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On the algorithmic complexity of twelve covering and independence parameters of graphs

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Abstract

The definitions of four previously studied parameters related to total coverings and total matchings of graphs can be restricted, thereby obtaining eight parameters related to covering and independence, each of which has been studied previously in some form. Here we survey briefly results concerning total coverings and total matchings of graphs, and consider the aforementioned twelve covering and independence parameters with regard to algorithmic complexity. We survey briefly known results for several graph classes, and obtain new NP-completeness results for the minimum total cover and maximum minimal total cover problems in planar graphs, the minimum maximal total matching problem in bipartite and chordal graphs, and the minimum independent dominating set problem in planar cubic graphs.

Keywords: covering, independence, total cover, total matching, independent domination

1 Introduction

In graph theory, the notion of covering vertices or edges of graphs by other vertices or edges has been extensively studied. For instance, covering vertices by other vertices leads to parameters concerned with vertex domination [45]. When edges are to be covered by vertices we obtain parameters connected with the classical vertex covering problem [41, p.94]. Covering vertices by edges, i.e. finding edge covers, is considered by Norman and Rabin [66]. Finally, when edges are to cover other edges, we obtain parameters associated with edge domination (introduced by Mitchell and Hedetniemi [62]). Independent sets of vertices [41, p.95] correspond to the case where vertices are chosen so as *not* to cover one another, and matchings [57] of a graph correspond to the similar restriction involving edges.

It is natural to extend this notion of covering by vertices and edges. Nordhaus [64], and also Alavi et al. [1], define the *elements* of a graph $G = (V, E)$ to be the set $V \cup E$. A vertex v is defined to cover itself, all edges incident on v and all vertices adjacent to v. An edge $\{u, v\}$ is said to cover itself, vertices u and v, and all edges incident on u or v. Two elements of $V \cup E$ are *independent* if neither covers the other. Thus, a *vertex cover* is a subset S of V that covers E, a *dominating set* is a subset S of V that covers V (in this paper, the term *dominating set* will only apply to a set of vertices), an *edge dominating set* is a subset S of E that covers E, and an *edge cover* is a subset S of E

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that covers V (assuming that G has no isolated vertices). A subset C of $V \cup E$ that covers all elements of G is said to be a *total cover* for G. Also, an *independent set* is a subset S of V whose elements are pairwise independent (in this paper, the term *independent set* will only apply to a set of vertices), and a *matching* is a subset S of E whose elements are pairwise independent (in this paper, the term *matching* will only apply to a set of edges). A subset M of $V \cup E$ whose elements are pairwise independent is said to be a *total matching* for G.

Suppose that $\mathcal{P}$ is some collection of sets. Denote by $\mathcal{P}^-$ the minimal elements of $\mathcal{P}$, i.e. $S \in \mathcal{P}^-$ if and only if $S \in \mathcal{P}$ and no proper subset of S is a member of $\mathcal{P}$. Similarly, denote by $\mathcal{P}^+$ the maximal elements of $\mathcal{P}$, i.e. $S \in \mathcal{P}^+$ if and only if $S \in \mathcal{P}$ and no proper superset of S is a member of $\mathcal{P}$. Let

$$\mathcal{C}(G) = \{C \subseteq V \cup E : C \text{ is a total cover for } G\}$$

and

$$\mathcal{M}(G) = \{M \subseteq V \cup E : M \text{ is a total matching for } G\}.$$

Nordhaus [64] and also Alavi et al. [1] define the following parameters[1]:

$$\begin{array}{ll} \alpha_2(G) = \min\{|C| : C \in \mathcal{C}^-(G)\}, & \alpha_2^+(G) = \max\{|C| : C \in \mathcal{C}^-(G)\}, \\ \beta_2^-(G) = \min\{|M| : M \in \mathcal{M}^+(G)\}, & \beta_2(G) = \max\{|M| : M \in \mathcal{M}^+(G)\}. \end{array}$$

1.1 Survey of non-algorithmic total covering and total matching results

Some upper and lower bounds involving each of these parameters separately are derived by Gupta [40], Nordhaus [64], Alavi et al. [1], Meir [61], Kulli et al. [56], Zhang et al. [75], Alavi et al. [2] and Gimbel and Vestergaard [37]. In particular, it is known [1] that

$$\alpha_2(G) \le \beta_2^-(G) \le \beta_2(G) \le \alpha_2^+(G).$$

Peled and Sun [67] derive exact values for these parameters in threshold graphs. Also, Alavi et al. [2] consider properties of those connected graphs on n vertices having $\alpha_2(G) = \lceil \frac{n}{2} \rceil$. Bounds for $\alpha_2(G) + \beta_2(G)$ are considered by Alavi et al. [1], Erdös and Meir [26] and Meir [61]. In addition, some Nordhaus–Gaddum [65] type results have been obtained, involving each of α_2 and β_2 [26, 61], and involving β_2^- [37]. Finally, Topp and Vestergaard [73] characterise those graphs in which every maximal total matching is maximum, and Topp [72] studies those graphs having a unique maximum total matching. The survey by Hedetniemi et al. [46] describes the inequalities involving the total covering and total matching parameters in more detail.

The terminology for total covers and total matchings does not seem to be universally agreed upon in the literature. Nordhaus [64] and Alavi et al. [1], who introduced these concepts, define a subset C of $V \cup E$ to be a total cover if C covers G and C is minimal. Similarly, they define C to be a total matching if the elements of C are pairwise independent and C is maximal. However, several authors [2, 37, 73] have defined total covers and total matchings without the minimality or maximality requirement, respectively, as is done

[1]The notation of the covering and independence parameters studied in this paper follows that of Harary [41] and Alavi et al. [1]. The convention these authors follow is that the α and β symbols refer respectively to covering and independence properties that are to be satisfied. The subscript of the parameter symbol is $0, 1, 2$ according to whether the parameter is associated with the optimum size of a set of vertices, edges or both, respectively. A superscript of '+' in the case of an α parameter refers to the 'maximum minimal' objective, and a superscript of '−' in the case of a β parameter refers to the 'minimum maximal' objective. When this superscript is missing from an α symbol, the objective is to minimise, and the objective is to maximise in the case that a β symbol is without superscript.

here. This can be advantageous, for example, when reasoning about a subset C of $V \cup E$ whose elements are pairwise independent, but C is not maximal. Following the terminology of Nordhaus [64], such a set is not a total matching. Referring to C as an *independent set* or a *matching* coincides with the usual notion of an independent set or matching when applied to sets containing vertices or edges only, respectively. Thus we choose to follow the terminology of [2, 37, 73]. We note in passing that total covers (as defined here) are referred to as *mixed dominating sets* by Hedetniemi et al. [46], *entire dominating sets* by Kulli et al. [56] and *total dominating sets* by Gimbel et al. [36]. The latter definition is quite distinct from the more widely accepted concept of a total dominating set, due to Cockayne et al. [13].

