Some endpoint inequalities for multilinear integral operators

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Abstract. In this paper, the endpoint estimates for some multilinear operators related to certain fractional singular integral operators are obtained. The operators include Calderón–Zygmund singular integral operator and fractional integral operator.

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1. Introduction

Let $T$ be the Calderón–Zygmund singular integral operator, the classical result by Coifman, Rochberg and Weiss (see [6]) states that the commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [1]) has proved a similar result when $T$ was replaced by the fractional integral operator; in [9], the endpoint boundedness of the commutators was obtained. The main purpose of this paper is to establish the endpoint boundedness of some multilinear operators related to certain non-convolution type fractional singular integral operators. As an application, the endpoint boundedness of the multilinear operators related to the Calderón–Zygmund singular integral operator and fractional integral operator is obtained.

2. Notations and results

Throughout this paper, $Q$ will denote a cube of $R^n$ with sides parallel to the axes. For a cube $Q$ and a locally integrable function $f$, let $f_Q = |Q|^{-1} \int_Q f(x) \, dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q| \, dy$. For
a weight function \( w \), \( f \) is said to belong to \( BMO(w) \) if \( f^# \in L^\infty(w) \). Set 
\[ \| f \|_{BMO(w)} = \| f^# \|_{L^\infty(w)}. \]
Note that \( BMO(w) = BMO(R^n) \) if \( w = 1 \).
A function \( a \) is called an \( H^1 \) atom if there exists a cube \( Q \) such that \( a \) is supported in \( Q \), 
\[ \| a \|_{L^\infty(w)} \leq w(Q)^{-1} \text{ and } \int a(x)dx = 0. \]
It is well known that the Hardy space \( H^1(w) \) has the atomic decomposition characterization (see [8, 12]).

In this paper, we consider a class of multilinear integral operators defined in the following way.

First, given a fixed locally integrable function \( K(x, y) \) on \( R^n \times R^n \), set 
\[ T_K(f)(x) = \int_{R^n} K(x, y)f(y) \, dy \]
for every bounded and compactly supported function \( f \). We write \( K \in \Sigma_\delta \) for \( \delta \geq 0 \)
if 
\[ |K(x, y)| \leq C|x - y|^{-n+\delta} \]
and 
\[ |K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leq C|y - z|\varepsilon|x - z|^{-n-\varepsilon+\delta} \]
and \( 2|y - z| \leq |x - z| \) for a fixed \( \varepsilon > 0 \). \( T_K \) is called a fractional singular integral operator if \( K \in \Sigma_\delta \) for some \( \delta \geq 0 \).

Now, let \( m \) be a positive integer and \( A \) be a function on \( R^n \). Set 
\[ R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha, \]
and 
\[ Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} D^\alpha A(x)(x - y)^\alpha. \]
The multilinear operator associated with the fractional singular integral operator \( T_K \) is defined by 
\[ T_K^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y)f(y) \, dy. \]
We also consider the variant of \( T_K^A \), which is defined by 
\[ \tilde{T}_K^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A; x, y)}{|x - y|^m} K(x, y)f(y) \, dy. \]
Note that $\tilde{T}_K^A$ is closely related to $T_K^A$, for

$$R_{m+1}(A; x, y) - Q_{m+1}(A; x, y) = \sum_{|\alpha|=m} \frac{1}{\alpha!} (x - y)^\alpha (D^\alpha A(x) - D^\alpha A(y)).$$

Note that when $m = 0$, $T_K^A$ is just the commutators of $T_K$ and $A$ (see [1, 6, 9]). It is well known that multilinear operator, as an extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see, e.g., [2–5]). In [7] and [10], the weighted $L^p$ ($p > 1$) and $H^p$ ($0 < p \leq 1$) boundedness of the multilinear operator related to the Calderón–Zygmund singular integral operator was obtained; in [2], the weak $(H^1, L^1)$ boundedness of the multilinear operator related to some singular integral operator was obtained.

Now we state our results as following.

**Theorem 2.1.** Let $0 \leq \delta < n$ and $D^\alpha A \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m$. Suppose $T_K$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p, q \in (1, +\infty)$ and $1/q = 1/p - \delta/n$. If $K \in \Sigma_\delta$, then

(a) $T_K^A$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$;

(b) $\tilde{T}_K^A$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$;

(c) $T_K^A$ is bounded from $H^1(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$.

