $P(\bar{q}) = 0$; (iii) $P'(q) < 0$; (iv) $-qP''(q)/P'(q) \leq 2$.*

Part (i) is a regularity condition that is satisfied by many well-known matching functions. Part (ii) is imposed for the convenience of working with bounded functions. Part (iii) is equivalent to $0 < \theta M_1/M < 1$, which is satisfied by all matching functions of constant returns to scale that are strictly increasing in the arguments. In the equilibrium, I will show $q'(V) > 0$. Then, part (iii) ensures $p'(V) < 0$. Part (iv) restricts the convexity of $P(q)$, which will be useful for ensuring uniqueness of a worker's application decision.[4] To see the different parts of the assumption more clearly, consider the matching function with a constant elasticity of substitution between searching workers and vacancies:

Example 2.1. *If $M(\theta, 1) = [\alpha\theta^\rho + 1 - \alpha]^{1/\rho}$, where $\alpha \in (0, 1)$ and $-\infty < \rho < 1$, then*

$$P(q) = q\left(\frac{q^\rho - 1}{\alpha} + 1\right)^{-1/\rho}$$

Parts (i) and (iii) of Assumption 2 are satisfied. Part (ii) is satisfied iff $-\infty < \rho < 0$, i.e., iff the elasticity of substitution between searching workers and vacancies is less than one. In this case, $\bar{q} = (1-\alpha)^{1/\rho}$. Part (iv) is satisfied iff $\alpha \geq 1-(1-\rho)q^\rho/2$. When $\rho \leq -1$, this condition is satisfied for all $\alpha > 0$. When $-1 < \rho < 0$, the condition puts a lower bound on α. Note that, for $\rho < 0$, the derivatives $P'(q)$ and $P''(q)$ are unbounded at $q = \bar{q}$.

3. Workers' and Firms' Optimal Decisions

In this section, I will characterize agents' optimal decisions and their value functions. Throughout this paper, denote $\dot{x} = dx/dt$ for any variable x.

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by aggregating agents' strategies, e.g., Peters (1991), Burdett et al. (2001), Julien et al. (2000) and Delacrox and Shi (2006). In this paper, I follow the approach in Moen (1997) and Acemoglu and Shimer (1999a) to take the matching function as given. This allows me to focus on the main feature of directed search, i.e., that agents take into account how their choices of offers and applications will affect their matching rates.

[4] For a general matching function, part (iv) of the assumption requires $1-\theta M_1/M \leq [-\theta M_{11}/(2M_1)]^{1/2}$, where the left-hand side of the inequality is the share of vacancies in the matching function.

3.1. Optimal Application

Workers' search is directed by the employment rate, $p(V)$, which gives the Poisson rate of getting an offer V. As emphasized before, this function is an equilibrium object. Before analyzing workers' search decisions, I describe the properties of this function by the following lemma, which is an implication of Lemma 5.1 later.

Lemma 3.1. *Under Assumption 2, $p(V)$ is bounded, continuous and concave for all V. Moreover, $p(V)$ is differentiable and strictly decreasing for all $V < \bar{V}$, with $p(\bar{V}) = 0$.*

Examine an applicant, who can be either employed or unemployed. Let $V(t)$ be the value that the worker can obtain at his current state. This notation suppresses the starting time of the contract, if the worker is employed. After receiving a job application opportunity, the expected increase in value to the worker is:

$$E(V(t)) = \max_{f \in [V(t), \bar{V}]} p(f) \left[f - V(t) \right]. \tag{3.1}$$

Denote the solution as $f(t) = F(V(t))$. Then F is given implicitly as follows:

$$V = F(V) + \frac{p(F(V))}{p'(F(V))}. \tag{3.2}$$

The following lemma is proven in Appendix A:

Lemma 3.2. $F(\bar{V}) = \bar{V}$. *For all $V < \bar{V}$, the following results hold: (i) There is a unique and interior solution to (3.1), $f = F(V)$; (ii) $F(.)$ is continuous and $E(V)$ is differentiable, with $E'(V) = -p(F(V)) < 0$; (iii) $F(.)$ is strictly increasing; (iv) if $p(.)$ is twice continuously differentiable, then $F(V)$ is differentiable with $0 < F'(V) \leq 1/2$, and $E(V)$ is twice differentiable.*

For a worker at a value V, applying to the offer $F(V)$ is the only optimal choice. This is true despite the fact that the worker observes all other offers. Offers higher than $F(V)$ are not sufficient for compensating for the lower probability of getting them. Offers lower than $F(V)$ have higher probabilities of being obtained, but these probabilities are not high enough for compensating for the low values. For workers at a value V, only the offer $F(V)$ provides the optimal tradeoff between the value and the probability of obtaining it.

Not only is a worker's optimal application unique, it is also monotonic in the worker's current value. That is, a worker with a higher current value applies for higher offers than a worker with a lower value. Thus, the workers choose to separate themselves in the application process according to their current values. This separation is optimal because an applicant's payoff function has the single-crossing property. That is, compared with an applicant with a low current value, an applicant with a high value can tolerate a higher risk of not getting an offer in exchange for a higher value of the offer. When an applicant with a high current value fails to get the offer to which he applies, his current job provides a good backup or insurance. As a result, he can afford to "gamble" on applying to higher offers than does a worker with a low current value. Therefore, the optimal application choice, $F(V)$, is an increasing function.

Figure 1 illustrates the single-crossing property between worker 1 at a value V_1 and worker 2 at value V_2, where $V_2 > V_1$. Worker i's indifference curve can be written as $f = V_i + E_i/p$, for $i = 1, 2$. Suppose that the two workers' indifference curves cross each other at a particular point (f_0, p_0), where $f_0 > V_2$. At this crossing point, the slope of worker i's indifference curve is $df/dp < 0$, and the absolute value of this slope decreases with V_i. This implies that, for the same increase in the offer, the high-value worker (worker 2) is willing to take a larger reduction in the probability of getting the offer than does worker 1. Equivalently, for the same reduction in the probability of getting an offer, worker 2 is willing to apply to a higher offer than worker 1.

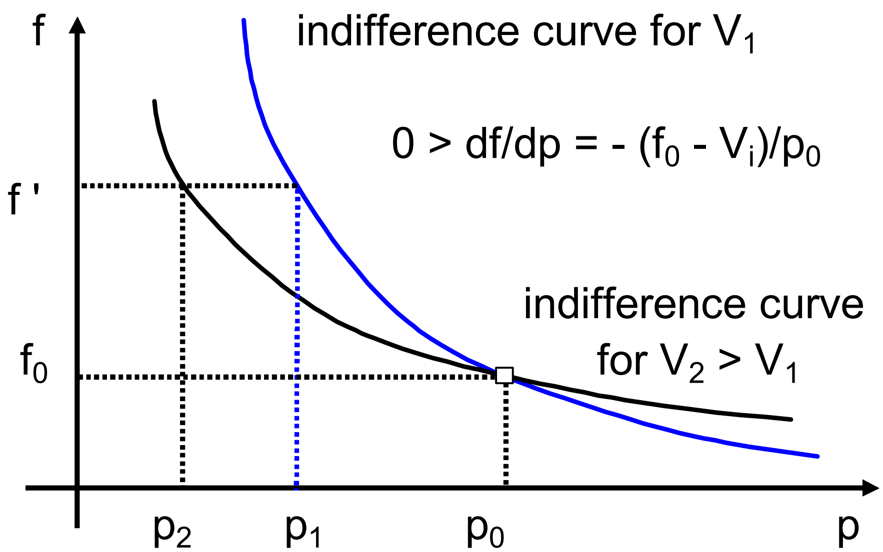

Figure 1. Monotonicity of the application decision

The optimality of the application decision is one of the key differences between this model and the BC model, or more generally, between directed search and undirected search. Models with undirected search have no counterpart to the above decision problem by an applicant; instead, an applicant is assumed to randomly apply to a value which is drawn from the offer distribution. Such an application is not optimal. First of all, the application may result in an offer which the worker will not accept. Second, even if the application results in an acceptable offer, there is a continuum of values from which the offer comes from. In contrast, with directed search, the set of values to which a worker optimally chooses to apply is a singleton.

This contrast between the two models leads to sharply different predictions on job-to-job transitions and wage mobility. Directed search predicts a definite pattern of transition and endogenously limited mobility in wages between jobs. For example, take two workers whose current wages are w_1 and w_2, respectively, with $w_2 > w_1$. Let w_A be the starting wage of the contract to which worker 1 will apply, and w_B be the starting wage of the contract to which worker 2 will apply. Then, for these two workers, the probability of transiting immediately to another job with a starting wage above w_B is zero. Moreover, for any $w' \in (w_A, w_B)$, the likelihood ratio between worker 2's and worker 1's probability of immediately transiting to another job with a starting wage above w' is infinite. In undirected search models, the probability of transiting to wages above w_B is positive for both workers, and the likelihood ratio is constant and finite.

In addition to limited wage mobility, directed search also yields predictions on the gain to a worker from an application.[5] First, a worker who has a high current value gains less from an application than a worker who has a low current value. This is true in terms of the expected gain from an application, because $E'(V) < 0$. The result is also true in terms of the actual gain in percentage, $(F - V)/V$, because $F'(V) \leq 1/2 < F(V)/V$. With risk aversion, however, this decreasing gain in the value does not necessarily translate into a decreasing gain in wages. The decreasing gain in the value partly reflects the worker's decreasing marginal utility as the wage increases. Second, $E''(V) > 0$. That is, the decrease in the expected gain from an application slows down as the worker's current value increases.

3.2. Value Functions of Workers and Firms

For an employed worker, the value can change over time for four possible reasons. The first is the change in wages during the contract with the same firm. The second is the event that the worker obtains a better offer and quits the current job.[6] The third is death. The fourth is the adjustment to the steady state. As in the literature, I abstract from the last

[5]Delacroix and Shi (2006) establish similar features in a model with directed, on-the-job search, but they restrict that offers must be a constant wage over time. Nevertheless, the similarity suggests that these features are common in directed search models.

[6]The worker can also choose to quit the job to become unemployed if the wage profile is sufficiently decreasing. However, this event will never occur in the equilibrium, because the optimal wage profile has increasing wages with tenure, as shown later.

source of changes in the value by focusing on a stationary equilibrium. Because the rate of time preference is zero, the value for an employed worker evolves as follows:

$$\dot{V}(t) = \sigma V(t) - u(w(t)) - \lambda_1 E(V(t)) \tag{3.3}$$

If wages were constant over tenure, then $\dot{V} = 0$.

In contrast to wages, the unemployment benefit does not change over time once it is drawn. Thus, the value to an unemployed worker with a given benefit, b, will be constant over time as long as he stays unemployed. Denote this value as $V_u(b)$. Then,

$$0 = \sigma V_u(b) - u(b) - \lambda_0 E(V_u(b)). \tag{3.4}$$

To characterize a firm's value function, consider a firm that has an employed worker at time t under a contract whose remaining value to the worker is $V(t)$. (Again, I suppress the starting time of the contract in this notation.) Let $J(t)$ denote this firm's value. Because the worker quits at rate $\lambda_1 p(F(V(t)))$ and dies at rate σ, then

$$\dot{J}(t) = [\sigma + \lambda_1 p(F(V(t)))]\, J(t) - y + w(t). \tag{3.5}$$

For dynamic optimization, it is useful to express the firm's values as the discounted sum of profits. To do so, let t_0 be an arbitrary point in $[s, t]$, where $s \leq t$ is the starting time of the contract. Let $\gamma(t, t_0)$ be the probability that a worker will still be with the firm at time t given that he is with the firm at t_0. Then,

$$\gamma(t, t_0) = e^{-\int_{t_0}^{t} [\sigma + \lambda_1 p(F(V(\tau)))] d\tau}. \tag{3.6}$$

Equivalently, γ is given by the solution to the following differential equation:

$$\frac{d\gamma(t, t_0)}{dt} = -\left[\sigma + \lambda_1 p(F(V(t)))\right] \gamma(t, t_0), \tag{3.7}$$

where $\gamma(t_0, t_0) = 1$ and $\gamma(\infty, t_0) = 0$. Because J is bounded, it satisfies the transversality condition $\lim_{t\to\infty} J(t)\gamma(t, t_0) = 0$. Integrating (3.5) yields:

$$J(t_0) = \int_{t_0}^{\infty} [y - w(t)]\, \gamma(t, t_0) dt.$$

For any $t_0 \geq s$, this value is determined by the remaining contract from t_0 onward.