1.2 More covering and independence parameters

Nordhaus [64] shows how we may use $\mathcal{C}$ and $\mathcal{M}$ to derive some existing graph parameters. Define

$$\mathcal{C}_0(G) = \{C \in \mathcal{C}(G) : C \subseteq V\} \text{ and } \mathcal{C}_1(G) = \{C \in \mathcal{C}(G) : C \subseteq E\}$$

and similarly define

$$\mathcal{M}_0(G) = \{M \in \mathcal{M}(G) : M \subseteq V\} \text{ and } \mathcal{M}_1(G) = \{M \in \mathcal{M}(G) : M \subseteq E\}.$$

Then we obtain, as in [64],

$$\begin{array}{ll} \alpha_0(G) = \min\{|C| : C \in \mathcal{C}_0^-(G)\} - I_G, & \alpha_0^+(G) = \max\{|C| : C \in \mathcal{C}_0^-(G)\} - I_G, \\ \beta_0^-(G) = \min\{|M| : M \in \mathcal{M}_0^+(G)\}, & \beta_0(G) = \max\{|M| : M \in \mathcal{M}_0^+(G)\}, \end{array}$$

where α_0 and α_0^+ are the minimum and maximum over all minimal vertex covers of G respectively, and β_0^- and β_0 are the minimum and maximum over all maximal independent sets of G respectively, and I_G denotes the number of isolated vertices of G. Similarly we obtain[2]

$$\begin{array}{ll} \alpha_1(G) = \min\{|C| : C \in \mathcal{C}_1^-(G)\}, & \alpha_1^+(G) = \max\{|C| : C \in \mathcal{C}_1^-(G)\}, \\ \beta_1^-(G) = \min\{|M| : M \in \mathcal{M}_1^+(G)\}, & \beta_1(G) = \max\{|M| : M \in \mathcal{M}_1^+(G)\}, \end{array}$$

where α_1 and α_1^+ are the minimum and maximum over all minimal edge covers of G respectively, and β_1^- and β_1 are the minimum and maximum over all maximal matchings of G respectively. Thus definitions relating to the total covering and total matching parameters $\alpha_2, \alpha_2^+, \beta_2^-, \beta_2$ can be restricted, in order to obtain the eight covering and independence parameters $\alpha_i, \alpha_i^+, \beta_i^-, \beta_i$ for $i = 0, 1$. This implies a possible framework for twelve covering and independence parameters of graphs. Each of α_i, α_i^+ and β_i^-, β_i has been studied previously in some form, for $0 \le i \le 2$.

Nordhaus [64] investigates relations between the parameters $\alpha_2, \alpha_2^+, \beta_2^-, \beta_2$ and α_i, α_i^+, β_i^-, β_i for $i = 0, 1$, and obtains the inequalities

$$\alpha_2(G) \le \alpha_i(G) \le \alpha_i^+(G) \le \alpha_2^+(G)$$

for $i = 0, 1$, and

$$\beta_2(G) \ge \max\{\beta_0(G), \beta_1(G)\} \text{ and } \beta_2^-(G) \ge \max\{\beta_0^-(G), \beta_1^-(G)\}.$$

Let $\gamma(G)$ and $\Gamma(G)$ denote respectively the minimum and maximum over all minimal dominating sets of a graph G. For a graph $G = (V, E)$, let $T(G)$ denote the *total graph* of G – this is the graph with vertex set $V \cup E$, and two vertices are adjacent in $T(G)$ if and only if the corresponding elements are adjacent or incident as vertices or edges of G. It is clear that $\alpha_2(G) = \gamma(T(G))$, $\alpha_2^+(G) = \Gamma(T(G))$, $\beta_2^-(G) = \beta_0^-(T(G))$ and $\beta_2(G) = \beta_0(T(G))$.

[2] In the case of α_1 and α_1^+, we assume that G has no isolated vertices, for the concept of edge covering is undefined for graphs with isolated vertices.

1.3 Organisation of the paper

We study the algorithmic complexity of the twelve decision problems related to α_i, α_i^+ and β_i^-, β_i for $0 \leq i \leq 2$ over several classes of graph. The classes that we consider include, in each case, four extensively studied classes of graphs, namely planar, bipartite, and chordal[3] graphs, and trees. Definitions of other graph classes mentioned here but not defined may be found in [38] and [52]. Henceforth we refer to 'the complexity of α' when we mean 'the complexity of the decision problem related to parameter α'. We survey briefly known results for graph classes that include at least the four mentioned above, and obtain new NP-completeness results for the following problems:

- Minimum total cover in planar graphs.
- Maximum minimal total cover in planar graphs.
- Minimum maximal total matching in bipartite and chordal graphs.
- Minimum independent dominating set in planar cubic[4] graphs.

In addition, we demonstrate that the complexities of the maximum minimal edge cover, maximum minimal vertex cover, and maximum total matching parameters are identical to the complexities of the minimum dominating set, minimum independent dominating set, and minimum edge dominating set parameters respectively, over all graph classes. These results do not appear to have been noted explicitly in the literature previously. Appropriate transformations are given for the new results, and references are supplied for the known results.

The remainder of this paper is organised as follows. In Sections 2-6, we consider each of the twelve covering and independence parameters. The total covering and total matching parameters are discussed in Sections 2 and 3 respectively, as their definition gives rise to the framework for the remaining parameters. Then, in Sections 4, 5 and 6, we consider the vertex covering and independence parameters, the edge covering parameters, and the matching parameters, respectively. In Section 7, we give a summary of the algorithmic results in this paper, and finally, we present some concluding remarks in Section 8.

2 Total covering

We begin by considering the total covering parameters. Majumdar [59, p.52] shows that α_2 is NP-complete for general graphs, using a transformation from 3-SAT [33, problem LO2], and gives a linear-time algorithm for trees. Hedetniemi et al. [46] show that α_2 remains NP-complete for bipartite and chordal graphs. The proof involves a transformation from EXACT COVER BY 3-SETS [33, problem SP2], which may be defined as follows:

> *Name:* EXACT COVER BY 3-SETS (X3C)
> *Instance:* Set $A = \{a_1, a_2, \ldots, a_{3q}\}$ of *elements*, for some $q \in \mathbb{Z}^+$, and a collection $C = \{c_1, c_2, \ldots, c_m\}$ of subsets of A (*clauses*), where $|c_i| = 3$ for each i.
> *Question:* Does C contain an exact cover for A, i.e. is there a set C' ($C' \subseteq C$) of pairwise disjoint sets whose union is A?