**Theorem 2.2.** Let $D^\alpha A \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m$ and $w \in A_1$. Suppose $T_K$ is bounded on $L^p(w)$ for all $1 < p \leq \infty$. If $K \in \Sigma_0$, then

(i) $T_K^A$ is bounded from $L^\infty(w)$ to $BMO(w)$;

(ii) $\tilde{T}_K^A$ is bounded from $H^1(w)$ to $L^1(w)$;

(iii) $T_K^A$ is bounded from $H^1(w)$ to weak $L^1(w)$.

**Remark 2.1.** The boundedness is uniform with respect to $K \in \Sigma_\delta$ and $K \in \Sigma_0$, respectively. In general, $T_K^A$ is not $(H^1, L^{n/(n-\delta)})$ or $(H^1(w), L^1(w))$ bounded.
3. Proofs of the theorems

To prove these theorems, we need the following lemmas.

**Lemma 3.1 (see [5, p. 448]).** Let $A$ be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{|Q(x, y)|} \int_{Q(x, y)} |D^\alpha A(z)|^q \, dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

**Lemma 3.2 (see [1, p. 8]).** Let $b \in \text{BMO}(\mathbb{R}^n)$ and $C_b$ be the commutator defined by

$$C_b(f)(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x - y|^{n-\delta}} f(y) \, dy.$$

1. If $0 \leq \delta < n$, $1 < p < \infty$ and $1/q = 1/p - \delta/n$, then $C_b$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and from $H^1(\mathbb{R}^n)$ to weak $L^{n/(n-\delta)}(\mathbb{R}^n)$.

2. If $\delta = 0$, $1 < p < \infty$ and $w \in A_1$, then $C_b$ is bounded on $L^p(w)$ and from $H^1(w)$ to weak $L^1(w)$.

**Lemma 3.3 (see [5, p. 454(28)] and [12, p. 222]).** Let $Q$ be a cube and $\tilde{A}(x) = A(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A)_{Q} x^\alpha$. Then $R_{m+1}(A; x, y) = R_{m+1}(\tilde{A}; x, y)$.

**Lemma 3.4 (see [3, p. 695, Lemma 2.2]).** Let $Q_1$ and $Q_2$ be the cubes with $Q_1 \subset Q_2$. Then

$$|b_{Q_1} - b_{Q_2}| \leq C (1 + |\log(|Q_1|/|Q_2|)|) \|b\|_{\text{BMO}}.$$

**Proof of Theorem 2.1.** (a) It suffices to prove that there exists a constant $C_Q$ such that

$$\frac{1}{|Q|} \int_{Q} |T_K^{A}(f)(x) - C_Q| \, dx \leq C \|f\|_{L^{n/\delta}}$$

holds for any cube $Q$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A)_{Q} x^\alpha$, then $R_{m+1}(A; x, y) = R_{m+1}(\tilde{A}; x, y)$ by induction and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for all $\alpha$ with $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus \tilde{Q}}$, 
Some endpoint inequalities...

\[ T^A_K(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x-y|^m} K(x, y) f(y) \, dy \]

\[ = \int_{\mathbb{R}^n} \frac{R_m(\tilde{A}; x, y)}{|x-y|^m} K(x, y) f_1(y) \, dy \]

\[ - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{K(x, y)(x-y)^{\alpha}}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y) \, dy \]

\[ + \int_{\mathbb{R}^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x-y|^m} K(x, y) f_2(y) \, dy, \]

then

\[ |T^A_K(f)(x) - T^A_K(f_2)(x_0)| \leq \left| T_K \left( \frac{R_m(\tilde{A}; x, \cdot)}{|x-\cdot|^m} f_1 \right)(x) \right| \]

\[ + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left| T_K \left( \frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1 \right)(x) \right| + \left| T^\tilde{A}_K(f_2)(x) - T^\tilde{A}_K(f_2)(x_0) \right| \]

\[ := I(x) + II(x) + III(x), \]

and, thus,

\[ \frac{1}{|Q|} \int_{Q} |T^A_K(f)(x) - T^A_K(f_2)(x_0)| \, dx \]

\[ \leq \frac{1}{|Q|} \int_{Q} I(x) \, dx + \frac{1}{|Q|} \int_{Q} II(x) \, dx + \frac{1}{|Q|} \int_{Q} III(x) \, dx \]

\[ := I + II + III. \]

