On 2-primal Ore extensions

Vijay K. Bhat

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Abstract. Let $R$ be a ring, $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$. We define a $\delta$ property on $R$. We say that $R$ is a $\delta$-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of $R$. We ultimately show the following.

Let $R$ be a Noetherian $\delta$-ring, which is also an algebra over $\mathbb{Q}$, $\sigma$ and $\delta$ be as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(P) = P$, $P$ any minimal prime ideal of $R$. Then $R[x, \sigma, \delta]$ is a 2-primal Noetherian ring.

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1. Introduction

A ring $R$ always means an associative ring. $\mathbb{Q}$ denotes the field of rational numbers. $\text{Spec}(R)$ denotes the set of prime ideals of $R$. $\text{MinSpec}(R)$ denotes the sets of minimal prime ideals of $R$. $P(R)$ and $N(R)$ denote the prime radical and the set of nilpotent elements of $R$ respectively. Let $I$ and $J$ be any two ideals of a ring $R$. Then $I \subset J$ means that $I$ is strictly contained in $J$.

This article concerns the study of ore extensions in terms of 2-primal rings. 2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [13], Greg Marks discusses the 2-primal property of $R[x, \sigma, \delta]$, where $R$ is a local ring, $\sigma$ is an automorphism of $R$ and $\delta$ is a $\sigma$-derivation of $R$.

Recall that a $\sigma$-derivation of $R$ is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In case $\sigma$ is the identity map, $\delta$ is called just a derivation of $R$. For example for any endomorphism $\tau$ of a ring $R$ and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$

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is a $\tau$-derivation of $R$. Also let $R = K[x]$, $K$ a field. Then the formal derivative $d/dx$ is a derivation of $R$.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [10]. 2-primal near rings have been discussed by Argac and Groenewald in [2]. Recall that a ring $R$ is 2-primal if and only if nil radical and prime radical of $R$ are same if and only if the prime radical is a completely semiprime ideal. An ideal $I$ of a ring $R$ is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$. We also note that a reduced is 2-primal and a commutative ring is also 2-primal.

For further details on 2-primal rings, we refer the reader to [7, 9, 10, 14]. Before proving the main result, we find a relation between the minimal prime ideals of $R$ and those of the Ore extension $R[x, \sigma, \delta]$, where $R$ is a Noetherian $Q$-algebra, $\sigma$ is an automorphism of $R$ and $\delta$ is a $\sigma$-derivation of $R$. This is proved in Theorem (2.1). Recall that $R[x, \sigma, \delta]$ is the usual polynomial ring with coefficients in $R$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x, \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x, \sigma, \delta]$ by $O(R)$.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 3, 4, 8, 11, 12]. Recall that in [11], a ring $R$ is called $\sigma$-rigid if there exists an endomorphism $\sigma$ of $R$ with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [12], Kwak defines a $\sigma(\ast)$-ring $R$ to be a ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(\ast)$-ring. The property is also extended to the skew-polynomial ring $R[x, \sigma]$.

Let $R$ be a ring, $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$. We introduce a property on $R$ and say that $R$ is a $\delta$-ring if $\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of $R$. We note that a ring with identity is not a $\delta$-ring.

Now let $R$ be a Noetherian $\delta$-ring, which is also an algebra over $Q$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $R[x, \sigma, \delta]$ is 2-primal. This is proved in Theorem (2.4).

2. Ore extensions

We begin with the following definition:

**Definition 2.1.** Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$. We say that $R$ is a $\delta$-ring if $\delta(a) \in P(R)$ implies $a \in P(R)$.
Recall that an ideal I of a ring R is called \(\sigma\)-invariant if \(\sigma(I) = I\) and is called \(\delta\)-invariant if \(\delta(I) \subseteq I\). If an ideal I of R is \(\sigma\)-invariant and \(\delta\)-invariant, then \(I[x, \sigma, \delta]\) is an ideal of \(R[x, \sigma, \delta]\). Also I is called completely prime if \(ab \in I\) implies \(a \in I\) or \(b \in I\) for \(a, b \in R\).

Gabriel proved in Lemma (3.4) of [5] that if R is a Noetherian Q-algebra and \(\delta\) is a derivation of R, then \(\delta(P) \subseteq P\), for all \(P \in \text{MinSpec}(R)\). We generalize this for \(\sigma\)-derivation \(\delta\) of R and give a structure of minimal prime ideals of O(R) in the following Theorem.