The restriction of X3C known as PLANAR EXACT COVER BY 3-SETS (PX3C) demands that the graph $G = (V, E)$, associated with an instance (A, C) of X3C, with vertex set $V = A \cup C$

[3] A graph G is *chordal* if every cycle in G of length four or more contains a *chord*, i.e. an edge connecting two non-adjacent points on the cycle

[4] A graph is *cubic* if every vertex has degree 3.

and edge set $E = \{(a, c) : a \in c \in C\}$, is planar. PX3C is NP-complete [23], even if each element occurs in either two or three clauses. It may be verified that the construction of Hedetniemi et al. [46], showing NP-completeness for α_2 in bipartite graphs, preserves the planarity of this graph G. Moreover, the maximum degree of the graph constructed is 4. Thus, by considering the same transformation, but from PX3C rather than X3C, we obtain the following result. (Let MINIMUM TOTAL COVER be the decision problem related to α_2, which takes a graph G and an integer $K \in \mathbb{Z}^+$ and asks whether $\alpha_2(G) \leq K$.)

Theorem 2.1. MINIMUM TOTAL COVER *is NP-complete, even for planar bipartite graphs of maximum degree 4.*

Investigating the computational complexity of α_2^+ is given as an open problem by Hedetniemi et al. [46]. Let MAXIMUM MINIMAL TOTAL COVER be the decision problem related to α_2^+, which takes a graph G and integer $K \in \mathbb{Z}^+$ and asks whether $\alpha_2^+(G) \geq K$. We show that MAXIMUM MINIMAL TOTAL COVER is NP-complete for planar graphs.

Theorem 2.2. MAXIMUM MINIMAL TOTAL COVER *is NP-complete, even for planar graphs.*

Proof. Clearly MAXIMUM MINIMAL TOTAL COVER is in NP. For, given $K \in \mathbb{Z}^+$ and a set S of at least K elements, it is straightforward to verify in polynomial time that S is a minimal total cover.

To show NP-hardness, we give a transformation from PX3C, defined above. Given an arbitrary instance of PX3C, we construct a planar graph G, with the property that there exists an exact cover for the PX3C instance if and only if there exists a minimal total cover of G with at least K elements, for a particular $K \in \mathbb{Z}^+$.

Suppose that a set of elements $A = \{a_1, a_2, a_3, \ldots, a_{3q}\}$ and a collection of clauses $C = \{c_1, c_2, c_3, \ldots, c_m\}$ (for some $q, m \in \mathbb{Z}^+$) is an arbitrary instance of PX3C. Suppose further that, for each j $(1 \leq j \leq m)$, $c_j = \{a_{i_{3j-2}}, a_{i_{3j-1}}, a_{i_{3j}}\}$, where $i_1, i_2, i_3, \ldots, i_{3m}$ is some sequence of integers such that

$$\{i_1, i_2, i_3, \ldots, i_{3m}\} = \{1, 2, 3, \ldots, 3q\}.$$

Construct an instance – graph $G = (V, E)$ and positive integer K – of MAXIMUM MINIMAL TOTAL COVER as follows:

- *Subset vertices:* For each j $(1 \leq j \leq m)$ create a *subset vertex* s_j.
- *Communication edges:* For each j $(1 \leq j \leq m)$, add three *communication edges*, $\{s_j, t_{i_{3j-2}}\}, \{s_j, t_{i_{3j-1}}\}$, $\{s_j, t_{i_{3j}}\}$, where $c_j = \{a_{i_{3j-2}}, a_{i_{3j-1}}, a_{i_{3j}}\}$.
- *Element components:* For each i $(1 \leq i \leq 3q)$, create an *element vertex* t_i. Form a clique among three vertices u_i, v_i, w_i, and join u_i to t_i. Create N (where N is to be defined) *leaf vertices* x_i^r, and join each x_i^r to t_i, for $1 \leq r \leq N$.
- *Target value:* Set $K = m + (3N + 8)q$.

Denote by S_i the following elements in the ith element component:

$$S_i = \{t_i, \{t_i, u_i\}, u_i, \{u_i, v_i\}, v_i, \{v_i, w_i\}, w_i, \{w_i, u_i\}\}.$$

Apart from the leaf vertices and their incident edges, G has a total of $12q + m$ vertices and $12q + 3m$ edges. Set N to be the sum of these totals, i.e. $N = 24q + 4m$. The construction is partly illustrated in Figure 1. Clearly, this construction is polynomial with respect to the size of the PX3C instance, and preserves the planarity of the graph constructed from this instance. First we show that if the PX3C instance has an exact cover, then G has a minimal total cover S, with $|S| = K$. From an exact cover C' for the PX3C instance, we construct a set S as follows. For each j $(1 \leq j \leq m)$:

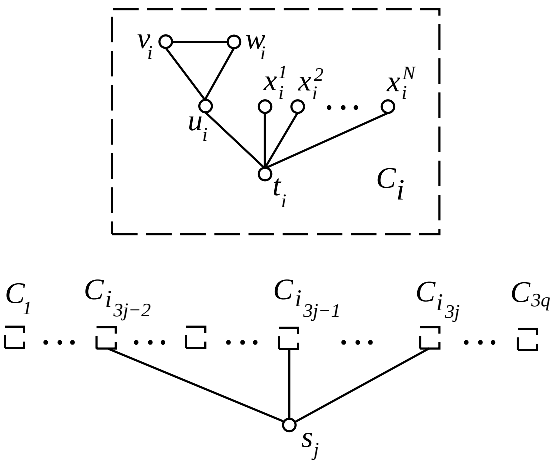

Figure 1: Part of the graph G constructed as an instance of MAXIMUM MINIMAL TOTAL COVER, showing typical subset and element components.

- If $c_j \in C'$, add to S the three edges $\{s_j, t_{i_{3j-2}}\}$, $\{s_j, t_{i_{3j-1}}\}$ and $\{s_j, t_{i_{3j}}\}$.
- If $c_j \notin C'$, add to S the vertex s_j.

