

**WEAKLY INJECTIVE AND WEAKLY PROJECTIVE
MODULES VERSUS EXTENDING AND LIFTING MODULES**

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DECLARATION

I certify that this thesis, submitted for the degree of Master of Science to the Department of Mathematics in Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis has not been submitted for a higher degree to any other university or institution.

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Abstract

In this thesis, we will study the relations between weakly injective, weakly projective versus extending and lifting modules and its generalization. Mainly, we study rings over which every weakly injective module is weakly projective and rings over which every weakly projective module is weakly injective. Also, we study weakly-injective modules versus extending modules.

Keywords: weakly-injective, weakly-projective, extending, lifting.

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Introduction

introduction

Preliminary results on injective modules, projective modules, perfect and semiperfect rings, artinian and noetherian rings are provided in chapter 1.

In chapter 2, we study weakly injective modules, we intend to begin with the basic facts on ring theory. The concept of weakly-injective modules was originally introduced to obtain a characterization of semiperfect rings over which each cyclic right module is embeddable as an essential submodule of a projective module (CEP rings). In analogy to a characterization of quasi-frobenius rings, a ring R is right CEP if and only if R is a right artinian and each indecomposable projective right R module is weakly R -injective. In section 2, we study among others the question: For what rings is it the case that each weakly-injective module is injective and when are the direct summand of weakly-injective modules again weakly-injective? In analogy to the Matlis Papp theorem on noetherian rings, rings over which direct sum of weakly injective module are weakly injective are also characterized [9]. These are precisely those rings over which each cyclic module has finite uniform dimension.

In chapter 3, we study weakly-projective modules a dual concept of weakly injective modules. We dualize all results given in chapter 2. Section 1 provides examples of non trivial weakly-projective modules. In section 2, we study rings over which every weakly-injective module is weakly-projective. In section 3, we study rings over which every weakly-projective module is weakly-injective.

In chapter 4, we study extending modules. This is a module for which every submodule is essential in a direct summand. Semisimple modules, uniform modules and injective modules are all examples of extending modules. But other examples can easily be given. For example, any finitely generated torsion free abelian group is an extending module over the ring of integers. Despite the fact that there are so many different examples, extending modules have many pleasant properties. For example, extending modules with every submodule projective have every finitely generated submodule noetherian. In addition, extending modules with every finitely generated submodule noetherian are direct sums of uniform submodules[6].

We study also lifting modules. This is a module for any submodule A there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll M$. Hollow modules are examples of lifting modules. We list basic lemmas and results that will be useful in characterizing lifting modules.

In chapter 5, we study rings over which every weakly injective module is extending. Under the hypothesis that each weakly injective module is extending every direct summand is weakly-injective, and we will give some examples and conditions for rings for which every weakly-injective extending modules are injective.

Chapter 1

Basic notation

1 Basic notation

In this chapter we collect material of general interest which is of particular importance for our investigations. Most of this is fairly well-known but also new notions and results are included.

1.1 Preliminaries

R will always denote an associative ring with unit 1, and $\text{Mod-}R$ the category of all unital right R -modules.

A submodule N of M is said to be a small (superfluous) submodule if the only submodule K of M such that $K + N = M$ is $K = M$. A small module N in M is denoted by $N \ll M$. A superfluous cover of a module M is a module P together with an epimorphism $p : P \rightarrow M$ such that $\ker p$ is small in P . Equivalently, one may think of superfluous cover of M as being a module P such that $P/K \cong M$ for some small submodule $K \subseteq P$.

A non zero R module M is said to be hollow if every proper submodule of M is small in M [13].

A submodule N of M is called essential (or large) in M , denoted by $N \lll M$, if $N \cap K \neq 0$ for every non-zero submodule $K \subset M$. A monomorphism $f : K \rightarrow M$ is said to be essential in case $\text{Im } f$ is essential in M [13].

We call an ideal of a ring semiprime if and only if it is an intersection of prime ideals of the ring, a semiprime ring is one in which the zero ideal is semiprime.

A ring R is said to satisfy the right ore condition with respect to a subset C of R if given $a \in R$ and $c \in C$ there exist $b \in R$ and $d \in C$ such that $ad = cb$.

Definition 1.1. R has classical right quotient ring if and only if R satisfies the right ore condition with respect to C , where C is the set of regular elements of R .

A ring $R \subset Q$ is a left classical ring of quotients for R if it satisfies

1. every regular element of R is invertible in Q ,
2. every element of Q can be written in the form $x^{-1}y$, with $x, y \in R$ and x is regular.

An extension ring S of a ring T is called the left quotient ring of T , if for any two element $x, y \neq 0$ of S there exist $a \in T$ such that $ax \neq 0$ and $ay \in T$.

Definition 1.2. The annihilators of an element $m \in M$ is $l(m) = \{r \in R : rm = 0\}$.

The annihilators of M $l(M) = \{r \in R : rm = 0, \forall m \in M\}$.

An R -module N is (finitely) generated by M or (finitely) M -generated if there exists an epimorphism

$$M^\Lambda \rightarrow N$$

for some (finite) index set Λ . N is finitely R -generated if and only if it is finitely generated in the usual sense [6, Def.1.2].

A module M is called semisimple if it is a direct sum of simple module.

Radical and socle

The sum of all simple submodules of M is called the socle of M and is denoted by $SocM$.

This is a fully invariant submodule of M and

$$SocM = \cap \{L \subset M : L \subset' M\}.$$

An R -module M is called finitely cogenerated if $SocM$ is finitely generated and essential in M .

The intersection of all maximal submodules of M is called the radical of M and is denoted by $RadM$. If M has no maximal submodules we set $RadM = M$. $RadM$ is also a fully invariant submodule of M and

$$RadM = \Sigma \{L \subset M : L \ll M\}.$$

If M is finitely generated or projective, then $RadM \ll M$. If every proper submodule of M is contained in a maximal submodule, then $RadM \ll M$. If $\bar{M} = M/RadM$ is semisimple and $RadM \ll M$, then every proper submodule of M is contained in a maximal submodule.

The radical of R_R is called the Jacobson radical of R and is denoted by $JacR$. If $R/JacR$ is a right semisimple ring, then R is said to be semilocal.

If a module M has a largest submodule, i.e. a proper submodule which contains all other proper submodules, then M is called a local module. Such a submodule has to be equal to the radical of M and in this case $Rad(M) \ll M$. M is local if and only if it is cyclic and every non-zero factor module of M is indecomposable. A cyclic and self-projective module M is local if and only if $End_R(M)$ is a local ring [6, 1.7].

Let M be any module. A submodule K of M is closed (in M). If K has no proper essential extension in M , i.e. whenever L is a submodule of M such that $K \subset' L$ then $K = L$.

Let N be any submodule of M . A submodule H of M is called a complement of N (in M) if H is maximal in the collection of submodules Q of M such that $Q \cap N = 0$. A submodule K of M is called a complement (in M) if there exists a submodule N of M such that K is a complement of N in M [6, 1.10].

Complements and closed submodules: Let K be a submodule of a module M and let L be a complement of K . Then K is closed if and only if K is a complement of L in M .

let N be a submodule of M if L is minimal with respect to $N + L = M$ then L is called a supplement of N in M .

Definition 1.3. *Injective module:* Let M, N, K be right modules, M is called N -injective (or injective relative to N), if for every monomorphism $f : K \rightarrow N$ and every homomorphism $g : K \rightarrow M$ there exists $g' : N \rightarrow M$ such that $g'f = g$. As in diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & K & \xrightarrow{f} & N \\
 & & \downarrow g & \swarrow \exists g' & \\
 & & M & &
 \end{array}$$

A right R -module M is called injective if its injective for every right module N .

Definition 1.4. *Projective module:* Let M, N be right modules, M is called N -projective (or projective relative to N), if for every epimorphism $f : N \rightarrow N/K$ and every homomorphism $g : M \rightarrow N/K$ there exists a homomorphism $g' : M \rightarrow N$ such that $fg' = g$.

$$\begin{array}{ccc}
 & & N \\
 & \nearrow \exists g' & \downarrow f \\
 M & \xrightarrow{g} & n/K \longrightarrow 0
 \end{array}$$

The module M is called regular in $\text{Mod} - R$ if every finitely presented module in $\text{Mod} - R$ is M -projective (see [26]).

A non-zero module M is said to be uniform if any two non-zero submodules of M have non-zero intersection, i.e., every non-zero submodule is essential in M [6, 5.1].

An R module M is called a free module if M admits a basis i.e., there exist a subset S of M such that M is generated by S and S is linearly independent over R . A free R module is projective and hence R_R is projective right R module, but the converse is not true. However, over a local ring an R module is projective if and only if it is free [26].

A ring R is called (semi) hereditary if every (finitely generated) right ideal is projective if and only if every (finitely generated) submodule of projective right module is projective.

We say that an R module M has finite Goldie dimension if M does not contain an infinite direct sum of indecomposable submodules. A module M has Goldie dimension equals n if and only if there exist an independent sequence H_1, H_2, \dots, H_n of uniform submodules of M such that $H_1 \oplus H_2 \oplus \dots \oplus H_n \subset M$ if and only if $E(M) = E_1 \oplus E_2 \oplus \dots \oplus E_n$ with each E_i indecomposable injective module (see [26]).

A ring R is called right Goldie if R has an ascending chain condition on right annihilators and has finitely Goldie dimension.

Also a ring R is called a right Goldie ring if it satisfies the ACC on right annihilators and R_R is a module of finite rank, we say that a module M has a finite rank if $E(M)$ is a finite direct sum of indecomposable submodules.

A ring R is called a right q.f.d. (Goldie finite dimension) ring if and only if every cyclic right R module has finitely generated (possibly zero) socle. This is equivalent to saying that every cyclic (finitely generated) right module has finite Goldie dimension. All rings with right Krull dimension are q.f.d.. In particular, right noetherian rings are right q.f.d.. A ring R is called a right

CEP-ring if every cyclic right R -module is essentially embeddable in a projective module (see[26]).

Definition 1.5. A ring R is Quasi Frobenius (QF) if R has descending chain condition on right ideals and R is right self injective.

Theorem 1.6. The following are equivalent:

1. R is QF ring.
2. Every right R module is embeddable in a projective module.
3. Every projective right module is injective.
4. Every injective right module is projective.

Indeed R is a QF ring if and only if R is both a right and left CEP ring.

A ring R is called CF ring if every cyclic right module is embeddable in a free module [26].

1.2 Injectivity and noetherian modules

Recall that an R -module M is called N -injective if every diagram in $\text{Mod-}R$ with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & N \\ & & \downarrow g & \swarrow \exists g' & \\ & & M & & \end{array}$$

can be extended commutatively by a morphism $N \rightarrow M$.

An R -module is injective if it is R -injective.

For example, the zero module is injective.

And, $2\mathbb{Z}$ is not an injective \mathbb{Z} -module, for if one tried to extend the identity map $2\mathbb{Z} \rightarrow 2\mathbb{Z}$ to a group map $f : \mathbb{Z} \rightarrow 2\mathbb{Z}$ one would be able to find $x \in 2\mathbb{Z}$ such that $2x = 2$ (namely, $x = f(1)$) but this is impossible. So, one cant extend the map $2\mathbb{Z} \rightarrow 2\mathbb{Z}$ to a map $\mathbb{Z} \rightarrow 2\mathbb{Z}$ and so $2\mathbb{Z}$ is not an injective \mathbb{Z} -module. For similar reasons \mathbb{Z} is not an injective \mathbb{Z} -module since the identity map $\mathbb{Z} \rightarrow \mathbb{Z}$ cannot be extended to a map $\mathbb{Q} \rightarrow \mathbb{Z}$ since $5x = 1$ is nt solvable in \mathbb{Z} .

Definition 1.7. *Injective hull [16, Def.5.6.1]: A monomorphism $\eta : M \rightarrow Q$ is called an injective hull of M if and only if Q is injective and η is a large monomorphism. ($\text{Im}\eta$ is large in Q)*

For any R -module X , we shall denote the injective hull of X by $E(X)$.

Theorem 1.8. *Characterization (see [6]). The following are equivalent for R -modules U, M :*

1. U is M -injective,
2. $f(M) \subset U$ for every morphism $f : E(M) \rightarrow E(U)$,
3. $\text{Hom}_R(-, U)$ is exact with respect to all exact sequences of the form $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$,
4. U is L -injective for every cyclic submodule L of M .
In this case, for every exact sequence $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$, U is N -injective and K -injective.

Lemma 1.9. [29, Lemma1] *Let M_1 and M_2 be modules, let X be a submodule of M_1 and let $M = M_1 \oplus M_2$. The following conditions are equivalent :*

1. M_2 is (M_1/X) injective,
2. For every (closed) submodule N of M such that $N \cap M_2 = 0$ and $\pi_1(N) \cap X \leq N$ there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$,
3. For every (closed) submodule N of M such that $N \cap M_2 = 0$ and $X \leq N$ there exist a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

Lemma 1.10. [26, Lemma1.3] *In the category of right R modules over a ring R*

- a) M is injective if and only if $M = E(M)$,
- b) if M is an essential submodule of N , then $E(M) = E(N)$,
- c) If M is a submodule of N with N injective, then $E(M)$ is a direct summand of N ,
- d) if M is a submodule of Q with M -injective, then M is a summand of Q ,
- e) the finite direct sum of injective module is injective,
- f) a summand of injective module is injective.

Definition 1.11. [16, Def.6.1.1] A module M is called right noetherian if and only if every non zero set of submodules (with respect to inclusion as ordering) possesses a maximal element.

A ring R is called right noetherian if and only if R_R is noetherian.

A chain of submodules of M

$$\dots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \dots$$

is called stationary if and only if the chain contains only finitely many different A_i .

Remark. A noetherian module is also called a module with maximal condition

Theorem 1.12. [16, Theorem6.1.2] Let $M = M_R$, and let A be a submodule of M . Then the following properties are equivalent

- a) M is noetherian,
- b) A and M/A are noetherian,
- c) every ascending chain $A_1 \subset A_2 \subset A_3 \subset \dots$ of submodules of M is stationary,
- d) every submodule of M is finitely generated,
- e) in every set $\{A_i : i \in I\} \neq \emptyset$ of submodules A_i of M there is a finite subset $\{A_i : i \in I_0\}$ with $\sum_{i \in I} A_i = \sum_{i \in I_0} A_i$.

Direct injectivity.[6, 2.11] The module M is called direct injective if, for every direct summand X of M , every monomorphism $X \rightarrow M$ splits. Obviously, any direct summand of a direct injective module is again direct injective

Definition 1.13. (see [26]) A module M is called (quasi-)continuous if it satisfies (1) and (2) ((3)) of the following conditions

1. Every submodule of M is essential in some direct summand of M .
2. If a submodule N of M is isomorphic to a direct summand of M , then N is a direct summand of M , and
3. If M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M .

Quasi continuous is also called π -injective.