\textbf{Theorem 2.1.} Let R be a Noetherian Q-algebra. Let \(\sigma\) be an automorphism of R and \(\delta\) be a \(\sigma\)-derivation of R such that \(\sigma(\delta(a)) = \delta(\sigma(a))\), for \(a \in R\). Then \(P \in \text{MinSpec}(O(R))\) such that \(\sigma(P \cap R) = P \cap R\) implies \(P \cap R \subseteq \text{MinSpec}(R)\) and \(P_1 \in \text{MinSpec}(R)\) such that \(\sigma(P_1) = P_1\) implies \(O(P_1) \in \text{MinSpec}(O(R))\).

\textit{Proof.} Let \(P_1 \in \text{MinSpec}(R)\) with \(\sigma(P_1) = P_1\). Let \(T = R[[t, \sigma]]\), the skew power series ring. Now it can be seen that \(e^{t\delta}\) is an automorphism of T and \(P_1T \in \text{MinSpec}(T)\). We also know that \(e^{t\delta}P_1T \in \text{MinSpec}(T)\) for all integers \(k \geq 1\). Now T is Noetherian by Exercise (12A(c)) of [6], and therefore Theorem (2.4) of [6] implies that \(\text{MinSpec}(T)\) is finite. So exists an integer an integer \(n \geq 1\) such that \((e^{t\delta})^n(P_1T) = P_1T\); i.e. \((e^{nt\delta})(P_1T) = P_1T\). But R is a Q-algebra, therefore, \(e^{t\delta}(P_1T) = P_1T\). Now for any \(a \in P_1\), \(a \in P_1T\) also, and so \(e^{t\delta}(a) \in P_1T\); i.e. \(a + t\delta(a) + (t^2/2!)(\delta(a) + \cdots) \in P_1T\), which implies that \(\delta(a) \in P_1\). Therefore \(\delta(P_1) \subseteq P_1\).

Now it can be easily seen that \(O(P_1) \in \text{Spec}(O(R))\). Suppose that \(O(P_1) \notin \text{MinSpec}(O(R))\), and \(P_2 \subset O(P_1)\) is a minimal prime ideal of O(R). Then we have \(P_2 = O(P_2 \cap R) \subset O(P_1) \in \text{MinSpec}(O(R))\). Therefore \(P_2 \cap R \subseteq P_1\), which is a contradiction as \(P_2 \cap R \in \text{Spec}(R)\). Hence \(O(P_1) \in \text{MinSpec}(O(R))\).

Conversely let \(P \in \text{MinSpec}(O(R))\) with \(\sigma(P \cap R) = P \cap R\). Then it can be easily seen that \(P \cap R \in \text{Spec}(R)\) and \(O(P \cap R) \in \text{Spec}(O(R))\). Therefore \(O(P \cap R) = P\). We now show that \(P \cap R \in \text{MinSpec}(R)\). Suppose that \(P_3 \subset P \cap R\), and \(P_3 \in \text{MinSpec}(R)\). Then \(O(P_3) \subset O(P \cap R) = P\). But \(O(P_3) \in \text{Spec}(O(R))\) and, \(O(P_3) \subset P\), which is not possible. Thus we have \(P \cap R \in \text{MinSpec}(R)\). \(\square\)

\textbf{Proposition 2.1.} Let R be a 2-primal ring. Let \(\sigma\) and \(\delta\) be as usual such that \(\delta(P(R)) \subseteq P(R)\). If \(P \in \text{MinSpec}(R)\) is such that \(\sigma(P) = P\), then \(\delta(P) \subseteq P\).

\textit{Proof.} Let \(P \in \text{MinSpec}(R)\). Now for any \(a \in P\), there exists \(b \notin P\) such that \(ab \in P(R)\) by Corollary (1.10) of [14]. Now \(\delta(P(R)) \subseteq P(R)\), and
therefore $\delta(ab) \in P(R)$; i.e. $\delta(a)\sigma(b)+a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies that $\delta(a)\sigma(b) \in P$. Also $\sigma(P) = P$ and by Proposition (1.11) of [14], $P$ is completely prime, we have $\delta(a) \in P$. Hence $\delta(P) \subseteq P$. $\square$

**Theorem 2.2.** Let $R$ be a $\delta$-ring. Let $\sigma$ and $\delta$ be as above such that $\delta(P(R)) \subseteq P(R)$. Then $R$ is 2-primal.

**Proof.** Define a map $\rho : R/P(R) \rightarrow R/P(R)$ by $\rho(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau : R/P(R) \rightarrow R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$, then it can be seen that $\tau$ is an automorphism of $R/P(R)$ and $\rho$ is a $\tau$-derivation of $R/P(R)$. Now $a\delta(a) \in P(R)$ if and only if $(a + P(R))\rho(a + P(R)) = P(R)$ in $R/P(R)$. Thus as in Proposition (5) of [8], $R$ is a reduced ring and, therefore $R$ is 2-primal. $\square$

**Proposition 2.2.** Let $R$ be a ring. Let $\sigma$ and $\delta$ be as usual. Then:

1. For any completely prime ideal $P$ of $R$ with $\delta(P) \subseteq P$, $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.

