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MODELLING OF SPACECRAFT UNDER RANDOM LOADING

by Joshua E. Greenspon

Prepared under Contract No. NAS5-3132 *by*
J G ENGINEERING RESEARCH ASSOCIATES
Baltimore, Md.

for

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ABSTRACT

This report gives modelling laws for space vehicles under random loading where the model is constructed of a different material than the full scale vehicle.

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CONTENTS

LIST OF SYMBOLS

ℓ'	characteristic length in prototype
t'	characteristic time in prototype
k'	characteristic force in prototype
ℓ	characteristic length in model
t	characteristic time in model
k	characteristic force in model
λ	length scale factor
τ	time scale factor
χ	force scale factor
$\overline{\lambda}, \overline{\mu}$	Lame's elastic constants
E	modulus of elasticity of model material
E'	modulus of elasticity of prototype material
ρ	mass density of model material
ρ'	mass density of prototype material
ν	Poisson's ratio
σ_E	scale factor for modulus of elasticity
σ_{ρ_s}	scale factor for solid density ρ_s'/ρ_s
f	frequency of model
f'	frequency of prototype
p	pressure
σ_x	direct stress in x direction
$\overline{v}$	velocity
a	acceleration
C	correlation
P.S.	power spectral density
S.S.	shear stress
N.S.	normal stress
$\overline{\overline{\mu}}$	coefficient of friction for model
$\overline{\overline{\mu}}'$	coefficient of friction for prototype
$\sigma_{\overline{\overline{\mu}}}$	scale factor for coefficient of friction
$(P.S.)_d$	power spectral density of displacement

$(P.S.)_v$	power spectral density of velocity
$(P.S.)_a$	power spectral density of acceleration
Y_r	Frequency response function for r^{th} mode
q_r	mode shape of r^{th} mode
$C_{rk}(\omega)$	correlation integral
M_r	generalized mass for n^{th} mode
β_r	material damping coefficient for n^{th} mode
ω_{0r}	natural frequency of n^{th} mode
δ_r	logarithmic decrement in n^{th} mode
$\overline{\beta}$	constant material damping factor
ξ	$= 1/\overline{\beta}$
ω	forcing frequency
X	body force per unit volume of solid
F_x	body force per unit mass of fluid
u, v, w	displacements of solid or fluid
T	temperature (absolute)
θ	scale ratio for temperature
K_s	thermal conductivity of the solid
C_s	specific heat of solid (per unit volume)
T_o	reference temperature above which temperature changes are measured
ρ_f	density of the fluid
$\dot{u}, \dot{v}, \dot{w}$	velocity components
μ	viscosity
C_{p_f}	specific heat at constant pressure for fluid (per unit mass)
k_f	thermal conductivity of fluid
P	force

GENERAL

All primed quantities refer to the prototype, all unprimed quantities refer to the model. All scale factors are denoted by σ ; e.g.

$\sigma_{C_{pf}}$ is the ratio of the specific heat at constant press of the fluid prototype material to the fluid model material. All s subscripts refer to solid; all f subscripts refer to fluid.

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I. INTRODUCTION

As spacecraft become larger and more complicated, it becomes more difficult and expensive to test the full scale structures. In addition, if specifications for a spacecraft are to be obtained before the vehicle and payload are constructed, the only way that predicted response can be obtained is for scale model tests to be conducted and calculations performed on the proposed design. If small models could be constructed of materials which were easier to handle in manufacturing and less expensive than the full scale, a great deal of information could be obtained efficiently on a proposed space vehicle. The purpose of this report is to present a derivation of the modelling laws for random dynamic loading on an elastic structure such as a spacecraft-booster combination which is constructed of a different material than the prototype.

II. Approximate Formulation of the Modelling Laws

A. Forces and Motions

The theory presented here is an application and extension of the basic theory of modelling presented by Weber (Ref. 1). Let the quantities associated with the full scale be primed and those associated with the model unprimed. Thus if ℓ' is the characteristic length in the full scale, t' the full scale time, and k' the force in the full scale,

$$\ell' = \ell\lambda, \quad t' = t\tau, \quad k' = k\chi \tag{1}$$

where λ is the linear dimension scale factor, τ is the time scale factor, and χ is the force scale factor.

The phenomenon that is to be scaled is random loading and response of an elastic structure. The only assumption that will be made is that the model is geometrically similar to the prototype and is subjected to a geometrically similar load distribution.

The basic equations governing the elastic motions of the structure in the absence of body forces are the three dimensional equations of elasticity (Ref. 2).

$$(\bar{\lambda} + \bar{\mu})\left(\frac{\partial\Delta}{\partial x}, \frac{\partial\Delta}{\partial y}, \frac{\partial\Delta}{\partial z}\right) + \bar{\mu}\nabla^2(u,v,w) = \rho_s\left(\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 w}{\partial t^2}\right) \tag{2}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are the Lame constants, Δ is the cubical dilatation, and ρ_s is the mass density of the solid material. The displacements are u, v, w.