Theorem 1.14. (see [6, 2.10]) The following statements are equivalent for a module M with injective hull E

1. M is quasi-continuous (π -injective),
2. whenever $E = E_1 \oplus E_2$ is a direct sum of submodules E_1, E_2 , then $M = (E_1 \cap M) \oplus (E_2 \cap M)$,
3. whenever L_1 and L_2 are submodules of M such that $L_1 \cap L_2 = 0$, there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$ and $L_i \subset M_i (i = 1; 2)$,
4. whenever L_1 and L_2 are submodules of M such that $L_1 \cap L_2 = 0$, there exists $f \in \text{End}(M)$ such that $L_1 \subset \text{Ker} f$ and $L_2 \subset \text{Ker}(1 - f)$,
5. whenever L_1 and L_2 are submodules of M such that $L_1 \cap L_2 = 0$, the following monomorphism splits $M \rightarrow (M/L_1) \oplus (M/L_2), m \rightarrow (m + L_1, m + L_2)$

Properties. Let M be quasi-continuous (π -injective) module. Then

1. every direct summand of M is again quasi-continuous ,
2. if $M = U \oplus V$, then V is U -injective,
3. if U, V are direct summands of M and $U \cap V = 0$, then $U \oplus V$ is also a direct summand of M ,
4. M is indecomposable if and only if it is uniform.

A module is said to be continuous if it is quasi-continuous(π -injective) and direct injective. Obviously direct summands of continuous modules are continuous. In particular, self-injective modules are continuous [6,2.12].

1.3 Projectivity and artinian modules

Recall that an R -module P is called M -projective if every diagram in $\text{Mod-}R$ with exact row

$$\begin{array}{ccccc}
 & & P & & \\
 & \exists & \swarrow \text{dotted} & \downarrow & \\
 M & \xrightarrow{g} & N & \longrightarrow & 0
 \end{array}$$

can be extended commutatively by a morphism $P \rightarrow M$. This is equivalent to the fact that $\text{Hom}_R(P; -)$ is exact with respect to all exact sequences

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0.$$

If P is M -projective and $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact, then P is M' and M'' -projective. If P is P -projective, then P is also called self- (or quasi-) projective. For a fully invariant submodule K of a self-projective P , P/K is also self-projective.

A module P is called projective if it is N -projective for every right module N . P is projective in $Mod - R$ if and only if P is M^Λ projective for every index set Λ . If P is finitely generated, then it is M -projective if and only if it is projective. For an arbitrary module P , M -projectivity need not imply projectivity. This does not even hold for $M = R$. For $M = \mathbb{Z}$ this is known as Whitehead's problem.

Every projective module P in $Mod - R$ is a direct summand of a direct sum of cyclic submodules of $M^{\mathbb{N}}$ and a direct summand of a direct sum of cyclic submodules of P .

A projective module P in $Mod - R$ together with an epimorphism $\pi : P \rightarrow N$ with $\ker \pi \ll P$ is called a projective cover of N in $Mod - R$ or a $Mod - R$ -projective cover of N [16, def. 5.6.2].

A projective module P is an M -projective cover of a simple module if and only if $End_R(P)$ is a local ring. In this case P is a cyclic R -module.

Definition 1.15. [16, Def.6.1.2] *A module M is called artinian if and only if every non zero set of submodules possesses (with respect to inclusion as ordering) minimal element. A ring R is called right artinian if and only if R_R is artinian.*

Remark: an artinian module is also called a module with minimal condition.

Theorem 1.16. *For M_R , and let A be submodule of M . Then the following properties are equivalent:*

- a) M is artinian,
- b) A and M/A are artinian,
- c) Every descending chain $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ of submodules of M is stationary,
- d) Every factor module of M is finitely cogenerated,
- e) In every set $A_i : i \in I \neq \emptyset$ of submodules A_i of M there is a finite subset $\{A_i : i \in I_0\}$ with $\bigcap_{i \in I} A_i = \bigcap_{i \in I_0} A_i$.

A module M is called semi-artinian if every non-zero homomorphic image of M has essential socle. A ring R is called right semi-artinian if the right R -module R_R is semi-artinian. Clearly artinian modules are semi-artinian, and right artinian rings are right semi-artinian. It is also clear that if M is a semi-artinian module with submodule $N \subset M$ then N and M/N are both semi-artinian [6, 3.3].

Chapter 2
Weakly injective modules

2 Weakly injective modules

The objective of this chapter is to introduce the concepts of weak injectivity of a right R module M . A module M is said to be weakly N -injective if for each homomorphism $\phi : N \rightarrow E(M)$, there exists a monomorphism $\sigma : M \rightarrow E(M)$ and a homomorphism $\phi' : N \rightarrow M$ such that $\phi = \sigma\phi'$, and it is called weakly injective if it is weakly N -injective for each finitely generated module N .

In section 1 we will list the basic facts and results from several articles. In section 2 we will show that arbitrary direct sum of weakly injective modules are weakly injective if and only if every cyclic right R -module has finite uniform dimension.

2.1 Basic results

Definition 2.1. [26, Def.1.2] A module M is said to be weakly N -injective, if for each homomorphism $\phi : N \rightarrow E(M)$ there exists a monomorphism $\sigma : M \rightarrow E(M)$ and a homomorphism $\phi' : N \rightarrow M$ such that $\phi = \sigma\phi'$, and it is called weakly injective if it is weakly N -injective for each finitely generated module N . Equivalently, A module M is called weakly injective relative to the module N or weakly N -injective if for each homomorphism $\phi : N \rightarrow E(M)$, $\phi(N) \subset X \cong M$ for some submodule X of $E(M)$.

$$\begin{array}{ccc}
 & & M \\
 & \nearrow \exists \phi' & \vdots \\
 N & \xrightarrow{\phi} & E(M) \\
 & & \downarrow \exists \sigma
 \end{array}$$

A ring R is right weakly N -injective if the right module R is weakly N -injective.

Notice that the ring of integers \mathbb{Z} is weakly- \mathbb{Z}^n injective for all $n \in \mathbb{Z}^+$, but \mathbb{Z} is not \mathbb{Z} -injective. Indeed, any commutative integral domain R which is not a field is weakly R^n -injective for all $n \in \mathbb{Z}^+$, but not self-injective.

Definition 2.2. A module M is said to be N -tight if any quotient of N which is embeddable in $E(M)$ is embeddable in M . A module M is said to be tight if for all finitely generated module N , N is embeddable in $E(M)$ implies N is embeddable in M .

Lemma 2.3. [14, Lemma1.2] Let M and N be R -modules. Then the following statements are equivalent:

1. M is weakly N -injective,
2. M is weakly N/K -injective for all $K \subset N$, and
3. for every submodule K of N and for every monomorphism $h : N/K \rightarrow E(M)$ there exists a monomorphism $\sigma : M \rightarrow E(M)$ and $h' : N/K \rightarrow M$ such that $h = \sigma h'$.

Proof. (1) \rightarrow (3). Let M be weakly N -injective. Then for each homomorphism $\phi : N \rightarrow E(M)$ there exists a monomorphism $\sigma : M \rightarrow E(M)$ and a homomorphism $\phi' : N \rightarrow M$ such that $\phi = \sigma\phi'$. Let $h : N/K \rightarrow E(M)$ be a homomorphism such that $h(n + K) = \phi(n)$, and define $h' : N/K \rightarrow M$ as $h'(n + K) = \phi'(n)$. Then obviously $h = \sigma h'$.

(3) \rightarrow (2). Obvious.

(2) \rightarrow (1). Since M is weakly N/K injective. Then for every homomorphism $h : N/K \rightarrow E(M)$ there exists a monomorphism $\sigma : M \rightarrow E(M)$ and $h' : N/K \rightarrow M$ such that $h = \sigma h'$. Define $\phi : N \rightarrow E(M)$ as $\phi(n) = h(n + K)$ and $\phi'(n) = h'(n + K)$. Then clearly $\phi = \sigma\phi'$. Hence M is weakly N injective. \square

Lemma 2.4. [14, Lemma1.3] Given two right modules M and N , M is weakly N -injective if and only if for every submodule Q of N and for every monomorphism $\sigma : N/Q \rightarrow E(M)$:

1. There exists a monomorphism $\sigma' : N/Q \rightarrow M$,
2. For every complement K of $\sigma'(N/Q)$ in M there exist $K' \subset E(M)$ such that $K' \cap \sigma(N/Q) = 0$ and $k' \cong k$.

Proof. Let $\sigma : N/Q \rightarrow E(M)$ be a monomorphism. By Lemma 2.4(3), there exists monomorphism $\alpha : M \rightarrow E(M)$ and $\sigma' : N/Q \rightarrow M$ such that $\sigma = \alpha\sigma'$. Thus (1) holds.

Let K be a complement of $\sigma'(N/Q)$ in M ; then $K' = \alpha(K)$ is isomorphic to

K and independent from $\sigma(N/Q)$ proving that (2) is also necessary. Conversely, let us assume that (1) and (2) hold and let $\sigma : N/Q \rightarrow E(M)$ be a monomorphism. By (1) there exists $\sigma' : N/Q \rightarrow M$. Let K be a complement of $\sigma'(N/Q)$ in M . Using (2), we get a monomorphism $\alpha : \sigma'(N/Q) \oplus K \rightarrow E(M)$. Since $\sigma'(N/Q) \oplus K \subset' M$, we may extend α to a monomorphism $\beta : M \rightarrow E(M)$. It is straightforward that $\beta\sigma' = \sigma$. Using Lemma 2.4(3) gives us that M is weakly N -injective. \square

The following remarks illustrate the meaning of weak relative injectivity in specific cases.

Remark 2.5. [9, Remark1.4] *An R -module M is weakly R^n -injective if and only if for all $x_1, \dots, x_n \in E(M)$ there exists a submodule X of $E(M)$ such that $x_i \in X \cong M$, $i = 1, \dots, n$.*

Proof. Since every homomorphism $\phi : R^n \rightarrow E(M)$ is determined by choosing arbitrary elements $x_1, x_2, \dots, x_n \in E(M)$, the proof follows from the definition of weak R^n -injective. \square

Remark 2.6. *Weak relative injectivity is closed under finite direct sums and under essential extension, but the direct summands of a weakly injective module need not be weakly injective.*

Lemma 2.7. 1. [14, Proposition1.7] *If L and M are weakly N -injective modules, then $L \oplus M$ is weakly N -injective.*

2. *If R is right noetherian, then arbitrary direct sums of weakly N -injective modules are weakly N -injective.*

3. *If M is weakly N -injective and L is an essential extension of M , then L is weakly N -injective.*

Proof. (1) Since L is weakly- N -injective. Then for each homomorphism $\phi_1 : N \rightarrow E(L)$, $\phi_1(N) \subset X \cong L$ for some submodule X of $E(L)$. And M is weakly N -injective, so for each homomorphism $\phi_2 : N \rightarrow E(M)$, $\phi_2(N) \subset Y \cong M$ for some submodule Y of $E(M)$. Let $\phi : N \rightarrow E(L) \oplus E(M)$. Then $\phi(N) \subset X \oplus Y \cong L \oplus M$, hence $L \oplus M$ is weakly N -injective.

(2) Follows from (1).

(3) Since $M \subset' L$, then $E(M) = E(L)$ (by lemma 1.10 (2)). It follows that L is weakly N -injective. \square

Lemma 2.8. [9, Lemma1.9] *Let U be M injective, and M be weakly R -injective. Then U is $E(M)$ injective.*

Proof: By contradiction, assume not, i.e., U is not $E(M)$ injective, then by Zorns lemma there exists a submodule A of $E(M)$ and homomorphism $f : A \rightarrow U$, which cannot be extended to any $f' : B \rightarrow U$ with B a submodule of $E(N)$ containing A properly. Let $b \in E(M)/A$. But A is essential submodule of $E(M)$ so $C = bR \cap A \neq 0$. Let $f_1 : C \rightarrow U$ be the restriction of f to C , as M is weakly R -injective, bR embeds in M therefore U is R -injective and f_1 extends to $g : bR \rightarrow U$. Define $f' : A + bR \rightarrow U$ by $f'(a + br) = f(a) + g(br)$ whenever $a \in A, r \in R$ so f' extends f , contradiction. Therefore U is $E(M)$ injective.

Proposition 2.9. [26, Proposition1.2.7] *Let M be quasi injective right module and weakly R -injective. Then M is injective.*

Proof : Take $U = M$ in previous lemma.

Lemma 2.10. [9, Example1.11.1] *A ring R is quasi Frobenius if and only if it is right artinian and right weakly injective.*

Proof. We will proof the converse. Let R be right artinian and weakly-injective. Let $x \in E(R)$. By weak injectivity, there exists $X \subseteq E(R)$, $X \cong R$ such that $1, x \in X$. Then $R \subseteq X$ and X is right artinian and isomorphic to R . Thus $R = X$ and, therefore, $x \in R$. So $R = E(R)$ is quasi frobenius. \square

Remark 2.11. [9, Remark1.5] *A ring R is weakly R^n -injective if and only if for all $x_1, x_2, \dots, x_n \in E(R)$, there exists an element $b \in E(R)$ such that $r.ann_R(b) = 0$ and $x_1, x_2, \dots, x_n \in bR$.*

Proof. As in the remark (2.5), if R is weakly R^n -injective, x_1, x_2, \dots, x_n must be contained in a submodule X of $E(R)$ which is isomorphic to R . Let $b = \phi(1)$. Then $x_1, x_2, \dots, x_n \in bR$ and $r.ann_R(b) = 0$. \square

Lemma 2.12. [9, Example1.11.3] *Every semiprime right and left noetherian ring is right and left weakly-injective.*

Proof. Let R be a right and left noetherian semiprime ring, and let $r.Q_{cl}, l.Q_{cl}$ respectively, denote right and left classical ring of quotients of R . Then $E(R_R) = E({}_R R) = r.Q_{cl}(R) = l.Q_{cl}(R)$. Let $q_1, q_2 \in Q$. There exist $r_1, r_2 \in R, s \in R - \{0\}$ such that $q_1 = s^{-1}r_1, q_2 = s^{-1}r_2$. By previous remark R is right weakly-injective. Similarly, R is left weakly-injective. \square

Lemma 2.13. [9, Example1.11.6] *Let R be commutative ring and E_R be an injective module where submodules are linearly ordered. Then for all $M \subset E$, $N = M \oplus E$ is weakly R -injective.*

Proof. Let $E(N) = E \oplus E$. Let $q = (a, b) \in E(N)$, $a \in E$, $b \in E$. Now either $aR \subset bR$ or $bR \subset aR$. Without loss of generality, let $bR \subset aR$. Hence $b = ax$ for some $x \in R$. Thus we have $q = (a, b) = (a, ax) \in \{(c, cx) : c \in E\} = Y \cong E$. Choose $X = Y \oplus \{(0, c) : c \in M\} \cong E \oplus M$. Therefore $N = M \oplus E$ is weakly R -injective. \square

Definition 2.14. [6, Def.4.6] *The singular submodule of an R -module M is $Z(M) = \{m \in M : Em = 0\}$ for some essential right ideal E of R .*

The module M is said to be singular if $M = Z(M)$, and is called nonsingular if $Z(M) = 0$. Clearly $Z(M)$ is singular, for any module M .