Now, let us estimate \( I, II \) and \( III \), respectively. First, we have known (see [12, p. 144]), for \( b \in BMO(\mathbb{R}^n) \),

\[ \|b\|_{BMO} \approx \sup_{Q} \left( \frac{1}{|Q|} \int_{Q} |b(y) - b_Q|^p \, dy \right)^{1/p}, \]

then, for \( x \in Q \) and \( y \in \tilde{Q} \), using Lemma 3.1 and Lemma 3.4, we get

\[ R_m(\tilde{A}; x, y) \leq C |x-y|^m \sum_{|\alpha|=m} \left[ \frac{1}{|Q(x, y)|} \int_{Q(x, y)} \left( |D^\alpha A(z) - (D^\alpha A)\tilde{Q}(x,y)| \right) \right]^{1/q} \]

\[ + \left| (D^\alpha A)\tilde{Q}(x,y) - (D^\alpha A)\tilde{Q} \right|^q \, dz \]
\[ \leq C|x - y|^m \sum_{|\alpha| = m} (\|D^\alpha A\|_{BMO} + 1 + |\log |Q(x,y)|/|\tilde{Q}||) \]

\[ \leq C|x - y|^m \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO}, \]

thus, by the \((L^{n/\delta}, L^\infty)\)-boundedness of \(T_K\), we have

\[ I \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO} \|T_\delta(f_1)\|_{L^\infty} \]

\[ \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO} f_{L^{n/\delta}}; \]

Secondly, by the \((L^p, L^q)\)-boundedness of \(T_K\) for \(1/q = 1/p - \delta/n\), \(p > 1\) and Hölder's inequality, we gain

\[ II \leq \frac{C}{|Q|} \int_{Q} T_\delta \left( \sum_{|\alpha| = m} (D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1 \right)(x) \, dx \]

\[ \leq C \sum_{|\alpha| = m} \left( \frac{1}{|Q|} \int_{Q} |T_\delta((D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1)(x)|^q \, dx \right)^{1/q} \]

\[ \leq C|Q|^{-1/q} \sum_{|\alpha| = m} \|(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1\|_{L^p} \]

\[ \leq C \sum_{|\alpha| = m} \left( \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha A(y) - (D^\alpha A)_{\tilde{Q}}|^q \, dy \right)^{1/q} \|f\|_{L^{n/\delta}} \]

\[ \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}. \]

To estimate \(III\), we write

\[ T_{K}(f_2)(x) - T_{K}(f_2)(x_0) \]

\[ = \int_{R^n} \left[ \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right] R_m(\tilde{A}; x, y) f_2(y) \, dy \]

\[ + \int_{R^n} \frac{K(x_0, y) f_2(y)}{|x_0 - y|^m} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)] \, dy \]
\[- \sum_{|\alpha| = m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left( \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} - \frac{K(x_0, y)(x_0 - y)^\alpha}{|x_0 - y|^m} \right) D^\alpha \tilde{A}(y) f_2(y) \, dy \leq III_1 + III_2 + III_3; \]

By Lemma 3.1 and Lemma 3.4, we know that, for \( x \in Q \) and \( y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q} \),

\[ |R_m(\tilde{A}; x, y)| \leq C |x - y|^m \sum_{|\alpha| = m} (\|D^\alpha A\|_{BMO} + |(D^\alpha A)\tilde{Q}(x, y) - (D^\alpha A)\tilde{Q}|) \leq C k |x - y|^m \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO}. \]

Note that \( |x - y| \sim |x_0 - y| \) for \( x \in Q \) and \( y \in \mathbb{R}^n \setminus \tilde{Q} \), we obtain, by the condition on \( K \),

\[ |III_1| \leq C \int_{\mathbb{R}^n} \left( \frac{|x - x_0|}{|x_0 - y|^{m + n + 1 - \delta}} + \frac{|x - x_0|^{\varepsilon}}{|x_0 - y|^{m + n + \varepsilon - \delta}} \right) |R_m(\tilde{A}; x, y)| |f_2(y)| \, dy \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k \left( \frac{|x - x_0|}{|x_0 - y|^{n + 1 - \delta}} + \frac{|x - x_0|^{\varepsilon}}{|x_0 - y|^{n + \varepsilon - \delta}} \right) |f(y)| \, dy \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}} \sum_{k=1}^{\infty} k (2^{-k} + 2^{-\varepsilon k}) \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}; \]