For each i $(1 \le i \le 3q)$:

- Add to S the vertices v_i, w_i.
- Add to S the vertices x_i^r for $1 \le r \le N$.

Now S is a total cover, for, clearly the leaf vertices cover themselves, their incident edges and t_i, for $1 \le i \le 3q$. Also s_j is covered either by itself or by an incident edge, for each j $(1 \le j \le m)$. As C' is an exact cover, then for each i $(1 \le i \le 3q)$, all edges incident on t_i are covered by some communication edge of S. Finally, all other vertices and edges in each element component are clearly covered.

S is minimal, for it is clear that each of the leaf vertices are are covered by no other element of S. Also $S\backslash\{v_i\}$ does not cover the edge $\{u_i, v_i\}$ and $S\backslash\{w_i\}$ does not cover the edge $\{u_i, w_i\}$, for any i $(1 \le i \le 3q)$. If $s_j \in S$ for any j $(1 \le j \le m)$, then no communication edge of S is incident on s_j, so that $S\backslash\{s_j\}$ does not cover s_j. Finally, if a communication edge $\{s_j, t_i\}$ is in S, for any i and j $(1 \le i \le 3q, 1 \le j \le m)$, then $S\backslash\{\{s_j, t_i\}\}$ does not cover $\{t_i, u_i\}$, since C' is an exact cover.

By construction of S, all $3q$ of the element vertices are covered by exactly one communication edge. As C' is an exact cover, these edges cover exactly q subset vertices. There are then $m - q = |C\backslash C'|$ subset vertices in S. Each element component contributes $N + 2$ vertices and no edges. Thus

$$\begin{aligned} |S| &= 3q + (m - q) + 3q(N + 2) \\ &= K \end{aligned}$$

as required.

Conversely, suppose that G has a minimal total cover S such that $|S| \ge K$. We show that the PX3C instance has an exact cover C'. From all minimal total covers for G with cardinality at least K, choose S to be such a set that has the fewest number of communication edges. We now establish a number of facts about the elements that S contains.

1. *S does not contain t_i, for any i ($1 \leq i \leq 3q$).* For, suppose $t_i \in S$ for some i ($1 \leq i \leq 3q$). Then by minimality $x_i^r \notin S$ for $1 \leq r \leq N$ and $\{t_i, x_i^r\} \notin S$ for $1 \leq r \leq N$. Thus, an upper bound for S in this case must be:

$$\begin{aligned} |S| &\leq N + (3q-1)N \\ &< K \end{aligned}$$

which is a contradiction.

2. *There are $3qN$ elements in S such that each element is either a leaf vertex or is an edge incident on a leaf vertex. Furthermore, these elements cover each of the vertices t_i, for $1 \leq i \leq 3q$.* This observation follows from Fact 1.

3. *$|S \cap S_i| = 2$, for any i ($1 \leq i \leq 3q$).* For, let $1 \leq i \leq 3q$ be given. From Fact 1, $t_i \notin S$. Suppose $\{t_i, u_i\} \in S$. If $S \backslash \{\{t_i, u_i\}\}$ does not cover some edge $\{s_j, t_i\}$, for some j ($1 \leq j \leq m$) then S does not cover s_j, since $t_k \notin S$, for any k ($1 \leq k \leq 3q$), a contradiction. Thus $S \backslash \{\{t_i, u_i\}\}$ covers all communication edges of G, but does not cover some element of S_i. It follows that exactly one more element of S_i is in S. In the case that $\{t_i, u_i\} \notin S$, it may also be verified that exactly two elements of S_i belong to S.

4. *S does not contain an edge $\{s_j, t_i\}$ together with vertex s_j, for any i and j ($1 \leq i \leq 3q$ and $1 \leq j \leq m$).* For, suppose S did. Since, by Fact 2, each of $t_{i_{3j-2}}, t_{i_{3j-1}}, t_{i_{3j}}$ is covered by a leaf vertex or an edge incident on a leaf vertex, then $S \backslash \{s_j\}$ also covers G, contradicting the minimality of S.

5. *S does not contain more than one communication edge incident on a vertex t_i, for any i ($1 \leq i \leq 3q$).* For, suppose S did – let $\{s_j, t_i\}$ and $\{s_k, t_i\}$ be two such edges, for some j, k ($1 \leq j \neq k \leq m$) and i ($1 \leq i \leq 3q$). Then by Fact 4, $s_k \notin S$, and by minimality, no edge incident on s_k other than $\{s_k, t_i\}$ is in S. Since, by Fact 2, each of $t_{i_{3k-2}}, t_{i_{3k-1}}, t_{i_{3k}}$ is already covered by a leaf vertex or an edge incident on a leaf vertex, then $S' = (S \backslash \{\{s_k, t_i\}\}) \cup \{s_k\}$ is a minimal total cover of G, with one fewer communication edge, and satisfies $|S'| = |S|$, contradicting the choice of S.

Let there be l communication edges in S. Then Fact 5 implies that these l edges are incident on exactly l of the element vertices, so that $l \leq 3q$. Suppose that S contains r subset vertices. Now suppose that the l communication edges in S are incident on a total of s subset vertices. Then $3s \geq l$ and by Fact 4, these s subset vertices are all distinct from the r subset vertices defined above. Thus $r + s \leq m$. But $r + s = m$, or else some s_j ($1 \leq j \leq m$) is not covered, since $t_i \notin S$, for any i ($1 \leq i \leq 3q$), by Fact 1. Finally, Facts 2 and 3 imply that S contains $N + 2$ elements from each of the $3q$ element components. Hence, having accounted for all the elements in S,

$$\begin{aligned} |S| &= r + l + 3q(N+2) \\ &= m + l - s + 3q(N+2) \qquad (\text{since } r + s = m). \end{aligned} \tag{1}$$

Assume firstly that $s < q$. Then by Equation 1,

$$\begin{aligned} |S| &\leq m + 2s + 3q(N+2) && (\text{since } 3s \geq l) \\ &< K && (\text{since } s < q) \end{aligned}$$

which is a contradiction. Thus $s \geq q$. Now assume for a contradiction that $l < 3q$. Then by Equation 1,

$$\begin{aligned} |S| &< m + 3q - s + 3q(N+2) && (\text{since } l < 3q) \\ &\leq K && (\text{since } s \geq q) \end{aligned}$$

which is also a contradiction. Hence $l = 3q$. Finally, assume for a contradiction that $s > q$. Then by Equation 1,

$$\begin{array}{rcll} |S| & = & m + 3q - s + 3q(N+2) & \text{(since } l = 3q\text{)} \\ & < & K & \text{(since } s > q\text{)} \end{array}$$

which gives a contradiction. Hence $s = q$ and $r = m - q$, so that exactly q of the subset vertices are covered by communication edges. Also, each of the $3q$ element vertices is covered by exactly one edge. Thus, for exactly q of the the subset vertices s_j ($1 \leq j \leq m$), we have $\{s_j, t_{i_{3j-2+r}}\} \in S$, for $0 \leq r \leq 2$; let C' contain the q corresponding c_j triples. Since the $m - q$ other subset vertices cover themselves, then C' is an exact cover. □