3.3. Optimal Recruiting Decisions and Contracts

Take an arbitrary time $s \geq 0$. A firm's recruiting decision at time s contains two parts. The first part is to choose a value $V(s)$ at which to recruit. The optimal choice maximizes the firm's expected value, $q(V(s))J(s)$, taking the function $q(V)$ as given. As I will explain later, the solution to this part of the firm's problem is a continuum of positive values of $V(s)$. The second part of a firm's problem is to choose a wage profile (i.e., a contract) to maximize $J(s)$ and to deliver the value $V(s)$. I characterize this decision below.

The optimal contract, $\{w(t)\}_{t=s}^{\infty}$, solves:

$$(\mathcal{P}) \;\; \max \;\; J(s) \text{ s.t. } (3.3) \text{ for all } t \geq s.$$

In this problem, $V(s)$ is taken as given, and so the maximized value of $J(s)$ is a function of $V(s)$. I express this fact by writing $J(s)$ as $J(V(s))$.

Treat $\gamma(t,s)$ as an auxiliary state variable in the dynamic optimization and (3.7) as the law of motion of γ. Then, the Hamiltonian of the dynamic optimization is:

$$\mathcal{H}(t,s) = (y-w)\gamma(t,s) - \Lambda_\gamma\left[\sigma + \lambda_1 p(F(V))\right]\gamma(t,s) + \Lambda_V\left[\sigma V - u(w) - \lambda_1 E(V)\right],$$

where Λ_γ and Λ_V are shadow prices of γ and V. I suppressed time on the right-hand side, except for γ. Following a similar argument to that in BC, it can be shown that the assumption $u(0) = -\infty$ implies $w(t) > 0$ for almost all t in all optimal contracts. Optimality conditions are: $\Lambda_\gamma = J$, $\Lambda_V = -\gamma/u'(w)$ and

$$\dot{w} = \frac{[u'(w)]^2}{u''(w)}\lambda_1 J(V)\left[\frac{dp(F(V))}{dV}\right]. \tag{3.8}$$

Optimal contracts have three important properties. First, an optimal contract provides optimal sharing of the value between a firm and its worker. To express this feature formally, note that the Hamiltonian is zero at the optimum.[7] Thus, an optimal contract satisfies:

$$-\dot{J} = \frac{1}{u'(w)}\dot{V}. \tag{3.9}$$

To explain, suppose that the contract increases the value to the worker by a marginal amount, $\dot{V}$. This will entail an increase in the wage by an amount, $\dot{V}/u'(w)$. The cost to the firm, in terms of profit, is $-\dot{J}$. The above condition requires that the marginal cost to the firm from increasing the wage should be equal to the marginal benefit to the worker.

[7] To obtain this result, differentiate the Hamiltonian with respect to time, and then substitute (3.3), (3.5) and the optimality conditions. This shows that the Hamiltonian, $\mathcal{H}(t,s)$, is constant over t. Because $\gamma(\infty,s)=0$, then $\mathcal{H}(t,s) = \mathcal{H}(\infty,s) = 0$ for all $t \geq s$.

For the analysis later, it is useful to substitute (3.5) and (3.3) to rewrite (3.9) as:

$$u'(w)(y-w)+u(w)=u'(w)\left[\sigma+\lambda_1 p(F(V))\right]J(V)+\left[\sigma V-\lambda_1 E(V)\right] \tag{3.10}$$

The best way to explain this condition is to view a worker-firm pair as a joint asset. With this view, the left-hand side of the above equation measures the flow of "dividends" to the asset, which consists of the firm's profit, evaluated with the worker's marginal utility, and the worker's utility of the wage. The right-hand side is the "permanent income" generated by the asset. In particular, the permanent income to the firm is $[\sigma+\lambda_1 p(F)]J$, which is translated into units of utility with the marginal utility of the worker. The permanent income to the worker is $[\sigma V-\lambda_1 E(V)]$. The optimal contract requires that the flow of dividends to the joint asset should be equal to the permanent income of the asset.

Second, an optimal contract provides an increasing wage profile that increases with tenure. This feature and the bounds on wages are stated in the following lemma (see Appendix B for a proof):

Lemma 3.3. *$\dot{w}(t)>0$ for all $V<\bar{V}$. Moreover, $\bar{w}=y-\sigma k/\bar{q}<y$, $\bar{V}=u(\bar{w})/\sigma$, $J(\bar{V})=k/\bar{q}>0$, and $q(\bar{V})=\bar{q}<\infty$.*

There are two forces that make an optimal wage profile smoothly increase with tenure. The first is a firm's incentive to retain a worker and the second is a worker's risk aversion. To retain a worker, it is optimal for a firm to backload wages so as to increase the worker's opportunity cost of quitting. As the wage and the value of the job to the worker rise with tenure, the probability with which the employee can find a better offer elsewhere falls, and so the worker's quit rate falls with tenure. Thus, a rising wage profile is less costly to the firm than a constant wage profile that provides the same expected value to the worker. However, if workers are risk neutral, then the best way for a firm to backload wages is to offer a step function as the wage contract (see Stevens, 2004). With risk aversion, workers prefer a smooth wage profile to a discontinuous profile, and so wages in an optimal contract are smoothly increasing with tenure. These two forces appear in the equation for wage dynamics, (3.8): the incentive to retain a worker appears through the negative derivative $dp(F(V))/dV$ (<0) and a worker's risk aversion through $u''<0$.

Because the wage is increasing with tenure and because the wage is bounded above, all optimal wage profiles increase toward the upper bound $\bar{w}$ as $t\to\infty$. Accordingly, the value for an employed worker converges to $\bar{V}$.[8] This convergence in the value is also monotonic, as I will show later in Corollary 5.3. As a result, a firm's value falls over time.

[8]Offers above $\bar{V}$ are not optimal because they generate expected values to the firm that are less than the recruiting cost.

The third property of optimal contracts is that all optimal contracts are sections of a baseline contracts. To describe this property, let the baseline optimal contract be $\{w_b(t)\}_{t=0}^{\infty}$, where $w_b(0)$ is the lowest wage equilibrium wage. Every other optimal contract, $\{w(t)\}_{t=0}^{\infty}$, traces out the baseline contract from a particular initial wage level. That is, the entire set of optimal contracts is:

$$\left\{\{w(t)\}_{t=0}^{\infty} : s \in [0, \infty);\ w(t) = w_b(t+s) \text{ for all } t\right\}.$$

This property is an implication of the principle of dynamic optimality. To explain why, note that the above problem of optimal contracts does not depend on the starting time s separately once V is given. Suppose that there are two contracts: contract 1 is offered at time s_1 and contract 2 offered at $s_2 > s_1$. The value offered by contract 2 is V_2. Suppose that contract 1 from s_2 onward also delivers V_2, then this remaining part of contract 1 must be the same as contract 2. Otherwise, the firm that offers contract 1 could replace the remaining part by contract 2 and, by the optimality of contract 2 at s_2, the replacement would improve the firm's expected value.

This property of dynamic optimality simplifies the analysis greatly. One simplification is that characterizing the entire set of optimal contracts at any time is equivalent to tracing out the baseline contract over time. Similarly, characterizing the set of offer values at any time is equivalent to tracing out the values provided by the baseline contract over time. From now on, I will focus on the baseline contract, suppress the subscript b, and suppress the starting point of a contract.

Another simplification is that the wage at any tenure can be written as a function of the value remaining in the contract, rather than a function of time. To do so, let $\mathcal{V}$ be the set of equilibrium lifetime utilities. Define $v_1 = \inf(\mathcal{V})$ and define T by

$$T(V(t)) = t, \text{ with } T(v_1) = 0. \tag{3.11}$$

Then, $T(x)$ is the length of tenure required for a worker to increase the value from v_1 to x according to the baseline wage contract. The wage level of a worker with tenure t on the baseline contract is $w(T(V(t)))$. With a slight abuse of the notation, I express this wage as $w(V)$ and refer to the function as the *wage function*. The above explanation makes it clear that $w(V(t))$ is also the starting wage of a contract that is offered at t with a value $V(t)$ to the worker. The notation $w(V)$ should be construed to mean that wage can only vary over time when the value to the worker changes over time.

Similarly, the notation $J(V)$ indicates that a firm's value can only change over time when the value to the worker changes over time. Thus, I can rewrite (3.9) as

$$J'(V) = -\frac{1}{u'(w(V))} < 0. \tag{3.12}$$

4. Definition and Configuration of the Equilibrium

Let n be the fraction of workers who are employed and $(1-n)$ the fraction of workers who are unemployed. Let G_e be the cumulative distribution function of employed workers over values and G_u be the distribution of unemployed workers over values.

An equilibrium is a set of lifetime utilities, $\mathcal{V}$, a Poisson rate of employment, $p(.)$, an application strategy, $F(.)$, a value function $J(.)$, a wage function $w(.)$, and distributions of workers, (G_e, G_u, n), that satisfy the following requirements:

(i) $F(V)$ solves (3.1), given $p(.)$;

(ii) Given $F(.)$ and $p(.)$, each value $V \in \mathcal{V}$ is delivered by a contract that solves $(\mathcal{P})$ for $s = 0$ with a starting wage $w(V)$, and the resulting value function of the firm is $J(V)$;

(iii) Zero expected profit of recruiting: $q(V)J(V) = k$ for all $V \in [\underline{V}, \bar{V}]$, and $q(V)J(V) < k$ for all $V > \bar{V}$, where $q(V) = P^{-1}(p(V))$;

(iv) G_e, G_u and n are stationary.

Most elements of this definition are self-explanatory, but requirement (iii) needs a clarification. This requirement asks the function $q(V)$ to induce zero expected profit from recruiting for all $V \in [\underline{V}, \bar{V}]$, not just for $V \in \mathcal{V}$. Since $\mathcal{V}$ is a strict subset of $[\underline{V}, \bar{V}]$, as I will show later, the requirement imposes a restriction on beliefs out of the equilibrium. The reason for imposing this restriction is as follows. For a non-equilibrium value $V \notin \mathcal{V}$, there can be two different reasons why the value is not in the equilibrium set. One is the self-fulfilling expectation that no worker will apply to that value: This expectation induces firms not to offer that value, in which case no worker will apply to that value, indeed. The second reason is that, even after firms offer that value, workers still find it optimal not to apply to it. The first reason for a "missing market" may not be robust to a trembling event that exogenously puts some recruiting firms at the value V. Requirement (iii) excludes such non-robust equilibria, and hence, refines the set of equilibria. This refinement resembles trembling-hand perfection.

Requirement (iii) can be viewed as a condition that determines the equilibrium function of firms' hiring rate, and hence, of applicants' employment rate. For given $J(.)$, the requirement yields $q(V) = k/J(V)$, and so $p(V) = P(k/J(V))$ for all $V \in [\underline{V}, \bar{V}]$.[9] For all $V > \bar{V}$, requirement (iii) states that recruiting at such values makes an expected loss to the firm. This part of the requirement is always satisfied, because Lemma 3.3 implies $q(V)J(V) \leq \bar{q}J(V) < \bar{q}J(\bar{V}) = k$.

I illustrate the likely configuration of the equilibrium in Figure 2. The set of equilibrium values for employed workers is $\mathcal{V} = [v_1, \bar{V}]$ and the set of equilibrium values for unemployed workers is $[v_0, \bar{V}]$, where $v_1 > v_0 > \underline{V}$. There are workers employed, and firms recruiting, at every level in $[v_1, \bar{V}]$. Similarly, there are unemployed workers at every level in $[v_0, \bar{V}]$. However, the arrows in Figure 2 depict only the applications of the workers at the special values v_j defined later, where $j = 0, 1, 2, ...$ For a worker whose value is v_j, his optimal choice of application is $v_{j+1} = F(v_j)$. Similarly, for a worker whose value lies in the interior of a segment, say $V \in (v_j, v_{j+1})$, he optimally applies to a unique value $F(V)$ in the interior of the next segment, (v_{j+1}, v_{j+2}). As remarked earlier, such optimality of workers' applications is the key difference between this model and an undirected search model, such as Burdett and Mortensen (1998) and BC. If search is undirected, a worker sends the application to a randomly selected value in $[v_1, \bar{V}]$.

