Searching for Fractal Structures in the Universal Steenrod Algebra at Odd Primes

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Abstract. Unlike the $p = 2$ case, the universal Steenrod algebra $\mathcal{Q}(p)$ at odd primes does not have a fractal structure that preserves the length of monomials. Nevertheless, when p is odd we detect inside $\mathcal{Q}(p)$ two different families of nested subalgebras each isomorphic (as length-graded algebras) to the respective starting element of the sequence.

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1. Introduction

Let p be any prime. The so-called universal Steenrod algebra $\mathcal{Q}(p)$ is an $\mathbb{F}_p$-algebra extensively studied by the authors (see, for instance, [2]-[12]). On its first appearance, it has been described as the algebra of cohomology operations in the category of H_∞-ring spectra (see [16]). Invariant-theoretic descriptions of $\mathcal{Q}(p)$ can be found in [11] and [15]. When p is an odd prime, the augmentation ideal of $\mathcal{Q}(p)$ is the free $\mathbb{F}_p$-algebra over the set

$$\mathcal{S}_p = \{\, z_{\epsilon,k} \mid (\epsilon, k) \in \{0,1\} \times \mathbb{Z} \,\} \tag{1.1}$$

subject to the set of relations

$$\mathcal{R}_p = \{\, R(\epsilon,k,n),\ S(\epsilon,k,n) \mid (\epsilon,k,n) \in \{0,1\} \times \mathbb{Z} \times \mathbb{N}_0 \,\}, \tag{1.2}$$

where

$$R(\epsilon,k,n) = z_{\epsilon,pk-1-n} z_{0,k} + \sum_{j \geq 0} (-1)^j \binom{(p-1)(n-j)-1}{j} z_{\epsilon,pk-1-j} z_{0,k-n+j}, \tag{1.3}$$

and

$$\begin{aligned} S(\epsilon,k,n) = z_{\epsilon,pk-n} z_{1,k} &+ \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(n-j)-1}{j} z_{\epsilon,pk-j} z_{1,k-n+j} \\ &+ (1-\epsilon) \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(n-j)}{j} z_{1,pk-j} z_{0,k-n+j}. \end{aligned} \tag{1.4}$$

Such relations are known as *generalized Adem relations*. In (1.3) and (1.4), as throughout the paper, binomial coefficients $\binom{a}{b}$ are understood to be 0 if $a < 0$, $b < 0$ or $a < b$.

The algebra $\mathcal{Q}(p)$ is related to many Steenrod-like operations. For instance to those acting on the cohomology of a graded cocommutative Hopf algebra ([6], [14]), or the Dyer-Lashof operations on the homology of infinite loop spaces ([1] and [17]). Details of such connections, at least for $p = 2$, can be found in [5]. In particular, the ordinary Steenrod algebra $\mathcal{A}(p)$ is a quotient of $\mathcal{Q}(p)$. At odd primes, the algebra epimorphism is determined by

$$\zeta : z_{\epsilon,k} \longmapsto \begin{cases} \beta^\epsilon P^k & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

The kernel of the map ζ turns out to be the principal ideal generated by $z_{0,0} - 1$.

All monic monomials in $\mathcal{Q}(p)$, with the exception of $z_\varnothing = 1$ have the form

$$z_I = z_{\epsilon_1,i_1} z_{\epsilon_2,i_2} \cdots z_{\epsilon_m,i_m}, \tag{1.6}$$

where the string $I = (\epsilon_1, i_1; \epsilon_2, i_2; \ldots; \epsilon_m, i_m)$ is the *label* of the monomial z_I. By *length* of a monomial z_I of type (1.6) we just mean the integer m, while the length of any $\rho \in \mathbb{F}_p \subset \mathcal{Q}(p)$ is defined to be 0. Since Relations (1.3) and (1.4) are homogeneous with respect to length, the algebra $\mathcal{Q}(p)$ can be regarded as a graded object.

A monomial and its label are said to be admissible if $i_j \geq p i_{j+1} + \epsilon_{j+1}$ for any $j = 1, \ldots, m-1$. We also consider $z_\varnothing = 1 \in \mathbb{F}_p \subset \mathcal{Q}(p)$ admissible. The set $\mathcal{B}$ of all monic admissible monomials forms an $\mathbb{F}_p$-linear basis for $\mathcal{Q}(p)$ (see [11]).