As pointed out in Section 1.2, $\alpha_2^+(G) = \Gamma(T(G))$ for a graph G. Yannakakis and Gavril [74] show that a connected graph is a tree if and only if its total graph is chordal. Jacobson and Peters [50] show that $\Gamma = \beta_0$ for chordal graphs. Hence, as β_0 is polynomial-time solvable for this class of graphs [35], the same is true for Γ, so that α_2^+ is polynomial-time solvable for trees. In addition, the remarks of this paragraph also imply that $\alpha_2^+ = \beta_2$ for trees.

3 Total matching

The total matching parameter β_2 is related to β_1^-: Gupta [40] shows that $\beta_2(G) + \beta_1^-(G) = n$ for any graph $G = (V, E)$, where $n = |V|$. Therefore we have the following result, which does not seem to have been explicitly noted in the literature previously.

Theorem 3.1. *The complexities of β_2 and β_1^- are identical, for any graph class.*

It is interesting to consider how we may construct a maximum total matching from a minimum maximal matching, and vice versa. Since Gupta's result is stated without proof, we provide, for completeness, one possible method. We use the following result, whose proof is straightforward, and is omitted.

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $M \subseteq V \cup E$ be a total matching. Then M is a maximal total matching if and only if M is a total cover.*

Proposition 3.3. *Let $G = (V, E)$ be a graph, where $n = |V|$. Then if $M \subseteq E$ is a maximal matching for G, where $m = |M|$, we may find a maximal total matching $M' \subseteq V \cup E$ for G, where $|M'| = n - m$, in polynomial time. Conversely, if $M \subseteq V \cup E$ is a maximum total matching for G, where $m = |M|$, we may find a maximal matching $M' \subseteq E$ for G, where $|M'| = n - m$, in polynomial time.*

Proof. Suppose that $M \subseteq E$ is a maximal matching for G, where $m = |M|$. Then M covers $2m$ vertices of V, so that there is a set V' of vertices not covered by M, where $|V'| = n - 2m$. Set $M' = M \cup V'$. Then M' is a total matching, since by maximality of M, no pair of vertices in V' are adjacent in G. Also M' is maximal by Proposition 3.2, since M' is a total cover of G. Finally $|M'| = m + (n - 2m) = n - m$.

Conversely, suppose that $M \subseteq V \cup E$ is a maximum total matching for G, so that $|M| = \beta_2(G)$. We may construct a set $M'' \subseteq V \cup E$, where $|M''| = |M|$, such that M'' is a total matching for G and for every edge $\{u, v\}$ of E, some edge of M'' is incident on u, or incident on v, or $\{u, v\} \in M''$. For, suppose there is an edge $\{u, v\}$ such that no edge of M is incident on u or v. Then as M is maximal, M covers the edge $\{u, v\}$, by Proposition 3.2. Thus, without loss of generality $u \in M$. Hence we may replace u with $\{u, v\}$ in M. Repeating this procedure with every such edge gives rise to M'', which is clearly a total matching, and must be maximal, since $|M''| = \beta_2(G)$. Now let $M' = M'' \cap E$. Then $M' \subseteq E$ is a matching and is maximal, since no two vertices that are not covered by M' are adjacent in G, by construction of M''. Let $|M'| = n - m$, for some $m > 0$. Then M' covers $2n - 2m$ vertices of G. Thus there are $2m - n$ elements (all vertices) in $M''\backslash M'$, since M'' is a total cover of G. Thus $|M| = |M''| = (n - m) + (2m - n) = m$. $\square$

Corollary 3.4. *There is a polynomial time algorithm to transform a minimum maximal matching into a maximum total matching, and vice versa.*

In order to resolve the complexity of β_2^-, we make the following definition. Given an arbitrary graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$, construct the *pendant graph* $G' = (V', E')$ of G by adding two new vertices u_i and w_i to V, for each i $(1 \leq i \leq n)$, and two new edges $\{u_i, v_i\}$ and $\{w_i, v_i\}$ to E, for each i $(1 \leq i \leq n)$.

Theorem 3.5 (Gimbel and Vestergaard [37]). *Given a graph $G = (V, E)$, where $n = |V|$,*

$$\beta_2^-(G') = 2n - \beta_0(G)$$

where $G' = (V', E')$ is the pendant graph of G.

By Theorem 3.5 and the complexity of β_0 (discussed in Section 4), we deduce that β_2^- is NP-complete for an arbitrary graph. In fact, as β_0 remains NP-complete for planar cubic graphs (see Section 4), we may deduce that β_2^- remains NP-complete for planar graphs of maximum degree 5.

We also note that it is possible to use the transformation of Hedetniemi et al. [46], showing NP-completeness for α_2 in bipartite and chordal graphs, in order to obtain NP-completeness for β_2^- in the same two classes of graphs. Let MINIMUM MAXIMAL TOTAL MATCHING be the decision problem related to β_2^-, which takes a graph G and integer $K \in \mathbb{Z}^+$ and asks whether $\beta_2^-(G) \leq K$.

Theorem 3.6. MINIMUM MAXIMAL TOTAL MATCHING *is NP-complete for bipartite and chordal graphs.*

Proof. Clearly, the problem is in NP for both graph classes. To show NP-hardness, we focus on the transformation of Hedetniemi et al. [46], showing NP-completeness for α_2 in bipartite or chordal graphs. The reduction begins from the NP-complete problem X3C [33, problem SP2], defined in Section 2. A bipartite/chordal graph G is constructed, and an integer K is defined, with the property that the X3C instance has an exact cover if and only if G has a total cover of size at most K.

Corresponding to an exact cover for the X3C instance, the total cover constructed by Hedetniemi et al. [46] is in fact a total matching, and hence a maximal total matching by Proposition 3.2. Conversely, if G has a maximal total matching M of size at most K, then M is a total cover for G by Proposition 3.2, and the corresponding argument of Hedetniemi et al. [46] shows that the X3C instance has an exact cover.