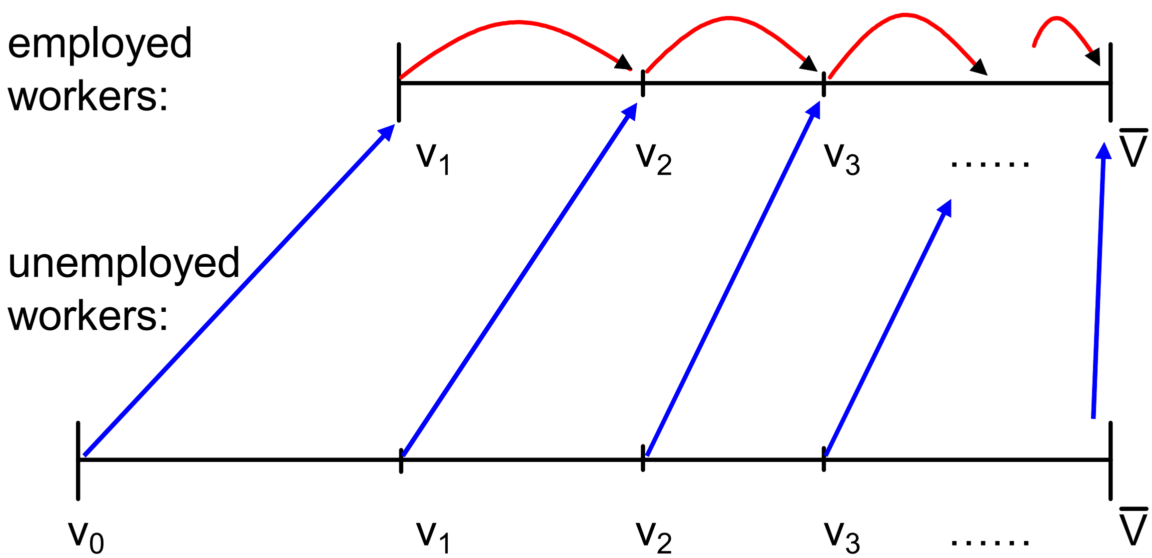

Figure 2. An illustration of the equilibrium

This difference in the nature of search implies a different procedure of finding an equilibrium. In an undirected search model, the most important equilibrium objects are the

[9]An alternative method of obtaining the function $p(.)$ is to require that a worker's expected surplus from applying to every offer (including a non-equilibrium offer) be the same. This method has been used in models of directed search with homogeneous applicants. However, the method is not practical when the applicants are heterogeneous. In this case, it is not possible to have one function $p(.)$ that induces all applicants to be indifferent between a non-equilibrium offer and an equilibrium offer.

distributions of values and workers. These distributions determine a worker's employment rate and a firm's hiring rate. To solve for an equilibrium under undirected search, one must solve for these distributions first. The procedure is almost reversed when search is directed. Now the most important equilibrium object is the function of the employment rate, $p(V)$. This function is critical for determining workers' optimal applications. It is also critical for determining firms' optimal recruiting decisions, because it implies the hiring rate, $q(V)$. One can determine the function, $p(V)$, by invoking requirements (i) – (iii) in the equilibrium definition, *without* any explicit reference to the distribution of offers or workers. Once this is done, the distributions of offers and workers can be calculated by invoking requirement (iv) in the equilibrium definition.

Before carrying out the analysis on the equilibrium in the next two sections, let me explain how heterogeneity in unemployment benefits simplifies the analysis. Let us see what will happen if all unemployed workers have the same unemployment benefit, say $\underline{b}$. In this case, all unemployed workers will have the same value, v_0. They will choose to apply to the same value v_1. The workers employed at v_1 will apply to v_2, and so on. Because there is a positive mass of unemployed workers, there will be a positive mass of workers at each of the values v_j, $j = 1, 2, ...$ Some of these workers will fail to get a better job in the next while and, according to the optimal contract, their values will drift up. Thus, there will be mass points at values $v_j + x < v_{j+1}$, for some $x > 0$ and all $j \geq 1$. With all these mass points, it is difficult to characterize the stationary distribution of workers. This difficulty is eliminated by introducing a continuous distribution of unemployment benefits. With this distribution, the value for unemployed workers will be dispersed over a continuum. As a result, their application targets will be dispersed over a continuum of values. This eliminates the mass points described above.[10]

5. Equilibrium Employment Rate and the Wage Function

The main step of determining an equilibrium is to determine the function of the employment rate, $p(V)$. For the analysis, however, it is more convenient to build the existence around the wage function, $w(V)$. The following procedure develops a mapping for w and obtains other functions such as $J(V)$, $p(V)$ and $F(V)$.

Start with any function $w(.)$ and add the subscript w to other functions constructed

[10] Undirected search models avoid this technical problem by assuming that workers randomly apply to all jobs. In such models, an unemployed worker accepts all offers in the equilibrium. Thus, the assumption makes workers dispersed over a continuum of values.

with this given function. First, given $w(.)$, I integrate (3.12) and use $J(\bar{V}) = k/\bar{q}$ to get:

$$J_w(V) = k/\bar{q} + \int_V^{\bar{V}} \frac{1}{u'(w(z))} dz \tag{5.1}$$

Second, the zero-profit condition for recruiting yields $q_w(V) = k/J_w(V)$ and hence

$$p_w(V) = P\left(\frac{k}{J_w(V)}\right) \tag{5.2}$$

Third, with $p_w(V)$ as the employment rate, I can express a worker's optimal application as $f = F_w(V)$ and the expected gain from the application as $E_w(V)$. Fourth, I explore (3.10), which is a requirement on a firm's optimal recruiting decision. Treat w on the left-hand side of (3.10) as a variable but substitute the given function $w(V)$ for w on the right-hand side. To avoid confusion, use w_1 instead of w on the left-hand side. Then,

$$u(w_1) + u'(w_1)(y - w_1) = u'(w(V))\left[\sigma + \lambda_1 p_w(F_w(V))\right] J_w(V) + \sigma V - \lambda_1 E_w(V) \tag{5.3}$$

Denote the solution for w_1 as $w_1(V)$. Then, $w_1(V) = (\Gamma w)(V)$. The equilibrium wage function $w(V)$ is a fixed point of the mapping Γ, i.e., $w(V) = (\Gamma w)(V)$ for all V.

Confirming an earlier statement, the above procedure does not involve the distributions of workers and offers. An implication is that optimal contracts and applications are independent of such distributions. I will explore this feature of the equilibrium later in section 7.

To characterize the fixed point for w, let me specify a few bounds on various functions. First, using the constant $\bar{w}$ to replace the function $w(V)$ in (5.1) and (5.2), I obtain $J_{\bar{w}}(V)$ and $p_{\bar{w}}(V)$. Because $J_w(.)$ and $p_w(.)$ are monotone in w, then $J_w(V) \leq J_{\bar{w}}(V)$ and $p_w(V) \leq p_{\bar{w}}(V)$ for all V. Second, define

$$\underline{q} = k/J_{\bar{w}}(\underline{V}). \tag{5.4}$$

Since $J_{\bar{w}}(V)$ is decreasing, $q(V) \in [\underline{q}, \bar{q}]$ for all V, and $\underline{q} \in (0, \bar{q})$. This lower bound on q is the one used in Assumption 2. Similarly, $p(V)$ is bounded in $[0, P(\underline{q})]$. Third, let $\underline{w}$ be a strictly positive number that is sufficiently close to 0.

Assumption 3. *Assume that $\underline{b}$, $\underline{V}$ and $\underline{w}$ satisfy:*

$$(0 <)\ \underline{b} < \bar{w} = y - \sigma k/\bar{q} \tag{5.5}$$

$$J_{\bar{w}}(\underline{V})\left[\sigma + \lambda_1 p_{\bar{w}}(\underline{V})\right] < y \tag{5.6}$$

$$u(\underline{w}) + u'(\underline{w})\left[y - \underline{w} - J_{\bar{w}}(\underline{V})(\sigma + \lambda_1 p_{\bar{w}}(\underline{V}))\right] \geq u(\underline{b}). \tag{5.7}$$

The condition (5.5) is a regularity condition: When it is violated, all workers will choose to stay out of employment. The condition (5.6) requires that the permanent income of a job to a firm is less than output even when the firm is providing the lowest value to the worker. To see which parameters this condition restricts, note that $J_{\bar{w}}(V)$ and $p_{\bar{w}}(V)$ are decreasing functions. Then, the left-hand side of (5.6) is decreasing in $\underline{V}$, and hence decreasing in $\underline{b}$. As a result, (5.6) is satisfied if $\underline{b}$ is bounded below by some number. If I set $\underline{b} = \bar{w}$, the left-hand side of (5.6) is equal to $\sigma k/\bar{q}$, which is less than y by (5.5). Thus, there exists $\hat{b} \in (0, \bar{w})$ such that (5.5) and (5.6) are satisfied if $\underline{b} \in (\hat{b}, \bar{w})$.

To see what (5.7) entails, note that the left-hand side of (5.7) is a decreasing function of $\underline{w}$ for sufficiently small $\underline{w}$. Thus, (5.7) imposes an upper bound on $\underline{w}$. Because $\underline{w}$ is chosen to be sufficiently close to 0, a sufficient condition for (5.7) is:

$$\lim_{w \downarrow 0} \left[u(w) + u'(w)(a - w)\right] = \infty \text{ for all } a > 0.$$

This sufficient condition is satisfied by the example $u(w) = \left(w^{1-\eta} - 1\right)/(1-\eta)$ with $\eta > 1$.

Define

$$\begin{aligned} \Omega = \quad & \{w(V) : w(V) \text{ is continuous and (weakly) increasing;} \\ & \qquad\qquad w(V) \in [\underline{w}, \bar{w}] \text{ for all } V;\, w(\bar{V}) = \bar{w}\Big\} \\ \Omega' = & \Big\{w \in \Omega : w(V) \text{ is strictly increasing for all } V < \bar{V}\Big\}. \end{aligned}$$

I establish the existence of a fixed point of Γ in Ω and then show that it lies in Ω'. First, the following lemma holds:

Lemma 5.1. *For any $w \in \Omega$, $J_w(V)$ defined by (5.1) is strictly positive, bounded, strictly decreasing and continuously differentiable for all V. The function $p_w(V)$ defined by (5.2) has all the properties stated in Lemma 3.1.*

Proof. Let $w(V) \in \Omega$. It is easy to verify that $J_w(V)$ defined by (5.1) is strictly positive, bounded, strictly decreasing and continuously differentiable, with $J'(V) = -1/u'(w(V)) < 0$. Because $w(V)$ is non-decreasing, then $J'(V)$ is non-increasing; i.e., $J(V)$ is (weakly) concave. Moreover, $J_w(\bar{V}) = k/\bar{q}$. Similarly, the function $p_w(V)$ defined by (5.2) is bounded and continuous for all V (including $V = \bar{V}$), with $p_w(\bar{V}) = P(\bar{q}) = 0$. For all $V < \bar{V}$, $p_w(V)$ is differentiable and strictly decreasing because

$$p'_w(V) = \left(P' \frac{k}{J_w^2}\right) \frac{1}{u'(w(V))} < 0,$$

where the argument of P' is $k/J_w(V)$ and $P' < 0$ under Assumption 2. Moreover, for any given value V,

$$\frac{d}{dJ_w}\left(P'\frac{k}{J_w^2}\right) = \frac{k}{J_w^3}\left(-\frac{k}{J_w}P'' - 2P'\right) \geq 0$$

where the inequality follows from part (iii) of Assumption 2. Because $J_w(V)$ is decreasing, the function $P'k/J_w(V)$ is non-increasing. Because $1/u'(w(V))$ is non-decreasing in V and $P' < 0$, then $p'_w(V)$ is non-increasing. That is, $p_w(V)$ is (weakly) concave. QED

The above lemma shows that $p_w(V)$ has all the properties that enable parts (i) - (iii) in Lemma 3.2 to hold. As a result, there is a unique and interior solution to (3.1), $F_w(V)$, which is continuous and strictly increasing for all $V < \bar{V}$. Moreover, $E'_w(V) = -p_w(F_w(V)) < 0$.

The following theorem states the existence of the fixed point of Γ (see Appendix C for a proof):

Theorem 5.2. *Maintain Assumptions 1, 2 and 3. Then, the mapping Γ has a fixed point, $w^* \in \Omega'$. That is, $w^*(V)$ is continuous on $[\underline{V}, \bar{V}]$, its values lie in $[\underline{w}, \bar{w}]$ with $w^*(\bar{V}) = \bar{w}$, and it is strictly increasing for all $V < \bar{V}$. The implied functions $J_{w^*}(V)$ and $p_{w^*}(V)$ are strictly concave, in addition to the properties stated in Lemma 5.1.*

In the remainder of this paper, I will suppress the * on the fixed point and the subscripts w^* on the functions J, p, F and E.