Through two different approaches, in [8] and [10] it has been shown that $\mathcal{Q}(2)$ has a fractal structure given by a sequence of nested subalgras $\mathcal{Q}_s$, each isomorphic to $\mathcal{Q}$. The interest in searching for fractal structures inside algebras of (co-)homology operations initially arouse in [18], where such structures were used as a tool to establish the nilpotence height of some elements in $\mathcal{A}(p)$. Results in the same vein are in [13].

Recently, in [7] the authors proved that no length-preserving strict monomorphisms turn out to exist in $\mathcal{Q}(p)$ when p is odd. Hence no descending chain of isomorphic subalgebras starting with $\mathcal{Q}(p)$ exists for $p > 2$. Results in [7] did not exclude the existence of fractal structures for proper subalgebras of $\mathcal{Q}(p)$. As a matter of fact, the subalgebras $\mathcal{Q}^0$ and $\mathcal{Q}^1$ generated by the $z_{0,h}$'s and the $z_{1,k}$'s respectively (together with 1) turn out to have self-similar shapes, as stated in our Theorem 1.1, our main result.

Theorem 1.1. *Let p be any odd prime. For any $\epsilon \in \{0,1\}$ there is a chain of nested subalgebras of $\mathcal{Q}(p)$*

$$\mathcal{Q}_0^\epsilon \supset \mathcal{Q}_1^\epsilon \supset \mathcal{Q}_2^\epsilon \supset \cdots \supset \mathcal{Q}_s^e \supset \mathcal{Q}_{s+1}^\epsilon \supset \ldots$$

each isomomorphic to $\mathcal{Q}_0^\epsilon = \mathcal{Q}^\epsilon$ as length-graded algebras.

Theorem 1.1 relies on the existence of two suitable algebra monomorphisms

$$\phi : \mathcal{Q}^0 \longrightarrow \mathcal{Q}^0 \quad \text{and} \quad \psi : \mathcal{Q}^1 \longrightarrow \mathcal{Q}^1. \tag{1.7}$$

Indeed, we just set $\mathcal{Q}^0_s = \phi^s(\mathcal{Q}^0)$ and $\mathcal{Q}^1_s = \phi^s(\mathcal{Q}^1)$, the restrictions $\phi|_{\mathcal{Q}^0_s}$ and $\psi|_{\mathcal{Q}^1_s}$ being the desired isomorphism between $\mathcal{Q}^\epsilon_s$ and $\mathcal{Q}^\epsilon_{s+1}$ ($\epsilon \in \{0,1\}$).

For sake of completeness we point out that the algebra $\mathcal{Q}(p)$ can also be filtered by the internal degree of its elements, defined on monomials as follows:

$$|\rho z_I| = \begin{cases} \sum_h (2i_h(p-1) + \epsilon_{i_h}), & \text{if } I = (\epsilon_1, i_1; \epsilon_2, i_2; \ldots; \epsilon_m, i_m) \\ 0 & \text{if } I = \varnothing. \end{cases} \tag{1.8}$$

In spite of its geometric importance, the internal degree will not play any role here.

We finally recall that the algebra $\mathcal{Q}(p)$ is not of finite type: for $k \geq 0$ the pairwise distinct monomials $z_{0,k}z_{0,-k}$ all have internal degree 0 and length 2, moreover they all belong to the basis $\mathcal{B}$ of monic admissible monomials.

2. A first descending chain of subalgebras

We first need to establish some congruential identities. Let $\mathbb{N}_0$ denote the set of all non-negative integers. Fixed any prime p, we write

$$\sum_{i\geq 0} \gamma_i(m)p^i \quad (0 \leq \gamma_i(m) < p) \tag{2.1}$$

to denote the p-adic expansion of a fixed $m \in \mathbb{N}_0$. The following well-known Lemma is a standard device to compute $\bmod p$ binomial coefficients.

Lemma 2.1 (Lucas' Theorem). *For any $(a,b) \in \mathbb{N}_0 \times \mathbb{N}_0$, the following congruential identity holds.*

$$\binom{a}{b} \equiv \prod_{i\geq 0} \binom{\gamma_i(a)}{\gamma_i(b)} \pmod p. \tag{2.2}$$

Proof. See [13, p. 260] or [19, I 2.6]. Equation 2.2 follows the usual conventions: $\binom{0}{0} = 1$, and $\binom{l}{r} = 0$ if $0 \leq l < r$. □

Congruence (2.2) immediately yields

$$\binom{p^r a}{p^r b} \equiv \binom{a}{b} \pmod p \qquad \text{for every } r \geq 0, \tag{2.3}$$

since, in both cases, we find on the right side of (2.2) the same products of binomial coefficients, apart from r extra factors all equal to $\binom{0}{0} = 1$.

