

# The Barber, Russell's Paradox, Catch-22, God, Contradiction and More

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## Introduction

When Crispin Wright, talking about Wittgenstein's views on consistency, says that 'the impression is not so much that of ordinary attitudes or assumptions questioned, as of good sense outraged' (Wright 1980, p.295), it is easy to see what he means. Wittgenstein speaks of the 'superstitious dread and veneration by mathematicians in face of a contradiction' and recommends that, instead of adopting this attitude, people might have *wanted* to produce a contradiction and 'would be glad to lead their lives in the *neighbourhood* of a contradiction' (Wittgenstein 1978: App. III-17; III-81). Gödel thought such claims 'nonsense' and few authors have dissented from that verdict. Wright cites a number of other passages from the *Remarks on the Foundations of Mathematics* where Wittgenstein makes a series of *prima facie* outrageous remarks about contradictions. Perhaps the most striking remark he makes is that *they are not false*. This claim first appears in his early notebooks (Wittgenstein 1960:108). In the *Tractatus*, Wittgenstein argued that contradictions (like tautologies) are not statements (Sätze) and hence are not false (or true). This is a consequence of his theory that genuine statements are pictures.<sup>1</sup>

The law of non-contradiction (LNC) may be formulated nonformally as follows:

(LNCN) The conjunction of a proposition and its negation is never true.

A question that can be debated is whether (LNCN) is itself true. A certain type of degree-of-truth theorist would answer that, since contradictions can be true to degree .5 or less, then (LNCN), if it is saying that that no conjunction of a proposition and its negation is ever true to any degree, is false. A dialetheist would say of (LNCN) that it is not true (to any degree). The early Wittgenstein, as we have just seen, would say that (LNCN) is true, this conclusion being a product of his distinctive logico-metaphysical system. The later Wittgenstein allows that some contradictions are true, but, as was his wont, argued that the question about (LNCN) rested on a false presupposition.

The view that contradictions lack content is not without historical precedent. A very distinguished list of subscribers includes Aristotle, Boethius and Abelard (Sylvan 1999: 316). Nor, despite its initial strangeness, is it wholly lacking in appeal. For example, adopting this deviant view of contradiction allows us to reject the classical ‘spread’ principle *ex falso quodlibet* (which licenses the inferring of any proposition from a contradiction), a principle that many have found deeply disturbing. From what lacks content, nothing that possesses content can be inferred. And the view comports nicely with a widely held view about propositions — that a proposition is the set of worlds in which it is true — for a contradiction is true in no world, and is thus to be identified with the empty set, in contrast to a false proposition, which is true in worlds other than this one.

Again, no object can both satisfy and fail to satisfy a certain predicate, or to satisfy a predicate if and only if it does not. It is, therefore, natural to say that ‘ $Fa \leftrightarrow \sim Fa$ ’ is necessarily about nothing and hence fails to state anything about something and hence is no statement. This squares satisfyingly with the view famously defended

by P.F. Strawson, that sentences with *contingently* vacuous descriptions as subjects fail to yield statements.

Despite these preliminary considerations in its favour, the claim that contradictions are not statements and lack truth-value is highly non-standard, and is liable to meet with strong resistance. I want here to do something to resist that resistance. My strategy will be to approach (LNC) by a circuitous route. I shall first look at some paradoxes, and shall try to show that an appealing solution to them depends on the claim that contradictions, and bi-conditionals of the form 'p iff not-p' are *not false*. Then I shall draw upon late-Wittgensteinian and other considerations to demonstrate, quite independently of the paradoxes, that this claim is plausible, and thus that a solution to these paradoxes is within our grasp.

### **Paradoxes as contradictions immersed in noise**

Were I to tell you that there is a barber, the sole survivor of a plague in a certain village, who shaves himself if and only if he does not shave himself, you would be right to reply 'No there isn't — there cannot be — it is a logical impossibility'. No individual can satisfy the condition:  $(S_{xx} \leftrightarrow \sim S_{xx})$ . This is not the same as saying that I told you something false about a certain barber. The Standard Barber Paradox speaks of a village barber who shaves all and only those male villagers of Alcala who do not shave themselves. Suppose there to be  $n$  male villagers,  $v_1, \dots, v_n$  and the barber  $b$ . We are given (with  $x$  ranging over the male villagers)

$$(x)(S_{bx} \leftrightarrow \sim S_{xx})$$

Expanded out, this becomes

$$(Sbv_1 \leftrightarrow \sim S_{v_1v_1}) \ \& \ (Sbv_2 \leftrightarrow \sim S_{v_2v_2}) \ \& \ \dots \ \& \ (Sbv_n \leftrightarrow \sim S_{v_nv_n}) \ \& \ (Sbb \leftrightarrow \sim Sbb)$$

or, more shortly,

$$(Sbb \leftrightarrow \sim Sbb) \ \& \ \text{NOISE}$$

The noise diminishes as plague ravages the village and its population dwindles. Now, just as, in the case of the ‘Lone Barber’ version, where we were right to say that there can be no such barber, so here we should say exactly the same — the presence of NOISE is only a distraction; since there is no individual  $b$  satisfying ‘ $(Sbb \leftrightarrow \sim Sbb)$ ’, there is *a fortiori* no such individual satisfying ‘ $(Sbb \leftrightarrow \sim Sbb) \ \& \ \text{NOISE}$ ’.