The above theorem establishes continuity, but not differentiability, of the wage function $w(V)$. However, differentiability is required for various parts of previous sections. For example, part (iv) of Lemma 3.2 requires $p(V)$ to be twice differentiable in order for $F'(V)$ to exist, which in turn requires $w(V)$ to be differentiable. Moreover, the exposition leading to the equilibrium definition relied on the supposition that optimal contracts provide increasing values, as well as increasing wages, to workers over the tenure of employment. These features are guaranteed if the focus is on wage profiles that are smooth over time, as the following corollary states (see Appendix D for a proof):

Corollary 5.3. *If $|\dot{w}(t)| < \infty$ for all t, then $w(V)$ is differentiable, with $0 < w'(V) < \infty$ for all V. Moreover, the following results hold for all $V < \bar{V}$: (i) the derivatives $J''(V)$, $p''(V)$ and $F'(V)$ exist and are finite; (ii) $\dot{V} > 0$ and $\dot{J}(V) < 0$.*

6. Equilibrium Distributions of Workers and Firms

The functions $p(V)$ and $F(V)$ induce equilibrium distributions of workers and firms. Let g_e be the density function corresponding to the distribution of employed workers, G_e, and g_u be the density function corresponding to the distribution of unemployed workers, G_u. Once these distributions of values are obtained, one can compute other distributions interested in the literature. For example, the distribution of employed wages, denoted as $G_w(w)$, is given by $G_w(w(V)) = G_e(V)$. Because $w'(V) > 0$, the density function of employed wages is $g_w(w) = g_e(V)/w'(V)$.

6.1. Unemployment Benefits versus Wages

Let me compare unemployment benefits with wages. To do so, I translate unemployment benefits into values, using (3.4). That equation solves $V_u = V_u(b)$. Denote the inverse of this function as $b = B(V_u)$, where

$$B(V) = u^{-1}\left(\sigma V - \lambda_0 E(V)\right). \tag{6.1}$$

I will refer to $B(V)$ as the *benefit function.* It gives the level of the unemployment benefit starting at which an unemployed worker can achieve the lifetime value V. It can be verified that $V_u'(b) > 0$, and so $B'(V) > 0$. Under the assumption $\bar{b} = \bar{w}$, $V_u(\bar{b}) = \bar{V}$.

Define

$$v_0 = V_u(\underline{b}) \text{ and } v_j = F^{(j)}(v_0), \quad j = 1, 2, ... \tag{6.2}$$

where $F^{(0)}(v_0) = v_0$ and $F^{(j)}(v_0) = F(F^{(j-1)}(v_0))$. The support of G_u is $[v_0, \bar{V}]$ and the support of G_e is $[v_1, \bar{V}]$, as depicted in Figure 2 earlier. Clearly, $v_1 > v_0$. Moreover, $v_0 = \underline{V} + \lambda_0 E(v_0)/\sigma > \underline{V}$. That is, the set of equilibrium values is a strict subset of $[\underline{V}, \bar{V}]$.

The following lemma compares the benefit function with the wage function:

Lemma 6.1. *$w(\bar{V}) = B(\bar{V})$. If $\lambda_0 \leq \lambda_1$, then $w(V) < B(V)$ for all $V \in [v_1, \bar{V})$.*

Proof. For all $V \in [v_1, \bar{V}]$, $\dot{V} \geq 0$, and so the following holds for all $\lambda_0 \leq \lambda_1$:

$$u(w(V)) = \sigma V - \lambda_1 E(V) - \dot{V} \leq \sigma V - \lambda_0 E(V) = u(B(V)).$$

Thus, $w(V) \leq B(V)$. The inequality is strict when $\dot{V} > 0$, and hence when $V < \bar{V}$. QED

For an unemployed worker to achieve the same value V as an employed worker, the unemployment benefit must be higher than the wage. This is true even if an unemployed worker has the same access to jobs as an employed worker, i.e., if $\lambda_0 = \lambda_1$. The reason for this result is that an employed worker enjoys the prospect of rising wages while an unemployed worker's benefit does not change over time. This disadvantage of an unemployed worker must be compensated by a higher unemployment benefit in order for the unemployed worker to achieve the same value as an employed worker. Note that the comparison between the unemployment benefit and wage may also hold for some $\lambda_0 > \lambda_1$. Of course, if $\lambda_0 < \lambda_1$, then an unemployed worker has more difficult access to jobs than an employed worker. In this case, there is an additional reason for $B(V) > w(V)$.

The benefit function transforms the exogenous distribution of new entrants over the benefits into an endogenous distribution over the values. To see this, define $\Phi(V) = H(B(V))$. Drawing a benefit b according to the distribution H is equivalent to drawing a value V according to Φ. Because $B'(V) > 0$, the density function corresponding to Φ is:

$$\phi(V) = h(B(V))B'(V) = \frac{\sigma + \lambda_0 p(F(V))}{u'(B(V))} h(B(V)). \tag{6.3}$$

Because $h(.)$, $F(.)$ and $B(.)$ are differentiable, $\phi(.)$ is differentiable.

6.2. Distribution of Unemployed Workers

Consider the group of unemployed workers whose values are greater than V, where $V \in [v_0, \bar{V}]$. Equating the flows into this group and out of this group in a small interval of time dt, I obtain the following equation:

$$\begin{aligned} & (\sigma dt)\left\{1-(1-n)\left[1-G_u(V)\right]\right\}\left[1-\Phi(V)\right] \\ = \;& (\sigma dt)(1-n)[1-G_u(V)]\Phi(V) + \lambda_0(1-n)\textstyle\int_V^{\bar{V}}[p(F(z))dt]dG_u(z). \end{aligned}$$

The left-hand side gives the flow into the group, which is generated by newborns who replace workers who were not in the group and who just died. The measure of agents who were not in the described group is $\{1-(1-n)\left[1-G_u(V)\right]\}$. When such an agent dies and is replaced by a new agent, the new agent belongs to the described group if the agent draws a value of leisure higher than V, which occurs with probability $1-\Phi(V)$. Note that if agents who just died were in the described group and are replaced by new agents who draw values above V, such newborns do not change the measure of the group.

The right-hand side of the equation gives the flows out of the group. The first term is caused by death in the group whose replacements draw values less than or equal to V. The second term is the flow of agents who just exited the group as the result of becoming employed (at higher values).

Dividing the two sides of the equation by dt and re-arranging, I obtain:

$$\sigma[n - \Phi(V) + (1-n)G_u(V)] = \lambda_0(1-n)\int_V^{\bar{V}} p(F(z))dG_u(z). \tag{6.4}$$

From this equation one can show that $G_u(V)$ is continuous and differentiable for all $V \in [v_0, \bar{V}]$. Differentiating with respect to V, I get:

$$g_u(V) = \frac{\sigma\phi(V)}{(1-n)\left[\sigma + \lambda_0 p(F(V))\right]}. \tag{6.5}$$

Integrating over V, I solve:

$$G_u(V) = \frac{\sigma}{1-n}\int_{v_0}^{V} \frac{\phi(z)}{\sigma + \lambda_0 p(F(z))} dz. \tag{6.6}$$

The requirement $G_u(\bar{V}) = 1$ determines the fraction of employed workers as:

$$n = 1 - \sigma\int_{v_0}^{\bar{V}} \frac{\phi(z)}{\sigma + \lambda_0 p(F(z))} dz. \tag{6.7}$$

The following lemma is proven in Appendix D:

Lemma 6.2. *$G_u(V) < \Phi(V)$ for all $V \in (v_0, \bar{V})$.*

The cause for the result in this lemma is the feature that the employment rate is decreasing in the worker's current value. This feature implies that a higher proportion of unemployed workers at low values transit from unemployment into employment than unemployed workers at high values. As a result, the distribution of the values of workers who remain unemployed in the steady state is more skewed toward high values than the distribution with which unemployed workers start their lives with. A particular implication of such first-order stochastic dominance is that, if H is uniform, then the density function g_u is increasing.

6.3. Distribution of Employed Workers

Examine the group of employed workers whose values are greater than V, where $V \in [v_1, \bar{V}]$. Since death is the only way to exit from this group, the measure of the outflow from this group in a small interval dt is $\sigma n[1 - G_e(V)](dt)$. There are three flows into the group. One is the group of workers who were employed at or below V and whose values increased above V according the contract. The size of this flow is $n[G_e(V) - G_e(V - \dot{V}dt)]$. The second inflow is the group of workers who were employed at or below V and who received offers above V. This inflow exists only if the workers' values before the application are equal to or greater than v_1, i.e., if $F^{-1}(V) \geq v_1$; otherwise, the workers were unemployed. The third inflow is the group of unemployed workers who received offers above V. Before receiving offers, these workers' values lie in $[F^{-1}(V), \bar{V}]$. Equating the outflows to the sum of inflows, and taking the limit $dt \to 0$, I get:

$$\begin{aligned} & \sigma n \left[1 - G_e(V)\right] \\ = \; & n \lim_{dt \downarrow 0} \tfrac{G_e(V) - G_e(V - \dot{V}dt)}{dt} + \lambda_1 n \textstyle\int_{\max\{v_1, F^{-1}(V)\}}^{V} p(F(z)) dG_e(z) \\ & + \lambda_0 (1-n) \textstyle\int_{F^{-1}(V)}^{\bar{V}} p(F(z)) dG_u(z). \end{aligned}$$

Using (6.4) to substitute for the last term and re-arranging, I get:

$$\begin{aligned} & \lim_{dt \downarrow 0} \tfrac{G_e(V) - G_e(V - \dot{V}dt)}{dt} \\ = \; & \sigma \Delta(F^{-1}(V)) - \sigma G_E(V) - \lambda_1 \textstyle\int_{\max\{v_1, F^{-1}(V)\}}^{V} p(F(z)) dG_e(z) \end{aligned} \tag{6.8}$$

Here, Δ is defined as $\Delta(V) = \left[\Phi(V) - (1-n) G_u(V)\right]/n$. Denote $\delta(V) = \Delta'(V)$. Then,

$$\delta(V) = \frac{\lambda_0 p(F(V)) \phi(V)}{n\left[\sigma + \lambda_0 p(F(V))\right]} = \frac{\lambda_0 p(F(V)) h(B(V))}{n u'(B(V))}. \tag{6.9}$$

Lemma 6.3. *G_e does not have a mass point and g_e is continuously differentiable.*

Proof. Suppose, to the contrary, that G_e has a mass point at a value $V \in [v_1, \bar{V}]$. Then

$$\lim_{dt \downarrow 0} \frac{G_e(V) - G_e(V - \dot{V}dt)}{dt} = \infty.$$

This violates (6.8), because the right-hand side of (6.8) is bounded. Thus, G_e does not have a mass point, and so it is a continuous function. Denote

$$g_e(V_-) = \lim_{dt \downarrow 0} \frac{G_e(V) - G_e(V - dt)}{dt}$$

The left-hand side of (6.8) is equal to $g_e(V_-)\dot{V}$. The right-hand side of (6.8) is a continuous function of V, because G_e, F^{-1} and $p(F(.))$ are continuous. Thus, $g_e(V_-)\dot{V}$ must be continuous. Because $\dot{V}$ is also continuous, g_e must be continuous. Continuity of g_e implies that G_e is continuously differentiable. Since F^{-1} and $p(F(.))$ are continuously differentiable, differentiability of G_e implies that the right-hand side of (6.8) is continuously differentiable. Therefore, g_e is continuously differentiable. QED

With the above lemma, I can rewrite (6.8) as

$$g_e(V)\dot{V} = \sigma \Delta(F^{-1}(V)) - \sigma G_E(V) - \lambda_1 \int_{\max\{v_1, F^{-1}(V)\}}^{V} p(F(z)) dG_e(z) \tag{6.10}$$

To solve for g_e, partition the support of G_e into subintervals $[v_j, v_{j+1})$, where v_j is defined by (6.2). Add a subscript j to $g_e(V)$ and $G_e(V)$ when $V \in [v_j, v_{j+1})$. Using the function $T(V)$ defined by (3.11) and $\delta(V)$ defined by (6.9), I summarize some previous results and the solution for g_e in the following theorem (see Appendix E for a proof):

Theorem 6.4. *The density of the equilibrium distribution of unemployed workers is given by (6.5), the measure of employed workers by (6.7), and the equilibrium density function of employed workers is given as follows:*

$$g_{e1}(V)\dot{V} = \sigma \int_{v_1}^{V} \gamma(T(V), T(z))\delta(F^{-1}(z))dF^{-1}(z), \tag{6.11}$$

$$\begin{aligned} & g_{ej}(V)\dot{V} - g_{ej}(v_j)\dot{v}_j\gamma(T(V), T(v_j)) \\ = \; & \textstyle\int_{v_j}^{V} \gamma(T(V), T(z)) \left\{\sigma\delta(F^{-1}(z)) + \lambda_1 p(z) g_{e(j-1)}(F^{-1}(z))\right\} dF^{-1}(z) \end{aligned} \tag{6.12}$$

where (6.12) holds for $j \geq 2$. Moreover, $g_{ej}(v_j) = \lim_{V \to v_j} g_{e(j-1)}(V)$ for all j.