Corollary 2.2. *For any $(\ell, t, h) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \{1, \ldots, p\}$, the following congruential identity holds.*

$$\binom{p\ell - h}{pt} \equiv \binom{\ell - 1}{t} \pmod p. \tag{2.4}$$

Proof. Since $p\ell - h = (p-h) + p(\ell - 1)$, we have $\gamma_0(p\ell - h) = p - h$. Note also that $\gamma_0(pt) = 0$. According to Lemma 2.1, we get

$$\binom{p\ell - h}{pt} \equiv \binom{p-h}{0}\binom{p(\ell-1)}{pt} \pmod p. \tag{2.5}$$

We now use Congruence 2.3 for $r = 1$, and the fact that $\binom{k}{0} = 1$ for all $k \in \mathbb{N}_0$. □

In order to make notation less cumbersome, we set

$$A(k,j) = \binom{(p-1)(k-j)-1}{j}. \tag{2.6}$$

Corollary 2.3. *Let (n,j) a couple of positive integers. Whenever $j \not\equiv 0 \pmod p$, the binomial coefficient $A(pn,j)$ is divisible by p.*

Proof. If a fixed positive integer j is not divisible by p, then there exists a unique couple $(l,h) \in \mathbb{N} \times \{1,\dots,p-1\}$ such that $j = pl - h$. Hence, setting

$$T = (p-1)(n-l) + h - 1,$$

we get

$$A(pn,j) = \binom{(p-h-1)+pT}{(p-h)+p(l-1)} \equiv \binom{p-h-1}{p-h} \cdot \binom{T}{l-1} \pmod p \tag{2.7}$$

by Lemma 2.1 and Equation (2.3). Since $p-h-1 < p-h$, the first factor on the right side of Equation (2.7) is zero, so the result follows. □

Lemma 2.4. *Let (s,n,j) a triple of positive integers. Whenever $j \not\equiv 0 \pmod{p^s}$, the binomial coefficient $A(p^s n, j)$ is divisible by p.*

Proof. We proceed by induction on s. The $s = 1$ case is essentially Corollary 2.3.

Suppose now $s > 1$. The hypothesis on j is equivalent to the existence of a suitable $(b,i) \in \mathbb{N} \times \{1,\dots,p^s - 1\}$ such that $j = p^s b - i$. Likewise, we can write $i = pl - r$, for a certain $(l,r) \in \{1,\dots,p^{s-2}\} \times \{0,\dots,p-1\}$.

We now distinguish two cases. If $r = 0$, the binomial coefficient $A(p^s n, j)$ has the form $\binom{p\ell-h}{pt}$ where

$$\ell = (p-1)(p^{s-1}n - p^{s-1}b + l), \qquad h = 1, \qquad \text{and} \qquad t = p^{s-1}b - l.$$

By Corollary 2.2, we get

$$A(p^s n, j) \equiv A(p^{s-1}n, p^{s-1}b - l) \pmod p,$$

and the latter is divisible by p by the inductive hypothesis.

Assume now $1 \le r \le p-1$. In this case,

$$A(p^s n, j) = \binom{r - 1 + pT'}{r + p(p^{s-1}b - l)} \tag{2.8}$$

where $T' = (p-1)(p^{s-1}n - p^{s-1}b + l) - r$. Therefore, by Lemma 2.1 we get

$$A(p^s n, j) \equiv \binom{r-1}{r} \cdot \binom{T'}{p^{s-1}b - l} \pmod p. \tag{2.9}$$

The right side of Equation 2.9 vanishes, since $r-1<r$, and the proof is over. □

Lemmas and Corollaries proved so far will be helpful to reduce, in some particular cases, the number of potentially non-zero binomial coefficients in (1.3) and in (1.4). For instance, for any $(h,n)\in\mathbb{Z}\times\mathbb{N}_0$, relations of type $R(\epsilon,p^sh-\alpha_s,p^sn)$, where

$$\alpha_s=\frac{p^s-1}{p-1}\qquad(s\geq 1),$$

only involve generators in the set

$$\mathcal{T}_{(\epsilon,s)}=\{z_{\epsilon,p^sm-\alpha_s}\,|\,m\in\mathbb{Z}\}\tag{2.10}$$

as stated in the following Proposition.