The cubical dilatation and the Laplacian are

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla^2 (u,v,w) = \frac{\partial^2(u,v,w)}{\partial x^2} + \frac{\partial^2(u,v,w)}{\partial y^2} + \frac{\partial^2(u,v,w)}{\partial z^2} \qquad (3)$$

The Lame constants $\bar{\lambda}$ and $\bar{\mu}$ can be written in terms of the modulus of elasticity E and Poisson's ratio ν as follows:

$$\bar{\lambda} = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \bar{\mu} = \frac{E}{2(1+\nu)} \qquad (4)$$

So the x component of the equations of motion becomes

$$\frac{E}{2(1+\nu)(1-2\nu)} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \, \partial y} + \frac{\partial^2 w}{\partial x \, \partial z} \right)$$

$$+ \frac{E}{2(1+\nu)} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho_s \frac{\partial^2 u}{\partial t^2} \qquad (5)$$

and there are two other equations for the y and z components. The scaling laws can be derived by merely working with one of these equations. In accordance with our notation, equation (5) is for the model since it contains unprimed quantities. The equivalent equation for the prototype, assuming a different material with approximately the same Poisson's ratio but different E and ρ,* is

$$\frac{E'}{2(1+\nu)(1-2\nu)}\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 v'}{\partial x'\,\partial y'}+\frac{\partial^2 w'}{\partial x'\,\partial z'}\right)$$

$$+\frac{E'}{2(1+\nu)}\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}+\frac{\partial^2 u'}{\partial z'^2}\right)=\rho_s'\,\frac{\partial^2 u'}{\partial t'^2} \tag{6}$$

Let the scale factors for E and ρ_s be

$$\frac{E'}{E}=\sigma_E\ ,\quad \frac{\rho_s'}{\rho_s}=\sigma_{\rho_s} \tag{7}$$

By the choice of model material and linear scale this will fix σ_E, σ_{ρ_s}, and λ. The rest of the scale factors will then be derived from these basic quantities. Thus the time scale factor τ will be determined as follows:

$$\frac{E'}{E}=\sigma_E=\frac{\dfrac{k'}{\ell'^2}}{\dfrac{k}{\ell^2}}=\frac{\dfrac{k\chi}{\ell^2\lambda^2}}{\dfrac{k^2}{\ell^2}}=\frac{\chi}{\lambda^2}$$

$$\frac{\rho_s'}{\rho_s}=\sigma_{\rho_s}=\frac{\dfrac{k't'^2}{\ell'^4}}{\dfrac{kt^2}{\ell^4}}=\frac{\dfrac{k\chi t^2\tau^2}{\ell^4\lambda^4}}{\dfrac{kt^2}{\ell^4}}=\frac{\chi\tau^2}{\lambda^4} \tag{8}$$

*Usually the Poisson ratio is of secondary importance. Moreover, the Poisson ratio for the model probably will not vary greatly from the full scale material.

Thus,

$$\chi = \sigma_E \lambda^2$$

and

$$\tau = \sqrt{\frac{\sigma_{\rho_s} \lambda^4}{\chi}} = \lambda \sqrt{\frac{\sigma_{\rho_s}}{\sigma_E}}$$

(frequency is therefore modelled as $1/\tau = (1/\lambda)\sqrt{\sigma_E/\sigma_{\rho_s}}$).

Equation (6) for the prototype can then be written

$$\frac{\sigma_E E}{2(1+\nu)} \frac{1}{\lambda} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{1-2\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 w}{\partial x\, \partial z}\right)\right]$$

$$= \sigma_{\rho_s} \rho_s \frac{\lambda}{\tau^2} \frac{\partial^2 u}{\partial t^2} \qquad (9)$$

Then all scaling terms are collected into a single coefficient on the left. In order for the model to obey the same dynamic equations as the prototype, the scaling relation must be as follows:

$$\sigma_E/\lambda = \sigma_{\rho_s} \lambda/\tau^2$$

which reduces to

$$\frac{E'/E}{\rho_s'/\rho_s} \frac{t'^2/t^2}{\ell'^2/\ell^2} = 1 \qquad (10)$$

or

$$\frac{E'}{f'^2 \rho_s' \ell'^2} = \frac{E}{f^2 \rho_s \ell^2}$$

or finally $f'\ell'/\sqrt{E'/\rho_s'} = f\ell/\sqrt{E/\rho_s}$ where f is the frequency.

The physical number $f\ell/\sqrt{E/\rho_s}$ is a type of Strouhal number. If the same material were used in the model and prototype, then Junger's relations (Ref. 3) would hold; i.e.,

$$f\ell = f'\ell' \tag{11}$$

For Junger's case, frequency is scaled inversely to length. Equations (8) and (10) are the basic relations for modelling with different materials. Variations of this scaling law are contained in References (4-7). The other quantities involved in the problem can be derived from these relations. Thus

$$\text{Pressure:}\quad p'/p = \chi/\lambda^2 \qquad p' = \sigma_E p$$

$$\text{Stress:}\quad \sigma_x'/\sigma_x = \chi/\lambda^2 \qquad \sigma_x' = \sigma_E\sigma_x \tag{12}$$

$$\text{Velocity:}\quad v'/v = \frac{\ell'/t'}{\ell/t} = \frac{\lambda\ell/t\tau}{\ell/t} = \frac{\lambda}{\tau} = \sqrt{\frac{\sigma_E}{\sigma_{\rho_s}}}$$

(for the same material, the velocities scale directly).

$$\text{Acceleration:}\quad a'/a = \frac{\ell'/t'^2}{\ell/t^2} = \frac{\ell\lambda/t^2\tau^2}{\ell/t^2} = \frac{\lambda}{\tau^2} = \frac{1}{\lambda}\,\frac{\sigma_E}{\sigma_{\rho_s}}$$

The statistical quantities will then be scaled as follows:

$$\text{Mean square pressure:}\quad \frac{\langle p'^2\rangle}{\langle p^2\rangle} = \sigma_E^2$$

Correlation function: $\left(\text{i.e., } \frac{C'}{C} = \frac{\text{Correlation full scale}}{\text{Correlation model}}\right)$

$$\text{(a) Pressure: } \langle p_1' \, p_2' \rangle / \langle p_1 p_2 \rangle = \sigma_E^2$$

$$\text{(b) Displacement: } \langle w_1' w_2' \rangle / \langle w_1 w_2 \rangle = \lambda^2 \tag{13}$$

$$\text{(c) Velocity: } \langle v_1' \, v_2' \rangle / \langle v_1 v_2 \rangle = \sigma_E / \sigma_{\rho_s}$$

$$\text{(d) Acceleration: } \langle a_1' a_2' \rangle / \langle a_1 a_2 \rangle = \frac{1}{\lambda^2} \left(\frac{\sigma_E}{\sigma_{\rho_s}}\right)^2$$