For \( III_2 \), by the formula (see (39) in [5]):

\[ R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x, x_0)(x - y)^\beta \]

and Lemma 3.1, we have

\[ |R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)| \leq C \sum_{|\beta| < m} \sum_{|\alpha| = m} |x - x_0|^{m - |\beta|} |x - y|^{\beta} \|D^\alpha A\|_{BMO}, \]
similar to the estimates of $III_1$, we get

$$|III_2| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} |f(y)| \, dy$$

$$\leq C \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}};$$

For $III_3$, by taking $r > 1$ such that $1/r + \delta/n = 1$, similar to the estimates of $III_1$, we get

$$|III_3| \leq C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \left( \frac{|x-x_0|}{|x_0-y|^{n+\epsilon-\delta}} \right) |D^\alpha \tilde{A}(y)||f(y)| \, dy$$

$$\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} (2^{-k} + 2^{-\epsilon k}) \left( |2^k \tilde{Q}|^{-1} \int_{2^kQ} |D^\alpha A(y) - (D^\alpha A)y|^r \, dy \right)^{1/r} \|f\|_{L^{n/\delta}}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}.$$

Thus

$$III \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}.$$

(b) It is only to show that there exists a constant $C > 0$ such that for every $H^1$-atom $a$ (that is that $a$ satisfies: supp $a \subset Q = Q(x_0, d)$, \|a\|_{L^\infty} \leq |Q|^{-1} and $\int a(y) \, dy = 0$ (see [8])), the following holds:

$$\|\tilde{T}_K^A(a)\|_{L^{n/(n-\delta)}} \leq C.$$

We write

$$\int_{R^n} \left[ \tilde{T}_K^A(a)(x) \right]^{n/(n-\delta)} \, dx$$

$$= \left[ \int_{|x-x_0| \leq 2r} + \int_{|x-x_0| > 2r} \right] \left[ \tilde{T}_K^A(a)(x) \right]^{n/(n-\delta)} \, dx := J + JJ.$$

For $J$, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!} (x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$
we have,

\[ |\tilde{T}^A_K(a)(x)| \leq |T^A_K(a)(x)| + C \sum_{|\alpha|=m} \int_{\mathbb{R}^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^{n-\delta}} |a(y)| \, dy, \]

thus, \( \tilde{T}^A_K \) is \((L^p, L^q)\)-bounded by Lemma 3.2 and (a), where \( 1/q = 1/p - \delta/n \). We see that

\[ J \leq C \|\tilde{T}^A_K(a)\|^{\frac{n}{n-\delta}}_{L^q} |2Q|^{1-\frac{n}{n-\delta}} \leq C \|a\|^{\frac{n}{n-\delta}}_{L^p} |Q|^{1-\frac{n}{n-\delta}} \leq C. \]

To obtain the estimate of \( JJ \), we denote \( \tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} \times (D^\alpha A)_{2Q} x^\alpha \). Then \( Q_m(A; x, y) = Q_m(\tilde{A}; x, y) \). We write, by the vanishing moment of \( a \),

\[ \tilde{T}^A_K(a)(x) = \int_{\mathbb{R}^n} \frac{K(x, y) R_m(A; x, y)}{|x - y|^m} a(y) \, dy \]

\[ - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{K(x, y) D^\alpha \tilde{A}(x) (x - y)^\alpha}{|x - y|^m} a(y) \, dy \]

\[ = \int_{\mathbb{R}^n} \left[ \frac{K(x, y)}{|x - y|^m} - \frac{K(x, x_0)}{|x - x_0|^m} \right] R_m(\tilde{A}; x, y) a(y) \, dy \]

\[ + \int_{\mathbb{R}^n} \frac{K(x, x_0)}{|x - x_0|^m} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)] a(y) \, dy \]

\[ - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left[ \frac{K(x, y) (x - y)^\alpha}{|x - y|^m} - \frac{K(x, x_0) (x - x_0)^\alpha}{|x - x_0|^m} \right] D^\alpha \tilde{A}(x) a(y) \, dy, \]

\[ := JJ_1 + JJ_2 + JJ_3. \]