Thus the same reduction may be used to prove NP-completeness for MINIMUM MAXIMAL TOTAL MATCHING in bipartite or chordal graphs. $\square$

As pointed out in Section 1.2, $\beta_2^-(G) = \beta_0^-(T(G))$ for a graph G. Majumdar [59, p.26] shows that a connected graph is a tree if and only if its total graph is strongly chordal[5]. Farber [28] shows that β_0^- is polynomial-time solvable for strongly chordal graphs. Hence β_2^- is polynomial-time solvable for trees.

[5]A graph G is *strongly chordal* if G is chordal and every cycle of length at least six has an 'odd' chord, i.e., a chord joining two vertices that are separated by an odd number of edges.

4 Vertex covering and vertex independence

The decision problems related to determining α_0 and β_0 are the well-known NP-complete problems MINIMUM VERTEX COVER and MAXIMUM INDEPENDENT SET (problems GT1 and GT20 of [33] respectively). The complexities of α_0 and β_0 for any class of graphs are identical, as is indicated by the following proposition, whose proof is trivial.

Proposition 4.1. *Given a graph $G = (V, E)$ and a set $V' \subseteq V$, V' is a vertex cover for G if and only if $V \backslash V'$ is an independent set for G.*

From Proposition 4.1, we deduce the classical result of Gallai [30], namely that for a graph G with n vertices, $\alpha_0(G) + \beta_0(G) = n$. The parameter β_0 is NP-complete, even for planar cubic graphs. This fact may be deduced from separate results due to Garey et al. [34], Garey and Johnson [31], and Maier and Storer [58]. On the other hand, β_0 is polynomial-time solvable for bipartite graphs (by matching – see Harary [41], for example), chordal graphs [35] and trees [22]. Many other classes of graphs for which β_0 remains NP-complete and for which β_0 is polynomial-time solvable are discussed in [33, problem GT20] and [52].

Similarly the complexities of α_0^+ and β_0^- are identical, as the following result shows. Again the proof is simple, and is omitted.

Lemma 4.2. *Given a graph $G = (V, E)$ and a set $V' \subseteq V$, V' is a minimal vertex cover for G if and only if $V \backslash V'$ is a maximal independent set for G.*

From Lemma 4.2 we may deduce another Gallai type identity, that for a graph G with n vertices, $\alpha_0^+(G) + \beta_0^-(G) = n$, as observed by McFall and Nowakowski [60]. In fact the complexities of α_0^+ and β_0^- are related to that of i, the cardinality of a minimum *independent dominating set*. A set of vertices S is an independent dominating set for a graph G if S is both an independent set and a dominating set for G. Independent dominating sets are related to maximal independent sets, as the following lemma demonstrates.

Lemma 4.3 (Berge [4, Thm.2, p.309]). *Given a graph $G = (V, E)$ and a subset V' of V, V' is a maximal independent set if and only if V' is an independent dominating set.*

Thus Lemma 4.3 implies that $i(G) = \beta_0^-(G)$ for any graph G. Lemmas 4.2 and 4.3 together give the following result.

Theorem 4.4. *α_0^+, β_0^-, i each have the same complexity, over every graph class.*

The parameter i is NP-complete for bipartite graphs [19, 49] and dually chordal graphs [8], though polynomial-time algorithms have been found for chordal graphs [27], interval and circular-arc graphs [11], permutation graphs [29, 9], cocomparability graphs [55], asteroidal triple-free graphs [10], k-polygon graphs (for fixed k) [25], series-parallel graphs [68, 39], partial k-trees (for fixed k) [71] and trees [6].

The complexity of i for planar graphs does not seem to be mentioned explicitly in the literature. However, the transformation of Corneil and Perl [19], showing NP-completeness for i in bipartite graphs, begins from MINIMUM DOMINATING SET (which is the decision problem associated with γ, taking a graph G and integer $K \in \mathbb{Z}^+$ as input, and asking whether $\gamma(G) \leq K$) in general graphs and preserves planarity. By transforming from the NP-complete restriction of MINIMUM DOMINATING SET to planar cubic graphs [53], we obtain NP-completeness for i in planar bipartite graphs, where all vertices in one part have degree at most 3, and all vertices in the other part have degree at most 2. An alteration to the transformation of Corneil and Perl gives NP-completeness for i in planar cubic graphs. To aid exposition, we present the proof in its entirety. (In what follows, we refer to the MINIMUM INDEPENDENT DOMINATING SET problem, which takes a graph G and integer $K \in \mathbb{Z}^+$ as input and asks whether $i(G) \leq K$.)

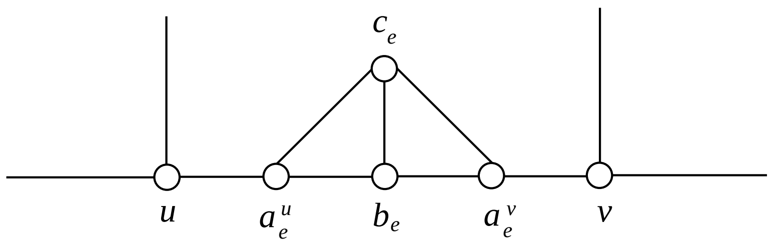

Figure 2: A typical edge component from the constructed instance of MINIMUM INDEPENDENT DOMINATING SET.

Theorem 4.5. MINIMUM INDEPENDENT DOMINATING SET *is NP-complete, even for planar cubic graphs.*

Proof. Clearly MINIMUM INDEPENDENT DOMINATING SET is in NP. For, given $K \in \mathbb{Z}^+$ and a set S of at most K vertices, it is straightforward to verify in polynomial time that S is an independent dominating set.

To show NP-hardness, we give a transformation from the NP-complete MINIMUM DOMINATING SET problem for planar cubic graphs, as discussed above. Hence let $G = (V, E)$ (a planar cubic graph) and K (a positive integer) be an instance of MINIMUM DOMINATING SET. Assume that $|E| = m$. We construct an instance $G' = (V', E')$ (planar cubic graph) and K' (positive integer) of MINIMUM INDEPENDENT DOMINATING SET. Corresponding to every edge $e = \{u, v\}$ of E, construct an *edge component* of G' as follows: replace the edge e by a path on five vertices, namely u, a^u_e, b_e, a^v_e, v, connected in that order. Create an additional vertex, c_e, adjacent to a^u_e, b_e, a^v_e. It may be verified that G' is planar and cubic. An example edge component is shown in Figure 2. Denote by V_e the vertices in the edge component corresponding to edge e, i.e.

$$V_e = \{u, a^u_e, a^v_e, b_e, c_e, v\}.$$

Denote by X_e the internal vertices in this edge component, i.e.

$$X_e = \{a^u_e, a^v_e, b_e, c_e\}.$$

Set $K' = K + m$. We claim that G has a dominating set of cardinality at most K if and only if G' has an independent dominating set of cardinality at most K'.