Proposition 2.15. [9, Proposition1.12] *Every nonsingular module over a noetherian prime ring is weakly-injective.*

Proof. Over a noetherian prime ring R , every torsion free right module contains an essential submodule which is a direct sum of uniform submodules. Since weakly-injective module over noetherian rings are closed under arbitrary direct sums and under essential extension, it suffices to show that every uniform nonsingular right R -module is weakly-injective. Let U be a uniform nonsingular right R -module and V be a finitely generated submodule of $E(U)$. Since R is prime and noetherian, it follows that V is isomorphic to a right ideal of R and that therefore it embeds in U via a monomorphism ϕ . By the injectivity and indecomposability of $E(U)$, ϕ extends to an automorphism $\phi' : E(U) \rightarrow E(U)$. Let σ be the restriction of ϕ'^{-1} to U . Then σ is monomorphism satisfying that $V \subseteq \sigma(U)$. Proving our claim. \square

Corollary 2.16. [9, Corollary1.13] *For any module A over a noetherian prime ring R , A is weakly injective if and only if its singular submodule $Z(A)$ is weakly injective.*

Proof. The injective hull of A may be written as $E(A) = E(Z(A)) \oplus K$ where $Z(A)$ is the torsion submodule of A and K is some nonsingular submodule of $E(A)$. If A is weakly injective and N is a finitely generated submodule of $E(Z(A))$, then there exists $X \cong A$ such that $N \subseteq X \subseteq E(A)$. But N is singular, hence $N \subseteq Z(X) \subseteq E(Z(A))$. Also $Z(X) \cong Z(A)$. Which prove our claim. \square

Proposition 2.17. [9, Proposition 1.14] *For a right nonsingular ring R , R is a right weakly injective ring if and only if for all $q_1, q_2 \in Q$, there exists $c \in R$ such that $q_1, q_2 \in c^{-1}R$. In particular Q is a classical left ring of quotients of R .*

Proof. \Rightarrow Let $Q = E(R_R)$ is a regular right self injective ring. Let $1, q_1, q_2 \in Q$. By remark 2.11, there exists $b \in Q$ such that $r.\text{ann}_R(b) = 0$ and $1 \in bR$, $q_1, q_2 \in bR$. Since $r.\text{ann}_R(b) = 0$, b has a left inverse say c in Q . Also $1 \in bR$ implies b has a right inverse in R . Thus $q_i \in c^{-1}R$, where $c \in R$, $i = 1, 2$. To prove that Q is classical left ring of quotients, we need to show in addition that every regular element in R is invertible in Q . We note that ${}_R R \subset'_R Q$. Next let $x \in R$ be a regular element. Then $r.\text{ann}_Q(x) = l.\text{ann}_Q(x) = 0$. Since ${}_R R \subset' Q_R$ and ${}_R R \subset'_R Q$. Hence x is invertible in Q . \square

Corollary 2.18. [9, Corollary 1.15] *If R is a Von-Neumann regular ring, then R is a right self-injective ring if and only if R is a right weakly-injective ring.*

Proof. Straightforward from previous proposition. \square

The next theorem shows that a right nonsingular ring right and left weak injectivity implies the coincidence of the classical ring of quotients with the maximal ring of quotients.

Theorem 2.19. [9, Theorem 1.16] *Let R be a right nonsingular ring. Then the following statements are equivalent:*

1. R is right and left weakly-injective;
2. $E({}_R R) = l.Q(R) = r.Q_{cl}(R) = l.Q_{cl}(R) = r.Q(R) = E(R_R)$.

Proof. (1) \Rightarrow (2). By previous proposition we have $Q = E(R_R) = l.Q_{cl}(R)$. Therefore, considering Q as a left R -module, we have ${}_R R \subset'_R Q$. Since Q is Von-Neumann regular, $Z({}_R R) = 0$. Therefore, applying previous proposition to the left weakly-injective left module R , we get $r.Q_{cl}(R) = l.Q(R) = E({}_R R)$. Since both classical right and left quotient rings exist, they must coincide. Hence

$$E({}_R R) = l.Q(R) = r.Q_{cl}(R) = l.Q_{cl}(R) = r.Q(R) = E(R_R)$$

. (2) \Rightarrow (1) This follows by the definition of weak-injectivity and remark 3. \square

2.2 Direct sums of weakly injective modules

In this section we study rings over which arbitrary direct sums of weakly injective modules are weakly injective. We show that this condition is equivalent to direct sums of tight modules be tight.

Recall that an R module M is called tight relative to the R -module N if whenever a quotient N/K of N is embeddable in $E(M)$, N/K is also embeddable in M . We call an R module M tight if M is N -tight for every finitely generated R modules N .

Remark 2.20. From the definition of tightness we notice that for a uniform module M , M is weakly N -injective if and only if M is tight.

We will consider when tight modules are weakly-injective, when weakly-injective modules are injective and when weakly-injective modules are closed under direct summands.

Theorem 2.21. [9, Theorem2.6] For a ring R , the following conditions are equivalent:

1. R is a right q.f.d. ring,
2. every direct sum of injective right R -modules is weakly- injective,
3. every direct sum of injective right R -modules is tight,
4. every direct sum of tight R -modules is tight,
5. every direct sum of weakly- injective right R -modules is tight,
6. every direct sum of weakly- injective right R -modules is R -tight,
7. every direct sum of indecomposable injective right R -modules is R -tight.

Proof. (1) \Rightarrow (2). Consider $M = \bigoplus_{i \in \Lambda} E_i$ where, for every $i \in \Lambda$, E_i is an injective right R -module. Let N be a finitely generated submodule of $E(M)$. By the hypothesis, N contains as an essential submodule a direct sum of uniform submodules $U_1 \oplus \dots \oplus U_k$. Since $M \subset' E(M)$, there exist $0 \neq q_i \in U_i \cap M$. So $\bigoplus_{i=1}^k q_i R$ is contained in a finite direct sum $E_{i_1} \oplus \dots \oplus E_{i_t}$ where, for $j = 1, \dots, t$, $i_j \in \Lambda$. This implies that $E_{i_1} \oplus \dots \oplus E_{i_t}$ contains an injective hull E of $\bigoplus_{i=1}^k q_i R$. Since E is injective and contained in M , we may write $M = E \oplus K$, for some submodule K of M . On the other hand, let $E(N)$ be an injective hull of N inside $E(M)$. Then $E(N) = \bigoplus_{i=1}^k E(U_i) = \bigoplus_{i=1}^k E(q_i R) \cong E$. Since $\bigoplus_{i=1}^k q_i R \subset' E(N)$. It follows that $E(N) \cap K = 0$.

So let $X = E(N) \oplus K \cong E \oplus K = M$. Then $N \subset X$, proving our claim.

(2) \Rightarrow (3) obvious.

(3) \Rightarrow (4). Consider now a direct sum $\bigoplus_{i \in \Lambda} M_i$ where, for each $i \in \Lambda$, M_i is tight. Let N be a finitely generated submodule of $E[\bigoplus_{i \in \Lambda} M_i] = E[\bigoplus_{i \in \Lambda} E_i]$ where for each $i \in \Lambda$, $E_i = E(M_i)$. By the hypothesis, N is embeddable in $\bigoplus_{i \in \Lambda} E_i$ via a monomorphism ϕ , say. Now, $\phi(N)$ is therefore, contained in a finite direct sum $E_{i_1} \oplus \dots \oplus E_{i_t}$ where, for $j = 1, \dots, t$, $i_j \in \Lambda$. Now $M_{i_1} \oplus \dots \oplus M_{i_t}$, being a finite direct sum of tight modules, is tight. So $N \cong \phi(N) \subset E_{i_t} = E(M_{i_1} \oplus \dots \oplus M_{i_t})$ is embeddable in $M_{i_1} \oplus \dots \oplus M_{i_t}$ and hence in $\bigoplus_{i \in \Lambda} M_i$, proving our claim.

(4) \Rightarrow (5), (5) \Rightarrow (6) and (6) \Rightarrow (7), obvious.

(7) \Rightarrow (1). We shall do this by proving that every cyclic right R -module has a finitely generated socle. Let M be a cyclic right R -module. If $\text{Soc}(M) = 0$, we are done. On the other hand, if $\text{Soc}(M) \neq 0$, Let k be a complement of $\text{Soc}(M)$ in M . Then $\text{Soc}(M)$ embeds as an essential submodule of the quotient of M by k (again acyclic right R -module). So we may assume, without loss of generality, that M has an essential socle. Let $\text{Soc}(M) = \bigoplus_{i \in A} S_i$ where, for each $i \in A$, S_i is simple. Then $E(M) = E[\bigoplus_{i \in A} E(S_i)]$. Now since $\bigoplus_{i \in A} E(S_i)$ is R -tight and M is a cyclic submodule of its injective hull, it follows that M embeds in $\bigoplus_{i \in A} E(S_i)$. Consequently, M embeds in a submodule $L = \bigoplus_{i \in B} E(S_i)$ where, B is a finite subset of A . Thus, since L has finitely generated socle M does also, concluding our proof. \square

In this theorem we are able to replace weakly injectivity by tight in every condition.

Corollary 2.22. [9, Theorem2.7] *A ring R is a right q.f.d. ring if and only if any one of the following conditions hold:*

1. *every direct sum of weakly-injective right modules is weakly-injective,*
2. *every direct sum of weakly injective right modules is weakly R -injective,*
3. *every direct sum of indecomposable injective right modules is weakly R -injective.*

Proof. Obviously (1) \Rightarrow (2) \Rightarrow (3), and (3) implies condition (4) of the previous theorem, and hence R is q.f.d. ring whenever(3) holds. So it is only left to show that every right q.f.d. ring R satisfies (1). Consider the module $M = \bigoplus_{i \in \Lambda} M_i$, a direct sum of weakly-injective modules $M_i, i \in \Lambda$. Let N be a finitely generated submodule of $E(M)$. By condition (2)of the previous

theorem, we know that the direct sum of injectives $\bigoplus_{i \in \Lambda} E(M_i)$ is weakly-injective. Also

$$M \subset' \bigoplus_{i \in \Lambda} E(M_i) \subset' E(M)$$

. Hence there exists a submodule $Y \subset E(M)$ such that $N \subset Y$ and $Y \cong \bigoplus_{i \in \Lambda} E(M_i)$. Write $Y = \bigoplus_{i \in \Lambda} E(Y_i)$ such that $Y_i \cong M_i$ for all $i \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma \subset \Lambda$ such that $N \subset \bigoplus_{i \in \Gamma} E(Y_i) = E[\bigoplus_{i \in \Gamma} Y_i]$. Since the Y_i 's are weakly-injective, the finite sum $\bigoplus_{i \in \Gamma} Y_i$ is weakly-injective and, therefore, there exists $X_1 \cong \bigoplus_{i \in \Gamma} M_i$ such that $N \subset X_1 \subset E[\bigoplus_{i \in \Gamma} Y_i]$. But then $N \subset X_1 \oplus \bigoplus_{i \in \Lambda - \Gamma} Y_i = X \cong M$, proving our claim. \square

Now we shall consider when tight modules are weakly-injective.

Theorem 2.23. [9, Lemma2.8] *Let Λ be a class of modules which is closed under submodules and under injective hulls. Let N be a finitely generated module. If every cyclic module in Λ has finite Goldie dimension, then for every $M \in \Lambda$, M is N -tight if and only if M is weakly N -injective.*

Proof. In (module whose quotients have finite Goldie dimension) it have been proved under these hypothesis every finitely generated module in Λ has finite Goldie dimension. It also follows using Zorns lemma, that every module in Λ contains an essential submodule, a direct sum of uniform submodules. Let M be an N -tight module in Λ and let $\phi : N \rightarrow E(M)$. Since $\phi(N)$ is a finitely generated module in Λ , we conclude that $\phi(N)$ has finite Goldie dimension and thus its injective hull is a sum of indecomposable injectives say $E(\phi(N)) = E_1 \oplus E_2 \oplus \dots \oplus E_n$ for $i = 1, \dots, n$. Let $W_i = E_i \cap \phi(N)$. Then $E_i = E(W_i)$. Since M is N tight there exists a map $\psi : N \rightarrow M$ such that $\psi(N)$ is isomorphic to $\phi(N)$. Let $\theta : \phi(N) \rightarrow \psi(N)$ be an isomorphism. Let K be a complement of $\psi(N)$ in M and let $\bigoplus_{i \in I} U_i$ essential in K , where for all $i \in I$, U_i is uniform. Similarly, let K' be a complement of $\phi(N)$ in $E(M)$ and let $\bigoplus_{j \in J} V_j$ is essential in K' . A direct sum of uniform submodule. It follows that

$$E_1 \oplus \dots \oplus E_n \oplus \bigoplus_{j \in J} E(V_j) \subset' E(M)$$

, and

$$E(\theta(W_1)) \oplus \dots \oplus E(\theta(W_n)) \oplus \bigoplus_{i \in I} E(U_i) \subset' E(M)$$

. Hence, (corollary in decomposition of injective module)

$$E_1 \oplus \dots \oplus E_n \oplus \bigoplus_{j \in J} E(V_j) \cong E(\theta(W_1)) \oplus \dots \oplus E(\theta(W_n)) \oplus \bigoplus_{i \in I} E(U_i)$$

. So there exist an isomorphism

$$\eta : \bigoplus_{i \in I} E(U_i) \rightarrow \bigoplus_{j \in J} E(V_j)$$

. Define η' the restriction of η to $\bigoplus_{i \in I} U_i$. Consider then the one to one map

$$\sigma = \theta^{-1} \oplus \eta' : \psi(N) \oplus \bigoplus_{i \in I} U_i \rightarrow E(M)$$

which extends to a monomorphism

$$\sigma' : M \rightarrow E(M) \text{ (since } \psi(N) \oplus \bigoplus_{i \in I} U_i \subset M \text{)}$$

. This monomorphism satisfies that

$$\phi(N) = \theta^{-1}(\psi(N)) = \sigma(\psi(N)) = \sigma'(\psi(N)) \subseteq \sigma'(M)$$

as desired. \square

Theorem 2.24. *Every tight module over a right q.f.d ring R (in particular over a right noetherian ring R) is weakly-injective.*

Proof. Let Λ be the class of all right R -module and use previous theorem. \square

We will consider direct summands of weakly-injective module, in particular weakly-injective modules are not necessarily closed under direct summands, the following two propositions illustrate to what extreme this condition fails.