Power spectral density: i.e.,

$$\frac{(P.S.)'}{(P.S.)} = \frac{\text{Power spectral density full scale}}{\text{Power spectral density model}}$$

$$\text{(a) Pressure: } \sigma_E^2 \tau = \sigma_E^2 \, \lambda \sqrt{\frac{\sigma_{\rho_s}}{\sigma_E}} = \lambda \sqrt{\sigma_{\rho_s} \sigma_E^3}$$

$$\text{(b) Displacement: } \lambda^2 \tau = \lambda^2 \lambda \sqrt{\frac{\sigma_{\rho_s}}{\sigma_E}} = \lambda^3 \sqrt{\frac{\sigma_{\rho_s}}{\sigma_E}}$$

$$\text{(c) Velocity: } \frac{\sigma_E}{\sigma_{\rho_s}} \tau = \frac{\sigma_E}{\sigma_{\rho_s}} \lambda \sqrt{\frac{\sigma_{\rho_s}}{\sigma_E}} = \lambda \sqrt{\frac{\sigma_E}{\sigma_{\rho_s}}} \tag{13a}$$

$$\text{(d) Acceleration: } \frac{1}{\lambda^2} \left(\frac{\sigma_E}{\sigma_{\rho_s}}\right)^2 \tau = \frac{1}{\lambda^2} \left(\frac{\sigma_E}{\sigma_{\rho_s}}\right)^2 \lambda \sqrt{\frac{\sigma_{\rho_s}}{\sigma_E}} = \frac{1}{\lambda} \sqrt{\frac{\sigma_E^3}{\sigma_{\rho_s}^3}}$$

1. Joint or Interface Damping

One of the big problems in using scale models is damping. Here we will analyze how Coulomb damping and material damping enter into the problem. Coulomb damping will arise as external damping through rubbing between attached pieces. The basic relation for Coulomb damping states that the frictional shearing stress is proportional to the normal stress; i.e.,

$$(S.S.) = \bar{\bar{\mu}}\,(N.S.) \tag{14}$$

where $\bar{\bar{\mu}}$ is the coefficient of sliding friction between the two surfaces. For the prototype this relation will be

$$(S.S.)' = \bar{\bar{\mu}}'\,(N.S.)' \tag{15}$$

The previous scaling laws derived in section A stated that the stress must scale by a factor σ_E; thus

$$(S.S.)' = \sigma_E\,(S.S.) \tag{16}$$

Thus the frictional shearing stress in the model must be adjusted to $1/\sigma_E$ of the frictional shearing stress in the prototype. This can be done by adjusting the coefficient of friction and the normal bearing stress in the model so that (16) is satisfied. The model could be constructed with bolt fittings at the joints so that the normal pressure could be adjusted.

2. Material and Air Damping

For material damping the problem is not as direct, and we must resort to some assumptions regarding the response of a structure to random loading. Take the case in which the material and air damping can be completely described by a viscous damping coefficient. The power spectral density of the displacement of a structure under random loading can, for practical purposes, be written as (Ref. 8)

$$(P.S.)_d = \sum_r \sum_k \frac{\vec{q}_m \cdot \vec{q}_n}{|Y_r| \; |Y_k|} \; C_{rk} \; (\omega) \tag{17}$$

where $\vec{q}$ is associated with the mode shape of the structure, C_{rk} is the correlation integral which for all practical purposes can be considered independent of material damping, and **Y** is the response function which is critically dependent on damping. (To make this paper self-contained, the origin of equation (17) is explained in the appendix.) The response function **Y** can be written

$$Y_r(\omega) = M_r \sqrt{(\omega_{0r}^2 - \omega^2)^2 + \beta_r^2 \; \omega^2} \tag{18}$$

where M_r is the generalized mass for the rth mode and β_r is the damping coefficient for this mode. The logarithmic decrement δ_r can be written

$$\delta_r = \frac{\pi \beta_r}{\omega_{0r}} \; , \quad \beta_r = \frac{\delta_r \, \omega_{0r}}{\pi} = \overline{\beta} \tag{19}$$

Fung et al (Ref. 9) found that β_r was approximately a constant for various modes of aluminum cylindrical shells. Thus at resonance, where $\omega_{0r} = \omega$,

$$Y_r(\omega) = M_r \beta_r \omega_{0r} = M_r \overline{\beta} \omega_{0r} \tag{20}$$

In a structure subjected to random loading, the most severe response in the spectrum will occur at the frequencies corresponding to resonances of the modes. Therefore, at these frequencies

$$(P.S.)_d \sim \frac{1}{\overline{\beta}^2} = \xi^2 \tag{21}$$

So, for a model of different material than the full scale,

$$\frac{(P.S.)_d'}{(P.S.)_d} \sim \frac{1/\overline{\beta}'^2}{1/\overline{\beta}^2} = \frac{\xi'^2}{\xi^2} \tag{22}$$

Thus, in order to scale material damping properly at the critical frequencies,

Displacement: $\dfrac{(P.S.)_d'}{(P.S.)_d} \approx \dfrac{\xi'^2}{\xi^2} \lambda \sqrt{\dfrac{\sigma_E}{\sigma_{f_s}}}$

$$\text{Velocity:} \quad \frac{(P.S.)'_v}{(P.S.)_v} \approx \frac{\xi'^2}{\xi^2} \frac{1}{\lambda} \left(\frac{\sigma_E}{\sigma_{\rho_s}}\right)^{3/2} \tag{23}$$

$$\text{Acceleration:} \quad \frac{(P.S.)'_a}{(P.S.)_a} \approx \frac{\xi'^2}{\xi^2} \frac{1}{\lambda^3} \left(\frac{\sigma_E}{\sigma_{\rho_s}}\right)^{5/2}$$

The ratio ξ'/ξ must be determined experimentally for the different materials. However, a relatively straightforward experiment might be used to do this; possibly an experiment such as the one performed by Fung et al. (Ref. 9) could be performed on cylindrical shell models made of the prototype and model materials.