2. For any completely prime ideal $U$ of $R[x, \sigma, \delta]$, $U \cap R$ is a completely prime ideal of $R$.

**Proof.** (1) Let $P$ be a completely prime ideal of $R$. Now let $f(x) = \sum_{i=0}^{n} x^i a_i \in R[x, \sigma, \delta]$ and $g(x) = \sum_{j=0}^{m} x^j b_j \in R[x, \sigma, \delta]$ be such that $f(x)g(x) \in P[x, \sigma, \delta]$. Suppose $f(x) \notin P[x, \sigma, \delta]$. We will show that $g(x) \in P[x, \sigma, \delta]$. We use induction on $n$ and $m$. For $n = m = 1$, the verification is easy. We check for $n = 2$ and $m = 1$. Let $f(x) = x^2a + xb + c$ and $g(x) = xu + v$. Now $f(x)g(x) \in P[x, \sigma, \delta]$ with $f(x) \notin P[x, \sigma, \delta]$. The possibilities are $a \notin P$ or $b \notin P$ or $c \notin P$ or any two out of these three do not belong to $P$ or all of them do not belong to $P$. We verify case by case.

Let $a \notin P$. Since $x^3\sigma(a)u + x^2(\delta(a)u + \sigma(b)u + av) + x(\delta(b)u + \sigma(c)u + bv) + \delta(c)u + cv \in P[x, \sigma, \delta]$, we have $\sigma(a)u \in P$, and so $u \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies $av \in P$, and so $v \in P$. Therefore $g(x) \in P[x, \sigma, \delta]$.

Let $b \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then $\sigma(a) \in P$ and therefore $a, \delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$ which in turn implies that $b \in P$, which is not the case. Therefore we have $u \in P$. Now $\delta(b)u + \sigma(c)u + bv \in P$ implies that $bv \in P$ and therefore $v \in P$. Thus we have $g(x) \in P[x, \sigma, \delta]$.

Let $c \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then as above $a, \delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$. Now $u \notin P$ implies that $\sigma(b) \in P$; i.e. $b, \delta(b) \in P$. Also $\delta(b)u + \sigma(c)u + bv \in P$ implies $\sigma(c)u \in P$ and therefore $\sigma(c) \in P$ which is not the case. Thus
we have $u \in P$. Now $\delta(c)u + cv \in P$ implies $cv \in P$, and so $v \in P$. Therefore $g(x) \in P[x, \sigma, \delta]$.

Now suppose the result is true for $k$, $n = k + 1$. We will prove for $n = k+1$. Let $f(x) = x^{k+1}a_{k+1} + x^ka_k + \cdots + xa_1 + a_0$, and $g(x) = xb_1 + b_0$ be such that $f(x)g(x) \in P[x, \sigma, \delta]$, but $f(x) \notin P[x, \sigma, \delta]$. We will show that $g(x) \in P[x, \sigma, \delta]$. If $a_{k+1} \notin P$, then equating coefficients of $x^{k+2}$, we get $\sigma(a_{k+1})b_1 \in P$, which implies that $b_1 \in P$. Now equating coefficients of $x^{k+1}$, we get $\sigma(a_k)b_1 + a_{k+1}b_0 \in P$, which implies that $a_{k+1}b_0 \in P$, and therefore $b_0 \in P$. Hence $g(x) \in P[x, \sigma, \delta]$.

If $a_j \notin P$, $0 \leq j \leq k$, then using induction hypothesis, we get that $g(x) \in P[x, \sigma, \delta]$. Therefore the statement is true for all $n$. Now using the same process, it can be easily seen that the statement is true for all $m$ also. The details are left to the reader.

(2) Let $U$ be a completely prime ideal of $R[x, \sigma, \delta]$. Suppose $a, b \in R$ are such that $ab \in U \cap R$ with $a \notin U \cap R$. This means that $a \notin U$ as $a \in R$. Thus we have $ab \in U \cap R \subseteq U$, with $a \notin U$. Therefore we have $b \in U$, and thus $b \in U \cap R$. □

**Corollary 2.1.** Let $R$ be a $\delta$-ring, where $\sigma$ and $\delta$ as usual such that $\delta(P(R)) \subseteq P(R)$. Let $P \in \text{MinSpec}(R)$ be such that $\sigma(P) = P$. Then $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.