Proposition 2.25. [9, Proposition 2.12] *Every completely reducible module over an arbitrary ring R is a direct summand of a weakly-injective R -module. (Completely reducible: A module A over an associative ring R which can be represented as the sum of its irreducible R submodule).*

Proof. Let M be a completely reducible R -module. Let us write $M = \bigoplus_{i \in I} [S_i]$, where $[S_i]$ represents the homogeneous component of M corresponding to the simple submodule $S_i \subset M$. It follows for every $i \in I$, there exists a cardinal χ_i such that $[S_i] \cong S_i^{\chi_i}$. Let N be an infinite cardinal greater than both the cardinality of R and the number of summands of M . In particular, for every $i \in I$, $N > \chi_i$. Notice that for every finitely generated right R -module N , if $\bigoplus_{\alpha \in \Gamma} U_\alpha$ is an internal direct sum of nonzero submodules of N , then the cardinality of Γ is less than χ . Let $V = M \oplus E(M^{(x)})$. We claim

that V is weakly-injective. Notice that $E(V) \cong E(M^{(\chi)})$ and $Soc(V) = Soc E(V) \cong \bigoplus_{i \in I} [S_i]^\chi \cong \bigoplus_{i \in I} ([S_i]^{\chi_i})^\chi \cong \bigoplus_{i \in I} [S_i]^\chi$. Let N be a finitely generated submodule of $E(V)$. Then the number of simple summands in any decomposition of $Soc N$ is less than χ . Let us say that $Soc(N) = \bigoplus_{i \in I} [[S_i]]$, where $[[S_i]]$ denotes the homogeneous component of $Soc(N)$ corresponding to S_i . Since for every $i \in I$ the number of simple summands in $[[S_i]]$ is less than χ , we conclude that the homogeneous component of $Soc E(V)$ corresponding to S_i equals $[[S_i]] \oplus K_i$, for some $K_i \cong S_i^\chi$, hence we get $Soc V = Soc N \oplus T$, for some $T \cong Soc V$, therefore, $E(Soc V) = E(V) = E(N) \oplus E(T)$ and $E(T) \cong E(V)$. Let Y be a submodule of $E(T)$ isomorphic to V and define $X = E(N) \oplus Y$. Then $X \cong E(N) \oplus M \oplus E(M^\chi) = M \oplus E(N \oplus M^\chi) \cong M \oplus E(\bigoplus_{i \in I} [[S_i]] \oplus \bigoplus_{i \in I} (S_i)^\chi) \cong M \oplus E(\bigoplus_{i \in I} (S_i)^\chi) \cong M \oplus E(M^\chi) = V$. Since $N \subset X$, this complete our proof. \square

Corollary 2.26. [9, Corollary2.13] *Over a right semi-artinian ring R , every right R -module is a summand of a weakly-injective right module.*

Proof. This follows from previous proposition since weak-injectivity is preserved by essential extension. \square

Proposition 2.27. [9, Proposition2.14] *Over arbitrary rings every module is a summand of a tight module. If R is a right q.f.d. ring every right R -module is a summand of a weakly-injective right module.*

Proof. Let M be a right module over the right q.f.d. ring R , and let χ be any infinite cardinal. Consider the module $N = M \oplus E(M^\chi)$, since $E(M)$ is isomorphic to a submodule of N , N is tight, then N is weakly injective. \square

Theorem 2.28. [9, Theorem2.15] *Let R be a ring. Then the following are true:*

1. *direct summand of weakly-injective (tight) right R -modules are weakly-injective (tight) if and only if every R -module is weakly-injective,*
2. *every weakly -injective (tight) right module is injective if and only if R is semisimple artinian.*

Proof. If weakly-injective (tight) right R -modules were closed under direct summands, proposition 2.25 implies that every completely reducible right R -module would be weakly-injective (tight) and thus injective. This implies that R is right noetherian. Then by proposition 2.27 and the hypothesis, R is right weakly-semisimple. One can argue in the same way to prove that if every weakly-injective module is injective. Then every right R -module is injective, and hence R is semisimple artinian. \square

Chapter 3
Weakly Projective Modules

3 Weakly projective modules

The object of this chapter is to introduce the concepts of weak relative projectivity of right R -module and to dualize most of the basic results of weakly injectivity stated in chapter 2 section 1. We study certain relations between the concepts of weakly injective and weakly projective modules.

3.1 Weakly projective modules

The object of this section is to introduce the concept of projective cover and weakly projective modules. An epimorphism $p : P \rightarrow M$ is called a projective cover of M if and only if P is projective and p is a small epimorphism ($\ker p$ is small in P). We denote projective cover of M by $P(M)$.

Theorem 3.1. [26, Theorem3.6] *If an R projective right R module M has a projective cover $P(M)$ then M is projective.*

Proof. Suppose $P(M) = \bigoplus \Sigma R$, via an epimorphism π , then consider $\pi_i : \bigoplus \Sigma R \rightarrow R$. Since M is R -projective, $\pi_i \ker \pi = 0$, for all i . Therefore $\ker \pi \subseteq \ker \pi_i$, for all i . Thus $\ker \pi \subseteq \bigcap \ker \pi_i = 0$. Thus π is one to one and onto, and so M is projective. Next, suppose that the projective cover $P(M)$ is not free, then there exist a free module $\bigoplus \Sigma R$ such that $P \oplus K = \bigoplus \Sigma R$. Consider $\pi_i \bigoplus \Sigma R \rightarrow R$, then $\pi_{i|P} : P \rightarrow R$, therefore $\ker \pi \subseteq \ker \pi_{i|P} = \ker \pi_i \cap P \subseteq \ker \pi_i$, for all i . Thus $\ker \pi \subseteq \bigcap \ker \pi_i \cap P = 0$. Thus $\ker \pi = 0$. This implies π is an isomorphism, and so M is projective. \square

Theorem 3.2. [11, Theorem2.0] *Let M and N be right R -module and assume M has a projective cover $P(M)$ via an epimorphism $\pi : P(M) \rightarrow M$. Then M is N -projective if and only if for every homomorphism $\phi : P(M) \rightarrow N$ there exists a homomorphism $\phi' : M \rightarrow N$ such that $\phi = \phi' \pi$. Equivalently, $\phi(\ker \pi) = 0$.*

Proof. Let $\phi : P \rightarrow N$ be a homomorphism. We shall first show that $\phi(\ker \pi) = 0$. Let $T = \phi(\ker \pi)$ and let $\pi_T : N \rightarrow N/T$ be the natural projection. Then ϕ induced $\phi' : M \rightarrow N/T$ defined by $\phi'(m) = \pi_T \phi(p)$, where $m = \pi(p)$. It follows that $\phi' \pi = \pi_T \phi$. Since M is N -projective, there exists a map $\beta : M \rightarrow N$ such that $\phi' = \pi_T \beta$. Clearly, $(\phi - \beta \pi)P \subseteq T$. We claim that $\phi = \beta \pi$.

let $X = \{p \in P : \phi(p) = \beta \pi(p)\}$. We shall show that $X = P$. Let $x \in P$, since $(\phi - \beta \pi)(x) \in T = \phi(\ker \pi)$, there exists $k \in \ker \pi$ such that $(\phi - \beta \pi)(x) = \phi(k)$. Therefore $\phi(x - k) = \beta \pi(x - k) = 0$, since $\beta \pi(k) = 0$. Thus $x - k \in X$. Therefore $\ker \pi + X = P$, which implies $X = P$, since $\ker \pi$ is small in

P . Therefore $(\phi - \beta\pi)P = 0$. In particular, $(\phi - \beta\pi)\ker\pi = 0$, yielding $\phi(\ker\pi) = 0$. Equivalently, there exists $\phi'' : M \rightarrow N$ such that $\phi''\pi = \phi$. Conversely, let $\psi : M \rightarrow N/K$ be a homomorphism. Then by the projectivity of P there exists a homomorphism $\psi'P \rightarrow N$ such that $\psi\pi = \pi_k\psi'$. By our hypothesis there exists $\psi'' : M \rightarrow N$ such that $\psi''\pi = \psi'$. It follows that $\pi_k\psi'' = \psi$ as desired. \square

Corollary 3.3. [26, Corollary3.9] *Let M be a right R module, assume M has a projective cover $P(M)$ via an epimorphism $p : P(M) \rightarrow M$, then M is quasi projective if and only if for every homomorphism $\phi : P(M) \rightarrow P(M)$, $\phi(\text{Ker } p) = 0$.*

Proof. Follows from previous theorem.

Definition 3.4. [11, Def.2.1] *Let M and N be modules, and assume M has a projective cover $p : P \rightarrow M$, we say that M is weakly N -projective, if for every map $\phi : P \rightarrow M$, there exist an epimorphism $\sigma : P \rightarrow M$ and homomorphism $\phi' : M \rightarrow N$ such that $\phi = \phi'\sigma$. If a module M is weakly N -projective for all finitely generated right R -module N , we say that M is weakly projective.*

$$\begin{array}{ccc}
 & & M \\
 & \nearrow \exists \sigma & \downarrow \exists \phi' \\
 P & \xrightarrow{\phi} & N
 \end{array}$$

For example, let R be a uniserial ring which is not a division ring, and $S = \text{soc}(R)$. Then as a right R -module, $R/S \times R$ is weakly R projective but not R projective.

We will conclude this section with basic results for weakly projective modules.

Theorem 3.5. [11, Theorem 2.2] *Let M and N be modules, and assume M has a projective cover $p : P(M) \rightarrow M$, then M is weakly N -projective if and only if for every map $\phi : P(M) \rightarrow M$ there exists a submodule $X \subset \ker\phi$, such that $P(M)/X \cong M$.*

Proof. Let $\phi : P \rightarrow N$ be a homomorphism. Assume first that M is weakly N -projective and let the homomorphism $\hat{\phi} : M \rightarrow N$ and the epimorphism

$\sigma : P \rightarrow M$ be as in the definition of weak relative-projectivity. Since $\phi = \hat{\phi}\sigma$, $\ker \sigma \subset \ker \phi$. Also, $P/\ker \sigma \cong M$. Thus, the implication is proven by choosing $X = \ker \sigma$. Conversely, if $X \subset P$ satisfies the condition in the statement of the theorem, then the isomorphism $P/X \cong M$, composed with the natural projection $p_X : P \rightarrow P/X$ is an epimorphism $\sigma : P \rightarrow M$ satisfying that $\ker \sigma = X \subset \ker \phi$. It follows that the map $\hat{\phi} : M \rightarrow N$ given by $\hat{\phi}(m) = \phi(p)$, whenever $\sigma(p) = m$ is well defined and satisfies $\phi = \hat{\phi}\sigma$, proving our claim. \square

Lemma 3.6. [26, Lemma2.1.3] *A right module M is weakly projective if and only if M is weakly R^n -projective for all $n \in \mathbb{Z}^+$.*

Proof. We only need to show that if M is weakly R^n -projective then it is weakly projective. Let N be a finitely generated module and let $\phi : P(M) \rightarrow N$. Since N is finitely generated, there exists an epimorphism $\rho : R^n \rightarrow N$ for some $n \in \mathbb{Z}^+$. The projectivity of $P(M)$ yields the existence of a homomorphism $\phi' : P(M) \rightarrow R^n$ such that $\rho\phi' = \phi$. Since M is weakly R^n -projective, there exists $X \subseteq \ker \phi'$ such that $P(M)/X \cong M$. However $\ker \phi' \subseteq \ker \phi$. Thus $X \subseteq \ker \phi$, proving M is weakly N -projective. \square

Domain of weak projectivity are closed under quotients and submodules as shown in the next proposition.

Proposition 3.7. [11, Proposition2.3] *Let M and N be modules and assume M has a projective cover $p : P(M) \rightarrow M$. Then the following statements are equivalent:*

1. M is weakly N -projective,
2. for every submodule $K \subset N$, M is weakly K -projective,
3. for every submodule $K \subset N$, M is weakly N/K projective.

Proof. Since either condition (2) or (3) trivially implies (1), we need only show that (1) implies both (2) and (3). (1) \Rightarrow (2) Assume M is weakly N -projective and let K be a submodule of N and $\phi : P \rightarrow K$ be a homomorphism. Then $\psi = i_K \phi : P \rightarrow N$ may be expressed as a composition $\psi = \hat{\psi}\sigma$, for some homomorphism $\hat{\psi} : M \rightarrow N$ and epimorphism $\sigma : P \rightarrow M$. Since σ is onto, the range of $\hat{\psi}$ equals the range of ψ and so it is contained in K . Thus, we may define $\hat{\phi} : M \rightarrow K$ via $\hat{\phi}(m) = \hat{\psi}(m)$ and then $\phi = \hat{\phi}\sigma$, proving that M is weakly K -projective, as claimed.

(1) \Rightarrow (3) Assume once again that M is weakly N -projective and let $f : P \rightarrow N/K$ be a homomorphism. Since P is projective, there exists a map

$g : P \rightarrow N$ such that $f = p_K g$. The weak N -projectivity of M yields an epimorphism $\sigma : P \rightarrow M$ and a homomorphism $\hat{g} : M \rightarrow N$ such that $g = \hat{g} \sigma$. Let $\hat{f} = p_K \hat{g}$. Then $\hat{f} \sigma = p_K \hat{g} \sigma = p_K g = f$, proving that M is indeed weakly N/K -projective. \square

One can characterize weak projectivity in terms of supplements of submodules as it is shown in the next proposition.

