Abstract. In this paper we consider some forbidden sublattices for $n$-distributive, but non-modular lattices. We define the new notion of $n$-modularity (weaker than $n$-distributivity). We also consider some forbidden sublattice for an $n$-modular lattice. We prove that $n$-modularity implies $(n+1)$-modularity. The counter-examples for the inverse implication are shown.

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1. Introduction

We recall the $n$-distributivity notion, which was introduced by G. M. Bergman (in [1]) and A. P. Huhn (in [4]) as a generalization of the ordinary distributivity (for $n = 1$), for modular lattices:

A lattice $(L, \lor, \land)$ is $n$-distributive if for every $x, y_0, \ldots, y_n \in L$ the condition is satisfied:

$$(D_n) \quad x \land \lor_{i=0}^n y_i = \lor_{j=0}^n (x \land \lor_{i=0; i\neq j}^n y_i).$$

A lattice $L$ is dually $n$-distributive if for every $x, y_0, \ldots, y_n \in L$ the following equality is satisfied:

$$x \lor \land_{i=0}^n y_i = \land_{j=0}^n (x \lor \land_{i=0; i\neq j}^n y_i).$$

A lattice $L$ is modular, if for every $x, y, z \in L$, $x \leq y$ implies $x \land (y \lor z) = x \lor (x \land z)$.

The condition $(D_n)$ is equivalent to the dual $n$-distributivity condition iff a lattice $L$ is modular (see [4]).

It is easy to show that every $n$-distributive lattice (dually $n$-distributive) is also $(n+1)$-distributive (dually $(n+1)$-distributive, respectively). For standard terminology, see [3].

We introduce two notions weaker than notion of $n$-distributivity and dual $n$-distributivity, respectively:
1) A lattice \((L, \lor, \land)\) is \(n\)-modular if for every \(x, y_0, \ldots, y_n \in L\) the following implication is true:
\[
[x \land \lor_{i=0}^{n-1} y_i \leq x] \Rightarrow [x \land \lor_{i=0}^{n} y_i = (\lor_{i=0}^{n-1} y_i) \lor \lor_{j=0}^{n-1} (x \land \lor_{i=0; i \neq j}^{n} y_i)].
\]

2) A lattice \((L, \lor, \land)\) is dually \(n\)-modular if for every \(x, y_0, \ldots, y_n \in L\) the implication:
\[
[\land_{i=0}^{n-1} y_i \geq x] \Rightarrow [x \lor \land_{i=0}^{n} y_i = (\land_{i=0}^{n-1} y_i) \land \land_{j=0}^{n-1} (x \lor \land_{i=0; i \neq j}^{n} y_i)]
\]
is valid.

The 1-modular lattices and dually 1-modular lattices are exactly modular.

If \(P\) is a poset and for \(a, b, c \in P\) the conditions \(a < b, a \leq c \leq b\) imply \(c = a\) or \(c = b\), then we say, that \(b\) covers \(a\) in the set \(P\) (or \(a\) is covered by \(b\)).

2. Some properties for \(n\)-distributive and \(n\)-modular lattices; Characterization of an \(n\)-modular lattice by the forbidden sublattice

In 1972 A. P. Huhn (see [4]) proved that a modular lattice \(L\) is not \(n\)-distributive iff it contains a sublattice \(B\) isomorphic to the \(2^{n+1}\)-element Boolean lattice and an element \(x\) such that \(x \land a = \land B, x \lor a = \lor B,\) for every atom \(a\) of \(B\). For \(n = 1\), it is the well-known criterion of distributivity.

The following proposition without the modularity assumption is some partial generalization for the above Huhn’s result.

**Proposition 1.** A lattice \((L, \lor, \land)\) is not \(n\)-distributive whenever it contains a sublattice \(B\) isomorphic to the \(2^{n+1}\)-element Boolean lattice and an element \(x\) such that \(x > b, \) for some \(b \in B\) and \(\lor B\) is the only element in \(B\), which covers \(x\) in \(L\).

