CORE
Metadata, citation and similar papers at core.ac.uk
Provided by NASA Technical Reports Server

N65 11630
(ACCESSION NUMBER) (THRU)
8 1
(PAGES) (CODE)
CR-59593 19
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)
FACILITY FORM 602

Technical Report No. 32-655
(Part I)

Mathematical Models of Missile Launching

Alfred C. Dahlgren

OTS PRICE

XEROX $ 1.00 FS
MICROFILM $ .50 MF

JPL
JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 21, 1964

Technical Report No. 32-655
(Part I)

Mathematical Models of Missile Launching

Alfred C. Dahlgren

T. Hamilton, Chief
Systems Analysis Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 21, 1964

Copyright © 1964
Jet Propulsion Laboratory
California Institute of Technology

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

CONTENTS

ABSTRACT

11630

By means of a mathematical model of the situation, including assumptions about the nature of the delays encountered, an estimate is presented of the expected number of days required to launch three missiles, each from its own pad, allowing no simultaneous (i.e., same-day) countdowns or launches.

author

I. INTRODUCTION

In space mission feasibility studies it is often necessary to have an estimate of the number of days required to launch a given number of missiles from a given (and possibly different) number of pads. This report (Part I of three parts) is one of a series, the purpose of which is to investigate this question with the aid of probability theory, for different launching configurations (see Refs. 1 and 2). The particular launching configuration examined here is that one of three missiles being launched, each from its own pad, allowing no simultaneous (i.e., same-day) counting-down or launching. The following assumptions constitute the launch model.

II. THE MATHEMATICAL LAUNCH MODEL

1. There are three identical pads and three identical missiles, each on its own pad.
2. For each missile, a complete countdown and launching in one day is possible, though not necessary. But once the counting has started on a given missile, the counting continues on that missile until it fires, no matter how many days are required. There is a probability p (a specified constant) that a missile will successfully count down and fire in one day. There is thus also a constant probability $q = 1 - p$ that the missile will fail to complete a countdown on that day, and will incur a one-day delay. In other words, if the countdown on a given missile stops on any day of counting, the counting must start over from the beginning the following morning.
3. Because p is a constant, failures to fire a given missile do not influence the value of p in subsequent attempts to fire the same missile or others. Each missile is thus probabilistically independent of the other two.
4. Prior to the start of the N-day period, the three missiles are erected, each on its own pad and ready to start counting. One of the three missiles is then chosen arbitrarily, because all are identical. On the first day of the N-day period, the counting down is started on this missile, and continues until the missile fires (the countdown started anew each day), no matter how many days this requires. On the day after the firing, counting down is initiated on one of the two remaining missiles (again the choice is

arbitrary). This second missile also counts down for as many days as are required for it to fire, the countdown starting anew each day. On the day after the second missile is launched, the remaining missile starts counting, continuing until the missile fires on the Nth day, by assumption. The third and last missile is always assumed to fire on the Nth day, whatever the positions of the first and second missile launchings within the N-day period. In this launching model, since no missiles are allowed to count down together on any day, no simultaneous (i.e., same-day) launches are possible. (The choosing at each firing of the next missile to be fired is logically equivalent to predefining the firing order before the start of the N-day period. The reader may take either point of view, depending on the practical application to which he intends to put the results of this report. See Section IV for further discussion of this point.

5. Since each missile is on its own pad, it is not necessary in this launch model to take into account any turnaround time between missile firings, as has been done in other reports in this series.

III. RESULTS

1. The probability $P(N)$ of launching (under the above conditions) three missiles from three pads in exactly N days from start of counting, is

$$P(N) = \frac{p^3 q^{N-3}}{2}(N-1)(N-2)$$

2. The mean day μ (counted from the first day of the N-day period), on which one can expect to launch the third and last missile, is found to be simply

$$\mu = \frac{3}{p} \text{ days}$$

3. The variance σ^2 of the $P(N)$ distribution (see Section IV) is given by

$$\sigma^2 = \frac{3q}{p^2}$$

4. The moment-generating function $M(\theta)$ is found to be

$$M(\theta) = \frac{p^3 e^{3\theta}}{(1 - qe^{\theta})^3}$$

IV. DERIVATION OF RESULTS

In order to construct a sample space for the problem, we ask in how many different ways the three missiles can be launched. Since we have made it a condition of this launch model that there be no simultaneous (i.e., same-day) countdowns, it will be seen that only one specific type of launch sequence is possible for the three missiles, for the following reasons. As outlined in the introduction, the first missile to be fired must be consciously chosen from the three, with the second missile to be fired then chosen from the two remaining. This order is in reality a type of 1, 2, 3 launch sequence. Let us number the individual missiles arbitrarily but permanently as 1, 2, 3. It might seem at first glance that the order 1, 2, 3 and permutations of this order would be distinct possible launch sequences, but this is not true. Granted, this actually would be the case if the conditions of the launch model did not require us to choose at each firing the missile to be launched next. But in so choosing, we remove at each firing the randomness otherwise associated with the question of which missile is next to be counted down. In other words, we constrain the actual launching order of the missiles to follow the single path that we choose for it. Because of this, only this one type of launch sequence is possible.