Proposition 3.8. [11, Proposition2.5] *Let M and N be modules, and assume M is supplemented and has a projective cover $p : P \rightarrow M$, then M is weakly N -projective if and only if for every submodule $K \subset N$ and for every epimorphism $\phi : P \rightarrow K$, there exists an epimorphism $\phi' : M \rightarrow K$ such that for every supplement L' of $\text{Ker} \phi'$ in M there exists a submodule $L \subset P$, such that $P/L \cong M/L'$ and $L + \text{ker} \phi = P$.*

Proof. Assume M is weakly N -projective and let $\phi : P \rightarrow K$ be an epimorphism onto a submodule $K \subset N$. Then there exists an epimorphism $\sigma : P \rightarrow M$ and $\hat{\phi} : M \rightarrow K$ such that $\phi = \hat{\phi} \sigma$. Let L' be a supplement of $\text{ker} \hat{\phi}$ in M and let $L = \sigma^{-1}(L')$. For an arbitrary $p \in P$, $\sigma(p)$ may be written as $\sigma(p) = l' + k'$, with $l' \in L'$ and $k' \in \text{ker} \hat{\phi}$. It follows then that $\phi(p) = \hat{\phi} \sigma(p) = \hat{\phi}(l') + \hat{\phi}(k') = \hat{\phi}(l')$. Choose $p_1 \in \sigma^{-1}(l') \subset L$. Then $\sigma(p_1) = l'$. On the other hand, $\phi(p_1) = \hat{\phi} \sigma(p_1) = \hat{\phi}(l') = \phi(p)$. So $p - p_1 \in \text{ker} \phi$ and so $L + \text{ker} \phi = P$. The fact that $P/L \cong M/L'$ follows since L is the kernel of the onto map $\pi_{L'} \sigma : P \rightarrow M/L'$. Conversely, let us assume that for every submodule $K \subset N$ and for every epimorphism $\phi : P \rightarrow K$ there exists an epimorphism $\hat{\phi} : M \rightarrow K$ such that for every supplement L' of $\text{ker} \hat{\phi}$ in M there exists a submodule $L \subset P$ such that $P/L \cong M/L'$ and $L + \text{Ker} \phi = P$. Let $\phi : P \rightarrow K$ be an epimorphism and $\hat{\phi} : M \rightarrow K$ be the corresponding epimorphism. All we need is to produce another epimorphism $\sigma : P \rightarrow M$ such that $\phi = \hat{\phi} \sigma$. Let L' be a supplement for $\text{ker} \hat{\phi}$ and let L be the corresponding submodule of P . Let $\theta : P/L \rightarrow M/L'$ be an isomorphism. The Chinese remainder theorem yields that the map $m + \text{ker} \hat{\phi} \cap L' \rightarrow (M + \text{ker} \hat{\phi}, m + L')$ is an isomorphism between $M/(\text{ker} \hat{\phi} \cap L')$ and $M/\text{ker} \hat{\phi} \times M/L'$. Also, $M/\text{ker} \hat{\phi} \cong K$ via $m + \text{ker} \hat{\phi} \rightarrow \hat{\phi}(m)$. So, one gets an isomorphism $\beta : M/\text{ker} \hat{\phi} \cap L' \rightarrow K \times M/L'$ such that $\beta(m + \text{ker} \hat{\phi} \cap L') = (\hat{\phi}(m), \pi_{L'}(m))$. The isomorphism θ induces an onto map $\psi = \theta \pi_L : P \rightarrow M/L'$. Since $\text{ker} \phi + L = P$, the map $\alpha : P \rightarrow K \times M/L'$ given by $\alpha(p) = (\phi(p), \psi(p))$ is onto. The inducted epimorphism $\alpha' : \beta^{-1} \alpha : P \rightarrow M/(\text{Ker} \hat{\phi} \cap L')$ may then be lifted to a map $\sigma : P \rightarrow M$. Since $\text{ker} \hat{\phi} \cap L' \ll M$, σ is indeed an epimorphism. It only remains to show that $\hat{\phi} \sigma = \phi$. Let us refer for the rest of this proof to

$\pi \ker \hat{\phi} \cap L'$. Simply as π . We do know that $\pi \sigma = \sigma' = \beta' \alpha$ Hence $\beta \pi \sigma = \alpha$. Let $p \in P$ be arbitrary. Then $\beta(\sigma(p) + \ker \hat{\phi} \cap L') = \alpha(p) = (\phi(p), \psi(p))$. On the other hand, $\beta(\sigma(p) + \ker \hat{\phi} \cap L') = (\hat{\phi}(\sigma(p), \sigma(p) + L')$. Comparing the first component in both expressions yields the desired equality. Thus, M is weakly N -projective. \square

Corollary 3.9. [11, Corollary2.6] *Let M be a hollow module with projective cover P and N be an arbitrary module, Then M is weakly N -projective if and only if any submodule K of M which is a homomorphic image of P is a homomorphic image of M .*

Proof. Straightforward from the above proposition. \square

Modules which are weakly-projective relative to a fixed module are closed under finite direct sum and under superfluous cover but not under direct summand. So, in particular, finite direct sums of weakly -projective modules are weakly-projective and superfluous cover of weakly-projective module are weakly-projective.

Proposition 3.10. [26, Proposition2.1.8]

1. *Let $M_i, i = 1, 2, \dots, n$ be a family of weakly N -projective modules. Then the direct sum $\bigoplus_{i=1}^n M_i$ is weakly N -projective.*
2. *Let M/N be weakly K -projective module where $N \ll M$. Then M is weakly K -projective.*
3. *If a module is weakly- projective relative to its own projective cover, then the module is indeed projective.*

Proof. (1) Let $p_i : P_i(M_i) \rightarrow M_i, (i = 1, \dots, n)$ be projective covers. By (Lemma 1.3.1 in [26]), $\bigoplus_{i=1}^n p_i : \bigoplus_{i=1}^n P_i(M_i) \rightarrow \bigoplus_{i=1}^n M_i$ is a projective cover. Let $\phi : \bigoplus_{i=1}^n P_i(M_i) \rightarrow N$, and let $i_{p_i} : P_i(M_i) \rightarrow \bigoplus_{i=1}^n P_i(M_i)$ be the inclusion map. Then by weak projectivity of M_i 's for each i , there exists an epimorphism $\sigma_i : P_i(M_i) \rightarrow M_i$ and $\hat{\phi}_i : M_i \rightarrow N$ such that $\hat{\phi}_i \sigma_i = \phi i_{p_i}$. Set $\hat{\phi} = \bigoplus_{i=1}^n \hat{\phi}_i$ and $\sigma = \bigoplus_{i=1}^n \sigma_i$. Then it follows that $\phi = \hat{\phi} \sigma$, as desired.

(2). Since $N \ll M, M$ and M/N have the same projective cover. Let $\phi : P(M) \rightarrow K$ and $\pi_N : M \rightarrow M/N$ be the natural projection. Since M/N is weakly K -projective, there exists an epimorphism $\sigma : P(M) \rightarrow M/N$ and a homomorphism $\hat{\phi} : M/N \rightarrow K$ such that $\hat{\phi} \sigma = \phi$. By the projectivity of $P(M)$, there exists $\sigma' : P(M) \rightarrow M$ such that $\pi_N \sigma' = \sigma$. Since $N \ll M$,

it follows that σ' is onto. It is easy to check that $\hat{\phi}\pi_N\sigma' = \phi$. Thus M is weakly K -projective.

(3) Consider a module M with projective cover $\pi : P \rightarrow M$. If we assume that M is weakly P -projective, then the identity map on P factors through M and this yields that $M \cong P$. \square

A finitely generated direct summand S of the projective cover of a weakly projective module M yields a direct summand (isomorphic to S) of M .

Lemma 3.11. [11, Lemma2.9] *Let M be a weakly projective module whose projective cover $P(M) = S \oplus K$, where S is finitely generated. Then M has a direct summand isomorphic to S .*

Proof. Since S is finitely generated, M is weakly S -projective (Proposition 3.7). Thus the projection map $p : P(M) \rightarrow S$ factors through M , yielding an epimorphism $\hat{p} : M \rightarrow S$. Since S is projective we get that $M \cong S \times \ker \hat{p}$, proving our claim. \square

Proposition 3.12. [11, Proposition2.10] *Every finitely generated projective module is indeed projective over a semiperfect ring R , a finite Goldie dimensional weakly projective module is indeed projective.*

Proof. If M is finitely generated, then $P(M)$ is also finitely generated and so, by Proposition 3.10, M is projective. Suppose $R = \bigoplus_{i=1}^n e_i R$ is the representation of the semiperfect ring R as a direct sum of indecomposable projective modules. Let N be a finite Goldie dimensional weakly projective R -module. Write $P(N) = \bigoplus_{i=1}^n (e_i R)^{\alpha_i}$ as a direct sum of indecomposable projective modules. If any of the α_i 's were infinite, by Lemma 3.11, N would contain sums of arbitrarily many submodules, contradicting that N has finite Goldie dimension. Therefore, $P(N)$ is finitely generated and hence N is projective. \square

An important fact in the theory of weakly-injective modules is that a quasi-injective weakly-injective module is indeed injective. The dual result is:

Proposition 3.13. [11, Proposition2.12] *Let N be a module, then any quasi-projective weakly N -projective module is indeed N -projective.*

Proof. Let M be a quasi-projective module M with projective cover $\pi : P \rightarrow M$ and assume that M is weakly N -projective. Consider a map $\phi : M \rightarrow N/K$, for some submodule $K \subset N$. The projectivity of P guarantees the

existence of a map $\hat{\phi} : P \rightarrow N$ such that $\phi\pi = \pi_K\hat{\phi}$. Now, since M is weakly N -projective there exists an epimorphism $\sigma : P \rightarrow M$ and a map $\hat{\psi} : M \rightarrow N$ such that $\hat{\psi}\sigma = \hat{\phi}$. Since M is quasi-projective there exists $\sigma' : M \rightarrow M$ such that $\sigma'\pi = \sigma$ (Theorem 3.2). One easily checks that the map $\hat{\psi}\sigma' : M \rightarrow N$ lifts ϕ , proving our claim. \square

The next result show that for right perfect rings it is true that every right module is a summand of a weakly projective module

Theorem 3.14. [11, Theorem3.1] *Over a right perfect ring R , there exists a module K such that the direct sum of K plus any other module yields a weakly projective module.*

Proof. Since R is right perfect, we may write $R = \bigoplus_{i=1}^k (e_i R)^{n_i}$, where $\{e_i R, \dots, e_k R\}$ is a complete set of representatives of indecomposable projective right R -modules. Let $L = \bigoplus I$, where $I \subset R^n$ for all $n \in \mathbb{Z}^+$ be the external sum of all submodules of finitely generated free right R -modules. Let \aleph be an infinite cardinal such that $\aleph > |R|$. Define $K = L \oplus |P(L)|^{(\aleph)}$, where $P(L)$ is the projective cover of L . Consider an arbitrary right R -module M and an integer $n \in \mathbb{Z}^+$. Our aim is to show that the direct sum $N = M \oplus K$ is weakly R^n -projective. Consider an epimorphism $\phi : P(N) \rightarrow I$, where $I \subset R^n$. Let $\pi : P(I) \rightarrow I$ be the projective cover map. The projectivity of $P(N)$ yields a map $\hat{\phi} : P(N) \rightarrow P(I)$ such that $\pi\hat{\phi} = \phi$. Furthermore, since $\ker\pi \ll P(I)$, one gets that $\hat{\phi}$ is an epimorphism.

Since $P(I)$ is projective, $\hat{\phi}$ splits and, therefore, we may write $P(N) = P \oplus \ker\hat{\phi}$, for some submodules $P \subset P(N)$ isomorphic to $P(I)$. Over a semiperfect ring all projective modules are decomposable as direct sums of indecomposable projective ones. So let us write $P(I) \cong \bigoplus_{i=1}^k (e_i R)^{\alpha_i} \cong P$, and $\ker\hat{\phi} \cong \bigoplus_{i=1}^k (e_i R)^{\beta_i}$. Suppose further that $P(L) \cong \bigoplus_{i=1}^k (e_i R)^{C_i}$. Then $P(K) \cong \bigoplus_{i=1}^k (e_i R)^{C_i \aleph} \cong \bigoplus_{i=1}^k (e_i R)^{D_i}$, where $D_i \geq \aleph$. Let $P(M) \cong \bigoplus_{i=1}^k (e_i R)^{F_i}$.

Since there exists an epimorphism $\psi : R^{\{t\}} \rightarrow I$, $P(I)$ is isomorphic to a summand of $R^{\{t\}} \cong \bigoplus_{i=1}^k (e_i R)^{\{n_i t\}}$. Therefore, $\alpha_i \leq n$, $|I| \leq n$, $|R^n| = n_i n |R| < \aleph$ for each i . The decomposition $P(N) = P(M) \oplus P(K)$ and $P(N) = P \oplus \ker\hat{\phi}$ imply that $\bigoplus_{i=1}^k (e_i R)^{(\alpha_i \cup \beta_i)} \cong \bigoplus_{i=1}^k (e_i R)^{(D_i \cup F_i)}$. Since each $\alpha_i < \aleph$, while $|D_i \cup F_i| \geq \aleph$, we must conclude that $|D_i \cup F_i| = \beta_i$. So, $\ker\hat{\phi} \cong P(N)$ and one can think of $\hat{\phi}$ as the projection $p : P(N) \times P(I) \rightarrow P(I)$. It then follows that $\ker\phi \cong P(N) \times \ker\pi$. Now, N is a homomorphic image of $P(N)$ and, by definition of K , there exists a submodule $N' \subset N$ such that $I \oplus N' = N$. So, there exists a submodule $K' \subset P(N)$ such that $P(N)/K' \cong N'$. \square

3.2 When weakly injective modules are weakly projective

The purpose of this section is to study some relations between the concepts of weakly-injective and weakly-projective modules.

Lemma 3.15. (a) [21, Proposition2.3] Let R be a ring and let M be a semisimple right module. Then there exists an infinite cardinal χ such that $M \oplus E(M^\chi)$ is weakly-injective. Consequently this result holds also for any right R -module with essential socle. Moreover, if R is a right q.f.d. ring, then result holds for any right module M .

(b)[21, Theorem3.1]. Over a right q.f.d. ring R , a module M is weakly-injective if and only if each finitely generated module N which is embeddable in $E(M)$ is indeed embeddable in M .

Lemma 3.16. [13, Lemma4.1] Let R be a ring such that every weakly injective right R -module has a projective cover. Then R is semiperfect.

Proof. Let S be a simple right R -module. By lemma 3.15(a) there exists an infinite cardinal χ such that $S \oplus E(S^\chi)$ is weakly-injective. Therefore by our hypothesis, $S \oplus E(S^\chi)$ has a projective cover. Our hypothesis also implies that $E(S^\chi)$ is weakly-projective, so it has projective cover. Consequently, S has projective cover. Thus R is semiperfect. \square

Lemma 3.17. [13, Lemma2.10.2] Let M be a weakly-projective right module over a semiperfect ring. If the projective cover $P(M)$ may be expressed as $S \oplus K$, with S finitely generated, then M has a direct summand isomorphic to S . In particular, a finitely generated or an indecomposable weakly-projective right module is projective.

Proposition 3.18. [13, Proposition4.2] Let R be a ring such that every weakly-injective right R -module is weakly-projective. Then R is right self-injective with a finitely generated and essential socle containing a copy of each simple right module, (R is a right PF ring).