For, suppose that D is a dominating set for G, where $|D| = k \leq K$. We construct an independent dominating set D' for G'. Initially, let D' contain the vertices of D. For any edge $e = \{u, v\}$ of G, we add vertices to D', according to four cases:

1. $u \notin D, v \notin D$. Add the vertex b_e to D'.
2. $u \in D, v \notin D$. Add the vertex a^v_e to D'.
3. $u \notin D, v \in D$. Add the vertex a^u_e to D'.
4. $u \in D, v \in D$. Add the vertex b_e to D'.

It may be verified that D' is an independent dominating set for G', and $|D'| = k+m \leq K'$.

Conversely, suppose that D' is an independent dominating set for G' of size at most K'. We construct a set D'' as follows. Initially let $D'' = D'$. For any edge $e = \{u, v\}$ of G, consider the elements of $Q_e = V_e \cap D'$. By domination, $|Q_e| \geq 1$, and if $|Q_e| = 1$, then

$Q_e = \{b_e\}$ or $Q_e = \{c_e\}$. By independence, $|Q_e| \leq 3$, and if $|Q_e| = 3$, then $Q_e = \{u, b_e, v\}$ or $Q_e = \{u, c_e, v\}$. If $|Q_e| = 2$, then either $|Q_e \cap X_e| = 1$, or $Q_e = \{a_e^u, a_e^v\}$. In the latter case, replace a_e^v by v in D''.

It may be verified that D'' is a dominating set for G', and $|D''| \leq |D'|$. Now let $D = D'' \cap V$. We claim that D is a dominating set for G. For, suppose that $u \in V \backslash D$. Then $u \notin D''$, so by the domination property of D'', there is some $e = \{u, v\} \in E$ such that $a_e^u \in D''$. By construction of D'', $|D'' \cap X_e| = 1$. Hence, $a_e^v \notin D''$, but as a_e^v must be dominated by D'', the only outcome is $v \in D''$. Hence $v \in D$ as required. Finally, $|D| = |D''| - m \leq |D'| - m \leq K' - m = K$. □

5 Edge covering

In this section, we consider only graphs with no isolated vertices, since the concept of edge covering is undefined for graphs with isolated vertices.

Norman and Rabin [66] demonstrate that there is a polynomial time algorithm to transform a maximum matching to a minimum edge cover, and vice versa. Hence the complexity of α_1 is identical to that of β_1. The proof of this result also demonstrates that a further Gallai type identity holds, i.e., for a graph G with n vertices, $\alpha_1(G) + \beta_1(G) = n$.

The parameter $\alpha_1^+(G)$, the cardinality of a maximum minimal edge cover, seems to have received relatively little attention in the literature. However, the parameter is considered by Hedetniemi [47], who shows that $\alpha_1^+(G) = \varepsilon(G)$ for a non-trivial connected graph G, where $\varepsilon(G)$ denotes the maximum number of *pendant edges* among all spanning forests for G. (Given a spanning forest F for G, $\{u, v\} \in F$ is a pendant edge for F if the degree of u or v in F is one.) Nieminen [63] shows that, for a non-trivial connected graph G with n vertices,

$$\gamma(G) + \varepsilon(G) = n \tag{2}$$

and hence $\gamma(G) + \alpha_1^+(G) = n$. It is clear that these results extend to arbitrary graphs with no isolated vertices. Hence we obtain the following theorem.

Theorem 5.1. *For graphs with no isolated vertices, the complexity of α_1^+ is identical to that of γ.*

The domination number, γ, remains NP-complete for planar cubic graphs [32, 53], bipartite graphs [5] and undirected path graphs (a subclass of chordal graphs) [7], though γ is polynomial-time solvable for strongly chordal graphs [28] and trees [16]. Polynomial-time algorithms and NP-completeness results for γ have been obtained for many other classes of graphs. Chapter 8 of [44] and Chapter 12 of [45] contain two recent algorithmic surveys of γ in various graph classes. See also [33, problem GT2] and [51, 52, 20].

It is also of interest to consider how we may construct a maximum minimal edge cover from a minimum dominating set, and vice versa. For a given graph G and a spanning forest F of G, let $e(F)$ denote the number of pendant edges of F. A spanning forest F of G such that $e(F) = \varepsilon(G)$ is called a *maximum spanning forest* of G. Nieminen's proof of Equation 2 involves constructing in polynomial time a maximum spanning forest $F(D)$ from a minimum dominating set D, where $e(F(D)) = |V| - |D|$. Hedetniemi's proof of $\alpha_1^+(G) = \varepsilon(G)$ involves constructing a maximum minimal edge cover from a maximum spanning forest. Together, these two constructions give a polynomial-time procedure for transforming a minimum dominating set into a maximum minimal edge cover. For the converse, we make the following observation about minimal edge covers (the proof is straightforward, and is omitted):

Proposition 5.2. *Let G be a graph with no isolated vertices and let $S \subseteq V \cup E$. Then S is a minimal edge cover if and only if S is a spanning forest for G that satisfies the following two properties:*

1. *$S \subseteq E$.*

2. *Every edge of S is a pendant edge.*

Thus a minimal edge cover of G is a spanning forest S such that each connected component of S is a non-trivial star (i.e. is a $K_{1,r}$ for some $r \geq 1$).

Given a graph $G = (V, E)$ with no isolated vertices, and a maximum minimal edge cover S of G, we construct a set of vertices $P \subseteq V$ as follows. For each edge $e \in S$, we know that e is a pendant edge, so that at least one endpoint vertex u of e has degree one in S; add u to P. Thus P contains exactly one vertex corresponding to every edge of S, so that $|P| = |S|$. Let $D = V \backslash P$. Then $|D| = \gamma(G)$, and it may be verified that D is a dominating set for G, by Proposition 5.2. Thus, we have the following result.

Theorem 5.3. *There is a polynomial time algorithm to construct a maximum minimal edge cover from a minimum dominating set and vice versa, for arbitrary graphs with no isolated vertices.*

6 Matching

Computation of $\beta_1(G)$ is the usual problem of finding a maximum matching of a graph. The famous algorithm due to Edmonds [24] is described in detail by Lovász and Plummer [57], for example.