A more basic assessment of material damping and its dependency on level of excitation is offered by Crandall (Ref. 10). However more recent experiments have been conducted by Granick (Ref. 11) (in an extension of Crandall's study) at Goddard Space Flight Center which should throw a great deal of additional light on the material and air damping problem.

III. More General Formulation

In the previous section the modelling laws for the structure were formulated in an approximate fashion considering material and air damping as lumped parameters in an effort to derive some practical modelling relationships. Consider now a more exact formulation by assuming that material damping arises out of a thermal relaxation phenomenon first discussed by Zener (Ref. 12) later generalized by Mason (Ref. 13) Biot (Ref. 14) and others and considered more recently by Lazan and Goodman (Ref. 15), Crandall (Ref. 10), and Granick (Ref. 11). Furthermore, consider that the vibrating structure is moving in a viscous heat conducting fluid so that fluid damping may arise out of radiation and viscous losses. The equations governing the behavior of the solid are

A. The equation of motion of the solid (Ref. 16) (x component with body forces)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{1-2\nu} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$- \rho_s \frac{2(1+\nu)}{E} \frac{\partial^2 u}{\partial t^2} + \frac{2(1+\nu)}{E} X = \frac{2(1+\nu)}{1-2\nu} \frac{\partial(\alpha T)}{\partial x} \qquad (24)$$

X = body force per unit volume

B. The heat conduction equation of the solid (Ref. 14) (which assumes material damping by the mechanism of thermal relaxation)

$$K_s\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = C_s \frac{\partial T}{\partial t} + T_0 \frac{\alpha E}{1-2\nu} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad (25)$$

C. The equations governing behavior of the fluid are

1. The equation of continuity (Ref. 17)

$$\frac{\partial \rho_f}{\partial t} + \frac{\partial(\rho_f \dot{u})}{\partial x} + \frac{\partial(\rho_f \dot{v})}{\partial y} + \frac{\partial(\rho_f \dot{w})}{\partial z} = 0 . \qquad (26)$$

2. The Navier Stokes equations of motion (Ref. 17) (x component with body forces)

$$\rho_f \left(\frac{\partial \dot{u}}{\partial t} + \dot{u}\frac{\partial \dot{u}}{\partial x} + \dot{v}\frac{\partial \dot{u}}{\partial y} + \dot{w}\frac{\partial \dot{u}}{\partial z}\right) = \rho_f F_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial \dot{u}}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial \dot{u}}{\partial x} + \frac{\partial \dot{v}}{\partial y} + \frac{\partial \dot{w}}{\partial z}\right)\right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial \dot{u}}{\partial z} + \frac{\partial \dot{w}}{\partial x}\right)\right] \qquad (27)$$

F_x = body force per unit mass

3. The equation of heat transfer in the fluid (Ref. 17, 18)

$$\rho_f C_{p_f}\left(\frac{\partial T}{\partial t} + \dot{u}\frac{\partial T}{\partial x} + \dot{v}\frac{\partial T}{\partial y} + \dot{w}\frac{\partial T}{\partial z}\right) - \left(\frac{\partial p}{\partial t} + \dot{u}\frac{\partial p}{\partial x} + \dot{v}\frac{\partial p}{\partial y} + \dot{w}\frac{\partial p}{\partial z}\right) = k_f\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$+ \mu\left[2\left(\frac{\partial \dot{u}}{\partial x}\right)^2 + 2\left(\frac{\partial \dot{v}}{\partial y}\right)^2 + 2\left(\frac{\partial \dot{w}}{\partial z}\right)^2 + \left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right)^2 + \left(\frac{\partial \dot{v}}{\partial z} + \frac{\partial \dot{w}}{\partial y}\right)^2 + \left(\frac{\partial \dot{w}}{\partial x} + \frac{\partial \dot{u}}{\partial z}\right)^2 - \frac{2}{3}\left(\frac{\partial \dot{u}}{\partial x} + \frac{\partial \dot{v}}{\partial y} + \frac{\partial \dot{w}}{\partial z}\right)^2\right] \qquad (28)$$

The main boundary conditions between fluid and solid to consider for modelling are

$(T)_{fluid} = (T)_{solid}$ at fluid-solid interface

$\left(k_f \frac{\partial T}{\partial n}\right)_{fluid} = \left(K_s \frac{\partial T}{\partial n}\right)_{solid}$ at fluid-solid interface

$(\text{Displacement})_{fluid} = (\text{Displacement})_{solid}$ at fluid-solid interface

$(\text{Normal velocity})_{fluid} = (\text{Normal velocity})_{solid}$ at fluid-solid interface

$(\text{Stress})_{fluid} = (\text{stress})_{solid}$ at fluid-solid interface

If modelling solids and fluids are chosen which are different from the prototype then the following scaling parameters must hold:

For the solid

$$\ell' = \ell\lambda,\ t' = t\tau,\ k' = k\chi,\ T' = T\theta$$

$$E'/E = \sigma_E,\ \rho_s'/\rho_s = \sigma_{\rho_s},\ K_s'/K_s = \sigma_{K_s},\ \alpha'/\alpha = \sigma_2,\ C_s'/C_s = \sigma_{C_s},\ x'/x = \sigma_x \qquad (29)$$

For the fluid

$$\rho_f'/\rho_f = \sigma_{\rho_f},\ \mu'/\mu = \sigma_\mu,\ C_{p_f}'/C_{p_f} = \sigma_{C_{p_f}},\ k_f'/k_f = \sigma_{k_f},\ p_f'/p_f = \sigma_{p_f},\ F_x'/F_x = \sigma_{F_x} \qquad (30)$$

These relations are the fundamental parameters once the model material and model fluid are chosen.