Now, similar to the proof of III, we obtain, for \( x \in (2Q)^c \)

\[ |JJ_1| \leq C \int_{\mathbb{R}^n} \left[ \frac{|y - x_0|}{|x - y|^{n+m+1-\delta}} + \frac{|y - x_0|^\varepsilon}{|x - y|^{n+m+\varepsilon-\delta}} \right] |R_m(\tilde{A}; x, y)||a(y)| \, dy \]

\[ \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |Q|^{1/n} |x - x_0|^{-n-1+\delta} + |Q|^{\varepsilon/n} |x - x_0|^{-n-\varepsilon+\delta}, \]

\[ |JJ_2| \leq C \int_{\mathbb{R}^n} \frac{|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)| |a(y)|}{|x - y|^{m+n-\delta}} \, dy \]
\[
\leq C \sum_{|\alpha|=m} \|D^{\alpha}A\|_{BMO} \int_{R^n} \frac{|x_0 - y| |a(y)|}{|x - x_0|^{n+1-\delta}} dy \\
\leq C \sum_{|\alpha|=m} \|D^{\alpha}A\|_{BMO} |Q|^{1/n} |x - x_0|^{-n-1+\delta}
\]

and
\[
|JJ_3| \leq C \int_{R^n} \frac{|x_0 - y|}{|x - y|^{n+1-\delta}} \sum_{|\alpha|=m} |D^{\alpha} \tilde{A}(x)| |a(y)| dy \\
\leq C \sum_{|\alpha|=m} |D^{\alpha} \tilde{A}(x)||Q|^{1/n} |x - x_0|^{-n-1+\delta} + |Q|^{\varepsilon/n} |x - x_0|^{-n-\varepsilon+\delta}. 
\]

Thus
\[
JJ \leq \int_{(2Q)^c} (|JJ_1 + JJ_2 + JJ_3|)^{n/(n-\delta)} dx \\
\leq C \left( \sum_{|\alpha|=m} \|D^{\alpha}A\|_{BMO} \right)^{n/(n-\delta)} \sum_{k=1}^{\infty} k^{2^{-kn/(n-\delta)}} + 2^{-kn\varepsilon/(n-\delta)} \leq C. 
\]

(c) By the following equality
\[
R_{m+1}(A; x, y) = Q_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!} (x - y)^{\alpha} (D^{\alpha}A(x) - D^{\alpha}A(y)), 
\]
we have
\[
|T^A_K(f)(x)| \leq |\tilde{T}^A_K(f)(x)| + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^{\alpha}A(x) - D^{\alpha}A(y)|}{|x - y|^{n-\delta}} |f(y)| dy, 
\]
thus, by Lemma 3.2 and (b), we obtain
\[
|\{x \in R^n : |T^A_K(f)(x)| > \lambda\}| \leq |\{x \in R^n : |\tilde{T}^A_K(f)(x)| > \lambda/2\}| \\
+ \left| \{x \in R^n : \sum_{|\alpha|=m} \int_{R^n} \frac{|D^{\alpha}A(x) - D^{\alpha}A(y)|}{|x - y|^{n-\delta}} |f(y)| dy > C\lambda \} \right| \\
\leq C(\|f\|_{H^1/\lambda})^{n/(n-\delta)}. 
\]
This completes the proof of Theorem 2.1. \qed
Proof of Theorem 2.2. (i) It is only to prove that there exists a constant $C_Q$ such that

$$ \frac{1}{w(Q)} \int_Q |T^A_K(f)(x) - C_Q|w(x) \, dx \leq C \|f\|_{L^\infty(w)} $$

holds for any cube $Q$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q}$ and $\tilde{A}(x)$ be the same as the proof of Theorem 2.1. We have, similar to the proof of Theorem 2.1, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{R^n \setminus \tilde{Q}}$,

$$ |T^A_K(f)(x) - T^A_K(f_2)(x_0)| \leq T_K \left( \frac{R_m(\tilde{A}; x, \cdot)}{|x - \cdot|^m} f_1 \right)(x) $$

$$ + \sum_{|\alpha|=m} \frac{1}{\alpha!} |T_K \left( \frac{(x - \cdot)^\alpha}{|x - \cdot|^m} D^\alpha \tilde{A} f_1 \right)(x)| + |T^A_K(f_2)(x) - T^A_K(f_2)(x_0)| $$

$$ := I(x) + II(x) + III(x), $$

and, thus,

$$ \frac{1}{w(Q)} \int_Q |T^A_K(f)(x) - T^A_K(f_2)(x_0)|w(x) \, dx $$

$$ \leq \frac{1}{w(Q)} \int_Q I(x)w(x) \, dx + \frac{1}{w(Q)} \int_Q II(x)w(x) \, dx $$

$$ + \frac{1}{w(Q)} \int_Q III(x)w(x) \, dx := I + II + III. $$

First, using Lemma 3.1 and the $L^\infty(w)$-boundedness of $T_K$, we have

$$ I \leq \frac{C}{w(Q)} \int_Q T_K \left( \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} f_1 \right)(x)w(x) \, dx $$

$$ \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|T f_1\|_{L^\infty(w)} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^\infty(w)}; $$