**Proof.** Let \(\{y_0, \ldots, y_n\}\) be the set of atoms in the algebra \(B\). Then
\[
\land_{i=0}^{n} y_i = \land B = x.
\]

According to the assumption there is an element \(b_0 \in B\) such that \(x\) covers \(b_0\) in the poset \(B \cup \{x\}\). Hence, \(x \land \lor_{i=0; i \neq j}^{n} y_i \leq b_0 < x,\) for \(0 \leq j \leq n\) and \(\lor_{j=0}^{n} (x \land \lor_{i=0; i \neq j}^{n} y_i) \leq b_0 < x,\) which contradicts the \(n\)-distributivity.

**Corollary 1.** A lattice \((L, \lor, \land)\) is not dually \(n\)-distributive whenever it contains a sublattice \(B\) isomorphic to the \(2^{n+1}\)-element Boolean lattice and an element \(x\) covering \(\land B\) in \(L\) such that \(x < b_0,\) for some \(b_0 \in B\).

The inverse implication in the above theorem seems true, but it is still an open problem.
Proposition 2. A lattice \((L, \lor, \land)\) is \(n\)-modular iff for comparable elements \(x\) and \(\bigvee_{i=0}^{n-1} y_i\) the following equality is satisfied:
\[
(M_n) \quad x \land \left[ (x \land \bigvee_{i=0}^{n-1} y_i) \lor y_n \right] = \bigvee_{j=0}^{n}(x \land \bigvee_{i=0; i \neq j}^{n} y_i).
\]

Proof. Assuming \(\bigvee_{i=0}^{n-1} y_i \leq x\) in \((M_n)\) we get \(x \land \bigvee_{i=0}^{n} y_{i} = (\bigvee_{i=0}^{n-1} y_{i}) \lor \bigvee_{j=0}^{n}(x \land \bigvee_{i=0; i \neq j}^{n} y_{i})\), which gives \(n\)-modularity. Let \(\bigvee_{i=0}^{n-1} y_i > x\), then we get \(n\)-modularity applying the absorption laws. Now, let \(\bigvee_{i=0}^{n-1} y_i \leq x\) and assume that \((M_n)\) fails, for some \(x, y_0, \ldots, y_n \in L\). Then
\[
x \land \bigvee_{i=0}^{n} y_{i} = x \land \left[ (x \land \bigvee_{i=0}^{n-1} y_{i}) \lor y_n \right] \neq \bigvee_{j=0}^{n}(x \land \bigvee_{i=0; i \neq j}^{n} y_{i}),
\]
which contradicts the \(n\)-modularity. \(\square\)

Corollary 2. A lattice \((L, \lor, \land)\) is dually \(n\)-modular iff for comparable elements \(x\) and \(\bigwedge_{i=0}^{n-1} y_i\) the following equality is valid:
\[
x \lor \left[ (x \lor \bigwedge_{i=0}^{n-1} y_{i}) \land y_n \right] = \bigwedge_{j=0}^{n}(x \lor \bigwedge_{i=0; i \neq j}^{n} y_{i}).
\]

Proposition 3. Let \(n \geq 1\). Then:

(i) Every \(n\)-distributive (dually \(n\)-distributive) lattice is \(n\)-modular (dually \(n\)-modular, respectively).

(ii) Every \(n\)-modular (dually \(n\)-modular) lattice is \((n+1)\)-modular (dually \((n+1)\)-modular, respectively).