The Epimenidean Liar is just a noisy version of the Eubulidean. If to the Eubulidean Liar (‘This statement is false’) one adds the noise ‘and all other statements made by me and my fellow Cretans are false’, one obtains the Epimenidean Liar (‘All Cretans are liars’, spoken by Epimenides, the Cretan). With paradoxes, it is often the case that examining the stripped down, noiseless versions helps us see to the heart of the problem. For example, it is useful to consider the ‘Surprise Examination’ paradox in a reduced form in which a teacher says to the class ‘There will be a surprise examination tomorrow’.

Most who have written on the subject agree that the conclusion to be drawn from the Standard Barber is that there is no barber answering to the description given, but most argue that it is just a matter of empirical fact that this is so.<sup>2</sup> Yet, as we have seen, it is a *logical* impossibility that there should be such a barber. Were this not the

case, there would be an unexplained asymmetry between the Standard Barber and the Russell Paradox despite the well-known fact that both paradoxes have a common structure: to obtain the Russell from the Standard Barber, substitute 'R' ('the Russell Class') for 'b', and 'x ∈ y' for 'Sx'

It has been overlooked, in the literature, that Russell's Paradox is not *essentially* infinitistic. Think of a universe in which there are just 4 physical objects, hence 16 sets of physical objects, together possibly with the set J of all those sets of physical objects which are non-self membered. If J exists, it contains all those 16 sets, but does it also contain itself as a member? To claim that there is such a set as J, we may write

$$1. (\exists!x)(y)(y \in x \leftrightarrow \sim(y \in y))$$

with 'y' ranging over all the classes in the given universe. If the 16 sets are named  $a_1, a_2, \dots, a_{16}$ , then 1. can be 'expanded out' as

$$2. (a_1 \in J \leftrightarrow \sim(a_1 \in a_1)) \& (a_2 \in J \leftrightarrow \sim(a_2 \in a_2)) \& \dots \& (J \in J \leftrightarrow \sim(J \in J))$$

Inspection of the last conjunct reveals that it is a rotten apple of the form 'p ↔ ~p', and the above expansion can be usefully contracted down to

$$(J \in J \leftrightarrow \sim(J \in J)) \& \text{NOISE}$$

A rotten apple spoils the whole barrel. What, at first sight, looked like a definition of J, a statement specifying the membership conditions for J, namely

$$3. (\forall y)(y \in J \leftrightarrow \sim (y \in y))$$

turns out to be not true and hence fails to define anything. Equally, the Russell class characterised by

$$4. (\forall y)(y \in R \leftrightarrow \sim (y \in y))$$

with 'y' ranging over all classes, is not true (for, when expanded out, it contains a contradictory clause) and hence is not a definition at all. No Russell class is defined, so the question does not arise as to whether *it* is or is not a member of itself.

It is one thing to say — and it is provable, and it is agreed on all sides — that the Russell Class does not exist, but quite another thing to see *why* it does not. But we now have the beginnings of an explanation — the biconditional purportedly defining the Russell class fails to satisfy a basic condition for being a definition because it is not true; it is of the form '(p ↔ ~p) & NOISE'. There cannot be a *false definition* or a definition that is neither true nor false.

As we have seen, the existence of the Barber of Alcalá and that of the Russell Class are both ruled out *a priori*. Yet it is true that there is generally a greater reluctance to accept the latter result. Why? Well, it is clear that there are non-self-membered classes — the class of horses, and the class of prime numbers are examples — and it may seem that nothing can prevent us collecting all such classes into a class called the Russell Class. That's how it may seem, but it is not the case. A comparison with the Standard Barber will, again, be instructive. If Miguel is an inhabitant of Alcalá, and he does not shave himself, then one might think that he must be shaved by

the village barber. But that is a mistaken thought. We have proved that there is no such barber, and hence Miguel is not 'his' client, and he remains unshaved. Similarly, since there can be nothing meeting the specification of the Russell Class, no such class exists for the class of horses and the class of primes to be members of. It is not, note, that we have succeeded in specifying a class that is empty; we have failed to specify a class.

Make the following substitutions in 4.:

'applies to\*' for ' $\epsilon$ '

'heterological' for 'R'

(where 'applies to\*' is the converse of the relation 'applies to') and have 'y' range over names of predicates. The result is Grelling's Paradox. Thus Grelling's Paradox and Russell's are structurally identical to the Standard Barber and may be handled in exactly the same way: there is no well-defined property heterologicality corresponding to the adjective 'heterological', just as there is no barber corresponding to the description 'the barber who shaves all and only those who do not shave themselves'. (And there is no number answering the description 'the greatest prime number'.)

In each of the paradoxes considered above (the Barber, Russell's and Grelling's), what seemed, at first sight, to be a specifying condition turned out to be a biconditional specifying nothing. In each case there is a *prima facie* plausible assumption that something exists which corresponds to a given specification. Once it is revealed that we do not have a true specification, we give up (and see why we should give up) the existence assumption. This diagnosis applies to an array of apparently diverse puzzles and paradoxes.

## Catch-22

Joseph Heller presents a conundrum so pleasing and amusing that to subject it to an analysis betrays a nerdishness for which I can only apologize. Here's Heller:

'You mean there's a catch?'

'Sure there's a catch,' Doc Daneeka replied. 'Catch-22. Anyone who wants to get out of combat duty isn't really crazy.'

There was only one catch and that was Catch-22, which specified that a concern for one's own safety in the face of dangers that were real and immediate was the process of a rational mind. Orr was crazy and could be grounded. All he had to do was ask; and as soon as he did, he would no longer be crazy and would have to fly more missions. Orr would be crazy to fly more missions and sane if he didn't, but if he was sane he had to fly them. If he flew them he was crazy and didn't have to; but if he didn't want to he was sane and had to. Yossarian was moved very deeply by the absolute simplicity of this clause of Catch-22 and let out a respectful whistle.