Secondly, since $w \in A_1$, $w$ satisfies the reverse of Hölder’s inequality:

$$ \left( \frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{1/q} \leq \frac{C}{|Q|} \int_Q w(x) \, dx $$
for all cube $Q$ and some $1 < q < \infty$ (see [12]), thus, taking $p > 1$ and \(1/p + 1/p' = 1\), by the $L^p(w)$-boundedness of $T_K$ and Hölder’s inequality, we gain

\[
II \leq \frac{C}{w(Q)} \int_Q \left| T \left( \sum_{|\alpha|=m} (D^\alpha A - (D^\alpha A)_Q) f_1 \right)(x) \right| w(x) \, dx
\]

\[
\leq C \sum_{|\alpha|=m} \left( \frac{1}{w(Q)} \int_Q |T((D^\alpha A - (D^\alpha A)_Q)f_1)(x)|^p w(x) \, dx \right)^{1/p}
\]

\[
\leq C \sum_{|\alpha|=m} \left( \frac{1}{w(Q)} \int_Q |(D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}) f_1(x)|^p w(x) \, dx \right)^{1/p}
\]

\[
\leq C \sum_{|\alpha|=m} w(Q)^{-1/p} \left( \int_Q |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}|^{pq'} \, dx \right)^{1/pq'}
\]

\[
\times \left( \int_Q w(x)^q \, dx \right)^{1/pq} \|f\|_{L^\infty(w)}
\]

\[
\leq C \sum_{|\alpha|=m} \left( \frac{1}{|Q|} \int_Q |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}|^{pq'} \, dx \right)^{1/pq'}
\]

\[
\quad \times \left( \frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{1/pq} \left( \frac{|Q|}{w(Q)} \right)^{1/p} \|f\|_{L^\infty(w)}
\]

\[
\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{1/p} \left( \frac{|Q|}{w(Q)} \right)^{1/p} \|f\|_{L^\infty(w)}
\]

\[
\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^\infty(w)} ;
\]

For $III$, similar to the proof of Theorem 2.1, we obtain

\[
III \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \frac{1}{w(Q)}
\]

\[
\times \left( \int_{k=0}^\infty \int_{Q_k} k \left( \frac{|x-x_0|}{|x_0-y|} + \frac{|x-x_0|^\epsilon}{|x_0-y|^{n+\epsilon}} \right) |f(y)| \, dy \right) w(x) \, dx
\]
\[ + C \sum_{|\alpha|=m} \frac{1}{w(Q)} \int \int_{\mathbb{R}^n} (\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}) \times |D^\alpha \tilde{A}(y)||f(y)| \, dy \, w(x) \, dx \]

\[ \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^\infty(w)} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-k\varepsilon}) \]

\[ \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^\infty(w)}. \]

(ii) It suffices to show that there exists a constant \( C > 0 \) such that for every \( H^1(w) \)-atom \( a \) (that is that \( a \) satisfy: \( \text{supp} \, a \subset Q = Q(x_0, r) \), \( \|a\|_{L^\infty(w)} \leq w(Q)^{-1} \) and \( \int a(y)dy = 0 \) (see [8])), we have

\[ \|\tilde{T}_K^A(a)\|_{L^1(w)} \leq C. \]

We write

\[ \int_{\mathbb{R}^n} \tilde{T}_K^A(a)(x)w(x) \, dx = \left[ \int + \int \right] \tilde{T}_K^A(a)(x)w(x) \, dx := J + JJ. \]

For \( J \), similar to the proof of Theorem 2.1, we get

\[ |\tilde{T}_K^A(a)(x)| \leq |T^A(a)(x)| + C \sum_{|\alpha|=m} \int_{\mathbb{R}^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)| \, dy, \]

thus, \( \tilde{T}_K^A \) is \( L^p(w) \)-bounded by Lemma 3.2 and (i). We see that

\[ J \leq C \|\tilde{T}_K^A(a)\|_{L^\infty(w)} w(2Q) \leq C \|a\|_{L^\infty(w)} w(Q) \leq C; \]