The theorem gives the following procedure to compute g_e. Starting with $j = 1$, (6.11) gives g_{e1}. Taking the limit $V \to v_2$ in the formula yields $g_{e2}(v_2)$. Then, setting $j = 2$ in (6.12) yields $g_{e2}(V)$. Taking the limit $V \to v_3$ in the result yields $g_{e3}(v_3)$. Continue this process until g_{ej} is obtained for all j.

The following corollary describes the shape of g_e (see Appendix E for a proof):

Corollary 6.5. *$g_e(v_1) = 0$ and $g_e'(v_1) > 0$. If $F'(\bar{V}) > 0$, then $g_e(\bar{V}) = 0$. In this case, there exists $\hat{V} \in (v_1, \bar{V})$ such $g_e'(V) < 0$ for $\hat{V} < V < \bar{V}$.*

Before discussing the result in this corollary, let us see how easily the condition $F'(\bar{V}) > 0$ can be satisfied. Consider the matching function in Example 2.1. Write the first-order condition of a worker's application as $F^{-1}(V) = V + p(V)/p'(V)$. Differentiating this condition and evaluating at $\bar{V}$ yields $dF^{-1}(V)/dV|_{V=\bar{V}} = 1 - \rho$. Thus, with the CES matching function, $F'(\bar{V}) > 0$ is automatically satisfied under Assumption 2.

The above corollary says that the density function of employed workers is an increasing function at low values. In addition, if $F'(\bar{V}) > 0$, the density function is decreasing at high values. In this case, the density function is non-monotonic. There are more workers employed at intermediate values than at values at the two ends.

The non-monotonic density of employed values implies a non-monotonic density of employed wages. To see this, recall that the density of employed wages is $g_w(w) = g_e(V)/w'(V)$. Because $0 < w'(V) < \infty$ by Lemma 5.3, the above corollary yields $g_w(w_1) =$

$g_w(\bar{w}) = 0$, where $w_1 = w(v_1)$. Thus, the shape of $g_w(w)$ at the two ends is similar to the shape of $g_e(V)$ at the two ends. That is, the density of employed wages is increasing when wage is low and decreasing when wage is high.

The non-monotonicity of the density function of employed values (or wages) is a robust feature of the data. However, this feature is not a prediction of the BC model or, more generally, of any undirected search model with on-the-job search and with homogeneous matches. Instead, such a model generates monotonically increasing density functions of employed values. The critical difference is in the shape of the density function at high values. While directed search induces the density function to decrease at high values, as in the data, undirected search induces the density function to increase at high values. To reverse this unrealistic prediction, an undirected search model needs to introduce sufficient heterogeneity across matches, such as heterogeneity in workers' or firms' productivity.

Directed search generates such a different result from undirected search because an applicant optimally chooses to apply to one target value. Since higher values are also available to the applicant, applying to the (lower) target value is optimal only if higher values are more difficult to be obtained than the target value. For this to be true, the measure of recruiting firms per applicant must be smaller at high values than at the target value. In particular, at values close to the upper bound $\bar{V}$, the measure of recruiting firms per applicant should be close to zero. In turn, as few workers apply to such high values, it is indeed optimal for only few firms to recruit at these values. The measure of workers who succeed in obtaining jobs at values near $\bar{V}$ is close to zero. This feature makes the density function of employed values decreasing near the upper end of the distribution.

This desirable feature does not exist under undirected search because, then, every applicant is assumed to send the application randomly and uniformly to the recruiting firms. Because firms cannot use offers to attract applications, they use offers to increase acceptance and retention. For these purposes, a high value is superior to a low value. Because all equilibrium offers must yield the same expected value to the firm, there must be more firms recruiting at high values than at low values. This results in more workers being employed at high values than at low values. In addition, since workers employed at high values are also less likely to quit than workers employed at low values, the density function of employed values is increasing under undirected search.

The above difference can be illustrated with the roles of $p(V)$ and $q(V)$, the Poisson rates of employment and hiring. A directed search model and an undirected search model both require zero net expected profit from recruiting at all values, i.e., $q(V) = k/J(V)$. With reasonable assumptions, both models produce a decreasing and concave function,

$J(V)$, which implies that $q(V)$ is increasing and convex. The difference arises in the link between $q(V)$ and the distribution of workers. This link is tight when search is undirected. In that case, a firm's hiring rate at V is equal to the rate at which the application comes from a worker employed below V. That is,

$$\lambda_1 n G_e(V) + \lambda_0(1-n)G_u(V) = q(V)$$

Because $q(V)$ is convex, then the density functions $g_e(V)$ and $g_u(V)$ are likely to be both increasing. Directed search breaks this close link between q and G_e. With directed search, the critical determinant of the equilibrium distribution of workers is not the hiring rate, but rather workers' application decisions which is governed by the function $p(V) = P(q(V))$. Although $q(V)$ is still convex, the function $p(V)$ is decreasing and concave. In particular, the employment rate at values close to $\bar{V}$ is almost zero. As a result, few workers are employed at such high values, and so the density of employed workers is decreasing at this high end of the distribution.

Now an alert reader may notice the heterogeneity in the distribution of unemployment benefits. Because this heterogeneity is not in the BC model, one might suspect that its presence also plays a role in creating the non-monotonic density function of employed workers. This suspicion is not supported. The non-monotonicity of g_e occurs regardless of whether $h(b)$ is increasing, decreasing, flat, or non-monotonic.

As a general matter, it is important to remark that the heterogeneity in unemployment benefits affects the details of the wage distribution, but eliminating it does not eliminate the wage dispersion. Even if all workers start their lives with the same unemployment benefit, the matching process results in only a fraction of them getting jobs initially. Those who luckily get jobs will continue to apply for higher wages in the future than those who do not have jobs. This precess continues, and so there will be wage dispersion in the equilibrium. Moreover, because wages increase with tenure, workers who obtain the same contract at different times will also see wages differ.

6.4. Distribution of Offer Values

Denote the distribution of recruiting firms as $R(V)$ and its density as $r(V)$. Because each firm has only one vacancy, R is also the distribution of offers. To compute this distribution, note that the relative measure of applicants for V to recruiting firms at V is the tightness $\theta(V) = q(V)/p(V)$. The measure of workers applying to V is:

$$A(V) = \lambda_0(1-n)g_u(F^{-1}(V)) + \lambda_1 n g_e\left(\max\{v_1, F^{-1}(V)\}\right).$$

The measure of firms recruiting at V is equal to $A(V)/\theta(V)$. Thus, the distribution of recruiting firms or offer values is:

$$R(V) = \int_{v_1}^{V} \frac{p(z)}{q(z)} A(z)dz \Big/ \left[\int_{v_1}^{\bar{V}} \frac{p(z)}{q(z)} A(z)dz \right]$$

The second integral is the total measure of recruiting firms.

As the distribution of employed values, the offer distribution is decreasing at high values. This can be verified by noting $r(\bar{V}) = 0$, because $p(\bar{V}) = 0$. The result implies that the distribution of offer wages is decreasing at high values.

7. Comparative Statics

In this section, I conduct two comparative statics, one with respect to the distribution of the unemployment benefit and the other with respect to the parameter λ_0. These comparative statics further illustrate the differences between the current model of directed search and undirected search models.

Suppose unemployment benefits increase in the sense that the distribution H changes in the first-order stochastic dominance. Suppose that the support of the distribution does not change. The effects of this change are summarized in the following corollary:

Corollary 7.1. *An increase of the first-order stochastic dominance in unemployment benefits has no effect on workers' optimal applications and equilibrium contracts. It does not affect the supports of the distributions $G_u(.)$ and $G_e(.)$, either, although it affects the shape of these distribution functions.*

Proof. The analysis on equilibrium contracts in section 5 are independent of the distributions, G_e, G_u and H. Thus, the functions, $w(.)$, $p(.)$, $q(.)$, $J(.)$, $F(.)$, and $E(.)$ after the change in H are all the same as before. The independence of the function $E(.)$ on H implies that the function $V_u(.)$ is independent of H. Under the assumption that $\underline{b}$ does not change, v_0 and v_1 do not change, either, because $v_0 = V_u(\underline{b})$ and $v_1 = F(v_0)$. Taken together, these results imply that the change in H has no effect on a worker's optimal application strategy or the equilibrium set of contracts. However, G_u and G_e change with H because Φ does. QED

The distributions of workers, G_e, G_u and H, do not play any role in the determination of optimal contracts and optimal application. Although this feature is evident from the procedure in section 5, it is worthwhile explaining why it arises. To do so, start with a worker's application. For a worker to decide on the optimal application, $F(V)$, he only needs the function of the employment rate. In turn, the employment rate must satisfy the requirement that recruiting should yield zero expected profit at all equilibrium offers. This requirement pins down $p(V)$, given a firm's value function, $J(V)$. However, a firm value function depends only on what happens after the hiring, that is, on the contract offered, $w(V)$, and the worker's quit rate, $p(F(V))$. By the proceeding argument, the two functions that determine the quite rate, $p(.)$ and $F(.)$, are only functions of optimal contracts. Thus, the only thing still to be determined is the set of optimal contracts, i.e., the function $w(V)$. The function $w(V)$ provides efficient sharing of the value between a firm and a worker, in the sense that $-\dot{J} = \dot{V}/u'(w)$. Again, $\dot{J}$ and $\dot{V}$ involve only the functions $F(V)$, $p(V)$, $J(V)$ and $w(V)$ (see (3.3) and (3.5)). The solution to this fixed-point problem is independent of the distributions of employed and unemployed workers.[11]

A strong (testable) implication of the above corollary is that changing unemployment benefits can change wage distributions and the average duration of unemployment, but it does not change individual workers' job-to-job transition rates or their wage paths.

The above predictions are markedly different from those in undirected search models.[12] An increase in unemployment benefits, in the way modelled here, reduces the probability with which a given offer will be accepted by a worker. This forces the equilibrium distribution of offers to increase. As more firms are offering higher values than before, the transition rate from low-value jobs to high-value jobs increases. That is, the quitting rate at low-value jobs increases. In order to mitigate this increase in the quitting rate, firms offer contracts in which wages increase more quickly with tenure than before. As is clear from this explanation, the main cause for this difference in the result is that the offer distribution plays a critical role in determining workers' quit rates under undirected search, but not so under directed search.

Now let me turn to the effects of an increase in λ_0, the probability with which an unemployed worker receives a job application opportunity. Again, the functions $w(.)$, $p(.)$, $q(.)$, $J(.)$, $F(.)$, and $E(.)$, are all unaffected, because the analysis in section 5 does not

[11] There are two qualifications. First, the distribution H can affect equilibrium contracts if the number of firms is fixed in ths short run, rather than being determined by competitive entry. In that case, a firm's expected value from recruiting is endogenous, rather than being given by the vacancy cost k. All effects of the distributions on equilibrium contracts come through this expected value of recruiting, and these effects vanish in the long run when entry becomes competitive. Second, if there is exogenous separation between a worker-firm pair and the worker returns to unemployment after such exogenous separation, then the increase in unemployment benefits will affect optimal contracts by affecting the equation for $\dot{V}$.

[12] One can verify the statements here by introducing a continuous distribution of unemployment benefits into BC or Burdett and Mortensen (1998).

depend on λ_0. However, the function $V_u(.)$ does depend on λ_0. An increase in λ_0 increases $V_u(b)$ for any given b. Thus, v_0 increases, and so does v_1. The distributions of unemployed and employed workers change as well.