'That's some catch, that Catch-22,' he observed.

'It's the best there is,' Doc Daneeka agreed.<sup>3</sup>

It looks as if an airman can avoid flying dangerous missions (A) on condition and only on condition that he is insane (I).

1.  $(x)(Ax \leftrightarrow Ix)$



All you need do is to establish your insanity. Now, it defines you as being insane if you *don't* request to be spared flying such missions (R):

$$2. (x)(Ix \leftrightarrow \sim Rx)$$

But you cannot be spared flying dangerous missions unless you request it:

$$3. (x)(\sim Rx \leftrightarrow \sim Ax)$$

Now, 1.,2. and 3. jointly entail

$$4. (x)(Ax \leftrightarrow \sim Ax) \text{ — one can avoid flying dangerous missions if and only if one cannot avoid it.}$$

Thus Catch-22 boils down to a biconditional of the sort that we have already encountered in paradoxes. Contrary to first appearances, airmen are not presented with a specification of the condition they have to meet in order to avoid flying dangerous missions, but merely with an empty form of words which specifies no condition at all.<sup>4</sup> I propose to call biconditionals of the form ' $p \leftrightarrow \sim p$ ' *vacuous*. A vacuous bicondition is clearly not the same as a condition that cannot be satisfied, such as 'You can avoid flying dangerous missions if and only if you can trisect an arbitrary angle using only straightedge and compass'; a vacuous biconditional just does not amount to the expression of any condition. The catch is this: what looks like a statement of the conditions under which an airman can be excused flying dangerous missions reduces not to the statement

(i) 'An airman can be excused flying dangerous missions if and only if Cont' (where 'Cont' is a contradiction)

(which could be a mean way of disguising an unpleasant truth), but to the worthlessly empty announcement

(ii) 'An airman can be excused flying dangerous missions if and only if it is not the case that an airman can be excused flying dangerous missions'

If the catch were (i), that would not be so bad – an airman would at least be able to discover that under no circumstances could he avoid combat duty. But Catch-22 is worse – a welter of words that amounts to nothing; it is without content, it conveys no information at all. (i) would be devilish, but (ii), like the characters in the book, and the plot, is zany. It does not state a truth or a falsity about the conditions under which danger can be avoided; on the Wittgensteinian view, it states nothing at all, though it has meaning, can be understood and may have the perlocutionary effect of engendering confusion. Catch-22 is an elaborate oxymoron.

### **Protagoras and Euathlus**

The ancient paradox of Protagoras and Euathlus turns out, perhaps surprisingly, to be related to Catch-22. The situation here is that Protagoras, the father of Sophistry, puts his pupil Euathlus through a training in law, and agrees not to be paid any fee for the instruction until Euathlus wins his first case. Euathlus, completes the course of instruction, but then, indolently, takes no cases. Eventually Protagoras

gets frustrated at not being paid, and sues him. So Euathlus's first case is this one — defending himself against Protagoras' suit. If Euathlus loses the case then, by the agreement he made with Protagoras, he does not have to pay him (for he has to pay only after his first *win*). However, if Euathlus wins, that means that Protagoras loses his suit to be paid; in other words, Euathlus does not have to pay him. It seems that Protagoras cannot recover his fee. On the other hand, it seems that Protagoras must recover his fee for, if he wins the suit, the court will order in his favour, but if he loses — i.e., if Euathlus wins — then, by the terms of their agreement, he gets paid. This paradox is somewhat simpler than Catch-22. For here there is a tension between just two conditions — the one generously agreed to by Protagoras, that he gets paid if and only if Euathlus wins:

$$5. \sim P \leftrightarrow \sim W$$

and the penalty code of the court which, in this particular case, enjoins

$$6. P \leftrightarrow \sim W$$

(where 'W' stands for 'Euathlus wins' and 'P' for 'Protagoras gets paid'). These two conditions entail

$$7. P \leftrightarrow \sim P$$

From this vacuous biconditional, we can, on the Wittgensteinian view, infer nothing; in particular, we cannot infer that Protagoras can or that he cannot recover his fee.

This seems correct. The case could be decided either by the court's rule or by Protagoras' rule. But, since these rules are in conflict, it cannot be decided by both together. In the same way, a football match could not get started were it bound by both rules 'The side winning the toss kicks off' and 'The side that loses the toss kicks off'. Note again our departure from classical principles, for, in classical logic, from 'p  $\leftrightarrow$  ~p', *everything* can be inferred.

### **God: a supposed proof of His non-existence**

Consider next a spoof proof of the non-existence of God, which starts from consideration of the sentence 'God can create a stone so heavy that He cannot lift it'. If this sentence (that we shall call the unliftability sentence) is false, then there is something that God cannot do — create the stone; if the sentence is true, then there is something that God cannot do — lift it. Either way, there is something that God cannot do, and this shows that the Judaeo-Christian God, defined *inter alia* by His omnipotence, does not exist. This is regarded as a paradoxical result because, to put it baldly and puritanically, so strong a conclusion just shouldn't be obtained with such little effort.

What creates this paradox is the unliftability claim that there is an omnipotent being — God — who can do everything, including creating a stone so heavy that He cannot lift it. The claim, suitably paraphrased, may be symbolized:

8.  $(\exists!x)(y)(x \text{ can do } y \ \& \ \sim (x \text{ can lift a certain stone}))$

(where 'y' ranges over tasks and 'x' over task-doers) and this entails the contradiction

X can lift a certain stone and X cannot lift that stone.