Proof. By lemma 3.16, R is semiperfect. So we may write $R = \bigoplus_{i=1}^n e_i R$, where $A = \{e_i R : i = 1, \dots, k\}$ is a complete set of representatives for the indecomposable projective right R -module. Then $B = \left\{ \frac{e_i R}{e_i J} : i = 1, \dots, k \right\}$ is a complete set of representatives for the simple right R -module. By the hypothesis and previous lemma, each indecomposable injective right R -module is projective. In particular, for each simple right R -module S the injective hull $E(S)$ is isomorphic to eR for some idempotent e in R . Therefore every simple right module embeds in R . Define $f : B \rightarrow A$ by $f(S) = e_i R$

where $e_i R \cong E(S)$. f is one to one and onto. Hence each indecomposable projective right module is injective and has a non zero socle. Consequently $R = \bigoplus_{i=1}^n e_i R$ is injective and the right socle of R is finitely generated and essential containing a copy of each simple right R -module. Thus R is a right PF ring. \square

Proposition 3.19. [13, Proposition4.3] *Let R be a ring such that every weakly-injective right R -module is weakly-projective. Then R is left perfect ring.*

Proof. By lemma 14, R is semiperfect. Thus we may write $R = \bigoplus_{i=1}^n e_i R$, where $\{e_i R : i = 1, \dots, k\}$ is a complete set of representative for indecomposable projective module. Let M be a right module. Then by hypothesis, $E(M)$ is weakly projective having projective cover $P(E(M)) = \bigoplus_{i=1}^n (e_i R)^{\alpha_i}$. By lemma 15, $e_i R$ is a summand of $E(M)$ for each i such that $\alpha_i \neq 0$. Hence $Soc(e_i R) \subseteq Soc(E(M)) = Soc(M)$. Thus by previous proposition each nonzero right module has a non zero socle and therefore R is left perfect. Now let N be a right module since $Soc(N)$ essential in N . Then $N \oplus E(N^\chi)$ is weakly-injective for some infinite cardinal χ . Therefore, by our hypothesis $N \oplus E(N^\chi)$ and $E(N^\chi)$ are weakly- projective. Thus $N \oplus E(N^\chi)$ and $E(N^\chi)$ have projective covers, yielding that N has a projective cover. Consequently R is a right perfect ring. \square

Lemma 3.20. [13, Lemma4.4] *Let R be a ring such that every weakly-injective right module is weakly-projective and let eR be an indecomposable projective right R -module. Then for any right ideal I , $soc(\frac{eR}{eI}) \cong [\frac{eR}{eJ}]^\alpha$ for some cardinal α .*

Theorem 3.21. [13, Theorem4.5] *Let R be a ring such that every weakly-injective right R -module is weakly-projective. Then R is a finite direct sum of matrix rings over local perfect right PF-ring.*

Proof. Since R is semiperfect, we may write $R = \bigoplus_{i=1}^n e_i R$ as a direct sum of indecomposable right ideals. We first show that if $\phi : e_i R \rightarrow e_j R$ is any nonzero R -homomorphism. Then $e_i R \cong e_j R$. Let $\phi : e_i R \rightarrow e_j R$ be nonzero homomorphism. Then $\frac{e_i R}{ker \phi}$ embeds in $e_j R$. Therefore by previous lemma embeds in $Soc(e_j R) \cong e_j R / e_j J$. So it follows that $e_i R \cong e_j R$. Let $[e_i R] = \sum e_j R$ where the summation runs over all j for which $e_j R \cong e_i R$. We may write $R = [e_1 R] \oplus \dots \oplus [e_k R]$ where $k \leq n$, $[e_i R]$ is an ideal in R and so $R \cong M_{n_1}(e_1 R e_1) \oplus \dots \oplus M_{n_k}(e_k R e_k)$. Where n_i is the number of summands in $[e_i R]$. \square

3.3 When weakly projective modules are weakly injective

It is an open question whether a (one-sided) perfect, one-sided self injective ring is QF. The next theorem is a result in that direction.

Lemma 3.22. [31, Theorem3.3] *Let R be a left or right perfect ring. Then R is right weakly-injective as a right R -module iff R is right self-injective.*

Lemma 3.23. [13, Lemma2.12] *A left perfect and right self-injective ring R is QF iff every cyclic right R -module embeds in a free module.*

Theorem 3.24. [26, Theorem3.2.1] *Let R be a left perfect ring such that every projective right R -module is weakly R -injective. Then R is a QF-ring.*

Proof. We first show that every cyclic right R -module embeds in a free module. Since R is projective then, by hypothesis, R is right weakly-injective. Using lemma 3.22 we conclude R is right self-injective. Then R is a right PF-ring. If C is a cyclic right R -module, then $S = Soc(C)$ is essential in C and S embeds in a free right R -module F , say. Now $E(C) = E(S) \subset E(F)$ and F is weakly R -injective. Thus $C \subset X \cong F$ for some submodule X . By lemma 3.23, we conclude that R is a QF-ring. \square

Proposition 3.25. [13, Proposition3.2] *Let R be a Qf-ring and let M_R be an R -module. Express $M = E \oplus K$, where $E = \bigoplus \Sigma_{i=1}^k (e_i R)^{\alpha_i}$ is a projective module and K is a singular module. If we write $K/RadK = \bigoplus \Sigma_{i=1}^k (e_i R/e_i J)^{\beta_i}$ and $Soc(K) = \bigoplus \Sigma_{i=1}^k (e_i R/e_i J)^{\gamma_i}$, then*

(a) *M is weakly-projective if and only if for all $i = 1, \dots, k$ if $\beta_i \neq 0$ then α_i is infinite.*

(b) *M is weakly-injective if and only if for all $i = 1, \dots, k$ if $\gamma_i \neq 0$ then α_i is infinite.*

Proof. (a) The necessity is clear. For if α_j is finite and $\beta_j \neq 0$, then $P(M) = (\bigoplus \Sigma_{i=1}^k (e_i R)^{\alpha_i}) \oplus (\bigoplus \Sigma_{i=1}^k (e_i R)^{\beta_i})$, yields by lemma 15, $e_j R^{\alpha_j+1} \subseteq^{\oplus} M$, a contradiction.

For the converse, let us start by writing $M = \bigoplus \Sigma_{i \in A} (e_i R)^{\alpha_i} \oplus \bigoplus \Sigma_{j \in B} (e_j R)^{\alpha_j} \oplus K$, where $A = \{i : \beta_i \neq 0\}$ and $B = \{i : \beta_i = 0\}$. It suffices to show $\bigoplus \Sigma_{i \in A} (e_i R)^{\alpha_i} \oplus K$ is weakly-projective. So we may assume $\beta_i \neq 0$ for all i . Consider an epimorphism $\phi : P(M) \rightarrow I$, where $I \subseteq R^n$. Let $\pi : P(I) \rightarrow I$ be a projective cover map. The projectivity of $P(M)$ yields a map $\phi' : P(M) \rightarrow P(I)$ such that $\pi \phi' = \phi$. Since $\text{Ker } \pi$ is small in $P(I)$, ϕ' is an epimorphism. Furthermore because $P(I)$ is projective, ϕ' splits and therefore we may write $P(M) = P \oplus \text{ker } \phi'$ for some submodule $P \subseteq P(M)$ isomorphic to $P(I)$.

Let us write $P(I) = \bigoplus_{i=1}^k (e_i R)^{n_i} \cong P$, and $\ker \phi' \cong \bigoplus_{i=1}^k (e_i R)^{\theta_i}$, and $P(M) \cong \bigoplus_{i=1}^k (e_i R)^{l_i}$, where l_i is infinite. It follows that $\ker \phi' \cong P(M)$. Thus there exists $X \subseteq \ker \phi'$ such that $\ker \phi'/X \cong M$. Now, $X \subseteq \ker \phi' \subseteq \ker \phi$, and $P(M)/X = \ker \phi' \oplus P/X \oplus 0 \cong (\ker \phi'/X) \oplus P \cong M \oplus P \cong M$, proving our claim.

(b) To prove necessity, assume on the contrary that there exists γ_i , such that $\gamma_i \neq 0$, and α_i is finite. Put $N = (e_i R)^{(\alpha_i+1)}$. By the weak injectivity of $M, (e_i R)^{(\alpha_i+1)}$ embeds in M . Contradicting our assumption.

To prove the converse, one can argue in a similar way as in part (a) and assume that, for all $i, \lambda_i \neq 0$. Consider a finitely generated submodule $N \subseteq E(M) = \bigoplus_{i=1}^k (e_i R)^{(\lambda_i)}$, where each λ_i is infinite. Then there exists positive integers n_1, \dots, n_k such that $N \subseteq \bigoplus_{i=1}^k (e_i R)^{(n_i)}$, and thus we conclude that M is tight, hence weakly-injective by lemma 3.a5(b). \square

Recall a ring R is called local if it has a unique maximal right ideal.

Theorem 3.26. [26, Theorem3.2.3] *Let R be a local QF-ring and let M be a right R -module. Then M is weakly-projective if and only if M is weakly-injective.*

Proof. By lemma 3.23, we may express $M = E \oplus K$, where K is a singular module and E is a free module. By previous proposition, M is weakly projective if and only if $K = 0$, or $E = R^{(\alpha)}$ with α infinite. This is equivalent to M being weakly-injective. \square

Corollary 3.27. [13, Corollary3.4] *Let R be a direct sum of matrix rings over local QF-rings, and let M be a right R -module. Then M is weakly-projective if and only if M is weakly-injective.*

Proposition 3.28. [26, Theorem3.2.5] *Let R be left perfect ring such that every weakly-projective right R -module is weakly R -injective. Then for indecomposable projective modules eR and fR , if fR/fJ embeds in eR/eI for some right ideal I , then $eR \cong fR$. Equivalently, $\text{Soc}(eR/eI) \cong (eR/eJ)^n$ for some positive integer n .*

Proof. Note that by theorem 3.24, R is QF. Suppose on the contrary that $\text{Soc}(fR)$ embeds in eR/eI and fR is not isomorphic to eR . We will show that $N = (eR)^w \oplus eR/eI$ is weakly-projective but not weakly-injective. Since $E(N) = (eR)^w \oplus fR \oplus K, fR \subseteq E(N)$. On the other hand, fR is not embeddable in N , and so M is not weakly-injective.

To show that N is weakly-projective, consider an epimorphism $\phi : P(N) \rightarrow K$, where $K \subseteq R^n$. Let $\pi : P(K) \rightarrow K$ be the projective cover map. The projectivity of $P(N)$ yields a map $\phi' : P(N) \rightarrow P(K)$ such that $\pi \phi' = \phi$.

Since $\ker\pi \ll P(K)$, we get that ϕ' is an epimorphism and therefore we may write $P(N) = P \oplus \ker\phi'$ for some submodule $P \subseteq P(N) \cong P(K)$. Therefore, $(eR)^{(w)} = \ker\phi' \oplus P$, where P is finitely generated. This implies that $(eR)^{(w)} \cong \ker\phi'$. Let $X = 0 \oplus eI$. Then $P(N)/X = (eR)^{(w)}/(0 \oplus eI) \cong (eR)^{(w)} \oplus (eR/eI) = N$, as desired. Hence N is weakly-projective which is not weakly-injective, a contradiction to our hypothesis. Therefore, $\text{soc}(eR)$ embeds in eR/eJ and thus, $\text{soc}(eR) \cong eR/eJ$. Consequently, $\text{soc}(eR/eI) \cong (eR/eJ)^n$ for some positive integer n . \square

Proposition 3.29. [13, Proposition3.6] *Let R be a QF-ring such that for any indecomposable projective right module eR and for any right ideals I , $\text{Soc}(eR/eI) = (eR/eJ)^n$ for some positive integer n . Then R is a direct sum of matrix rings over local QF-rings.*

Proof. Write $R = \bigoplus_{i=1}^n e_i R$, where $\{e_i : i = 1, \dots, n\}$ is a complete set of orthogonal primitive idempotents, and let $A = \{e_i R : i = 1, \dots, k\}$ be a complete set of representatives for the indecomposable projective right R -module. Let $[e_i R] = \sum e_j R$, where the summation runs over all j for which $e_j R \cong e_i R$. Renumbering if necessary we may write $R = [e_1 R] \oplus \dots \oplus [e_k R]$, where $k \leq n$. By our hypothesis $[e_i R]$ is an ideal in R and so $R \cong M_{n_1}(e_1 R e_1) \oplus \dots \oplus M_{n_k}(e_k R e_k)$, where n_i is the number of summands in $[e_i R]$. \square

Theorem 3.30. [13, Theorem3.7] *Let R be a left perfect ring such that every weakly-projective right module is weakly-injective. Then R is a direct sum of matrix rings over local QF-rings.*

Proof. The proof follows directly from two previous propositions. \square

Chapter 4
Extending and Lifting Modules

4 Extending and Lifting modules

In this chapter extending modules and lifting modules are introduced and elementary properties are given. In particular, injectivity properties and the relationship with modules of finite uniform dimension are investigated.

4.1 Extending modules

Recall that a submodule K of an R -module M is said to be closed or a complement(in M), if K has no proper essential extension in M .

Definition 4.1. The module M is called extending, or a CS-module, if every closed submodule is a direct summand. Equivalently, M is an extending module if and only if every submodule is essential in a direct summand of M (see[6, 7]). This notion is the key one in this monograph and in this section we explore some of the basic properties of extending modules.

More generally M is called uniform-extending if every uniform submodule is essential in a direct summand of M .

To clarify some notation also in use consider the following conditions on M :

- (C1) Every submodule of M is essential in a direct summand of M ;
- (C2) Every submodule isomorphic to a direct summand of M is a direct summand;
- (C3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M .

Modules satisfying (C1) are called extending (or CS) modules, M is π -injective (or quasi-continuous) if and only if it has (C1) and (C3). Finally, M is continuous if and only if it satisfies conditions (C1) and (C2). It is easy to see that (C2) \Rightarrow (C3) and the hierarchy is as follows:

injective \Rightarrow self-injective \Rightarrow continuous \Rightarrow π -injective \Rightarrow extending (see[6,2.12]).

Proof. injective \Rightarrow self-injective: M is called injective if it is N -injective for every right module N . Let $N = M$. Then M is M -injective, so it is self-injective (quasi-injective).

self-injective \Rightarrow continuous: Since M is self-injective, so M is M injective, and $fM \subset M$ for every endomorphism f of $E(M)$. But M is continuous if $fM \subset M$ for every idempotent endomorphism f of $E(M)$, it follows that M is continuous (see [6, 2.12]).

continuous \Rightarrow π -injective: Clear from definition. Write $M = M_1 \oplus M_1^*$,

and let π denote the projection map $M_1 \oplus M_1^* \rightarrow M_1^*$. Then $M_1 \oplus M_2 = M_1 \oplus \pi|M_2$. Since $\pi|M_2$ is monomorphism, we get $M_2 \subset^\oplus M$ by (C2). As $\pi|M_2 \leq M_1^*$, $M_1 \oplus \pi|M_2 \subset^\oplus M$ [23, Proposition 2.2].

π -injective \Rightarrow extending: See proposition 4.6. \square

Definition 4.2. A module U is called nearly (resp. essentially) M -injective if every diagram in $\text{Mod-}R$ with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M \\ & & \downarrow g & & \\ & & U & & \end{array}$$

and $\ker g \neq 0$ (resp. $\ker g$ is essential in K) can be extended commutatively by some homomorphism $M \rightarrow U$.

Obviously, nearly M -injective modules are essentially M -injective. For a uniform module M the two notions coincide (see [6, 2.14]).

The modules M_1, M_2 are relatively nearly injective (resp. relatively essentially injective), if M_i is nearly (essentially) M_j injective for $i, j \in \{1, 2\}, i \neq j$.