Proposition 2.5. *Let* (ϵ,k,n,s) *a fixed* 4*-tuple in* $\{0,1\}\times\mathbb{Z}\times\mathbb{N}_0\times\mathbb{N}$*. The polynomial* $R(\epsilon,p^sk-\alpha_s,p^sn)$ *in* (1.3) *is actually equal to*

$$z_{\epsilon,p^s(pk-1-n)-\alpha_s}z_{0,p^sk-\alpha_s}+\sum_j(-1)^jA(n,j)\,z_{\epsilon,p^s(pk-1-j)-\alpha_s}z_{0,p^s(k-n+j)-\alpha_s}.$$

Proof. By definition (see (1.3)), $R(\epsilon,p^sk-\alpha_s,p^sn)$ is equal to

$$z_{\epsilon,p(p^sk-\alpha_s)-1-p^sn}z_{0,p^sk-\alpha_s}+\sum_l(-1)^lA(p^sn,l)\,z_{\epsilon,p(p^sk-\alpha_s)-1-l}z_{0,p^sk-\alpha_s-p^sn+l}.$$

According to Lemma 2.4, the only possible non-zero coefficients in the sum above occur when $l\equiv 0\pmod{p^s}$. Thus, after setting $l=p^sj$, we write $R(\epsilon,p^sk-\alpha_s,p^sn)$ as

$$\begin{aligned}&z_{\epsilon,p(p^sk-\alpha_s)-1-p^sn}z_{0,p^sk-\alpha_s}+\\&\quad+\sum_j(-1)^{p^sj}A(p^sn,p^sj)z_{\epsilon,p(p^sk-\alpha_s)-1-p^sj}z_{0,p^sk-\alpha_s-p^sn+p^sj}.\end{aligned}\tag{2.11}$$

In such polynomial we can replace $z_{\epsilon,p(p^sk-\alpha_s)-1-p^sn}$ and $z_{\epsilon,p(p^sk-\alpha_s)-1-p^sj}$ by

$$z_{\epsilon,p^s(pk-1-n)-\alpha_s}\quad\text{and}\quad z_{\epsilon,p^s(pk-1-j)-\alpha_s}$$

respectively, since $p\alpha_s+1=p^s+\alpha_s$. Finally, applying Equation (2.4) as many times as necessary, and recalling that we are supposing p odd, we get

$$(-1)^{p^sj}A(p^sn,p^sj)\equiv(-1)^jA(n,j)\pmod p.\tag{2.12}$$

□

As a consequence of Proposition 2.5, the admissible expression of any non-admissible monomial with label $(\epsilon,p^sh_1-\alpha_s;0,p^sh_2-\alpha_s;\ldots;0,p^sh_m-\alpha_s)$ involves only generators in $\mathcal{T}_{(\epsilon,s)}$.

That's the reason why, for any non-negative integer s, there is a well-defined $\mathbb{F}_p$-algebra $\mathcal{Q}^0_s$ generated by the set $\{1\}\cup\mathcal{T}_{(0,s)}$ and subject to relations

$$R(0,p^sh-\alpha_s,p^sn)=0\quad\forall n\in\mathbb{N}_0.$$

Thus $\mathcal{Q}^0_0$ and $\mathcal{Q}^0_1$ are the subalgebras of $\mathcal{Q}(p)$ generated by the sets

$$\{1\}\cup\{z_{0,h}\,|\,h\in\mathbb{Z}\}\quad\text{and}\quad\{1\}\cup\{z_{0,ph-1}\,|\,h\in\mathbb{Z}\}$$

respectively. The former has been simply denoted by $\mathcal{Q}^0$ in Section 1. The arithmetic identity

$$p^{s+1}h - \alpha_{s+1} = p^s(ph-1) - \alpha_s, \tag{2.13}$$

implies that $\mathcal{Q}^0_s \supset \mathcal{Q}^0_{s+1}$.

Lemma 2.6. *A monomial of type*

$$z_I = z_{\epsilon,p^s h_1 - \alpha_s} z_{0,p^s h_2 - \alpha_s} \cdots z_{0,p^s h_m - \alpha_s} \tag{2.14}$$

is admissible if and only if $h_i \geq p h_{i+1}$ *for any* $i = 1, \dots, m-1$.