There is obviously a connection between vacuous biconditionals and contradictions. In our discussion of the paradox of Protagoras and Euathlus, we showed that the merging of Protagoras' rule and the court's rule issues in a vacuous biconditional ' $P \leftrightarrow \sim P$ '. One could continue this line of thought as follows: Given that biconditional, and accepting that Protagoras must either get paid or not get paid (no middle way), we infer the contradiction 'Protagoras gets paid and Protagoras does not get paid'. The unliftability sentence also reduces, as we have just seen, to a contradiction. It seems most natural to say that contradictions too (unless they are being used for one or other rhetorical purpose) are vacuous, not false. And this is exactly the view that Wittgenstein defended throughout his philosophical career.

From the apparent truth that it is either true or false that God can create a stone so heavy that He cannot lift it, a conclusion is apparently validly inferred that contradicts the claim that God is omnipotent. Some might want to say that this does indeed give us good reason to deny the existence of an omnipotent being. Others might want to say that, by *ex falso quodlibet*, we can infer anything, including the existence of an omnipotent being. At this point, we could either go 50-50 or dial a theist. The dialetheist might respond to the derivation of the contradiction by asserting that the claim that the unliftability sentence is either true or false is itself one of those claims that is both true and false. But, if the Dialetheist maintains that it is true (let's isolate that from the 'and false' part) then it seems that he and she will derive, too cheaply, an assurance of God's existence.

There is a better alternative. On the Wittgensteinian view, 8. is an illicit, because contradictory, specification; no individual can satisfy it. But equally, no individual can satisfy the condition of being able to paint a vase both red and green all

over. Failure to do the impossible is not a real failing, hence nothing so far tells against the possibility of a God who is omnipotent, in the sense of being able to do everything that is possible. And this seems to be all that can be properly extracted from the Paradox of Omnipotence. On the Wittgensteinian account, the contradiction is without content and *nothing* can be inferred from it. One way of solving a paradox is to show that the plausibility of one of the premises can be undermined. In the case of the Paradox of Omnipotence, the relevant premise is that the unliftability sentence is either true or false. Hence, if we can convince ourselves that the unliftability sentence is vacuous, and thus *neither* true nor false, then the paradox is solved.

### **The Liar**

Finally, the Liar Paradox. Where 'S' is the name of a statement, the statement 'S is not true' obviously has a truth-value different from (classically: opposite to) that of S. We can, therefore, no more *identify* S with 'S is not true' than we can identify 2 with -2. No such stipulation is admissible. The letter 'S' was, of course, one of any that could have been used instead in this argument. The conclusion we just drew can be formulated without the use of any particular letter — it is the conclusion that no statement can state of itself that it is not true. So, initial appearances to the contrary, 'This statement is not true' is not a statement; it states nothing; in particular, it does not state that it is not true. Do not be fooled by the presence of the phrase 'This statement' — the description 'the number four less than itself' does not describe a number. Similarly, the token of the *sentence* 'S is not true' mentioned above does not yield a statement; it has no truth-value. It has a character, but no content; it is discontent.<sup>5</sup>

The argument just given concerned statements, not sentences. A sentence is the material typically used to make contentful acts of speech; the sentence by itself (i.e. not in use) does not have a truth-value. We can think of a statement as a sentence together with an interpretation. But, where we have a sentence consisting of a singular term followed by 'is not true', then that singular term can be given no consistent interpretation if it is also styled as the name of the putative statement.

It is sometimes said that the meaning of a sentence is its truth-conditions. That cannot be correct, otherwise the meaning of all tautologies would be the same, and that of all contradictions the same, and, on the usual understanding of 'meaning', 'Either he is heavy or not heavy' differs in meaning from 'Either 7 is greater than 3 or it is not' — they would translate differently into German, for example. What can be said, however, is that the content of an utterance may be given by stating its truth conditions. The content of 'Schnee ist weiss' can be explained to someone who knows no German by telling him or her that 'Schnee ist weiss' is true in all situations (in all possible worlds) in which snow is white. So, where 'C' is a name of the target statement to be interpreted, we give someone the content of C by means of the equivalence

C is true iff p

Where 'p' is in a language intelligible to the hearer and has the same truth-conditions as C. Now, let 'S' be a name of 'S is not true'

Then, following the above prescription, we specify the content of S by

(SpecS) S is true iff S is not true.

And, on the Wittgensteinian view recommended above, (SpecS) is not false, for it says nothing; in particular, it assigns no content to S. So, if there is no independent way of assigning content to S, we can say that S too has no content.

Let us put the point in a slightly different way. Suppose that we have a statement A to the effect that some other statement B is not true. If we know the truth-conditions of B, then the truth-conditions of A can be specified as follows:

A is true if and only if B is not true

But now consider

S is true if and only if S is not true

This is vacuous and so specifies nothing. Yet note that this *would be* the result of specifying the truth-conditions for S, where S is to the effect that S is not true, i.e., for the (strengthened) Liar. It follows that, while there is a Liar *sentence*, there is no Liar *statement*, no truth-valued claim made by that sentence. Thus the Liar paradox, which starts with the assumption that there is such a statement, cannot get off the ground.