The increases in λ_0 has only a limited effect on the job-to-job transition rate and the wage path. To see this, let $\hat{v}_1$ be the new level of v_1 after the increase in λ_0. Because $w(.)$, $p(.)$ and $q(.)$ are unaffected, the optimal baseline contract after the increase in λ_0 is the part of the original baseline contract from $\hat{v}_1$ onward. Put differently, the new set of optimal contracts is identical to the subset of the original contracts whose starting values are equal to or greater than $\hat{v}_1$. Therefore, the job-to-job transition rate and the wage path of a worker to whom the contract provides $\hat{v}_1$ or more do not change. Again, these results contrast with those in undirected search models, where an increase in λ_0 increases the job-to-job transition rate and the steepness of the wage path.

The comparative statics above have obvious policy implications. If policymakers attempt to affect the labor market outcomes of employed workers, changing the aspects of the market for unemployed workers would be a wrong place to put the resource. Instead, the policy should directly target the aspects of the labor market relevant for employed workers, such as λ_1.

8. Conclusion

In this paper, I analyze the equilibrium in a labor market where firms offer wage-tenure contracts to direct the search of employed and unemployed workers. Each applicant observes all offers and there is no coordination among individuals. Because search is directed, workers' applications (as well as firms' recruiting decisions) must be optimal. This optimality requires the equilibrium to be formulated differently from the that in the large literature of undirected search. I provide such a formulation and show that the equilibrium exists. In the equilibrium, individuals explicitly tradeoff between an offer and the matching rate at that offer. This tradeoff yields a unique offer which is optimal for each worker to apply. Despite this uniqueness and directed search, the stationary equilibrium has a non-degenerate and continuous distribution of wages.

One cause of wage dispersion in the model is the feature that the optimal application increases with the value that a worker's current state yields. Even if all workers were initially identical, those who obtained jobs earlier will apply to higher wages than those who obtain jobs later. In the stationary equilibrium, a continuum of values are offered, each of which is tailored to workers who have a particular current value. The other cause of the wage distribution is the wage-tenure contract. With risk-averse workers and imperfect capital markets, it is optimal for a firm to offer a wage profile that increases smoothly with tenure. Such a contract provides partial insurance to the worker and backloads wages to increase retention of the worker. The positive wage-tenure relationship implies that workers who are employed under the same contract but at different times may earn different wages. It also implies that the quit rate falls with tenure.

The model generates several novel implications. First, because applicants separate themselves according to their current values, wage mobility is endogenously limited by the workers' current wages. Second, the density function of the wage distribution of employed workers over wages is increasing at low wages and decreasing at high wages, even when all worker-firm pairs are equally productive. Finally, an increase in unemployment benefits has no effect on the set of equilibrium contracts or individual workers' job-to-job transition rates, although it affects wage distributions of workers.

These differences are clearly testable. In addition, the model provides a tight link between the unobserved distribution of offers and the observed distribution of employed wages. Because each worker's application is optimal, every match results in a formation of a firm-worker relationship. There is no offer which is received by a worker but which is not taken. If a worker applies to an offer, then the offer is acceptable. This means that a worker's duration with a job is simply equal to the duration in which the worker fails to get an offer. This property allows one to use the duration data and the observed distribution of wages to back out the distribution of offers. The procedure may even be workable when there is unobserved heterogeneity across matches. As Barlevy (2003) discusses, the distribution of offers is useful for a range of issues, but it cannot be easily identified in undirected search models. One cause of the difficulty is that, with random search, an offer that is received by a worker may not be acceptable to the worker.

A useful extension will be to incorporate heterogeneity across firms and workers. If workers differ in productivity, then firms may rank the applicants according to productivity, as Shi (2002) and Shimer (2005) have shown in simpler environments. If firms are different in productivity, then they may provide different wage-tenure contracts. Such an extension will undoubtedly be challenging, but it will bring the model closer to the data.

Appendix

A. Proof of Lemma 3.2

The result $F(\bar{V})=\bar{V}$ is evident. Let $V<\bar{V}$ in the following proof. Temporarily denote $K(f,V)=p(f)(f-V)$. Because $p(.)$ is continuous and bounded, as stated in Lemma 3.1, $K(f,V)$ is continuous and bounded. Thus, the maximization problem in (3.1) has a solution. Because $p(\bar{V})=0$, I have $K(\bar{V},V)=0=K(V,V)$. Since any interior value of f gives a positive value of $K(f,V)$, then the solution is interior. To show that the solution is unique, I show that $K(f,V)$ is strictly concave in f for all $f\in(V,\bar{V})$. To do so, let $\alpha\in(0,1)$. Let f_1 and f_2 be two arbitrary interior values with $f_2>f_1>V$. Denote $f_\alpha=\alpha f_1+(1-\alpha)f_2$. Then,

$$\begin{aligned}
K(f_\alpha,V) &= p(f_\alpha)\left[\alpha(f_1-V)+(1-\alpha)(f_2-V)\right] \\
&\geq \left[\alpha p(f_1)+(1-\alpha)p(f_2)\right]\left[\alpha(f_1-V)+(1-\alpha)(f_2-V)\right] \\
&= \alpha K(f_1,V)+(1-\alpha)K(f_2,V)+\alpha(1-\alpha)[p(f_1)-p(f_2)][f_2-f_1] \\
&> \alpha K(f_1,V)+(1-\alpha)K(f_2,V).
\end{aligned}$$

The two equalities come from rewriting, the first inequality from the concavity of p, and the last inequality from the strictly decreasing property of p. Thus, $K(f,V)$ is strictly concave in f and part (i) in the Lemma is established.

For part (ii), uniqueness of the solution implies that $F(.)$ is continuous by the Theorem of the Maximum. To show that $E(.)$ is differentiable, let V_1 and V_2 be two arbitrary values with $V_1<V_2<\bar{V}$. Express $F_i=F(V_i)$ for $i=1,2$. Uniqueness of the solution implies $K(F_1,V_1)>K(F_2,V_1)$ and $K(F_2,V_2)>K(F_1,V_2)$. Thus,

$$E(V_2)-E(V_1)>K(F_1,V_2)-K(F_1,V_1)=-p(F_1)(V_2-V_1);$$

$$E(V_2)-E(V_1)<K(F_2,V_2)-K(F_2,V_1)=-p(F_2)(V_2-V_1).$$

Divide the two inequalities by (V_2-V_1) and take the limit $V_2\to V_1$. Because $F(.)$ is continuous, the limit shows that $E(V)$ is differentiable at V_1 and that $E'(V_1)=-p(F_1)$. Since V_1 is arbitrary, this argument establishes part (ii) for all $V<\bar{V}$.

For part (iii), again take two arbitrary values V_1 and V_2, with $V_1<V_2\leq\bar{V}$. Then, $p(F_j)(F_j-V_i)<p(F_i)(F_i-V_i)$ for $j\neq i$. I have:

$$\begin{aligned}
0 &> [p(F_2)(F_2-V_1)-p(F_1)(F_1-V_1)]+[p(F_1)(F_1-V_2)-p(F_2)(F_2-V_2)] \\
&= p(F_2)(V_2-V_1)+p(F_1)(V_1-V_2)=[p(F_2)-p(F_1)](V_2-V_1).
\end{aligned}$$

This result implies $p(F_2)<p(F_1)$. Because $p(.)$ is strictly decreasing, $F(V_2)>F(V_1)$.

Because p is continuously differentiable, $F(V)$ is given by the first-order condition of the maximization problem, which leads to (3.2). Furthermore, if p is twice differentiable, then differentiating the first-order condition generates the derivative $F'(V)>0$. Concavity of p yields $F'(V)\leq 1/2$. Finally, $E''(V)=-p'(F(V))F'(V)$. QED

B. Proof of Lemma 3.3

Lemmas 3.1 and 3.2 yield $p'(F(V)) < 0$ and $F'(V) > 0$ for all $V < \bar{V}$. Because $J(V) > 0$ for all V, as shown later, then (3.8) implies $\dot{w}(t) > 0$ for all $V < \bar{V}$.

Because $\bar{V}$ is the highest value offered, then $p(F(\bar{V})) = 0$ and $\dot{V} = 0$ at $V = \bar{V}$. Then $E(\bar{V}) = 0$, and (3.3) implies $\bar{V} = u(\bar{w})/\sigma$. Similarly, because $\dot{J}(\bar{V}) = 0$, (3.5) implies $J(\bar{V}) = (y - \bar{w})/\sigma$. Because recruiting at $\bar{w}$ should yield zero net profit, $q(\bar{V})J(\bar{V}) = k$; that is, $\bar{w} = y - \sigma k/q(\bar{V})$. If $q(\bar{V}) = \bar{q}$, then the stated expressions for $\bar{w}$ and $J(\bar{V})$ follow. Since $\bar{q} < \infty$ by Assumption 2, then $\bar{w} < y$ and $J(\bar{V}) > 0$.

To show $q(\bar{V}) = \bar{q}$, suppose that $q(\bar{V}) = \bar{q} - \delta$ to the contrary, where $\delta > 0$. Because $q(\bar{V})J(\bar{V}) = k > 0$ and $J(\bar{V}) = (y - \bar{w})/\sigma$, then $\bar{w} = y - \sigma k/(\bar{q} - \delta)$. Consider a firm that deviates from $\bar{w}$ to $\bar{w} + \varepsilon$, where $\varepsilon > 0$, which generates a value to a worker as $\hat{V} = u(\bar{w} + \varepsilon)/\sigma$. Because the firm is the only one that offers a wage higher than $\bar{w}$, the workers who are employed at $\bar{w}$ will all apply to this firm, which yields $q(\hat{V}) = \bar{q}$. The firm's expected value of recruiting is $q(\hat{V})J(\hat{V}) = (y - \bar{w} - \varepsilon)\bar{q}/\sigma$, which exceeds k for sufficiently small $\varepsilon > 0$. This result contradicts the statement that $\bar{V}$ is an equilibrium value. Thus, $q(\bar{V}) = \bar{q}$. QED

C. Proof of Theorem 5.2

The sets Ω and Ω' are defined prior to Lemma 5.1 and the mapping Γ is defined by $w_1(V) = (\Gamma w)(V)$, where w_1 is the solution to (5.3). It can be verified that Ω is a closed and convex set. Lemmas C.1 and C.2 below state that $\Gamma : \Omega \to \Omega'$ is a continuous mapping in the supnorm. Then, Γ has a fixed point in Ω, denoted as w^*. Because $w^*(V) = (\Gamma w^*)(V) \in \Omega'$, then $w^*(V)$ is strictly increasing for all $V < \bar{V}$. This implies that $J_{w^*}(V)$ and $p_{w^*}(V)$ are strictly concave, in addition to the properties stated in Lemma 5.1.

Lemma C.1. $\Gamma : \Omega \to \Omega' \subset \Omega$.

Proof. Temporarily denote the left-hand side of (5.3) as $L(w_1)$ and the right-hand side $R(V)$. Recall that $\bar{w} < y$. Because $L(w)$ is continuous and strictly decreasing for all $w < y$, it is invertible for all $w \in [\underline{w}, \bar{w}]$. Then, $w_1 = L^{-1}(R(V)) \equiv w_1(V)$. I show that $w \in \Omega \Longrightarrow w_1 \in \Omega'$. This is done in the following steps.

First, $w_1(V)$ is continuous because $J_w(.)$, $p_w(.)$ and $F_w(.)$ are all continuous.

Second, $w_1(V)$ is strictly increasing for all $V < \bar{V}$. To establish this property is equivalent to showing that $R(V)$ is strictly decreasing for all $V < \bar{V}$. Pick arbitrary values V_1 and V_2, with $\underline{V} \le V_1 < V_2 < \bar{V}$. Then,

$$\begin{aligned} R(V_2) - R(V_1) = \; & [u'(w(V_2)) - u'(w(V_1))] \left[\sigma + \lambda_1 p_w(F_w(V_2))\right] J_w(V_2) \\ & + u'(w(V_1))\lambda_1 J_w(V_2) \left[p_w(F_w(V_2)) - p_w(F_w(V_1))\right] + D \end{aligned} \tag{C.1}$$

where

$$D = u'(w(V_1)) \left[\sigma + \lambda_1 p_w(F_w(V_1))\right] J_w(V_2) + \sigma(V_2 - V_1) - \lambda_1 \left[E_w(V_2) - E(V_1)\right]$$

Because $p_w(V)$ is strictly decreasing as in Lemma 5.1, $F_w(V)$ is strictly increasing, and so $u'(w(V_2)) \leq u'(w(V_1))$ and $p_w(F(V_2)) < p_w(F(V_1))$. The first term on the right-hand side of (C.1) is non-positive and the second term is negative. If $D \leq 0$, then $R(V_2) < R(V_1)$, as desired. To show $D \leq 0$, note that $J_w(V)$ and $E_w(V)$ are differentiable by Lemmas 5.1 and 3.1. , Then, D is differentiable with respect to V_2. Using $E_w'(V) = -p_w(F_w(V))$ and $J_w'(V) = -1/u'(w(V))$, I can calculate the derivative as:

$$\frac{\partial D}{\partial V_2} = \sigma + \lambda_1 p_w(F(V_2)) - [\sigma + \lambda_1 p_w(F_w(V_1))] \frac{u'(w(V_1)}{u'(w(V_2))}.$$

Because $u'(w(V_1)) \geq u'(w(V_2))$ and $p_w(F_w(V_1)) > p_w(F_w(V_2))$, then $\partial D / \partial V_2 < 0$ for all $V_2 < \bar{V}$. Thus, $D > D|_{V_2 = V_1} = 0$ for all $V_2 \in (V_1, \bar{V})$.