Proof. First implication is obvious. Now we prove that the usual modularity implies \(n\)-modularity for \(n > 1\). Let \(\bigvee_{i=0}^{n-1} y_i \leq x\). Then using modularity, we get
\[
x \land \bigvee_{i=0}^{n} y_{i} = x \land (\bigvee_{i=0}^{n-1} y_{i} \lor \bigvee_{i=0; i \neq j}^{n} y_{i}) = \bigvee_{i=0}^{n-1} y_{i} \lor \left( x \land \bigvee_{i=0; i \neq j}^{n} y_{i} \right),
\]
for every \(0 \leq j \leq n-1\). Hence, \(x \land \bigvee_{i=0}^{n} y_{i} = (\bigvee_{i=0}^{n-1} y_{i}) \lor \bigvee_{j=0}^{n}(x \land \bigvee_{i=0; i \neq j}^{n} y_{i})\), what gives \(n\)-modularity. Now, let \(x, y_0, \ldots, y_n, y_{n+1} \in L\), \(\bigvee_{i=0}^{n} y_i \leq x\) and let \(0 \leq l, k \leq n\) be fixed indices. Then assuming \(n\)-modularity and treating \(y_l \lor y_k\) as a single element we get the equality:
\[
x \land \bigvee_{i=0}^{n} y_{i} = \bigvee_{i=0}^{n} y_{i} \lor \left[ \bigvee_{j=0; j \neq l, k}^{n} (x \lor \bigvee_{i=0; i \neq j}^{n} y_{i}) \right] \lor (x \land \bigvee_{i=0; i \neq l, k}^{n} y_{i}).
\]
The supremum over all \(0 \leq l, k \leq n\) of the right-hand side of this equality is exactly equal to \(\bigvee_{i=0}^{n} y_{i} \lor \bigvee_{j=0}^{n}(x \land \bigvee_{i=0; i \neq j}^{n+1} y_{i})\). Hence we get \((n+1)\)-modularity. Analogously, inverting operations we prove the dual theorem. \(\square\)

Remark. The inverse implications in the Proposition 3 are not always true!

The lattices \(L_1, L_2, L_3\) (see Figure 1) are not modular;
\(L_2, L_3\) are not 2-distributive, but they are 2-modular;
\(L_1\) is not 2-distributive and not 2-modular.
Proposition 4. A lattice $(L, \lor, \land)$ is not $n$-modular whenever it contains a sublattice $B$ isomorphic to the $2^{n+1}$-element Boolean lattice and an element $x$ such that $c_0 \leq x < \lor B$, for some coatom $c_0$ of $B$. A lattice $L$ is not 2-modular if and only if it contains an isomorphic copy of $L_1$ as a poset (see Figure 1).

Proof. Let $A = \{y_0, y_1, \ldots, y_n\} \subseteq B$ be the set of atoms of $B$. Since $c_0 = \lor_{i=0}^{n-1} y_i < x$, hence $x \land \lor_{i=0}^{n-1} y_i = x$. An element $\lor_{i=0; j \neq i}^{n} y_i$ is a coatom of $B$, for $0 \leq j \leq n$. Hence $x \land \lor_{i=0; j \neq i}^{n} y_i \leq c_0 < x$, for every $0 \leq j \leq n$ and $(\lor_{i=0}^{n-1} y_i) \lor (\lor_{j=0}^{n-1} (x \land \lor_{i=0; j \neq i}^{n} y_i)) = \lor_{i=0}^{n-1} y_i = c_0$, which contradicts the $n$-modularity.

Now, we prove the inverse implication, in the case $n = 2$. Assume that $L$ is not 2-modular. Then for some $x, y_1, y_2, y_3 \in L$, $y_1 \lor y_2 < x$ we get the inequality (*) $x \land (y_1 \lor y_2 \lor y_3) > (y_1 \lor y_2) \lor [x \land (y_1 \lor y_3)] \lor [x \land (y_2 \lor y_3)]$.

Notice that comparability of every pair of elements $y_1, y_2, y_3$ contradicts this inequality. Now, let $y_1 \lor y_2 = y_1 \lor y_3$. Then $(y_1 \lor y_2) \geq (y_2 \lor y_3)$, $x \land (y_1 \lor y_2 \lor y_3) = y_1 \lor y_2$ and $(y_1 \lor y_2) \lor [x \land (y_1 \lor y_3)] \lor [x \land (y_2 \lor y_3)] = (y_1 \lor y_2) \lor [x \land (y_2 \lor y_3)] = (y_1 \lor y_2)$.

If $(y_1 \lor y_3) = (y_2 \lor y_3)$, then $(y_1 \lor y_2) \leq (y_2 \lor y_3)$, $x \land (y_1 \lor y_2 \lor y_3) = x \land (y_1 \lor y_3)$ and $(y_1 \lor y_2) \lor [x \land (y_1 \lor y_3)] \lor [x \land (y_2 \lor y_3)] = (y_1 \lor y_2) \lor [x \land (y_1 \lor y_3)] = x \land (y_1 \lor y_3)$. Hence, elements $y_1 \lor y_2, y_1 \lor y_3, y_2 \lor y_3$ must be different. Similarly, if any two of the following elements $y_1 \lor y_2, y_1 \lor y_3, y_2 \lor y_3$ are comparable, then it contradicts the inequality (*). Three incomparable elements $y_1 \lor y_2, y_1 \lor y_3, y_2 \lor y_3$ generate a lattice isomorphic to the $2^3$-element Boolean lattice (see. [3], p. 48).