The Liar Paradox trades on the mistaken assumption that 'This statement is not true' *does* state that it itself is not true. The argument goes: "'This statement is not true' cannot, on pain of contradiction, be either true or false. Therefore it is neither. But that's one of the things it states, since *it states that it is not true*. So, after all, and paradoxically, it is true'. The italicized claim is, as we have seen, what needs to be rejected in this argument. It is often claimed that if 'This statement is not true' is neither true nor false, then, being not true and not false it is (by '&-Elimination') not true and that therefore, since this is what it states itself to be, it is (paradoxically) true. But this is a mistaken line of reasoning, for it does not state itself to be anything;<sup>6</sup> it does not state anything at all and nothing may be inferred from it.



This approach to the Liar is not new. It flourished in the early middle ages under the name `cassatio' which translates into computer lingo, as `crash'. When your computer crashes you keep hitting the keys in the usual way, but nothing happens on the screen. Likewise with the Liar sentence — you produce a flurry of words that belong to the vocabulary of a certain language and conform to the syntax of that language; you go through the sort of motions that you would normally go through for producing a statement, but, on this occasion, no statement results. That may seem curious, even suspicious, until we reflect that nothing is stated by a vacuous biconditional, and the Liar is only a vacuous biconditional in disguise — in other words, a vacuous biconditional can be extracted (derived) from it by a proof or a piece of reasoning. The Liar sentence has a literal meaning; it can be translated into other natural languages. But it lacks content, fails to express a proposition.<sup>7</sup>

The Liar is the simplest of a large family of paradoxes, and it is interesting to observe that many of them can be diagnosed as arising from illegitimate stipulation. Consider the following example, a version of which is published (on T-shirts) by the American Philosophical Association:

$S_1$ :  $S_2$  is false

$S_2$ :  $S_1$  is true

We *can* legitimately assign the name  $S_1$  to the statement `S<sub>2</sub> is false' but, in so doing, we are stipulating that  $S_1$  and  $S_2$  have *opposite* truth-values. Therefore, we are not free, in the same context, to assign the name  $S_2$  to `S<sub>1</sub> is true', for that stipulation would guarantee that  $S_1$  and  $S_2$  have the *same* truth-value. There is thus a *restriction* on what we can stipulate concerning the names  $S_1$  and  $S_2$  once the initial stipulation

that  $S_1$  is to be the name of ' $S_2$  is false' has been made. Another way of putting this would be to say that, after the initial assignment has been made, the names ' $S_1$ ' and ' $S_2$ ' are no longer *free for* indiscriminate use, if logical perspicuity is to be respected.

This kind of restriction, and the notion of being *free for*, are familiar in first-order logic, and it is important to see that the restriction on the assignment of names to which we have just alluded is not *ad hoc*, but is of a kind that is familiar both in logic and in everyday life. In everyday life, if you sign up with an e-mail provider, you will not be assigned a name that has already been assigned to another user, for such duplication would facilitate duplicity, infringe privacy and foster piracy. In logic, a name, once it has been introduced in the course of a natural deduction, is no longer available for replacing the variable when applying the rule of inference 'Existential Instantiation' (EI). In Quine's natural deduction system of *Methods of Logic*, for example, the rule EI that licences the inference from ' $(\exists x)Fx$ ' to ' $Fy$ ' is annotated by flagging the variable ' $y$ ', and the restriction on proofs incorporating EI is given as 'no flagged variable retains free occurrences in premises or conclusion' (Quine 1952: 161). Equivalent restrictions hold in all the common deductive systems of first-order logic.

In most texts, including Quine's, the restriction on the rule is justified merely by showing that ignoring it exposes you to the risk of deriving a false conclusion from true premises, without explaining the rationale for the restriction. Yet the rationale is easy enough to explain. Suppose that somewhere in a proof you have established that some object has property F, i.e.,  $(\exists x)Fx$ . Applying EI then amounts, roughly speaking, to the stipulation: 'Let that object be called "N"'. Now, suppose, later in the proof, it is established that some object has property G. It would obviously be rash to suppose that *that* object too is N. Therefore, formally, the prophylactic is not to use ' $N$ ' when

that name has already been assigned to some object earlier in the proof. Likewise, with the paradoxes we have been considering. We do not refrain from stipulating a name for a given statement just because to do so generates a contradiction, but because there are readily intelligible limits on our freedom to stipulate. We can stipulate that 'S' is the name of a statement. But, if we do, then, should we wish to assign a name to the statement 'S is not true', that name must be something other than 'S', for reasons given above.

### **Yablo and circular variants**

Another example to illustrate how a member of the Liar family is revealed as discontent through illegitimate stipulation is

J: J and K are untrue

K: J and K are untrue

You can easily verify that this is paradoxical: Look at the first line. If J is true, then it is untrue, because J says that itself (and K) are untrue. Conversely, if J is untrue, then, since it is saying the truth about itself (viz., that J is untrue), it must be saying something untrue about K. What it says about K is that K is untrue, so if that's untrue, it follows that K is true. But (now look at the second line) K *cannot* be true because K says of itself that it is untrue. Doh!!! The paradox is broken, however, once you work out that the initial assignment of names renders the two sentences discontent.

It is instructive to see how this last paradox is related to Yablo's:<sup>8</sup>

*Yablo's Paradox*

Y1. For all  $k > 1$ ,  $Y_k$  is not true

Y2. For all  $k > 2$ ,  $Y_k$  is not true

.....

Yi. For all  $k > i$ ,  $Y_i$  is not true

.....

Here we have an infinite list of putative statements, no members of which refer to themselves, yet together they generate a paradox (Yablo 1993).