Third, $w_1(V) \in [\underline{w}, \bar{w}]$ for all V, with $w_1(\bar{V}) = \bar{w}$. Examine $w_1(\bar{V})$. Because $w(\bar{V}) = \bar{w}$, then (5.3) implies:

$$L(w_1(\bar{V})) = R(\bar{V}) = u'(\bar{w})(y - \bar{w}) + u(\bar{w}) = L(\bar{w}).$$

Because $L(w)$ is strictly decreasing, the above equation implies $w_1(\bar{V}) = \bar{w}$. Since $w_1(V)$ is strictly increasing for $V < \bar{V}$, then $w_1(V) < \bar{w}$ for all $V < \bar{V}$.

Finally, I show $w_1(V) \geq \underline{w}$. Since $L'(w) < 0$, $w_1(V) \geq \underline{w}$ if and only if $L(\underline{w}) \geq R(V)$. A sufficient condition is $L(\underline{w}) \geq R(\underline{V})$, because $R(V)$ is decreasing function. Note that the following holds:

$$\begin{aligned} R(\underline{V}) &= u'(w(\underline{V})) [\sigma + \lambda_1 p_w(F_w(\underline{V}))] J_w(\underline{V}) + \sigma \underline{V} - \lambda_1 E_w(\underline{V}) \\ &< u'(\underline{w}) [\sigma + \lambda_1 p_w(F_w(\underline{V}))] J_w(\underline{V}) + u(\underline{b}) \\ &\leq u'(\underline{w}) [\sigma + \lambda_1 p_w(\underline{V})] J_w(\underline{V}) + u(\underline{b}) \\ &\leq u'(\underline{w}) [\sigma + \lambda_1 p_{\bar{w}}(\underline{V})] J_{\bar{w}}(\underline{V}) + u(\underline{b}) \end{aligned}$$

The first inequality follows from the facts that $w(\underline{V}) \geq \underline{w}$, $\underline{V} = u(\underline{b})/\sigma$ and $E_w(\underline{V}) > 0$. The second inequality follows from the facts that $F_w(\underline{V}) \geq \underline{V}$ and that $p_w(.)$ is decreasing. To obtain the third inequality, note that $J_w(V) \leq J_{\bar{w}}(V)$ and $p_w(V) \leq p_{\bar{w}}(V)$ for all V. Therefore, a sufficient condition for $w_1(V) \geq \underline{w}$ is:

$$L(\underline{w}) \geq u'(\underline{w}) [\sigma + \lambda_1 p_{\bar{w}}(\underline{V})] J_{\bar{w}}(\underline{V}) + u(\underline{b})$$

This condition can be re-arranged as (5.7), which is assumed to hold. This completes the proof of Lemma C.1.

Lemma C.2. Γ *is continuous in the supnorm.*

Proof. To show that the mapping Γ is continuous in the supnorm, I show that the following holds for all $w_a, w_b \in \Omega$ and all V:

$$|(\Gamma w_a)(V) - (\Gamma w_b)(V)| \leq A \|w_a - w_b\|, \tag{C.2}$$

where the norm is the supnorm and $A > 0$ is a finite constant. Once this is done, then

$$\|\Gamma w_a - \Gamma w_b\| = \sup |(\Gamma w_a)(V) - (\Gamma w_b)(V)| \leq A \|w_a - w_b\|,$$

which implies that Γ is continuous in the supnorm.

To show (C.2), take arbitrarily $w_a, w_b \in \Omega$ and $V \in [\underline{V}, \bar{V}]$. Without loss of generality, assume $w_a(V) \geq w_b(V)$ for the given value V. Shorten the subscript w_i on J, p, F, and E to i, where $i = a, b$. Also, denote the right-hand side of (5.3) with $w = w_i(V)$ as $R_i(V)$. Because $w \geq w_L > 0$, Assumption 1 implies that there are positive and finite constants ω_1 and ω_2 such that $\omega_1 \leq |u''(w)| \leq \omega_2$ for all $w \in [\underline{w}, \bar{w}]$. Then

$$|L'(w)| = (y - w)\,|u''| \geq (y - \bar{w})\,\omega_1 \equiv A_1.$$

Note that A_1 is bounded above 0. Since $L(w)$ is decreasing, then

$$|R_a(V) - R_b(V)| = |L(\Gamma w_a(V)) - L(\Gamma w_b(V))| \geq A_1\,|\Gamma w_a(V) - \Gamma w_b(V)|\,.$$

Now, consider $|R_a(V) - R_b(V)|$. Suppressing the given V, I have:

$$\begin{aligned} |R_a - R_b| &= |\{[u'(w_a) - u'(w_b)]\,J_a + u'(w_b)(J_a - J_b)\}\,[\sigma + \lambda_1 p_a(F_a)] \\ &\qquad + \lambda_1 u'(w_b) J_b\,[p_a(F_a) - p_b(F_b)] - \lambda_1\,[E_a - E_b]| \\ &\leq [|u'(w_a) - u'(w_b)|\,J_a + u'(w_b)\,|J_a - J_b|]\,[\sigma + \lambda_1 p_a(F_a)] \\ &\quad + \lambda_1 u'(w_b) J_b\,|p_a(F_a) - p_b(F_b)| + \lambda_1\,|E_a - E_b| \end{aligned}$$

I find the bound on each of the absolute values in the above expression.

Because $u'' < 0$, then

$$|u'(w_a) - u'(w_b)| \leq |w_a - w_b| \max\{|u''(w_a)|\,, |u''(w_b)|\} \leq \omega_2\,\|w_a - w_b\| \tag{C.3}$$

By the definition of J_w,

$$\begin{aligned} |J_a - J_b| &= \left| \int_{\underline{V}}^{\bar{V}} \tfrac{u'(w_a(z)) - u'(w_b(z))}{u'(w_a(z))u'(w_b(z))} dz \right| \\ &\leq \tfrac{1}{[u'(\bar{w})]^2} \int_{\underline{V}}^{\bar{V}} |u'(w_a(z)) - u'(w_b(z))|\,dz \\ &\leq \tfrac{\omega_2}{[u'(\bar{w})]^2} \int_{\underline{V}}^{\bar{V}} |w_a(z) - w_b(z)|\,dz \leq \tfrac{\omega_2(\bar{V} - \underline{V})}{[u'(\bar{w})]^2}\,\|w_a - w_b\| \end{aligned} \tag{C.4}$$

The coefficient of $\|w_a - w_b\|$ is bounded because $u'(\bar{w}) > 0$ and $\omega_2 < \infty$.

To develop bounds on $|p_a(F_a) - p_b(F_b)|$ and $|E_a - E_b|$, assume $\|w_a - w_b\| = \varepsilon > 0$ with loss of generality. (If $\|w_a - w_b\| = 0$, then $w_a = w_b$ for all V, in which case $|p_a(F_a) - p_b(F_b)| = |E_a - E_b| = \|w_a - w_b\|$; these provide the required bounds.) I examine two cases separately: the case where V is close to $\bar{V}$ and the case where V is sufficiently away from $\bar{V}$. The separation is necessary because $P'(q)$ and $P''(q)$ might be unbounded at $q = \bar{q}$ (i.e., at $V = \bar{V}$).

Consider first the case where V is close to $\bar{V}$. In this case, $F_a(V)$ and $F_b(V)$ are close to $\bar{V}$. Because $p_w(V)$ is continuous for all V, including $V = \bar{V}$. then for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\bar{V} - V < \delta \Longrightarrow \left| p_i(F_i) - p_i(\bar{V}) \right| < \varepsilon/2, \quad \text{for } i \in \{a, b\}$$

Because $p_i(\bar{V}) = 0$, the following holds for $V > \bar{V} - \delta$:

$$|p_a(F_a) - p_b(F_b)| \leq |p_a(F_a)| + |p_b(F_b)| < \varepsilon = \|w_a - w_b\| \tag{C.5}$$

$$|E_a - E_b| \leq |p_a(F_a)|\,(F_a - V) + |p_b(F_b)|\,(F_b - V) < (\bar{V} - \underline{V})\,\|w_a - w_b\| \tag{C.6}$$

For the last inequality, I used the facts that $|p_i(F_i)| < \varepsilon/2$ and that $F_i - V_i \leq \bar{V} - \underline{V}$. (C.5) and (C.6) provide the required bounds when $V > \bar{V} - \delta$.

Now consider the case where $V \leq \bar{V} - \delta$, where $\delta > 0$. In this case, $q < \bar{q}$, and hence Assumption 2 implies that $|P'(q)|$ and $|P''(q)|$ are bounded for $q \in [\underline{q}, \bar{q})$. Because $p(V) = P\left(\frac{k}{J(V)}\right)$, then

$$\left|\frac{dP(k/J)}{dJ}\right| = \left(-\frac{k}{J^2}\right) P'\left(\frac{k}{J}\right)$$
$$\left|\frac{d^2P(k/J)}{dJ^2}\right| = \left(\frac{k}{J_w^3}\right)\left(-\frac{k}{J_w}P'' - 2P'\right)$$

These absolute values are bounded above in the current case. Let A_2 and A_3 be the upper bounds. Define

$$A_4 = A_2 \frac{\omega_2(\bar{V} - \underline{V})}{[u'(\bar{w})]^2} < \infty$$

For any $x \in [\underline{V}, \bar{V} - \delta]$,

$$|p_a(x) - p_b(x)| \leq A_2\,|J_a(x) - J_b(x)| \leq A_4\,\|w_a - w_b\|$$

$$\left|\frac{dP_a}{dJ_a} - \frac{dP_b}{dJ_b}\right| \leq A_3\,|J_a - J_b|$$

These results lead to the following result:

$$\begin{aligned}
|p'_a(x) - p'_b(x)| &\leq \left|\tfrac{dP_a/dJ_a}{u'(w_a)} - \tfrac{dP_b/dJ_a}{u'(w_b)}\right| \\
&\leq \left|\tfrac{dP_a}{dJ_a}\right|\left|\tfrac{1}{u'(w_a)} - \tfrac{1}{u'(w_b)}\right| + \tfrac{1}{u'(w_b)}\left|\tfrac{dP_a}{dJ_a} - \tfrac{dP_b}{dJ_b}\right| \\
&\leq \tfrac{A_2}{[u'(\bar{w})]^2}\,|u'(w_a) - u'(w_b)| + \tfrac{A_3}{u'(\bar{w})}\,|J_a - J_b| \\
&\leq \tfrac{A_4}{\bar{V} - \underline{V}}\,\|w_a - w_b\| + \tfrac{A_4 A_3/A_2}{u'(\bar{w})}\,\|w_a - w_b\|
\end{aligned}$$

Suppose first that $F_a \geq F_b$. If $p_a(F_a) \geq p_b(F_b)$, then

$$0 \leq p_a(F_a) - p_b(F_b) \leq p_a(F_a) - p_b(F_a) \leq A_4\,\|w_a - w_b\|$$

The second inequality comes from the fact that p is decreasing and the last inequality from the bound on $|p_a - p_b|$ just derived. If $p_a(F_a) < p_b(F_b)$, then

$$\begin{aligned}
0 &< p_b(F_b) - p_a(F_a) = -p'_b(F_b)(F_b - V) + p'_a(F_a)(F_a - V) \\
&\leq (F_a - V)\,[p'_a(F_a) - p'_b(F_b)] \leq (\bar{V} - \underline{V})\,[p'_a(F_b) - p'_b(F_b)] \\
&\leq \left[1 + \tfrac{A_3(\bar{V} - \underline{V})}{A_2 u'(\bar{w})}\right] A_4\,\|w_a - w_b\|
\end{aligned}$$

The equality follows from the first-order condition for F, the second inequality from the supposition $F_a \geq F_b$, the third inequality from the facts that p' is a decreasing function and that $F_a - V \leq \bar{V} - \underline{V}$, and the last inequality from the bound on $|p_a' - p_b'|$. Thus, if $F_a \geq F_b$, then

$$|p_a(F_a) - p_b(F_b)| \leq \left[1 + \frac{A_3(\bar{V} - \underline{V})}{A_2 u'(\bar{w})}\right] A_4 \|w_a - w_b\| \tag{C.7}$$

Suppose now that $F_a < F_b$. By switching the roles of F_a and F_b, it can be shown that (C.7) continues to hold. Thus, (C.7) holds for arbitrary $F_a(V)$ and $F_b(V)$ with $V \leq \bar{V} - \delta$.