Equation (24) and (25) can then be written for the solid model as (see Eq. (9) and again assume same ν in model and prototype)

$$\frac{\sigma_E E}{2(1+\nu)}\frac{1}{\lambda}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{1-2\nu}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)\right]+\sigma_x X$$

$$= \sigma_{\rho_s}\rho_s\frac{\lambda}{\tau^2}\frac{\partial^2 u}{\partial t^2}+\frac{\sigma_E E}{1-2\nu}\ \sigma_\alpha\,\alpha\frac{\theta}{\lambda}\frac{\partial T}{\partial x} \qquad (31)$$

$$\frac{\sigma_{K_s}K_s\theta}{\lambda^2}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\sigma_{C_s}C_s\frac{\theta}{\tau}\frac{\partial T}{\partial t}+\theta T_o\frac{\sigma_\alpha\alpha\sigma_E E}{1-2\nu}\frac{1}{\tau}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \qquad (32)$$

Thus the following relations must hold among the scaling factors in order that the basic differential equations will be the same for model and prototype

$$\frac{\sigma_E}{\lambda}=\frac{\sigma_{\rho_s}\lambda}{\tau^2}=\frac{\sigma_E\sigma_\alpha\theta}{\lambda}=\sigma_x \qquad (33)$$

$$\frac{\sigma_{K_s}}{\lambda^2} = \frac{\sigma_{C_s}}{\tau} = \frac{\sigma_\alpha \sigma_E}{\tau} \tag{34}$$

The relation $\sigma_E/\lambda = \sigma_{\rho_s} \lambda/\tau^2$ leads to equation (10). The additional relation $\sigma_E/\lambda = \sigma_E \sigma_2 \theta/\lambda$ leads to

$$\theta = \frac{1}{\sigma_\alpha} \tag{35}$$

This says that the temperature of the model will be scaled by $1/\sigma_\alpha$.

From (34) we have the relations

$$\frac{\sigma_{K_s}}{\lambda^2} = \frac{\sigma_{C_s}}{\tau} \qquad \sigma_{C_s} = \sigma_\alpha \sigma_E \tag{36}$$

It is plainly seen that if the same materials are used in model and prototype and testing is done in the same fluids, the first of the equations in (36) is inconsistent with the elastic scaling law (33). In fact the only way that thermoelastic damping can be scaled is by choosing the length scale and the material for the model in such a manner that the coefficient of expansion, heat transfer coefficient and specific heat of the model and prototype obey equation (36). This does not seem to offer much hope. Thus the only hope for obtaining sensible results for material damping scaling seems to be by use of a relation similar to (23). In cases of built up structures containing many connections, the joint or interface damping will undoubtedly overshadow the material damping so that this inconsistency will play a minor role.

Now consider the fluid. If the equations of the model fluid are written in a similar manner as those for the solid, we obtain

$$\frac{\sigma_{\rho_f}}{\tau}\frac{\rho_f}{\partial t} + \sigma_{\rho_f}\frac{\lambda}{\lambda\tau}\left[\frac{\partial}{\partial x}\left(\rho_f\,\dot{u}\right) + \frac{\partial}{\partial y}\left(\rho_f\,\dot{v}\right) + \frac{\partial}{\partial z}\left(\rho_f\,\dot{w}\right)\right] = 0 \qquad (37)$$

$$\sigma_{\rho_f}\left(\frac{\lambda}{\tau^2}\right)(\text{Left Side}) = \sigma_{\rho_f}\,\sigma_{F_x}\,\rho_f\,\mathbf{F}_x - \frac{\sigma_{p_f}}{\lambda}\frac{\partial p}{\partial x}$$

$$+\frac{\sigma_\mu}{\lambda\tau}\left\{\left[2\mu\frac{\partial\dot{u}}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial\dot{u}}{\partial x} + \frac{\partial\dot{v}}{\partial y} + \frac{\partial\dot{w}}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\ldots\right] + \frac{\partial}{\partial z}\left[\ldots\right]\right\} \qquad (38)$$

$$\sigma_{\rho_f}\,\sigma_{C_{p_f}}\,\rho_f\,C_{p_f}\frac{\sigma_\theta}{\tau}\left(\frac{\partial T}{\partial t} + \ldots\right) - \frac{\sigma_{pf}}{\tau}\left(\frac{\partial p}{\partial t} + \ldots\right)$$

$$= \sigma_{k_f}\,k_f\frac{\theta}{\lambda^2}\left(\frac{\partial^2 T}{\partial x^2} + \ldots\right) + \sigma_\mu\frac{\mu}{\tau^2}\left(2\left(\frac{\partial\dot{u}}{\partial x}\right) + \ldots\right) \qquad (39)$$

The basic scaling relations that must hold are therefore

$$\frac{\sigma_{\rho_f}}{\tau^2} = \frac{\sigma_{p_f}}{\lambda} = \frac{\sigma_\mu}{\lambda\tau} = \sigma_{\rho_f}\,\sigma_{F_x} \qquad (40)$$

$$\frac{\sigma_{\rho_f}\,\sigma_{C_{p_f}}\,\sigma_\theta}{\tau} = \frac{\sigma_{p_f}}{\tau} = \frac{\sigma_{k_f}\,\theta}{\lambda^2} = \frac{\sigma_\mu}{\tau^2} \qquad (41)$$

The relation $\sigma_{\rho_f}\lambda/\tau^2 = \sigma_\mu/\lambda\tau$ leads to the constant Reynold's number scaling

$$\frac{\rho_f'\,\overline{v}'\,\ell'}{\mu'} = \frac{\rho\,\overline{v}\,\ell}{\mu} \qquad (42)$$

(noting that $\sigma_{\overline{v}} = \frac{\lambda}{\tau}$)