Step 1 – Sorensenize the paradox

Following an idea of Roy Sorensen's (1982), make the paradox more homely by viewing it as an infinite queue of people, each of whom just says 'Everyone further down the queue is saying something untrue'

Step 2 — Manufacture a Finite, Circular Version

Chop the queue at the  $n$ th person (for some finite  $n$ ) and send the remaining infinite number of people back to their hotel. Now, with your finite queue, bring the tail round to the head, thus forming a circular queue of speakers, each saying what he or she was saying before. Of course, what each speaker is now saying is self-referential since he or she is further down the queue from him/herself each time we go full circle.<sup>9</sup>

Step 3 — Tighten the circle

For each finite  $n$ , you get a paradox. Consider a very tight circle, where  $n = 2$ . So here we have just two persons, each of whom is saying 'What each of us is saying is untrue'.

#### Step 4 — De-Sorensenize

This gives us the ‘pair paradox’ we were just considering, viz.

J: J and K are untrue

K: J and K are untrue

It now becomes very natural to suggest that the Yablo is to be dissolved by refusing to accept that there can be any *statement* of the form ‘For all  $k > i$ ,  $Y_i$  is not true’ occurring in the list that can be assigned the name ‘ $Y_i$ ’.

In the foregoing discussion, we have observed how vacuous biconditionals and contradictions are implicated in various paradoxes and conundrums, but have said little about their truth value, save to point out that they are not true. Classical principles dictate that they are simply (and necessarily) false, but our treatment of the paradoxes has already indicated that a principled denial of this ascription will deliver a solution to a bundle of logico-semantical paradoxes. As we mentioned at the outset, Wittgenstein (for reasons quite independent of considerations about paradox) held that contradictions are empty of content and bereft of truth-value. If he is right, then our approach to these paradoxes acquires real backbone. Is he?

#### **Wittgenstein’s (and Aristotle’s?) position on contradiction**

Wittgenstein urges that we not think of a contradiction as a ‘wrong proposition’ (Wittgenstein 1976: 223); contradictions and tautologies are not propositions at all; they have ‘the mere *ring of a statement*’. ‘The basic evil of Russell’s logic, as also of mine in the *Tractatus*’, he confesses, ‘is that what a

statement is is illustrated by a few commonplace examples and then presupposed as understood in full generality' (Wittgenstein 1980: §38). It is, he believed, a mistake to assume that, just because tautologies and contradictions are well-formed sentences, they can be used to make statements that have truth-value.

In late writings, Wittgenstein argued that, although there may be certain surroundings (Umgebungen) in which the utterance of a straight (undisguised) contradiction makes sense, in the absence of such surroundings, the speaker could not have understood the meanings of some or all of his words and no meaning can be attached to his utterance. Since what has no meaning is neither true nor false, we may say that what is common to Wittgenstein's early and late positions is the thesis that contradictions (with the exception of cases like 'It is and it isn't raining' to report very light drizzle) do not express propositions.

Aristotle, in the *Metaphysics*, appears to be committed to the same conclusion as Wittgenstein's. In his discussion of The Principle of Non-Contradiction (PNC), he first makes the uncontroversial point that a *fundamental* logical principle does not admit of proof (*Met. 1006a10*). He argues, though, that no rational person can fail to accept LNC. The ability to speak demands the ability to identify and name objects and this implies being able to recognize the boundary between an object and its background — the line (possibly a blurred one) between what is the object and what is *not* the object. From his ability to speak about things, we can transcendently deduce that an individual must acknowledge that what is a particular object is separated by a boundary from what is *not* that object, that what is that object cannot be what is *not* that object. Aristotle says not that a contradiction is false, but wonders of someone who asserts a contradiction 'how would his state be different from a vegetable's?' (*Met 1008b11*). If I tell you that I am both going and not going to

Macy's tonight, you cannot figure out what I am saying; you assume that you have misheard, or that I have gone nuts (this may have been the vegetable that Aristotle had in mind) or am playing some kind of trick. You would be foolish to plan to meet up with me in the evening on the basis of the words I uttered, for no content can be ascribed to them. This certainly seems to be Aristotle's view, but I shall indicate with an `?' some slight caution about ascribing it to him.

The Wittgenstein/Aristotle view is that no rational person can undermine, can speak against (*contra-dicere*), a proposition that he or she is asserting; someone who sincerely utters a contradictory form of words — assuming that he or she is not being deceptive, ironical, or anything like that — simply has not gained a mastery of all the words that he or she is employing. Stripped of anthropological accretion, this becomes the view that contradictions are not false (and not true either — they are in a different ballpark).

An impatient response to this suggestion might be that we can understand tautologies and contradictions perfectly well, and that even small children recognize them as paradigm examples of truth and falsity respectively. But we have already mentioned that to grant that a sentence has meaning (and can thus be understood) is not yet to reckon that sentence capable of yielding a proposition. And, interestingly, it is empirically false that, in an untutored state, we recognize tautologies and contradictions as having truth-value (Osherson and Markman 1975). When a sentence is used, in a given context, to express a proposition, the context typically contributes to the determination of what proposition is expressed. But contradictions are not *used*, except in exceptional surroundings, to say anything (Wittgenstein 1980: §1132). And, where there is no use, there is no proposition and no truth-value. This is Wittgenstein's view and, of course, it will exercise little persuasive influence on those

broadly unsympathetic to Wittgenstein's later philosophy. But the conclusion may be defended independently of this standpoint. Some preliminary considerations have been offered in the Introduction and in the preceding section, and we have already considered an 'impatient response' to the Wittgenstein/Aristotle position. It may now be helpful to observe how a variety of objections can be taken care of.