Now let us examine $|E_a - E_b|$ for the case $V \leq \bar{V} - \delta$. If $E_a \geq E_b$, then

$$\begin{aligned} 0 \quad &\leq E_a - E_b = p_a(F_a)(F_a - V) - p_b(F_b)(F_b - V) \\ &\leq p_a(F_a)(F_a - V) - p_b(F_a)(F_a - V) \\ &= (F_a - V)\left[p_a(F_a) - p_b(F_a)\right] \leq (\bar{V} - \underline{V}) A_4 \|w_a - w_b\| \end{aligned}$$

The first equality comes from the definition of $E(V)$, the second inequality from the fact that $p_b(f)(f - V)$ is maximized at $f = F_b$, the last inequality from the bound on $|p_a - p_b|$ derived above and the fact $F_a - V \leq \bar{V} - \underline{V}$. The same result holds if $E_a < E_b$. Thus,

$$|E_a - E_b| \leq (\bar{V} - \underline{V}) A_4 \|w_a - w_b\| \tag{C.8}$$

Defining $A_5 = \max\{A_4, 1\}$ and replace A_4 in (C.7) and (C.8) with A_5. The resulting bounds on $|p_a - p_b|$ and $|E_a - E_b|$ apply for both the case $V > \bar{V} - \delta$ and $V \leq \bar{V} - \delta$. Substituting these bounds, (C.3) and (C.4), I have:

$$\begin{aligned} |R_a - R_b| \quad &\leq \Big\{ \left[\omega_2 J_a + u'(w_b)\tfrac{A_4}{A_2}\right] \left[\sigma + \lambda_1 p_a(F_a)\right] \\ &\quad + \lambda_1 A_5 \left[u'(w_b)\left(1 + \tfrac{A_3(\bar{V} - \underline{V})}{A_2 u'(\bar{w})}\right) J_b + \lambda_1(\bar{V} - \underline{V})\right] \Big\} \|w_a - w_b\| \end{aligned}$$

Let A_6 be the maximum value of the coefficient of $\|w_a - w_b\|$ in the above expression, taken over $V \in [\underline{V}, \bar{V}]$. Then, A_6 is bounded above. Setting $A = A_6/A_1$ establishes the inequality (C.2), which shows that Γ is continuous in the supnorm. This completes the proof of Lemma C.2, and hence of Theorem 5.2. QED

D. Proofs of Corollary 5.3 and Lemma 6.2

To prove Corollary 5.3, suppose that $|\dot{w}(t)| < \infty$ for all t. That is, $\dot{w}(V(t))$ is finite. If $\dot{V} \neq 0$, then $w'(V) = \dot{w}/\dot{V}$ exists and is finite. If $\dot{V} = 0$ at V_1, then, $\sigma V_1 - u(w(V_1)) - \lambda_1 E(V_1) = 0$. Differentiating this equation with respect to V_1 yields:

$$w'(V_1) = \frac{\sigma + \lambda_1 p(F(V_1))}{u'(w(V_1))} \in (0, \infty). \tag{D.1}$$

That is, $w(V)$ is differentiable at V_1. This argument applies to $\bar{V}$, because $\dot{V} = 0$ at $V = \bar{V}$. Thus, $w'(V_1)$ exists and is finite for all V. From (5.1), (5.2) and Lemma 3.2, one can then verify that $J''(V)$, $p''(V)$ and $F'(V)$ all exist and are finite for all $V < \bar{V}$.

I still need to show that $w'(V) > 0$, $\dot{V} > 0$ and $\dot{J}(V) < 0$ in the case $V < \bar{V}$. In this case, $F(V) < \bar{V}$. Lemma 3.2 implies $dp(F(V))/dV < 0$. The right-hand side of (3.8) is positive and finite, which implies $\dot{w}(V) > 0$. Thus, $w'(V)\dot{V} \in (0, \infty)$ for all $V < \bar{V}$. Because $w(V)$ is strictly increasing for all $V < \bar{V}$ and $\dot{V}$ is bounded (see (3.3)), then $w'(V) \in (0, \infty)$ and $\dot{V} \in (0, \infty)$ for all $V < \bar{V}$. Finally, $\dot{J}(V) = J'(V)\dot{V} \in (0, \infty)$ for all $V < \bar{V}$. This completes the proof of Corollary 5.3.

To prove Lemma 6.2, temporarily denote $D(V) = \Phi(V) - G_u(V)$. Then, $D(v_0) = D(\bar{V}) = 0$. Moreover, computing $D'(V)$ and using (6.7), I have:

$$D'(V) = \frac{\sigma\phi(V)}{1-n}\left[\int_{v_0}^{\bar{V}} \frac{\phi(z)dz}{\sigma + \lambda_0 p(F(z))} - \frac{1}{\sigma + \lambda p(F(V))}\right].$$

Examine the expression in [.]. Because $p(F(V))$ is strictly decreasing in V for all $V < \bar{V}$, the expression is strictly decreasing in V for such V. Similarly, the expression is positive at $V = v_0$ and negative at $V = \bar{V}$. Thus, there exists $v_a \in (v_0, \bar{V})$ such that $D'(V) > 0$ for $v_0 \leq V < v_a$, and that $D'(V) < 0$ for $v_a < V \leq \bar{V}$. Thus, for all $V \in (v_0, \bar{V})$, $D(V) > \min\{D(v_0), D(\bar{V})\} = 0$. QED

E. Proofs of Theorem 6.4 and Corollary 6.5

I prove Theorem 6.4 first. Given the analysis leading to the theorem, it is only necessary to establish (6.11) and (6.12). For $V \in [v_1, v_2)$, $F^{-1}(V) < v_1$, and so (6.10) becomes:

$$g_{e1}(V)\dot{V} = \sigma\Delta(F^{-1}(V)) - \sigma G_{e1}(V) - \lambda_1 \int_{v_1}^{V} p(F(z))dG_{e1}(z) \tag{E.1}$$

Setting $V = v_1$ in (E.1) leads to $g_{e1}(v_1) = 0$. Differentiate (E.1) and divide the result by $\gamma(T(V), 0)$, where γ is defined by (3.6) and T by (3.11). I have:

$$\frac{g'_{e1}(V)\dot{V} + a(V)g_{e1}(V)}{\gamma(T(V), 0)} = \frac{b(V)}{\gamma(T(V), 0)} \tag{E.2}$$

where

$$a(V) = \sigma + \lambda_1 p(F(V)) + \frac{d\dot{V}}{dV}, \quad b(V) = \sigma\frac{d}{dV}\Delta(F^{-1}(V))$$

The definition of $T(V)$ implies $T'(V) = 1/\dot{V}$. Then, it can be verified that the left-hand side of (E.2) is equal to the derivative of the function, $g_{e1}(V)\dot{V}/\gamma(T(V), 0)$, with respect to V. Integrate (E.2) from v_1 to V. Using the fact $\gamma(T(V), 0)/\gamma(T(z), 0) = \gamma(T(V), T(z))$ to rewrite the result, I have (6.11). Since g_e is continuous, taking the limit $V \uparrow v_2$ in (6.11) gives $g_e(v_2)$.

Now examine the case $V \in [v_j, v_{j+1})$, where $j \geq 2$. In this case, $F^{-1}(V) \geq v_1$, and so (6.10) becomes

$$g_{ej}(V)\dot{V} = \sigma\Delta(F^{-1}(V)) - \sigma G_{en}(V) - \lambda_1 \int_{F^{-1}(V)}^{V} p(F(z))dG_e(z) \tag{E.3}$$

I do not add the subscript j to G_e on the right-hand side of the equation because, if $v_j < V < v_{j+1}$, some applicants to values above V come from the interval $[v_j, V)$ while others come from the interval $[F^{-1}(V), v_j)$. Differentiating (6.10) yields:

$$g'_{ej}(V)\dot{V} + a(V)g_{ej}(V) = b(V) + \lambda_1 p(V) g_{e(j-1)}(F^{-1}(V)) \frac{dF^{-1}(V)}{dV} \tag{E.4}$$

where $a(V)$ and $b(V)$ are defined as before. Using the same procedure as the one used to solve for g_{e1} above, I obtain:

$$g_{ej}(V) = \frac{1}{\dot{V}} \int_{v_1}^{V} \gamma(T(V), T(z)) \left\{ \sigma\delta(F^{-1}(z)) + \lambda_1 p(z) g_{e(j-1)}(F^{-1}(z)) \right\} dF^{-1}(z)$$

To obtain (6.12), set $V = v_j$ in the above solution:

$$g_{ej}(v_j) = \frac{1}{\dot{v}_j} \int_{v_1}^{v_j} \gamma(T(v_j), T(z)) \left\{ \sigma\delta(F^{-1}(z)) + \lambda_1 p(z) g_{e(j-1)}(F^{-1}(z)) \right\} dF^{-1}(z)$$

Note that

$$\gamma\left(T(V), T(v_j)\right) \gamma(T(v_j), T(z)) = \gamma(T(V), T(z)).$$

Using this fact and the above formulas for $g_{ej}(V)$ and $g_{ej}(v_j)$, one can compute the left-hand side of (6.12) and show that it is equal to the right-hand side. Because g_e is continuous, then $g_{ej}(v_j) = \lim_{V \to v_j} g_{e(j-1)}(V)$, all j. This completes the proof of Theorem 6.4.

Now, turn to Corollary 6.5. I have shown $g_e(v_1) = 0$ in the above proof. Because $0 < F'(V) \leq 1/2$ for all $V < \bar{V}$, then $dF^{-1}(V)/dV > 0$ for all $V < \bar{V}$. Also, $\dot{V} > 0$ and $\delta(V) > 0$ for all $V < \bar{V}$. These features imply that $b(V) > 0$ for all $V < \bar{V}$. Substituting this result and $g_e(v_1) = 0$ into (E.4) yields $g'_e(v_1) > 0$.

Supposing $F'(\bar{V}) > 0$, I now show that $g_e(\bar{V}) = 0$. The supposition $F'(\bar{V}) > 0$ implies that $dF^{-1}(V)/dV$ is bounded. Because $p(\bar{V}) = 0$, then regardless of whether $\bar{V} = v_2$, the following holds (see (E.2) and (E.4)):

$$g'_e(\bar{V})\dot{V}|_{V=\bar{V}} + a(\bar{V})g_e(\bar{V}) = b(\bar{V})$$

The first term is zero because $\dot{V} = 0$ at $V = \bar{V}$. Since (D.1) holds for $V_1 = \bar{V}$, then

$$\left. \frac{d\dot{V}}{dV} \right|_{V=\bar{V}} = \sigma + \lambda_1 p(F(\bar{V})) - u'(w(\bar{V}))w'(\bar{V}) = 0.$$

This implies $a(\bar{V}) = \sigma$. Because $p(F(\bar{V})) = 0$, then $\delta(\bar{V}) = 0$. In addition, $dF^{-1}(V)/dV$ is finite at $V = \bar{V}$. Thus, $b(\bar{V}) = 0$. The above form of (E.4) at $V = \bar{V}$ becomes $0 = -\sigma g_e(\bar{V})$, i.e., $g_e(\bar{V}) = 0$.

The feature $g'_e(v_1) > 0$ implies that $g_e(v_1 + \varepsilon) > 0$, where $\varepsilon > 0$ is sufficiently small. Because $g_e(V)$ is continuous and $g_e(\bar{V}) = 0$, then $g_e(V)$ must be decreasing when V is close to $\bar{V}$. That is, there exists $\hat{V} \in (v_1, \bar{V})$ such that $g'_e(V) < 0$ for $\hat{V} < V < \bar{V}$. QED

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