This relation inherently contains the viscous damping offered by the fluid to the solid. Noting the fluid-solid boundary conditions and introducing (33)

$$\frac{\sigma_{\rho_f}}{\sqrt{\sigma_{\rho_s}/\sigma_E}} = \sigma_\mu/\lambda \tag{43}$$

If the same fluids and solids are used for the model and prototype, this will lead again to an inconsistency in (43) if $\lambda \neq 1$. The only way that we can scale fluid viscous damping is to choose a length scale and model materials which obey (43). The second scaling relation from (40) is

$$\frac{\sigma_{\rho_f}\lambda}{\tau^2} = \frac{\sigma_{p_f}}{\lambda}$$

$$\frac{\sigma_{\rho_f}\sigma_{\bar{v}}^2}{\sigma_{p_f}} = 1 \quad \text{or} \quad \frac{\frac{\rho_f'}{\rho_f}\frac{\bar{v}'^2}{\bar{v}^2}}{p'/p} \quad \text{or} \quad \frac{P'}{\rho_f'\,\bar{v}'^2\,\ell'^2} = \frac{P}{\rho_f\,\bar{v}^2\,\ell^2} \tag{44}$$

where p denotes pressure and P denotes force.

This is Newton's Universal Similitude Law (Ref. 1) which must hold between the model and full scale fluids. It inherently contains the scaling relations for mass loading and radiation damping offered by the fluid to the solid.

Now going back to the boundary condition between solid and fluid, it is seen that stresses and pressures must be scaled by σ_E. Thus

$$\frac{\sigma_{\rho_f} \lambda}{\tau^2} = \frac{\sigma_E}{\lambda} \tag{45}$$

and noting that $\lambda^2/\tau^2 = \sigma_E/\sigma_{\rho_s}$ from elastic scaling, we obtain finally

$$\sigma_{\rho_f} = \sigma_{\rho_s} \tag{46}$$

This says that the scale factors for the density of the fluids and solids for model and prototype must be the same in order for mass loading and radiation damping to scale properly. If the same density fluids are used for the model testing then the same density solids must be used.

The fluid heat transfer equation gives scaling parameters which must satisfy

$$\sigma_{\rho_f} \sigma_{c_{p_f}} \theta = \sigma_{p_f} \quad \text{or} \quad \sigma_{\rho_f} \sigma_{c_{p_f}} \theta = \sigma_E$$

$$\frac{\sigma_{\rho_f} \sigma_{c_{p_f}}}{\tau} = \frac{\sigma_{k_f}}{\lambda^2} \quad \text{or} \quad \sigma_{\rho_f} \sigma_{c_{p_f}} = \frac{\sigma_{k_f}}{\lambda} \sqrt{\sigma_{\rho_s}/\sigma_E}$$

$$\sigma_{\rho_f} \sigma_{c_{p_f}} \theta = \sigma_\mu/\tau \quad \text{or} \quad \sigma_{\rho_f} \sigma_{c_{p_f}} \theta = \frac{\sigma_\mu}{\lambda} \sqrt{\sigma_E/\sigma_{\rho_s}} \tag{47}$$

The same inconsistency between these relations and the elastic scaling relation is obtained when the same materials are used for model and full scale. In cases where heating of the fluid is not a primary problem, the heat transfer characteristics of the fluid can be neglected completely and it is only necessary to consider the fluid-solid problem without Eq. (28).

IV. DISCUSSION

The use of very small scale models of different material than the prototype offers a very efficient and inexpensive way to obtain order of magnitude answers which would ordinarily be very difficult or impossible to obtain. This type of scaling also offers an opportunity to use much lower pressure excitation levels than the full scale. There might also be a possibility of using light weight, easy to construct, plastic models to perform initial tests, in the same manner as those performed by Sankey and Wright (Ref. 19) but with a more complete assessment of damping.

Invariably, damping offers a big problem in using scale model results to extrapolate full scale response. If we limit our discussion to the resonance response, then damping could be introduced approximately in the scaling as shown in the previous section if ξ were constant for a given material. This resonant limitation is not serious in random loading, since the major response at each frequency of interest is usually composed of primatily resonant contributions in modes which are close to this frequency.

As shown in the third section of the report material damping of the thermoelastic type cannot easily be scaled nor can viscous air damping arising from air viscosity unless tests are conducted in "thinner" air or in different fluids. The main hope with model testing built-up structures under random loading is that joint or interfacial slip will be the main source of damping both in the model and in the full scale whether they be of the same or different materials. If this were true and if frictional stresses were scaled properly according to (16), then relations (13) and (13a) will hold for the random response. If one uses model materials with high material damping such as plastics, he must be sure that the joint damping is modelled properly before resorting to approximate equations of the form of (23). Equations (23) must be used with extreme caution since they will hold at resonant frequencies and they will hold only if ξ can be shown to be a function of material only.

APPENDIX A

The general variational equation of motion for any elastic structure in the absence of body forces can be written as (Ref. 8)

$$\iiint_V [\rho(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w) + \delta w]\,dv - \iint_S (x_\nu \delta u + y_\nu \delta v + z_\nu \delta w)dS = 0 \qquad \text{(A1)}$$

where

ρ = mass density of body

u, v, w = displacements at any point

$\delta_u, \delta_v, \delta_w$ = variations of displacements

x_ν, y_ν, z_ν = surface forces

dS = elemental surface area

dv = elemental volume

δw = variation of strain energy function.