## **Objections and Replies**

### *Objection 1*

Contradictions *are* false in virtue of the meanings of 'and' and 'not' and the composition of these into the meaning of the whole. The classical truth-tables inform us that contradictions are false. Reply: Truth-tables are supposed to reflect the semantical properties of the connectives, unless we are simply inventing connectives that have no independent established use. The simplest way (though arguably not the best way) of reflecting that contradictions are not false would be to accept the classical truth-tabular characterization of 'A and B', except for when the sentence substituting for 'A' is the negation of that substituting for 'B'. That may seem to be untidy but, as Wittgenstein pointed out, the demand for 'crystalline purity' in logic is ill-founded.<sup>10</sup> To insist that 'p and not-p' must take the value 'false' because that is what is dictated by clean and exceptionless truth-tabular requirements is surely to let the tail wag the dog.

There is already a huge literature on the senses of 'not', but not quite so much on 'and', so let us say a little about the latter here. What should we say about the meaning of 'and' as it occurs in 'p and not-p'? My wife received a letter from her Uncle Jimmy, in which he wrote: 'Auntie Ivy had two strokes and a heart attack and died, but luckily she was in hospital at the time and they managed to revive her'. I



expect that most readers would say that Jimmy had misused the word `died' even though the word `revive', in its original sense, means `to bring back to life'. It would be correct to say that the meaning Jimmy attached to `dead' was incorrect. Meanings change over time. We might now truly say `Her heart stopped beating but she had not died', yet, thirty years ago, that would have been a solecism for to say then that someone had died *implied* that their heart had stopped beating.

The meaning of an expression at any historical time is just the ambient use of the expression at that time, although not everyone's use carries the same weight. Coiners with charisma and authority can effect rapid meaning innovation — in the sense that a new expression they introduce, or a new use they suggest for an existing expression, can be swiftly taken up by the population and by standard dictionaries (recent examples include `burnout' and `bootstrap'). Equally, some expressions (particularly scientific ones) may be widely *misused*, and reputable dictionaries may refuse to follow a popular trend. New meanings do not spring into existence unsolicited. Scientific terminology may be invented and embraced via a relatively simple process, but most new meanings are products of complex social interaction. The important point to recognise is that meanings are not super-human; they do not exist independent of our sociolinguistic practices.

There is a controversy between those who say that the word `and' has a unique meaning characterised by the classical truth table or the classical laws of inference, and those who deny this. A consequence of the `classical' (or `purist') view is that `and' is commutative — `A and B' entails and is entailed by `B and A'. This consequence seems unacceptable. Ordinarily, we take `I am going to drink and drive' *not* to entail `I am going to drive and drink'. In a case like this, the `and' has the sense of `and then'. In other contexts, the `and' is non-temporal. If I say to the waiter in my

local café, 'I'll have soup and liver and bacon', then I should expect the soup to come first but not the liver to precede the bacon or vice-versa, though, in a different culinary setting — in a Chinese restaurant, say — I would spell out 'and then' or 'and, when that's finished' if I wished to avoid all three items coming at the same time.

The order of sentences describing a sequence of events generally reflects the order in which the corresponding events occurred. This is a convention with a perfectly obvious rationale, and we speak not just misleadingly but falsely if we breach this convention. 'I got dressed and had a bath' *entails* that I bathed clothed. Since it is a fact about ordinary use that people take my announcement 'I got dressed and had a bath' as indicating, if true, that I bathed clothed, and there is no higher court of appeal to determine the meaning of the sentence than how people use those words, then, in this context, the 'and' means 'and then'.<sup>11</sup> Classicists, following Grice, claim that the order of the component sub-sentences *conveys* information about the order of the corresponding events, and that, *strictly speaking*, this has nothing to do with the *meaning* of 'and' as it occurs in the molecular sentence. But what could be a clearer indication of the ambiguity of 'and' than that it sometimes can, and sometimes cannot, be replaced by 'and then', as in our sentence above about soup and liver and bacon? In some languages the distinct senses are borne by distinct words.<sup>12</sup> The word 'and' is also frequently used in the sense 'and, in consequence', as in 'He ran 9.05 seconds and broke the World 100m. record' or 'She betrayed her friend and was never trusted again'. It is as futile to try wishing away ambiguity or non-classical connectives as it is to try wishing away irregular verbs.

We have seen that the word 'and' does not have a unique meaning that it carries with it to any context in which it is used. Rather, what the word means on any occasion of use is read off its occurrence in that context.<sup>13</sup> It is, *inter alia*, worldly

knowledge (e.g. that liver usually accompanies bacon, that soup in a Chinese restaurant does not normally precede the other dishes) that enables you to interpret, to read off the sense of a word in the context in which it is used. What, then, must we say about the meaning of the word 'and' as it occurs in a *contradiction*? Only question-beggingly could one assert that its meaning, as read off from this occurrence, is such as to deliver the truth-value 'false' to the contradiction.

The (literally) correct response I should make to someone who tells me 'I am going to the theatre and I am not going to the theatre' is 'You can't mean that'. There are innumerable grammatically correct sentences to which (unless apprised of extraordinary surroundings) we can attach no content. A (not particularly good) example of Wittgenstein's is 'Milk me sugar.' (Wittgenstein 1953: § 498). So it should not be assumed that a contradiction has content. And there is no question of importing a particular meaning of 'and' into a contradiction of the form 'p and not p' and of that imported meaning *dictating* a sense and a truth-value for the contradiction.