**Proof.** $R$ is 2-primal by Theorem (2.2), and so by Proposition (2.1) $\delta(P) \subseteq P$. Further more $P$ is a completely prime ideal of $R$ by Proposition (1.11) of [10]. Now use Proposition (2.2). □

We now prove the following Theorem, which is crucial in proving Theorem 2.4.

**Theorem 2.3.** Let $R$ be a $\delta$-ring, where $\sigma$ and $\delta$ as usual such that $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in \text{MinSpec}(R)$. Then $R[x, \sigma, \delta]$ is 2-primal if and only if $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.

**Proof.** Let $R[x, \sigma, \delta]$ be 2-primal. Now by Corollary (2.1) $P(R[x, \sigma, \delta]) \subseteq P(R)[x, \sigma, \delta]$. Let $f(x) = \sum_{j=0}^{n} x^ja_j \in P(R)[x, \sigma, \delta]$. Now $R$ is a 2-primal subring of $R[x, \sigma, \delta]$ by Theorem (2.2), which implies that $a_j$ is nilpotent and thus $a_j \in N(R[x, \sigma, \delta]) = P(R[x, \sigma, \delta])$, and so we have $x^ja_j \in P(R[x, \sigma, \delta])$ for each $j$, $0 \leq j \leq n$, which implies that $f(x) \in P(R[R[x, \sigma, \delta])$. Hence $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.

Conversely suppose $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$. We will show that $R[x, \sigma, \delta]$ is 2-primal. Let $g(x) = \sum_{i=0}^{n} x^ib_i \in R[x, \sigma, \delta]$, $b_n \neq 0$, be such that $(g(x))^2 \in P(R[x, \sigma, \delta]) = P(R)[x, \sigma, \delta]$. We will show that $g(x) \in P(R[x, \sigma, \delta])$. Now leading coefficient $\sigma^{2n-1}(a_n)a_n \in P(R) \subseteq P$, for all
\( P \in \text{MinSpec}(R) \). Now \( \sigma(P) = P \) and \( P \) is completely prime by Proposition (1.11) of \([10]\). Therefore we have \( a_n \in P \), for all \( P \in \text{MinSpec}(R) \); i.e. \( a_n \in P(R) \). Now since \( \delta(P(R)) \subseteq P(R) \) and \( \sigma(P) = P \) for all \( P \in \text{MinSpec}(R) \), we get \((\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x, \sigma, \delta]) = P(R)[x, \sigma, \delta]\) and as above we get \( a_{n-1} \in P(R) \). With the same process in a finite number of steps we get \( a_n \in P[R[x, \sigma, \delta]] \) and as above we get \( a_{n-1} \in P(R) \). Thus we have \( P(R[x, \sigma, \delta]) \) is completely semiprime. Hence \( R[x, \sigma, \delta] \) is 2-primal.

**Theorem 2.4.** Let \( R \) be a Noetherian \( \delta \)-ring, which is also an algebra over \( Q \) such that \( \sigma(\delta(a)) = \delta(\sigma(a)) \), for all \( a \in R \); \( \sigma(P) = P \) for all \( P \in \text{MinSpec}(R) \) and \( \delta(P(R)) \subseteq P(R) \), where \( \sigma \) and \( \delta \) are as usual. Then \( R[x, \sigma, \delta] \) is 2-primal.

**Proof.** We use Theorem (2.1) to get that \( P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta]) \), and now the result is obvious by using Theorem (2.3).

**Corollary 2.2.** Let \( R \) be a commutative Noetherian \( \delta \)-ring, which is also an algebra over \( Q \) such that \( \sigma(\delta(a)) = \delta(\sigma(a)) \), for all \( a \in R \); \( \sigma(P) = P \) for all \( P \in \text{MinSpec}(R) \), where \( \sigma \) and \( \delta \) are as usual. Then \( R[x, \sigma, \delta] \) is 2-primal.

**Proof.** Using Theorem (1) of \([15]\) we get \( \delta(P(R)) \subseteq P(R) \). Now rest is obvious.

The above gives rise to the following questions:

If \( R \) is a Noetherian \( Q \)-algebra (even commutative), \( \sigma \) is an automorphism of \( R \) and \( \delta \) is a \( \sigma \)-derivation of \( R \). Is \( R[x, \sigma, \delta] \) 2-primal? The main problem is to get Theorem (2.3) satisfied.

**References**


**Contact information**

**Vijay Kumar Bhat**

School of Applied Physics and Mathematics, SMVD University, P/o Kakryal, Udhampur, J and K, India 182121

*E-Mail:* vijaykumarbhat2000@yahoo.com