In accordance with Love's analysis let the displacements in the normal modes be described by

$$u = u_r \varphi_r', \quad v = v_r \varphi_r', \quad w = w_r \varphi_r' \qquad \text{(A2)}$$

where $\varphi_r = A_r \cos p_r t$, p_r being the natural frequency of the r^{th} mode. Now let the forced motion of the system be described by

$$u = \sum_r u_r \varphi_r, \quad v = \sum_r v_r \varphi_r, \quad w = \sum_r w_r \varphi_r \qquad \text{(A3)}$$

where u_r, v_r, w_r, are the mode shapes and φ_r is a function of time. In accordance with Love let

$$\begin{aligned} u &= u_r \varphi_r & \delta u &= u_s \varphi_s \\ v &= v_r \varphi_r & \delta v &= v_s \varphi_s \\ w &= w_r \varphi_r & \delta w &= w_s \varphi_s \end{aligned} \qquad \text{(A4)}$$

Substituting into the variational equation of motion, we obtain

$$\iiint_V \rho(u_r \ddot{\varphi}_r u_s \varphi_s + v_r \ddot{\varphi}_r v_s \varphi_s + w_r \ddot{\varphi}_r w_s \varphi_s)dV + \iiint_V \delta w \; dV$$
$$= \iint_S (x_\nu u_s \varphi_s + y_\nu v_s \varphi_s + z_\nu w_s \varphi_s)dS \qquad \text{(A5)}$$

However, since the modal functions satisfy the equation for free vibration

$$\iiint_V \delta w \, dV = \iiint_V \rho(p_r^2 u_r \varphi_r u_s \varphi_s + p_r^2 v_r \varphi_r v_s \varphi_s + p_r^2 w_r \varphi_r w_s \varphi_s) dV \tag{A6}$$

and the orthogonality of the normal modes implies that

$$\iiint_V \rho(u_r u_s + v_r v_s + w_r w_s) dV = 0 \quad (\text{if } r \neq s) \tag{A7}$$

So the final equation of motion becomes

$$\ddot{\varphi}_r(t) + p_r^2 \varphi_r(t) = F_r(t) \tag{A8}$$

where

$$M_r = \iiint_V \rho(u_r^2 + v_r^2 + w_r^2) dV \text{ (the generalized mass for the } r^{th} \text{ mode)}$$

and

$$F_r(t) = \frac{1}{M_r} \iint_S [x_\nu(t) u_r + y_\nu(t) v_r + z_\nu(t) w_r] dS \tag{A9}$$

(the generalized force for the r^{th} mode)

If structural damping is taken into account, it can be written as another generalized force which opposes the motion

$$(F_r)_{damping} = -K\dot{\varphi}_r \iiint_V (u_r^2 + v_r^2 + w_r^2) dV \tag{A10}$$

where K is the damping force per unit volume per unit velocity. Finally, the equation of motion becomes

$$\ddot{\varphi}_r + \beta_r \dot{\varphi}_r + p_r^2 \varphi_r = F_r$$

$$\beta_r = \frac{K}{M_r} \iiint\limits_V (u_r^2 + v_r^2 + w_r^2)\, dV \tag{A11}$$

It will be convenient to employ the vector notation; thus, let the displacement functions in the r^{th} mode be written as

$$\vec{q}_r = u_r \vec{i} + v_r \vec{j} + w_r \vec{k} \tag{A12}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors in the x, y, z directions, respectively.

Let

$$\vec{F}(S, t) = x_\nu \vec{i} + y_\nu \vec{j} + z_\nu \vec{k} \tag{A13}$$

Thus,

$$M_r = \iiint\limits_V \rho\, \vec{q}_r \cdot \vec{q}_r\, dV; \; \vec{q}_r = \vec{q}_r\, (V)$$

$$F_r(t) = \frac{1}{M_r} \iint\limits_\sigma \vec{F} \cdot \vec{q}_r\, d\sigma \;;\; \vec{F} = \vec{F}\,(\sigma, t) \tag{A14}$$

where σ means surface.

Now the Fourier transform of $\vec{F}(\sigma, t)$ is

$$S_{\vec{F}}(\sigma, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \vec{F}(\sigma, t)\, e^{-i\omega t}\, dt \tag{A15}$$

and the Fourier transform of φ_r can be written

$$\begin{aligned} S_{\varphi_r}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi_r(t)\, e^{-i\omega t}\, dt \\ S_{\dot{\varphi}_r}(\omega) &= i\omega S_{\varphi_r}(\omega) \\ S_{\ddot{\varphi}_r}(\omega) &= -\omega^2 S_{\varphi_r}(\omega) \end{aligned} \tag{A16}$$

and

$$\begin{aligned} S_{\vec{q}}(\omega, V) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum_r \vec{q}_r(V)\, \varphi_r(t) e^{-i\omega t}\, dt \\ &= \sum_r \vec{q}_r \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi_r(t)\, e^{-i\omega t} dt \\ &= \sum_r \vec{q}_r(V)\, S_{\varphi_r}(\omega) \end{aligned} \tag{A17}$$

Thus the transform of the equation of motion becomes

$$-\omega^2 S_{\varphi_r} + i\omega\beta_r S_{\varphi_r} + p_r^2 S_{\varphi_r} = S_{\vec{F}_r} \tag{A18}$$

$$S_{\vec{F}_r} = \frac{1}{M_r} \iint_\sigma S_{\vec{F}} \cdot \vec{q}_r\, d\sigma$$

Thus

$$S_{\varphi_r}(\omega) = \frac{\iint_{\sigma} S_{\vec{F}} \cdot \vec{q}_r \, d\sigma}{(p_r^2 - \omega^2) + i\,\omega\,\beta_r} \tag{A19}$$

Now, write the auto correlation function of the displacement in vector notation as follows:

$$A_{\vec{q}\vec{q}}(V, \tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \vec{q}(t, V) \cdot \vec{q}(t+\tau, V)\, dt \tag{A20}$$

where $\vec{q}$ denotes the complete displacement vector

$$\vec{q} = \sum_r \vec{q}_r\, \varphi_r = \sum_r \varphi_r (u_r \vec{i} + v_r \vec{j} + w_r \vec{k}) \tag{A21}$$