### *Objection 2*

If a contradiction is not false (and not true either), should not the same be said of its negation, a tautology? Wittgenstein of the *Tractatus* replies in the affirmative. He regards both tautology and contradiction as the disintegration (Auflösung) of the combination of signs (Wittgenstein 1961a: 4.466). This position has something to recommend it, but one problem (a particularly severe problem for Wittgenstein) is that we do *use* tautologies (e.g., in dilemma arguments).<sup>14</sup> If this is a persuasive consideration for acknowledging that tautologies are true, then there are options as to how to reflect this in the formal semantics. At a minimum, one requires a negation connective that, when adjoined to a contradiction with no truth-value (or with the

value 'GAP') delivers a truth. The rationale is that if it would be absurd to say such-and-such, then to say the opposite makes perfectly good sense and is indeed true. Again in Wittgensteinian terms, an *unsinnig* combination of words is not going to acquire *sinnigkeit* by sticking a 'not' in front of it, but, arguably, the same should not be said of a combination of words that is merely *sinnlos*.

### *Objection 3*

*Reductio ad absurdum* arguments that occur in mathematics, but also in many other areas of discourse, depend on ascribing falsity to contradictions. Reply: A *reductio* can be described in the following way: if an argument leads to a contradiction, then one of the premises is to be rejected. This description of *reductio* would hold irrespective of whether one is inclined to term a contradiction 'false' or 'vacuous', but the real difficulty, it might be said, is that if the conclusion of an argument is neither true nor false, but vacuous, then at least one of the premises must be not false but vacuous too. And how could we have reasoned to a true conclusion from a vacuous premise — surely nothing but nothing comes from nothing; there is no such thing as a free lunch.

This line of reasoning is clearly mistaken. For consider a perfectly good and true proposition *p*. It and its negation could be the two premises of an argument with a contradictory conclusion. But what about when a contradiction is derived from a *single* premise? Some sentences that have literal meaning can be used in inferences even if they are vacuous. If someone says to me 'The man who lives in the moon is cheerful', then, if I am gullible, I will reason that one way to meet a cheerful man would be to travel to the moon. But my impish (mis-)informant was literally speaking of nothing. Wittgenstein held that the offending premise in a *reductio* (e.g. 'The square

root of 2 is  $m/n$ , where  $m$  and  $n$  are integers') is, like 'A triangle has four sides', vacuous, but that the vacuity in the former case is not immediately transparent, and is demonstrated by the proof. When you reject the offending premise, you reject it not as false, but as senseless. We are happy to say that the notion of a 4-sided triangle is a conceptual confusion and, in the case of the claim 'The square root of 2 is  $m/n$ , where  $m$  and  $n$  are integers' we could say that the proof *unmasks* a conceptual confusion. Certainly, we should distinguish a mere *reductio ad falsum* (given that  $p$ , possibly together with some innocuous premises, entails  $q$  which is *false*, then  $p$  is false) from *reductio ad absurdum*, which, as the name implies, reduces an assumption to an *absurdity* or, as I should say, to a vacuity.<sup>15</sup>

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## Notes

<sup>1</sup> For a detailed account of Wittgenstein's early views on tautology and contradiction, see Goldstein 1986 and 1999.

<sup>2</sup> See, for example, Priest 1998: 836.

<sup>3</sup> Heller 1994: 62-3. The suggestion that Catch-22 bears comparison with the paradox of Protagoras and Euathlus is made in Poundstone 1988:128.

<sup>4</sup> Formalizing Catch-22 would be an interesting exercise for an introductory logic class. I do not claim that my formalization is the only possible one and, if someone claims that it is inaccurate, I am happy, for the sake of making the point I want to make here, to say that it is instead the accurate formalization of Catch-23.

<sup>5</sup> See Kripke 1975: 56 on why we can have self-referential *sentences* — for example, if the name has not been assigned already, we can stipulate that the name of the sentence 'Jack is short' is 'Jack' — without it following that we can have self-referential *propositions*.

<sup>6</sup> Though a token of the same type may state something about it, see Goldstein 1992, 2001, and Clark 1999.

<sup>7</sup> For a discussion of the distinction between *meaning* and *proposition*, see Soames 1999: 16-19.

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<sup>8</sup> (Yablo 1993). There has been some debate in the literature over whether Yablo's Paradox is genuinely non-self-referential. We show below how to recast this paradox so that it is clearly self-referential.

<sup>9</sup> For exploration and applications of this technique for forming circular queues, see Goldstein 1999a.

<sup>10</sup> Speaking of the unwarranted demand ('requirement') for an 'ideal' language, Wittgenstein writes 'The more narrowly we examine actual language, the sharper becomes the conflict between it and our requirement. (For the crystalline purity of logic was, of course, not a result of investigation: it was a requirement.)' (Wittgenstein 1953: § 107). In the preceding few sections, Wittgenstein argues for abandoning preconceptions about logic that he, like many others, embraced at the time of the *Tractatus*.

<sup>11</sup> Kent Bach (2002) has questioned the reliability of such arguments that appeal to the semantic intuitions of ordinary speakers.

<sup>12</sup> There is a large literature, including Atlas 1989, controverting such ambiguity claims and with which, in a much longer paper, it would be good to engage.

<sup>13</sup> For more doubt on the view that understanding a sentence involves a rule-governed composition of the meanings of the component words, see Sayward 2000.

<sup>14</sup> This is presented as an *ad hominem* argument in Goldstein 1999b.

<sup>15</sup> Some penetrating queries of David Papineau's prompted a beneficial re-shaping of this paper.