Proof. Admissibility for a monomial of type (2.14) is tantamount to the condition

$$p^s h_i - \alpha_s \geq p(p^s h_{i+1} - \alpha_s) \quad \forall i \in \{1, \dots, m-1\}.$$

Inequalities above are equivalent to

$$h_i \geq p h_{i+1} - \frac{p^s - 1}{p^s} \quad \forall i \in \{1, \dots, m-1\},$$

and the ceiling of the real number on the right side is precisely $p h_{i+1}$. □

Proposition 2.7. *An* $\mathbb{F}_p$*-linear basis for* $\mathcal{Q}^0_s$ *is given by the set* $\mathcal{B}_{\mathcal{Q}^0_s}$ *of its monic admissible monomials.*

Proof. In [11] it is explained the procedure to express any monomial in $\mathcal{Q}(p)$ as a sum of admissible monomials. As Proposition 2.5 shows, the generalized Adem relations required to complete such procedure starting from a monomial in $\mathcal{Q}^0_s$ only involve generators actually available in the set at hands. □

So far, we have established the existence of the following descending chain of algebra inclusions:

$$\mathcal{Q}^0 = \mathcal{Q}^0_0 \supset \mathcal{Q}^0_1 \supset \mathcal{Q}^0_2 \supset \cdots \supset \mathcal{Q}^0_s \supset \mathcal{Q}^0_{s+1} \supset \dots,$$

On the free $\mathbb{F}_p$-algebra $\mathbb{F}_p\langle\{1\} \cup \mathcal{T}_{(0,0)}\rangle$ we now define a monomorphism Φ acting on the generators as follows

$$\Phi(1) = 1 \qquad \text{and} \qquad \Phi(z_{0,k}) = z_{0,pk-1}. \tag{2.15}$$

We set $\Phi^0 = 1_{\mathbb{F}_p\langle \mathcal{S}_p \rangle}$ and $\Phi^s = \Phi \circ \Phi^{s-1}$ for $s \geq 1$.

Proposition 2.8. *For each* $s \geq 0$,

$$\Phi^s(z_{0,i_1} \cdots z_{0,i_m}) = z_{0,p^s i_1 - \alpha_s} \cdots z_{0,p^s i_m - \alpha_s}, \tag{2.16}$$

and

$$\Phi^s(R(0,k,n)) = R(0, p^s k - \alpha_s, p^s n). \tag{2.17}$$

Proof. Equations (2.16) and (2.17) are trivially true for $s = 0$. For $s \geq 1$ use an inductive argument taking into account (2.13) and Proposition 2.5. □

Proposition 2.9. *Let* $\pi : \mathbb{F}_p\langle\{1\} \cup \mathcal{T}_{(0,0)}\rangle \to \mathcal{Q}^0$ *be the quotient map. There exists an algebra monomorphism* ϕ *such that the diagram*

$$\begin{array}{ccc} \mathbb{F}_p\langle\{1\} \cup \mathcal{T}_{(0,0)}\rangle & \xrightarrow{\Phi} & \mathbb{F}_p\langle\{1\} \cup \mathcal{T}_{(0,0)}\rangle \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{Q}^0 & \xrightarrow[\phi]{\cdots\cdots} & \mathcal{Q}^0 \end{array} \tag{2.18}$$

commutes.

Proof. By Equation (2.17), it follows in particular that

$$\Phi(R(0,k,n)) = R(0, pk-1, pn).$$

Therefore there exists a well-defined algebra map

$$\phi : z_{0,i_1} z_{0,i_2} \cdots z_{0,i_m} \in \mathcal{Q}^0 \longmapsto z_{0,pi_1-1} z_{0,pi_2-1} \cdots z_{0,pi_m-1} \in \mathcal{Q}^0.$$

Such map is injective since the set $\mathcal{B}_{\mathcal{Q}^0_s}$ – an $\mathbb{F}_p$-linear basis for $\mathcal{Q}^0$ according to Proposition 2.7 – is mapped onto admissibles by Lemma 2.6. □

Corollary 2.10. *The algebra* $\mathcal{Q}^0_s$ *is isomorphic to its subalgebra* $\mathcal{Q}^0_{s+1}$.

Proof. By Propositions 2.8 and 2.9, we can argue that $\phi^s(\mathcal{Q}^0) = \mathcal{Q}^0_s$. Hence the map

$$\phi|_{\mathcal{Q}^0_s} : \operatorname{Im} \phi^s \longrightarrow \operatorname{Im} \phi^{s+1}$$

gives the desired isomorphism. □

Corollary 2.10 proves Theorem 1.1 for $\epsilon = 0$.

3. A second descending chain of subalgebras

The aim of this Section is to provide a proof for the $\epsilon = 1$ case of Theorem 1.1. We choose to follow as close as possible the line of attack put forward in Section 2.