The decision problem related to the parameter β_1^-, the cardinality of a minimum maximal matching, is problem GT10 of [33]. In fact, β_1^- is equal to γ', the cardinality of a minimum edge dominating set, as we now show. Two propositions follow, the proof of the first of which is trivial. Both propositions involve the concept of an *independent edge dominating set*, which is a set of edges that is both a matching and an edge dominating set.

Proposition 6.1. *Given a graph $G = (V, E)$ and a set $E' \subseteq E$, E' is a maximal matching for G if and only if E' is an independent edge dominating set for G.*

Proposition 6.2 (Yannakakis and Gavril [74]). *Given a graph $G = (V, E)$ and an edge dominating set E' for G, we may construct, in polynomial time, an independent edge dominating set E'' for G, with $|E''| \leq |E'|$.*

From Propositions 6.1 and 6.2, it follows that $\beta_1^-(G) = \gamma'(G)$ for any graph G, which implies that the complexities of β_1^- and γ' are identical. The minimum edge domination parameter, γ', remains NP-complete for planar or bipartite graphs of maximum degree 3 [74], planar bipartite graphs, their subdivision, line and total graphs, perfect claw-free graphs, planar cubic graphs and iterated total graphs [48]. The problem of computing γ' is polynomial-time solvable for bipartite permutation graphs and cotriangulated graphs [70], trees [62, 74], k-outerplanar graphs [3] and a number of other classes of graphs including claw-free chordal graphs [48].

Proposition 6.1, together with the algorithm implied by the proof of Proposition 6.2, indicates how we may construct a minimum maximal matching from a minimum edge dominating set in polynomial time. The converse is trivial, since any minimum maximal matching is, of course, a minimum edge dominating set.

Graph class ⇒	Arbitrary	Planar	Bipartite	Chordal	Tree
Parameter ⇓					
α_0	as for β_0				
α_0^+	as for β_0^-				
β_0^-	NPC(†)	NPC(*)	NPC[19]	P[27]	P[6]
β_0	NPC(†)	NPC[34]	P[41]	P[35]	P[22]
α_1	as for β_1				
α_1^+	as for γ, i.e.				
	NPC(†)	NPC[33]	NPC[5]	NPC[7]	P[16]
β_1^-	as for γ', i.e.				
	NPC(†)	NPC[74]	NPC[74]	?	P[62]
β_1	P[24]	P(†)	P(†)	P(†)	P(†)
α_2	NPC[59]	NPC(*)	NPC[46]	NPC[46]	P[59]
α_2^+	NPC(†)	NPC(*)	?	?	P[50, 35]
β_2^-	NPC(†)	NPC[37]	NPC(*)	NPC(*)	P[28]
β_2	as for β_1^-				

Table 1: Summary of complexity results in this paper.

7 Summary of results

Table 1 summarises the complexity results for the decision problems associated with each of the parameters discussed in this paper. In a table entry, 'NPC' denotes NP-completeness and 'P' denotes polynomial-time solvability. Appropriate references are indicated. The symbol '†' denotes the fact that either NP-completeness follows by restriction from another result in the same table row, or polynomial-time solvability follows by noting polynomial-time solvability from a class of graphs that contain the class in question. An asterisk indicates that the result is new and the proof is given here, and a question mark indicates that the corresponding problem is open. The classes of graphs dealt with in the table are of course far from exhaustive, but extending our attention beyond planar, bipartite and chordal graphs and trees would give rise to many additional open problems.

8 Conclusion and open problems

The twelve covering and independence parameters studied in this paper are treated collectively as a result of a framework suggested by Nordhaus [64]. However, alternative characterisations of covering and packing parameters in graphs are possible – for example Majumdar [59] presents a framework for such parameters in terms of neighbourhood hypergraphs and Slater [69] considers graphical subset parameters in terms of linear programming and integer programming constructions, using certain matrices defined on a graph.

Relatively speaking, the parameters $\alpha_2, \alpha_2^+, \beta_2, \beta_2^-$ have not been extensively studied, despite their very natural definitions. In particular, there is scope for investigating whether Gallai type identities hold [30]. A survey of such results involving the parameters α_i, α_i^+, β_i, β_i^- for $i = 0, 1$ appears in [17]. As mentioned in Section 1.1, bounds for $\alpha_2(G) + \beta_2(G)$ have been investigated [1, 26, 61], and the identity $\beta_2(G) + \beta_1^-(G) = n$ holds [40], but it is open as to whether bounds exist involving $\alpha_2^+(G) + \beta_2^-(G)$ that improve on those obtained by simply considering the sum of known upper and lower bounds for α_2^+ and β_2^- separately.

Similarly, the existence of Nordhaus–Gaddum [65] type inequalities are of interest. Such results have been obtained for the parameters β_0 and β_1 [12], β_0^- [18, 14, 15, 43], γ and γ' (see [42] for a survey). As reported in Section 1.1, Nordhaus–Gaddum inequalities involving α_2, β_2 and β_2^- have been obtained [26, 61, 37], but there is still scope for investigating such bounds involving the other parameters treated in this paper.

Table 1 indicates some of the open problems regarding the complexity of the parameters considered in this paper when restricted to certain classes of graphs. One, perhaps significant, open problem is the complexity of MINIMUM EDGE DOMINATING SET for chordal graphs – that this problem is open is noted by Horton and Kilakos [48].

The NP-completeness results for the parameters considered here imply that their properties of approximability are of interest. Results have been obtained for the parameters $\alpha_0, \beta_0, \beta_0^-$ and are surveyed in [21]. Regarding the approximability of β_1^-, any maximal matching is a 2-approximation to β_1^- [54]. Proposition 6.2 implies that we may construct, in polynomial time, a maximal matching E'' from an edge dominating set E', such that $|E''| \leq |E'|$. Thus, since $\beta_1^- = \gamma'$, and MINIMUM EDGE DOMINATING SET admits a PTAS for planar graphs [3], then MINIMUM MAXIMAL MATCHING also admits a PTAS for planar graphs. Also, β_1^- is APX-complete, even for graphs of maximum degree 3 [76] (definitions of the terms *2-approximation*, *PTAS* and *APX-complete* may be found in [21], for example). However it appears that the approximability of the parameters $\alpha_0^+, \alpha_1^+, \alpha_2, \alpha_2^+, \beta_2^-, \beta_2$ is open.

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