Hence, $L$ must contain $L_1$. \hfill \blackslug

Corollary 3. A lattice $(L, \lor, \land)$ is not dually $n$-modular whenever it contains a sublattice $B$ isomorphic to the $2^{n+1}$-element Boolean lattice and an element $x$ such that $\land B < x < a_0$, for some atom $a_0$ of $B$ (the inverse implication is true for $n = 1$ and $n = 2$).
The inverse implication of Proposition 4 seems true also for \( n > 2 \), but it is still an open problem (for \( n = 1 \) it is the well-known criterion of modularity).

A. P. Huhn proved, that for a modular lattice \( L \) the equality:
\[
\bigwedge_{j=0}^{n+1} \bigvee_{i=0; i \neq j}^{n+1} y_i = \bigwedge_{k=0}^{n+1} \bigvee_{j=0; j \neq k}^{n+1} \bigvee_{i=0; i \neq j,k}^{n+1} y_i,
\]
for \( y_0, \ldots, y_{n+1} \in L \) is equivalent to \((D_n)\) (see [4], [5]). The next proposition gives the equality condition implying \((D_n)\) without modularity assumption:

**Proposition 5.** A lattice \( L \) is \( n \)-distributive whenever for every \( y_0, \ldots, y_{n+1} \in L \) the following equality is satisfied:
\[
\bigwedge_{j=0}^{n+1} \bigvee_{i=0; i \neq j}^{n+1} y_i = (\bigwedge_{i=0}^{n} y_i) \lor \bigvee_{j=0}^{n+1} (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i).
\]

**Proof.** Denote the left-hand side of the above equality by \( a \), and the right-hand one by \( b \). Assuming \( \bigvee_{i=0}^{n-1} y_i \leq y_{n+1} \) in \( a = b \) and using the absorption laws we get \( y_{n+1} \land \bigvee_{i=0}^{n} y_i = (\bigvee_{i=0}^{n-1} y_i) \lor (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i) \), what gives \( n \)-modularity. Notice, that \( y_{n+1} \land a = y_{n+1} \land (\bigwedge_{j=0}^{n+1} \bigvee_{i=0; i \neq j}^{n+1} y_i) = y_{n+1} \land \bigvee_{i=0}^{n} y_i \land \bigwedge_{j=0}^{n} (y_{n+1} \lor \bigvee_{i=0; i \neq j}^{n} y_i) = y_{n+1} \land \bigvee_{i=0}^{n} y_i. \)

Since \( L \) is \( n \)-modular and \( \bigvee_{j=0}^{n} (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i) \leq y_{n+1} \), hence
\[
y_{n+1} \land b = y_{n+1} \land [(\bigwedge_{i=0}^{n} y_i) \lor \bigvee_{j=0}^{n} (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i)] = [\bigvee_{j=0}^{n} (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i)] \lor \bigvee_{j=0}^{n} \{y_{n+1} \land [(\bigwedge_{i=0}^{n} y_i) \lor (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i)]\}.
\]

Because of the inequality
\[
\{y_{n+1} \land [(\bigwedge_{i=0}^{n} y_i) \lor (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i)]\} \leq (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i), 0 \leq j \leq n,
\]
which is valid for an arbitrary lattice, we deduce
\[
y_{n+1} \land b = \bigvee_{j=0}^{n} (y_{n+1} \land \bigvee_{i=0; i \neq j}^{n} y_i).
\]
The equality \( y_{n+1} \land a = y_{n+1} \land b \) gives \( n \)-distributivity. \( \square \)

There are some useful applications for \((D_n)\) condition in lattices of closed sets with respect to a given closure operator. For example, the \( n \)-distributivity property can be associated to the Carathéodory number, which is some parameter describing a closure operator on a given set (see [2], [6–8]).

**References**


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