Lemma 4.3. [29, Lemma 3] Let M_1 and M_2 be modules and let $M = M_1 \oplus M_2$. Then the following conditions are equivalent :

1. M_2 is nearly M_1 -injective;
2. M_2 is (M_1/X) -injective for every nonzero submodule X of M_1 ;
3. for every (closed) submodule N of M such that $N \cap M_1 \neq 0$, $N \cap M_2 = 0$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

Lemma 4.4. [29, Lemma 4] Let M_1 and M_2 be modules and let $M = M_1 \oplus M_2$. Then the following conditions are equivalent:

1. M_2 is essentially M_1 injective;
2. M_2 is (M_1/X) -injective for every essential submodule X of M_1 ;
3. for every (closed) submodule N of M such that $N \cap M_1 \subset' M_1$ and $N \cap M_2 = 0$ there exists a submodule N' of M such that $N \subseteq N'$ and $M = N' \oplus M_2$;

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4. for every closed submodule N of M such that $N \cap M_1 \subset' M_1$ and $N \cap M_2 = 0$, $M = N \oplus M_2$;
 5. for every (closed) submodule N of M such that $N \cap M_1 \subset' N$ there exists a submodule N' of M such that $N \subseteq N'$ and $M = N' \oplus M_2$.

An R -module U is uniform provided $U \neq 0$ and $V \cap W \neq 0$ for all non-zero submodules V, W of U . Examples of such modules are, for any ring R , simple modules, non-zero submodules of uniform modules, and indecomposable extending modules (see [6, 5.1]).

Lemma 4.5. [6, Lemma7.1] *Any direct summand of a (uniform) extending module is also (uniform) extending.*

A submodule N of a module M is called essentially finitely generated (respectively essentially cyclic) if N contains a finitely generated (cyclic) essential submodule.

It is well known that the module M has finite uniform dimension if and only if every submodule is essentially finitely generated.

The module M is called a CEF-module (respectively, CEC-module) if every closed submodule is essentially finitely generated (cyclic).

Clearly CEC-modules are CEF. Modules with finite uniform dimension are CEF, and so too are finitely generated extending modules [6, 7.11].

Examples of extending modules

1. Every semisimple module is extending, because every submodule is a direct summand.
2. Every uniform module is extending, because every non-zero submodule is essential.
3. Consider the special case of \mathbb{Z} -modules. Any finitely generated torsion-free \mathbb{Z} -module is extending. For, if A is a finitely generated torsion free \mathbb{Z} -module and B is any submodule of A , let C be the submodule of A containing B such that C/B is the torsion submodule of A/B . Then A/C is finitely generated, torsion-free, whence free, and C is a direct summand of A . Moreover, C/B torsion and A torsion-free together give B essential in C . Thus A is extending (see[6, 7.1]).

We also have the following basic fact:

Proposition 4.6. [6, Proposition7.2] *For any ring R , quasi continuous R -modules are extending.*

Proof. Let M be any quasi continuous R -module and N any submodule of M . Consider a complement L of N in M . Then $N \oplus L$ essential in M . Taking injective hulls, $E(M) = E(N) \oplus E(L)$ (by theorem 1.14). $M = (M \cap E(N)) \oplus (M \cap E(L))$ and N is essential in the direct summand $M \cap E(N)$ of M . Thus M is extending. \square

In particular, this proposition shows that every (quasi-)injective R -module is extending, for any ring R

Theorem 4.7. 1 [Krull-Schmidt-Azumaya Theorem] *Let M be a module that is a direct sum of modules with local endomorphism rings. Then M is a direct sum of indecomposable modules in an essentially unique way in the following sense. If*

$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ *where all the M_i ($i \in I$) and all the N_j ($j \in J$) are indecomposable modules, then there exists a bijection $\phi' : I \rightarrow J$ such that $M_i \cong N_{\phi'(i)}$ for every $i \in I$.*

Lemma 4.8. [6, Lemma7.3] *Let A and B be uniform modules with local endomorphism rings such that $M = A \oplus B$ is extending. Let C be a submodule of A and let $f : C \rightarrow B$ be a homomorphism. Then the following hold.*

1. *If f cannot be extended to a homomorphism from A to B , then f is a monomorphism and B is embedded in A .*
2. *If any monomorphism $B \rightarrow A$ is an isomorphism, then B is A -injective.*
3. *If B is not embedded in A , then B is A -injective.*

Proof. (1) Suppose f cannot be extended to A . Let $U = \{x - f(x) : x \in C\} \subset A \oplus B$: Then $U \cong C$ is a uniform submodule of M and clearly $U \cap B = 0$. Hence there is a direct summand U^* of M such that U is essential in U^* . By the Krull-Schmidt-Azumaya Theorem, we have $M = U^* \oplus A$ or $M = U^* \oplus B$. Suppose that $M = U^* \oplus B$. Let $\pi : U^* \oplus B \rightarrow B$ be the projection. Then it is easy to see that $\pi|_A$ extends $f : C \rightarrow B$, a contradiction. Thus $M = U^* \oplus A$ which implies that $f(x) \neq 0$ for $x \neq 0$, i.e. f is a monomorphism. Since $U^* \cap B = 0$, clearly B is embedded in A .

(2) As in the proof of (1), given any homomorphism $f : C \rightarrow B$ with $C \subset A$, suppose that $M = U^* \oplus A$. Let $\pi : U^* \oplus A \rightarrow A$ be the projection. Then clearly $\pi|_P$ is a monomorphism (because U is essential in U^*), hence an isomorphism by the hypothesis. It follows easily that $M = U^* \oplus B$, so that, as in (1), f can be extended to a homomorphism from A to B . It follows

that B is A -injective.

(3) Immediate by (1). □

We have seen that every quasi continuous module is extending. Which extending modules are quasi continuous? To answer this question we first prove the following result:

Lemma 4.9. [6, Lemma7.5] *Let M_1 and M_2 be R -modules and let $M = M_1 \oplus M_2$. Then M_1 is M_2 -injective if and only if for every submodule N of M such that $N \cap M_1 = 0$ there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $N \subset M'$.*

Proof. . Suppose first that M_1 is M_2 -injective. Let $\pi_i : M \rightarrow M_i (i = 1; 2)$ denote the canonical projections. Let N be a submodule of M such that $N \cap M_1 = 0$. Consider the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & N \xrightarrow{\alpha} M_2 \text{ exact} \\ & & \beta \downarrow \\ & & M_1 \end{array}$$

where $\alpha = \pi_{2|N}$ and $\beta = \pi_{1|N}$. By hypothesis, there exists an $f : M_2 \rightarrow M_1$ such that $\alpha f = \beta$. Define $M' = \{f(m) + m : m \in M_2\}$. It is easy to check that M' is a submodule of M , $M = M_1 \oplus M'$ and $N \subset M'$.

Conversely, suppose that for every submodule N of M with $N \cap M_1 = 0$, there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $N \subset M'$. Let L be a submodule of M_2 and $g : L \rightarrow M_1$ be a homomorphism. Put $H = \{-g(x) + x : x \in L\}$. Then H is a submodule of M and $H \cap M_1 = 0$. There exists a submodule H' of M such that $M = M_1 \oplus H'$ and $H \subset H'$. Let $\pi : M \rightarrow M_1$ denote the projection with kernel H' . Then $\pi|_{M_2} : M_2 \rightarrow M_1$ and for any x in L

$$\pi(x) = \pi(g(x) + (-g(x) + x)) = g(x)$$

It follows that M_1 is M_2 -injective. □

Corollary 4.10. [6, Corollary7.6] *A module M is quasi continuous if and only if M is an extending module such that whenever $M = M_1 \oplus M_2$ is a direct sum of submodules, then M_1 and M_2 are relatively injective.*

Proof. Suppose first that M is quasi continuous. By proposition 4.6, M is extending. Suppose that $M = M_1 \oplus M_2$. By theorem 1.14 and lemma 4.9, M_1 and M_2 are relatively injective. Conversely, suppose that M is an

extending module with the stated property. Let L_1 and L_2 be submodules of M such that $L_1 \cap L_2 = 0$. There exist submodules P_1, P_2 of M such that $M = P_1 \oplus P_2$ and L_1 essential in P_1 . Clearly $P_1 \cap L_2 = 0$. By hypothesis, P_1 is P_2 -injective. Hence, by lemma 4.9, there exists a submodule P_0 of M such that $M = P_1 \oplus P_0$ and $L_2 \subset P_0$. Then M is quasi continuous. \square

Lemma 4.11. [6, Lemma1.10] (*properties for closed modules*): Let L, K, N be submodules of a module M with $K \subset L$,

1. There exists a closed submodule H of M such that N is essential in H .
2. The submodule K is closed in M if and only if whenever Q is essential in M such that $K \subset Q$ then Q/K is essential in M/K .
3. If L is closed in M , then L/K is closed in M/K .
4. If K is closed in L and L is closed in M , then K is closed in M .

Lemma 4.12. [6, Lemma7.7] Let M be a uniform-extending module and $K \subset M$ a closed submodule with finite uniform dimension. Then K is a direct summand of M .

Proof. . Let U be a uniform closed submodule of K . By previous properties (4), U is a uniform closed submodule of M , and hence $M = U \oplus U'$ for some submodule U' . Then $K = U \oplus (K \cap U')$. Again by previous properties (4), $K \cap U'$ is a closed submodule of M . Clearly $K \cap U'$ has smaller uniform dimension than K . By induction, $K \cap U'$ is a direct summand of M , and hence also of U' . Thus K is a direct summand of M . \square

Corollary 4.13. [6, Corollary7.8] A module with finite uniform dimension is extending if and only if it is uniform-extending.

4.2 Lifting modules

In this section we will introduced lifting modules and it's elementary properties. We start with providing some basic definitions and results, we will study some relations between lifting modules and other modules.

Basic definitions and results:

Recall that a submodule A of M is called essential in M if whenever $X \leq M, 0 = A \cap X$ implies $X = 0$.

If $A \leq B \leq M$ and $A \subset' B$, then B is called an essential extension of A in M .

If $A \leq B \leq M$, then A is called a coessential submodule of B (or the inclusion $A \subseteq B$ is called cosmall in M) (denoted by $A \leq_{ce} B$) if $B/A \ll M/A$. In that case, B is called coessential extension of A in M .

Also we have defined submodule A as closed submodule, if A has no proper essential extension in M .

A is called coclosed in M (denoted by $A \leq_{cc} M$), if A has no proper coessential submodule in M . i.e., if $X \leq_{ce} A$, then $X = A$.

Let A, B be submodules of a module M , A is called coclosure of B in M , if A is a coessential submodule of B and A is a coclosed in M .

M is called amply supplemented if for every $A, B \leq M$, $M = A + B$ implies A has a supplement in M contained in B .

Definition 4.14. *Lifting module:* We say that M is a lifting module if for any submodule A of M , there exists a direct summand B of M such that $B \leq A$ and A/B is small in M/B . Equivalently, a module M is lifting if for every submodule $N \leq M$, there exist $K \subseteq^{\oplus} M$ such that $K \leq N$ and $K \leq_{ce} N$ in M .

Now we list some lemmas and corollaries that will be so valuable in characterizing lifting modules.

Lemma 4.15. [32, Lemma41.11] Let M be a module and $A \leq M$. Then the following are equivalent.

- (a) There is $X \subseteq^{\oplus} M$ with $X \leq_{ce} A$ in M .
- (b) There is $X \subseteq^{\oplus} M$ and $Y \ll M$ with $A = X \oplus Y$.
- (c) there is a decomposition $M = X \oplus X'$ with $X \leq A$ and $A \cap X' \ll X'$.

Lemma 4.16. [32, Lemma41.1] Let M be a module with submodules A and B . Assume that A is a supplement of B in M . Then

- (a) if $A + C = M$ for some $C \leq B$, then A is a supplement of C in M .
- (b) if $C \ll M$, then A is supplement of $B + C$ in M .
- (c) if $C \leq B$, then $(A + C)/C$ is a supplement of B/C in M/C .

Lemma 4.17. [20, Proposition1.2.1] Let M be a module and $N \leq M$. Consider the following conditions:

- (a) N is supplement submodule of M .
 - (b) $N \leq_{cc} M$.
 - (c) for all $X \leq N$, $X \ll M$ implies $X \ll N$.
- Then (a) \Rightarrow (b) \Rightarrow (c).

Proposition 4.18. [17, Proposition1.5] If M is an amply supplemented module, then every submodule of M has a coclosure in M .

Proposition 4.19. *Let M be a lifting module. Then the following are true:*

1. *Any coclosed submodule of M is a direct summand.*
2. *M is amply supplemented.*
3. *M is hollow iff it is indecomposable.*

Proof. (1) Let $A \leq_{cc} M$. Since M is lifting, there exists $K \subseteq^{\oplus} M$ such that $K \leq_{ce} A$ in M . But A has no proper coessential submodules, hence $A = K$.

(2) Let $A, B \leq M$ with $M = A + B$. We will show that B contains a supplement of A in M . From the previous lemma $B = X \oplus Y$, where $X \ll M$ and $Y \subseteq^{\oplus} M$. Therefore, $M = A + Y$. Again by the same lemma $A \cap Y = N \oplus S$ with $S \ll M$ and $N \subseteq^{\oplus} M$. Hence $S \ll Y$ and $N \subseteq^{\oplus} Y$. Let $Y = N \oplus N'$ for some $N' \leq Y$. Clearly, N' is a supplement of N in Y . But $S \ll Y$. Therefore by lemma 4.16 N' is a supplement of $N + S$ in Y which means $Y = N' + N + S$ and $N' \cap (N + S) \ll N'$ implies that $(A \cap Y) + N' = N + S + N' = Y$. Consequently, $M = A + N + S + N' = A + N'$. Moreover, $A \cap N' = (A \cap Y) \cap N' = (N + S) \cap N' \ll N'$. Hence N' is a supplement of A in M with $Y' \leq B$.

(3) If M is hollow and $M = A \oplus B$, then either $A = M$ or $B = M$, hence M is indecomposable. Conversely, suppose M is an indecomposable lifting module and let A be a proper submodule of M . Since M is lifting, there exists a direct summand K of M with $A/K \ll M/K$. But M is indecomposable, hence $K = 0$ and so $A \ll M$. Thus M is hollow. \square

Theorem 4.20. [3, 22.3] *Let M be a module. Then the following are equivalent:*

1. *M is lifting.*
2. *for every $A \leq M$, there is a decomposition $M = X \oplus X'$ with $X \leq A$ and $A \cap X' \ll M$.*
3. *every $A \leq M$ can be written as $A = X \oplus Y$ with $X \subseteq^{\oplus} M$ and $Y \ll M$.*
4. *M is amply supplemented and every coclosed submodule of M is a direct summand.*
5. *M is amply supplemented and every supplement submodule of M is a direct summand.*

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) Follows from lemma 4.a5.

(1) \Rightarrow (4) Follows from proposition 4.19.

(4) \Leftrightarrow (5) Follows from lemma 4.17.

(5) \Leftrightarrow (4) Follows from proposition 4.18. \square

Example. Any hollow module is lifting.