Proposition 3.1. *Let* (k,n,s) *a fixed triple in* $\mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}$. *In* (1.4) *the polynomial* $S(1, p^s k, p^s n)$ *is actually equal to*

$$z_{1,p^s(pk-n)} z_{1,p^s k} + \sum_j (-1)^{j+1} A(n,j)\, z_{1,p^s(pk-j)} z_{1,p^s(k-n+j)}.$$

Proof. By definition (see 1.4),

$$S(1, p^s k, p^s n) = z_{1,p^s(pk-n)} z_{1,p^s k} + \sum_l (-1)^{l+1} A(p^s n, l) z_{1,p^{s+1}k-l} z_{1,p^s k - p^s n + l}. \tag{3.1}$$

According to Lemma 2.4, the only possible non-zero coefficients in the sum above are those with $l \equiv 0 \mod p^s$. Setting $l = p^s j$, the polynomial (3.1) becomes

$$z_{1,p^{s+1}k-p^s n} z_{1,p^s k} + \sum_j (-1)^{p^s j+1} A(p^s n, p^s j) z_{1,p^{s+1}k-p^s j} z_{1,p^s k - p^s n + p^s j}.$$

The result now follows from Equation (2.12). □

Proposition 3.1 implies that relations of type $S(1,p^s h,p^s n)$ only involve generators of type $z_{1,p^s m}$. therefore the admissible expression of any non-admissible monomial with label $(1,p^s h_1;1,p^s h_2;\dots;1,p^s h_m)$ only involves generators in the set

$$\mathcal{T}'_{(1,s)} = \{z_{1,p^s m} \,|\, m \in \mathbb{Z}\}. \tag{3.2}$$

So it makes sense to define $\mathcal{Q}^1_s$ as the $\mathbb{F}_p$-algebra generated by the set $\{1\} \cup \mathcal{T}'_{(1,s)}$ and subject to relations

$$S(1,p^s h,p^s n) = 0 \quad \forall n \in \mathbb{N}_0.$$

Each $\mathcal{Q}^1_s$ is actually a subalgebra of $\mathcal{Q}(p)$. We have inclusions $\mathcal{Q}^1_s \supset \mathcal{Q}^1_{s+1}$. In Section 1, the algebra $\mathcal{Q}^1_0$ has been simply denoted by $\mathcal{Q}^1$.

Lemma 3.2. *A monomial of type*

$$z_{1,p^s h_1} z_{1,p^s h_2} \cdots z_{1,p^s h_m} \tag{3.3}$$

in $\mathcal{Q}^1_s \subset \mathcal{Q}(p)$ is admissible if and only if $h_i \geq p h_{i+1} + 1 \quad \forall i \in \{1,\dots,m-1\}$.

Proof. By definition, the monomial (3.3) is admissible if and only if

$$p^s h_i \geq p(p^s h_{i+1}) + 1 \quad \forall i \in \{1,\dots,m-1\}.$$

Inequalities above are equivalent to

$$h_i \geq p h_{i+1} + \frac{1}{p^s} \quad \forall i \in \{1,\dots,m-1\},$$

and the ceiling of the real number on the right side is precisely $p h_{i+1} + 1$. □

Proposition 3.3. *An $\mathbb{F}_p$-linear basis for $\mathcal{Q}^1_s$ is given by the set $\mathcal{B}_{\mathcal{Q}^1_s}$ of its monic admissible monomials.*

Proof. Follows verbatim the proof of Proposition 2.7, just replacing "Proposition 2.5" by "Proposition 3.1" and $\mathcal{Q}^0_s$ by $\mathcal{Q}^1_s$. □

We are now going to prove that the subalgebras in the descending chain

$$\mathcal{Q}^1 = \mathcal{Q}^1_0 \supset \mathcal{Q}^1_1 \supset \mathcal{Q}^1_2 \supset \cdots \supset \mathcal{Q}^1_s \supset \mathcal{Q}^1_{s+1} \supset \dots,$$

are all isomorphic. To this aim we consider the injective endomorphism Ψ on the free $\mathbb{F}_p$-algebra $\mathbb{F}_p\langle\{1\} \cup \mathcal{T}'_{1,0}\rangle$ by setting

$$\Psi(1) = 1 \qquad \text{and} \qquad \Psi(z_{1,k}) = z_{1,pk}. \tag{3.4}$$

Proposition 3.4. *Let $\pi' : \mathbb{F}_p\langle\{1\} \cup \mathcal{T}'_{(1,0)}\rangle \to \mathcal{Q}^1$ be the quotient map. There exists an algebra monomorphism ψ such that the diagram*

$$\begin{array}{ccc} \mathbb{F}_p\langle\{1\} \cup \mathcal{T}'_{(1,0)}\rangle & \xrightarrow{\ \Psi\ } & \mathbb{F}_p\langle\{1\} \cup \mathcal{T}'_{(1,0)}\rangle \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathcal{Q}^1 & \xrightarrow[\psi]{\cdots\cdots} & \mathcal{Q}^1 \end{array} \tag{3.5}$$

commutes.