For \( JJ \), notice that if \( w \in A_1 \), then \( \frac{w(Q_2)}{|Q_2|} \frac{|Q_1|}{w(Q_1)} \leq C \) for all cubes \( Q_1, Q_2 \) with \( Q_1 \subset Q_2 \). Thus, by Hölder’s inequality and the reverse of Hölder’s inequality for \( w \in A_1 \) and some \( 1 < q < \infty \), taking \( p > 1 \) and \( 1/p + 1/p' = 1 \), similarly, we obtain

\[ JJ \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \left( \frac{|Q|}{w(Q)} \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right) \]

\[ + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \frac{|Q|}{w(Q)} \left( \frac{1}{|2^{k+1}Q|} \right) \int |D^\alpha \tilde{A}(x)|^p \, dx \right)^{1/p}. \]
\[
\times \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^{p'} \, dx \right)^{1/p'} \\
\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \left( \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \frac{|Q|}{w(Q)} \right) \leq C.
\]

(iii) Similarly, we know
\[
|T^A_K(f)(x)| \leq |	ilde{T}^A(f)(x)| + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |f(y)| \, dy,
\]
by Lemma 3.2 and (ii), we obtain
\[
w(\{x \in R^n : |T^A_K(f)(x)| > \lambda \}) \leq w(\{x \in R^n : |	ilde{T}^A(f)(x)| > \lambda/2 \}) \\
+ w \left( \left\{ x \in R^n : \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |f(y)| \, dy > C\lambda \right\} \right) \\
\leq C \|f\|_{H^1(w)}/\lambda.
\]
This completes the proof of Theorem 2.2.

4. Applications

In this section we shall apply the Theorem 2.1 and 2.2 to some particular operators such as the Calderón–Zygmund singular integral operator and fractional integral operator.

Application 1 (Calderón–Zygmund singular integral operator). Let $T$ be the Calderón–Zygmund operator defined by (see [8, 12])
\[
T(f)(x) = \int K(x, y) f(y) \, dy,
\]
the multilinear operator related to $T$ is defined by
\[
T^A(f)(x) = \int \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y) \, dy.
\]
Then it is easily to see that $T_K$ satisfies the conditions in Theorem 2.2, thus that $T^A$ is bounded from $L^\infty(w)$ to $BMO(w)$ and from $H^1(w)$ to weak $L^1(w)$ and that $\tilde{T}^A$ is bounded from $H^1(w)$ to $L^1(w)$ for $w \in A_1$ and $D^\alpha A \in BMO(R^n)$ with $|\alpha| = m$. 
Aplication 2 (Fractional integral operator with rough kernel). For $0 \leq \delta < n$, let $T_\delta$ be the fractional integral operator with rough kernel defined by (see [7, 9, 10])

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x - y)}{|x - y|^{n-\delta}} f(y) \, dy,$$

the multilinear operator related to $T_\delta$ is defined by

$$T_\delta^A f(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^{m+n-\delta}} \Omega(x - y) f(y) \, dy,$$

where $\Omega$ is homogeneous of degree zero on $R^n$, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$ and $\Omega \in \text{Lip}_\gamma(S^{n-1})$ for $0 < \gamma \leq 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma$. Then $T_\delta$ satisfies the conditions in Theorem 3.1. In fact, for $\text{supp} f \subset (2Q)^c$ and $x \in Q = Q(x_0, d)$, by the condition of $\Omega$, we have (see [12])

$$\left| \frac{\Omega(x - y)}{|x - y|^{n-\delta}} - \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-\delta}} \right| \leq C \left( \frac{|x - x_0|^\gamma}{|x_0 - y|^{n+\gamma-\delta}} + \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} \right),$$

thus, similar to the proof of Theorem 2.1,

$$|T_\delta^A(f)(x) - T_\delta^A(f)(x_0)| \leq C \sum_{k=1}^\infty k(2^{-\gamma k} + 2^{-k}) \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}} \leq C \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}.$$

Therefore that $T_\delta^A$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$ and from $H^1(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$ and $T_\delta^A$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$ for all $D^\alpha A \in BMO(R^n)$ with $|\alpha| = m$.

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References


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