Proof. Let $A \leq M$. If $A = M$, then $A = M \oplus 0$ where $0 \ll M$ and $M \subseteq^{\oplus} M$. And if $A \neq M$, then $A = A \oplus 0$ where $A \ll M$ and $0 \subseteq^{\oplus} M$. Hence M is lifting. \square

4.3 Some conditions for a direct sum of extending module to be extending

We will state some conditions for a direct sum of extending module to be extending.

Lemma 4.21. [29, Lemma6] Let M_1 be an extending (resp, uniform extending) module, let M_2 be any module and let $M = M_1 \oplus M_2$. If M_2 is essentially (res. U -essentially) M_1 -injective, then every closed (res, closed uniform) submodule K of M such that $K \cap M_1 \subset' K$ is a direct summand of M .

Proof. : Suppose that M_2 is essentially M_1 -injective, and let K be a closed submodule of M such that $K \cap M_1 \subset' K$. Then there exists a submodule K' of M such that $K \subset K'$ and $M = K' \oplus M_2$. As K' is isomorphic to M_1 , K' is extending and K , being a closed submodule of K' is a direct summand of K' , thus K is also a direct summand of M . \square

Lemma 4.22. [6, Lemma7.9] Let $M = M_1 \oplus M_2$ where M_1 and M_2 are both extending modules. Then M is extending if and only if every closed $K \subset M$ with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M .

Proof. The necessity is clear. Conversely, suppose that every closed $K \subset M$ with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand. Let $L \subset M$ be closed. There exists a complement H in L such that $L \cap M_2$ is essential in H . By lemma 4.11(4), H is closed in M . Clearly $H \cap M_1 = 0$. By hypothesis, $M = H \oplus H'$ for some submodule H' of M . Now $L = H \oplus (L \cap H')$. By lemma 4.11(4) again, $L \cap H'$ is closed in M . Also, clearly, $(L \cap H') \cap M_2 = 0$. By hypothesis, $L \cap H'$ is a direct summand of M , and hence also of H' . It follows that L is a direct summand of M . Thus M is extending. \square

Proposition 4.23. [6, Proposition 7.10] *Let $M = M_1 \oplus \dots \oplus M_n$ be a finite direct sum of relatively injective modules M_i . Then M is extending if and only if all M_i are extending.*

Theorem 4.24. [29, Theorem 8] *Let M_1 and M_2 be extending (resp. Uniform extending) modules and let $M = M_1 \oplus M_2$. If one of the following conditions holds, then M is extending (resp. uniform extending):*

1. M_2 is essentially (resp. u -essentially) M_1 -injective and every closed (resp. closed uniform) submodule K of M such that $K \cap M_1 = 0$ is a direct summand of M .
2. M_1 and M_2 are relatively essentially (resp. u -essentially) injective, and every closed (resp. closed uniform) submodule K of M such that $K \cap M_1 = K \cap M_2 = 0$ is a direct summand of M .
3. M_1 is M_2 -injective and M_2 is essentially (resp. u essentially) M_1 -injective.
4. M_2 is semisimple and essentially (resp. u -essentially) M_1 -injective.

Chapter 5
Weakly injective modules
versus extending modules

5 Weakly injective modules versus extending modules

In this chapter we study several properties of weakly-injective extending modules. We state some cases for which weakly-injective modules are extending. We use some results for weakly-injective and extending modules.

5.1 Weakly injective modules versus extending modules

In this section we will use the fact that in quasi injective module every weakly-injective modules are injective. We will use some definitions.

Definition 5.1. *Let R be a ring with identity not equal to zero. A right R -module is said to be quasi injective (pseudo injective), if for every submodules N of M every R -homomorphism (R - monomorphism) of N into M can be extended to an R -endomorphism of M (see[15, 1]).*

Lemma 5.2. *A direct sum of finitely many copies of a quasi-injective module is quasi-injective.*

Lemma 5.3. [15, Lemma1] *A direct summand of a pseudo-injective modules is pseudo-injective.*

Corollary 5.4. *Let $N_1 \oplus N_2$ be a pseudo injective module and $\sigma : N_1 \rightarrow N_2$ be a monomorphism. Then if N_1 is weakly-injective module, then N_1 is extending module.*

Proof. Let $N_1 \oplus N'_1 = N_2$, so $N_1 \oplus N_2 = N_1 \oplus N_1 \oplus N'_1$ and $T = N_1 \oplus N_1$ is pseudo-injective. Write $T = M_1 \oplus M_2$, $M_1 = M_2 = N_1$. Let N be any submodule of N_1 and $\sigma : N \rightarrow N_1$ be an R -homomorphism. If we treat N as a submodule of T contained in M_1 , then the mapping $\eta : N \rightarrow T$ given by $\eta(x) = (x, \sigma(x))$, $x \in N$, is a monomorphism. Hence it can be extended to an endomorphism λ of T . If $q_1 : M_1 \rightarrow T$ and $p_2 : T \rightarrow M_2$ are natural injection and projection respectively. Then $\mu = p_2 \lambda q_1$ is an endomorphism of N_1 which extend σ , hence N_1 is quasi-injective (see [15, Lemma 2]). But N_1 is weakly-injective, so N_1 is injective (in quasi-injective modules every weakly injective modules are injective). Hence N_1 is extending. \square

Corollary 5.5. *If $M \oplus M$ is pseudo-injective and M is weakly-injective module. Then M is extending module.*

Proof. M is quasi-injective, by previous corollary, and hence M is injective, so it is extending module. \square

We state the definition of uniserial module and generalized uniserial module.

Definition 5.6. [6, 1.8] *Uniserial modules.* An R -module M is called uniserial if its submodules are linearly ordered by inclusion.

If R is R_R uniserial, we call R right uniserial.

We recall: Properties. For M the following are equivalent:

- (a) M is uniserial;
- (b) any submodule of M has at most one maximal submodule;
- (c) for every factor module L of M , $\text{Soc } L$ is simple or zero.

Definition 5.7. [15, Section3] *Generalized uniserial module:* a right and left artinian ring R is said to be generalized uniserial if for every primitive idempotent e of R , $eR(Re)$ have a unique composition series as right (left) R -module. A module X of finite composition length is said to be uniserial if it has a unique composition series.

Theorem 5.8. [15, theorem2] (Nakayama) Let R be a generalized uniserial ring. Then every R -module is a direct sum of uniserial modules.

The above theorem shows that any indecomposable module over a generalized uniserial ring is a uniserial module.

Let E and F be two indecomposable modules over a generalized uniserial ring R , and let $m(E, F)$ denote the submodule of E which is minimal among the kernel of all homomorphism of E into F . As E is uniserial, $m(E, F)$ is well defined and unique. $m(E, F) = 0$ if and only if there exists a monomorphism of E into F . For any module X , let $E(X)$ denote its injective hull and $l(X)$ denote the composition length.

Theorem 5.9. Let N be a module over a generalized uniserial ring R . If $N = \bigoplus_{i \in \Lambda} N_i$ where N_i are uniserial and $l(N_i) \leq l(N_j) + l(m(E(N_i), E(N_j)))$.

$i, j \in \Lambda$ and N is weakly-injective then N is extending module.

Proof. Let the inequality hold. Let $\sigma \in \text{Hom}_R(E_i, E_j)$. Then using the inequality and fact that $m(E_i, E_j) \subseteq \ker \sigma$, we immediately get $\sigma(N_i) \subseteq N_j$. Hence N is quasi injective. But N is weakly-injective, so as before N is extending. \square

Theorem 5.10. *A weakly -injective pseudo injective modules over a generalized uniserial ring R is extending.*

Proof. Let N be a pseudo injective R -module. By [15, heorem2] we can write $N = \bigoplus_{i \in \Lambda} N_i$, where, N_i are non zero uniserial module. Let $E_i = E(N_i)$.

If we prove that for all $i, j \in \Lambda$ and $l(N_i) \leq l(N_j) + l(m(E(N_i), E(N_j)))$ then [15,theorem3] shows that N is quasi injective. Clearly we only need to consider the case when $m(E_i, E_j) \subset N_i$. Now by lemma 5.2, $N_i \oplus N_j$ is pseudo injective. Let $\sigma : E_i \rightarrow E_j$ be an R homomorphism with $\ker \sigma = m(E_i, E_j)$. Let F_j be the simple submodule of E_j , since E_j is uniserial, $F_j \subset N_j$, also, then $\sigma^{-1}(F_j) \subset N_j$. Define $\eta : \sigma^{-1}(F_j) \rightarrow N_i \oplus N_j$, by $\eta(x) = (x, \sigma(x))$, $x \in \sigma^{-1}(F_j)$. η is an R -monomorphism, thus it can be extended to an R -endomorphism η^* of $N_i \oplus N_j$. If $\lambda_i : N_i \rightarrow N_i \oplus N_j$, and $p_j : N_i \oplus N_j \rightarrow N_j$ are natural injection and projection. Then $p_j \eta^* \lambda_i : N_i \rightarrow N_j$ is such that its restriction to $\sigma^{-1}(F_j)$ is equal to the restriction of σ to $\sigma^{-1}(F_j)$. Thus $\ker(p_j \eta^* \lambda_i) = \ker \sigma = m(E_i, E_j)$. Hence $N_i/m(E_i, E_j) \cong (p_j \eta^* \lambda_i)(N_i) \subseteq N_j$ gives that $l(N_i) \leq l(N_j) + l(m(E(N_i), E(N_j)))$. Thus N is quasi-injective weakly-injective module and so it is extending. \square

A ring R is said to be right (left) bounded if each of its essential right (left) ideals contains a non zero two-sided ideal. A ring R which is both right and left bounded is called bounded. A prime ring which is left noetherian, left hereditary as well as right noetherian, right hereditary is called a hereditary noetherian prime ring (hnp ring) (see[15, Section3]).

Theorem 5.11. *Any weakly-injective torsion pseudo-injective module M over abounded hnp- ring R is extending module.*

Proof. Let N be a submodule of M and $\sigma : N \rightarrow M$ be an R -homomorphism. We shall show that σ can be extended to an R -endomorphism of M . By an application of Zorn's lemma we suppose that $N \neq M$ and σ cannot be extended to any submodule N' of M containing N properly. Choose $x \in M$ such that $x \notin N$. Now $\text{ann}(x) = \{a \in R : xa = 0\}$ is an essential right ideal, and so it is contain a nonzero two sided ideal. Set $A = \text{ann}(xR)$ which is a nonzero two sided ideal, and $L = \{y \in M : yA = 0\}$. Then L is a module over a generalized uniserial ring R/A . As L is fully invariant submodule of M . L is also pseudo injective. Hence L is quasi injective. Define an R homomorphism $\lambda : xR \cap N \rightarrow L$ by $\lambda(z) = \sigma(z)$, $z \in xR \cap N$. As $xR \subset L$ and L is quasi injective λ can be extend to an R -endomorphism λ^* of L . Define $\sigma^* : N + xR \rightarrow M$ by $\sigma^*(n + xu) = \sigma(n) + \lambda^*(xu)$. Then σ^* is a well defined

R -homomorphism and is a proper extension of σ . This is a contradiction. Hence M is quasi injective. But M is weakly-injective so it is injective. As before it is extending module. \square

5.2 Weakly injective extending module

The purpose of this section is to study modules over which every weakly-injective modules are extending. Recall that for given two modules M and N we say that M is weakly N injective if for every homomorphism $\phi : N \rightarrow E(M)$ there exists a submodule $X \subset E(M)$ which is isomorphic to M such that $\phi(N) \subset X$. If a module M is weakly N injective for every finitely generated module N we say that M is weakly-injective. A module M is said to be weakly self-injective if it is weakly M injective (see[10]).

Theorem 5.12. *Let M be a weakly K -injective extending module. Then every direct summand of M is weakly K -injective.*

Proof. Let $M = N \oplus T$ be a weakly K -injective extending right R -module. Want to show N is weakly K -injective Consider a map $\phi : K \rightarrow E(N)$. Let $K_1 = \phi(K) \cap N$. Notice that K_1 is essential in $\phi(K)$. Since N is extending there exist a summand $K^* \subset^\oplus N$ such that $K_1 \subset' K^*$, so $N = K^* + H$, but M is weakly K -injective, so there exist $M' \subset E(M)$, $\phi(K) \subset M'$, $M' \cong M$, and M is extending, then $\phi(K)$ is essential in $K' \subset^\oplus M'$.

Let $\mu : M' \rightarrow M$ be an isomorphism. Then $\mu(K') \subset^\oplus M$, but $E(K^*) \cong E(\mu(K'))$, therefore, $K^* \cong \mu(K') \cong K'$.

Take $i : \phi(k) \rightarrow \phi(K)$ extend to an embedding $i^* : K' \rightarrow E(\phi(K)) \subset E(N)$. K_1 is essential in $\phi(K)$ is essential in $i^*(K')$, we get $i^*(K') \cap H = 0$. Therefore, $\phi(K) \subset i^*(K') \oplus H \cong K^* + H = N$. So $i^*(K') \oplus H \cong N$, and $\phi(K) \subseteq i^*(K') \oplus H$, then N is weakly K injective. \square

Corollary 5.13. *Every direct summand of a weakly injective extending module is weakly-injective.*

Recall that the module M is called direct injective if for every direct summand X of M , every monomorphism $X \rightarrow M$ splits.

In the next examples we will define a condition.*

Condition * *Let M be a direct injective module such that $M = M_1 \oplus M_2$ is a direct sum of submodules, such that M_1 and M_2 are relatively injective.*

Example. Let R be any commutative noetherian ring. Then every weakly-injective extending Module M that satisfies condition* is injective.

Proof. Let R be commutative noetherian ring and let M be weakly-injective extending module satisfies condition*. Let L_1 and L_2 be submodules of M such that $M = P_1 \oplus P_2$ and $L_1 \subset P_1$ (by extending). Clearly $P_1 \cap L_2 = 0$. By hypothesis, P_1 is P_2 -injective. Hence, by theorem4, there exists a submodule P' of M such that $M = P_1 \oplus P'$ and $L_2 \subset P'$. Hence M is quasi -continuous module. quasi-continuous direct injective module is continuous (see[6, 2.12]). So M is quasi injective [25, Corollary 5], Quasi-injective weakly injective modules are always injective. \square

Example. Let R be any commutative perfect ring with $J^2 = 0$ (where J is the Jacobson radical of R), and let M be weakly-injective extending R -module that satisfies condition*. Then M is injective.

Proof. If R is a commutative perfect ring with $J^2 = 0$, as before M is continuous module. Then every continuous module satisfies our hypothesis is quasi-injective (see[25, Corollary 10]). Consequently, every weakly injective continuous R -module is injective. \square

Example. Let R be a perfect ring such that every uniform factor ring (\overline{R}) of R is artinian, and let M be weakly-injective extending module that satisfies condition*. Then M is injective.

Proof. M is extending satisfies*, then M is continuous (by[25, Theorem 9]), M is quasi injective, and so M is injective. \square

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