Proof. Since $\Psi^s(z_{1,i_1}\cdots z_{1,i_m}) = z_{1,p^s i_1}\cdots z_{1,p^s i_m}$, by Proposition 3.1 we argue that

$$\Psi^s(S(1,k,n)) = S(1,p^s k, p^s n). \tag{3.6}$$

Therefore there exists a well-defined algebra map

$$\psi : z_{1,i_1}\cdots z_{1,i_m} \in \mathcal{Q}^1 \longmapsto z_{1,pi_1}\cdots z_{1,pi_m} \in \mathcal{Q}^1.$$

Such map is injective since the set $\mathcal{B}_{\mathcal{Q}^1_s}$ – an $\mathbb{F}_p$-linear basis for $\mathcal{Q}^1$ according to Proposition 3.3 – is mapped onto admissibles by Lemma 3.2. □

Corollary 3.5. *The algebra $\mathcal{Q}^1_s$ is isomorphic to its subalgebra $\mathcal{Q}^1_{s+1}$.*

Proof. By Equation (3.6) and Proposition 3.4, we can argue that $\psi^s(\mathcal{Q}^1) = \mathcal{Q}^1_s$. Thus, the desired isomorphism is given by

$$\psi|_{\mathcal{Q}^1_s} \colon \operatorname{Im}\psi^s \longrightarrow \operatorname{Im}\psi^{s+1}.$$

□

4. Further substructures

For each $s \in \mathbb{N}_0$, we define V_s to be the $\mathbb{F}_p$-vector subspace of $\mathcal{Q}(p)$ generated by the set of monomials

$$\mathcal{U}_s = \{z_{1,p^s h_1-\alpha_s} z_{0,p^s h_2-\alpha_s}\cdots z_{0,p^s h_m-\alpha_s} \mid m \geq 2,\ (h_1,\dots,h_m) \in \mathbb{Z}^m\}.$$

Equation 2.13 implies that $V_s \supset V_{s+1}$. None of the V_s's is a subalgebra of $\mathcal{Q}(p)$, nevertheless, by Proposition 2.5 and the nature of relations (1.3) it follows that V_s can be endowed with a right $\mathcal{Q}^0_s$-module structure just by considering multiplication in $\mathcal{Q}(p)$. By using once again Lemma 2.6 and the argument along the proof of Proposition 2.7, we get

Proposition 4.1. *An $\mathbb{F}_p$-linear basis for V_s is given by the set $\mathcal{B}_{V_s}$ of its monic admissible monomials.*

Proposition 4.2. *The map between sets*

$$z_{1,i_1} z_{0,i_2}\cdots z_{0,i_m} \in \mathcal{U}_0 \ \longmapsto\ z_{1,pi_1-1} z_{0,pi_2-1}\cdots z_{0,pi_m-1} \in \mathcal{U}_0$$

can be extended to a well-defined injective $\mathbb{F}_p$-linear map $\lambda : V_0 \longrightarrow V_0$. Moreover

$$\lambda^s(V_0) = V_s \subset V_0. \tag{4.1}$$

Proof. As in the proof of Proposition 2.8, Equation 2.13 and Proposition 2.5 show that the s-th power of the $\mathbb{F}_p$-linear map

$$\Lambda : z_{\epsilon_1 1,i_1} z_{\epsilon_2,i_2}\cdots z_{\epsilon_m,i_m} \in \mathbb{F}_p\langle\mathcal{S}_p\rangle \longmapsto z_{\epsilon_1,pi_1-1} z_{\epsilon_2,pi_2-1}\cdots z_{\epsilon_m,pi_m-1} \in \mathbb{F}_p\langle\mathcal{S}_p\rangle$$

maps the polynomial $R(\epsilon,k,n) \in \mathbb{F}_p\langle\mathcal{S}_p\rangle$ onto $R(\epsilon, p^s k - \alpha_s, p^s n)$. Hence there are two maps $\bar{\Lambda}$ and λ such that the diagram