Now

$$\vec{q}(t+\tau, V) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_{\vec{q}}(\omega, V) e^{i\omega(t+\tau)}\, d\omega \tag{A22}$$

$$A_{\vec{q}\vec{q}}(\tau, V) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \vec{q}(t, V) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_{\vec{q}}(\omega, V) e^{i\omega(t+\tau)}\, d\omega\, dt \tag{A23}$$

or

$$A_{\vec{q}\vec{q}} = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{+\infty} S_{\vec{q}}(\omega, V) e^{i\omega\tau} \cdot \frac{1}{\sqrt{2\pi}} \int_{-T}^{+T} \vec{q}(t, V) e^{i\omega t}\, dt \;\; d\omega \tag{A24}$$

So

$$A_{\vec{q}\vec{q}} = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} S_{\vec{q}}(w,V) \cdot S_{\vec{q}}^{*}(w,V) e^{i\omega\tau} d\omega \tag{A25}$$

Now the power spectral density of the displacement* $P_{\vec{q}\vec{q}}$ is defined in terms of the autocorrelation function as

$$A_{\vec{q}\vec{q}}(\tau,V) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P_{\vec{q}\vec{q}}(\omega,V) e^{i\omega\tau} d\omega \tag{A26}$$

Thus,

$$\frac{1}{\sqrt{2\pi}} P_{\vec{q}\vec{q}}(\omega,V) = \lim_{T\to\infty} \frac{1}{2T} S_{\vec{q}}(\omega,V) \cdot S_{\vec{q}}^{*}(\omega,V) \tag{A27}$$

or

$$P_{\vec{q}\vec{q}}(\omega,V) = \lim_{T\to\infty} \sqrt{\frac{\pi}{2T^2}} S_{\vec{q}}(\omega,V) \cdot S_{\vec{q}}^{*}(\omega,V) \tag{A28}$$

But

$$S_{\vec{q}}(\omega,V) = \sum_{r} \vec{q}_r(V) S_{\varphi_r}(\omega) \tag{A29}$$

*The notation $(P.\ S.)_d$ is used for power spectral density in the main body of the report.

So

$$P_{\vec{q}\,\vec{q}} = \lim_{T\to\infty} \sqrt{\frac{\pi}{2T^2}} \sum_r \sum_k \vec{q}_r(V) S_{\varphi_r}(\omega) \cdot \vec{q}_k(V) S_{\varphi_k}{}^*(\omega) \qquad \text{(A30)}$$

Therefore,

$$P_{\vec{q}\,\vec{q}}(\omega,V) = \sum_r \sum_k \frac{\vec{q}_r(V) \cdot \vec{q}_k(V)}{|Y_r(\omega)|\ |Y_k(\omega)|} \iint_{\sigma_u} \iint_{\sigma_v} \lim_{T\to\infty} \sqrt{\frac{\pi}{2T^2}} [S_{\vec{F}}(\omega,\sigma_n) \cdot \vec{q}_r(\sigma_u)]$$

$$[S_{\vec{F}}{}^*(\omega,\sigma_v) \cdot \vec{q}_v(\sigma_v)]\ e^{i(\theta_r-\theta_k)} d\sigma_u\, d\sigma_v \qquad \text{(A31)}$$

where

$$|Y_r(\omega)| = M_r \sqrt{(\omega^2 - p_r^2)^2 + \omega^2 \beta_r^2} \qquad \text{(A32)}$$

$$\theta_r = \tan^{-1} \frac{\beta_r^\omega}{p_r^2 - \omega^2} \qquad \text{(A33)}$$

Let $C_{rk}(\omega)$ represent the double surface integral in (A31). $C_{rk}(\omega)$, when written out, gives

$$\iint_{\sigma_u} \iint_{\sigma_v} \lim_{T\to\infty} \sqrt{\frac{\pi}{2T^2}} \left[S_{\vec{F}}(\omega,\sigma_n) \cdot \vec{q}_r(\sigma_u)\right] \left[S_{\vec{F}}{}^*(\omega,\sigma_v) \cdot \vec{q}_k(\sigma_v)\right] e^{i(\theta_r-\theta_k)} d\sigma_u\, d\sigma_v$$

$$= \iint_{\sigma_u} \iint_{\sigma_v} \lim_{T\to\infty} \sqrt{\frac{\pi}{2T^2}} \left[S_X u_r + S_Y v_r + S_Z w_r\right]\left[S_X{}^* u_k + S_Y{}^* v_k + S_Z{}^* w_k\right] e^{i(\theta_r - \theta_k)} d\sigma_u d\sigma_v \qquad \text{(A34)}$$

The terms in the brackets become

$$\lim_{T\to\infty} \sqrt{\frac{\pi}{2T^2}} \begin{bmatrix} S_X S_X{}^* u_r u_k + S_X S_Y{}^* u_r v_k + S_X S_Z{}^* u_r w_k \\ + S_Y S_X{}^* v_r u_k + S_Y S_Y{}^* v_r v_k + S_Y S_Z{}^* v_r w_k \\ + S_Z S_X{}^* w_r u_k + S_Z S_Y{}^* w_r v_k + S_Z S_Z{}^* w_r w_k \end{bmatrix} \qquad \text{(A35)}$$

For the special case of only vertical forces in Z direction, all terms disappear except the last ($S_Z S_Z{}^* w_r w_k$). This last equation gives the coupling between the forces when there are exciting forces in more than one direction. It is seen that the general solution for forces in any direction involves cross correlations between all pairs of components of the forces in all directions.

Equation (31) can now be written as

$$P_{\vec{q}\,\vec{q}}(\omega, V) = \sum_r \sum_k \frac{\vec{q}_r(V) \cdot \vec{q}_k(V)}{|V_r(\omega)|\ |Y_k(\omega)|} C_{rk}(\omega)$$

which corresponds to equation (17) in the body of the report.

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