$$\begin{array}{ccc} \mathbb{F}_p\langle\mathcal{S}_p\rangle & \xrightarrow{\Lambda} & \mathbb{F}_p\langle\mathcal{S}_p\rangle \\ \uparrow & & \uparrow \\ \mathbb{F}_p\langle\mathcal{U}_0\rangle & \overset{\bar{\Lambda}}{\dashrightarrow} & F_p\langle\mathcal{U}_0\rangle \\ \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi''} \\ V_0 & \overset{\lambda}{\dashrightarrow} & V_0 \end{array} \tag{4.2}$$

commutes, where $\pi'' : \mathbb{F}_p\langle\mathcal{U}_0\rangle \to V_0$ is the quotient map. Finally, taking into account Equation 2.13, one checks that

$$\lambda^s(z_{1,i_1} z_{0,i_2} \cdots z_{0,i_m}) = z_{1,p^s i_1 - \alpha_s} z_{0,p^s i_2 - \alpha_s} \cdots z_{0,p^s i_m - \alpha_s}. \tag{4.3}$$

Since Equation (4.3) implies (4.1), the proof is over. □

We now introduce a category $\mathcal{K}$ whose objects are couples (M, R), with R being any ring, and M any right R-module. A morphism between two objects (M, R) and (N, S) is given by a couple (f, ω) where $f : M \to N$ is a group homomorphism and $\omega : R \to S$ is a ring homomorphism, furthermore

$$f(mr) = f(m)\,\omega(r) \qquad \forall (m, r) \in (M, R).$$

The category $\mathcal{K}$ is partially ordered by "inclusions". More precisely we say that

$$(M, R) \subseteq (M', R')$$

if M is a subgroup of M' and R is a subring of R'.

Theorem 4.3. *The objects in $\mathcal{K}$ of the descending chain*

$$(V_0, \mathcal{Q}_0^0) \supset (V_1, \mathcal{Q}_1^0) \supset \cdots \supset (V_s, \mathcal{Q}_s^0) \supset (V_{s+1}, \mathcal{Q}_{s+1}^0) \supset \ldots$$

are all isomorphic.

Proof. By Proposition 4.2 it follows that $\lambda|_{V_s} : V_s \longrightarrow V_{s+1}$ is an isomorphism between $\mathbb{F}_p$-vector spaces. Thus, recalling Corollary 2.10, the desired isomorphism in $\mathcal{K}$ is given by

$$(\lambda|_{V_s}, \phi|_{\mathcal{Q}_s^0}) : (V_s, \mathcal{Q}_s^0) \longrightarrow (V_{s+1}, \mathcal{Q}_{s+1}^0).$$

□

5. A final remark

Theorem 1.1 in [7] says that no strict algebra monomorphism in $\mathcal{Q}(p)$ exists when p is odd. Hence there is no chance to find algebra endomorhisms over $\mathcal{Q}(p)$ extending the maps ϕ and ψ defined in Sections 2 and 3 respectively. Just to give an idea about the obstructions you come up with, consider the $\mathbb{F}_p$-linear map

$$\Theta : \mathbb{F}_p\langle\mathcal{S}_p\rangle \longrightarrow \mathbb{F}_p\langle\mathcal{S}_p\rangle$$

defined on monomials as follows

$$\Theta(z_{\epsilon_1,i_1}\cdots z_{\epsilon_m,i_m}) = z_{\epsilon_1,pi_1}\cdots z_{\epsilon_m,pi_m}.$$

Neither the map Θ nor the map Λ introduced in Section 4 stabilizes the entire set (1.2). Indeed, take for instance

$$R(0,0,0) = z_{0,-1}z_{0,0} \quad \text{and} \quad S(1,0,0) = z_{1,0}z_{1,0}.$$

The polynomial

$$\Theta(R(0,0,0)) = z_{0,-p}z_{0,0} \tag{5.1}$$

does not belong to the set $\mathcal{R}_p$. In fact, the only polynomial in $\mathcal{R}_p$ containing (5.1) as a summand is

$$R(0,0,p-1) = z_{0,-1-(p-1)}z_{0,0} + z_{0,-1}z_{0,-p+1}.$$

Similarly, the polynomial

$$\Lambda(S(1,0,0)) = z_{1,-1}z_{1,-1}$$

does not belong to the set $\mathcal{R}_p$, since it consists of a single admissible monomial, whereas each element in $\mathcal{R}_p$ always contains a non-admissible monomial among its summands.

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