It may further be noted that there is no difference logically between choosing the next missile at each launching as we go along, or equivalently predefining the same firing order before the N-day period even starts. (The more general case of choosing at each step is treated in this report for completeness.) If we did predefine the order, then it is obvious that this defined order would be the only sequence that could occur in the firing of the three missiles. Thus from another angle we have the same conclusion that if we choose at each firing the missile to count down next, there are no other possible launch sequences than this one. Thus in the calculations below we need take account only of this one type of launch sequence in the sample space for the problem.

Keeping the above in mind, let us assume that we choose at each firing such that the missiles do result in launching in the order 1, 2, 3 (with the numbering from above). The probability p_1 that missile 1 will fire on the kth day (counted from the first day of the N-day period), is equal to the probability that it will fail to fire for $k-1$ days, times the probability that it will fire in one day, or $p_1 = pq^{k-1}$. Similarly, if missile 2 counts for m days (starting with the $(k+1)$st day), $p_2 = pq^{m-1}$. Missile 3 by assumption fires on the Nth day, and thus it counts for the number of days remaining in the N-day period after the $(k+m)$th day, or thus for $N-k-m$ days. The probability p_3 of missile 3 firing on the Nth day is thus $p_3 = pq^{N-k-m-1}$.

Out of a given N-day period, k and m can assume various different values, the limits on these values being determined by N. In order to express this situation mathematically, we will make m dependent on k, and the days for missile 3 dependent on m and k. We will then permit k to range over its maximum and minimum values within the N-day period. Doing this will automatically take care of the maximum and minimum values for m and the days for missile 3. The limits on k are as follows. The quantity k must be at least 1 day, but if k is almost as large as N, at the end of the N-day period we must leave at least one day each to launch missiles 2 and 3. Thus out of a given N days, k can range only over $1 \leq k \leq N-2$. The m days for missile 2 starts on the $(k+1)$st day, and m is at least one day, but again we must leave at least the Nth day in order to launch missile 3. Thus $1 \leq m \leq N-k-1$. From above, missile 3 counts for $N-k-m$ days, and thus k and m ranging over their allowed values takes care of the maximum and minimum values for the days missile 3 counts.

Further, the situation that missile 1 will take a particular k_0 days, with missile 2 taking a particular m_0 days and missile 3 taking the remaining $N-k_0-m_0$ days, is a compound event whose own compound probability pc therefore equals the product of the component event probabilities, or $p_c = p_1p_2p_3$. There can be different specific k_0 and m_0 combinations within the given N-day period. The compound event associated with a particular k_0 and m_0 is an event mutually exclusive of those events associated with other k_0 and m_0 sets. Thus, since there are various values available to k_0 and m_0, we must sum the compound probability $p_c = p_1p_2p_3$ over all possible values of k and m within a given N-day period. By doing this we will arrive at the general probability expression $P(N)$ of launching 3 missiles in N days under the launch model conditions, or

$$\begin{aligned} P(N) &= \sum_{k=1}^{N-2} \sum_{m=1}^{N-k-1} p_1p_2p_3 \\ &= \sum_{k=1}^{N-2} \sum_{m=1}^{N-k-1} pq^{k-1}pq^{m-1}pq^{N-k-m-1} \\ &= p^3q^{N-3} \sum_{k=1}^{N-2} \sum_{m=1}^{N-k-1} (1) \end{aligned}$$

or

$$P(N) = \frac{p^3q^{N-3}}{2}(N-1)(N-2)$$

By assumption in this launch model, the possibility of failing to launch all three missiles was not included in the sample space for the problem. We would therefore keep counting until all three missiles were launched, which is equivalent to assuming an upper limit of infinity for N. At the other end of the spectrum, the smallest admissible value of N is three days, one day being the minimum necessary for counting down and launching each of the three missiles.

We have a sample space for this launch model in which we have concluded that only one type of launch sequence is possible. If this sample space has actually taken into account all possible launch sequences, the total probability over the whole sample space must equal 1. The total probability P_t equals the sum of $P(N)$ over all admissible avlues of N, or

$$P_t = \sum_{N=3}^{\infty} P(N) = \sum_{N=3}^{\infty} \frac{p^3 q^{N-3}}{2} (N-1)(N-2)$$
$$= \frac{p^3}{2} \frac{d^2}{dq^2} \left(\frac{q^2}{1-q} \right) = 1$$

Thus we conclude that in the sample space, we have accounted for all possible launch sequences, this model having only one type. Looked at from another angle, the result $P_t = 1$ indicates that the expression for $P(N)$ is correct, since on other grounds (as explained above), we had concluded that there can be only this one type of launch sequence in the sample space.

Lastly, $P(N)$ in this launch model plays the role of the probability function for a discrete random variable n, where n is the number of the day on which the third and last missile is launched. In other words, $P(N)$ in reality is the probability that $n = N$. Thus the mean day μ (counted from the first day of the N-day period), on which one expects to launch the third and last missile, and the variance σ^2 of the $P(N)$ distribution may be calculated by the usual series:

$$\mu = E[N] = \sum_{N=3}^{\infty} NP(N)$$

and

$$\sigma^2 = \nu_2 - \mu^2 = \sum_{N=3}^{\infty} N^2 P(N) - \mu^2.$$

μ and σ^2 may also be calculated from the moment-generating function $M(\theta)$, where

$$M(\theta) = E[e^{\theta N}] = \sum_{N=3}^{\infty} P(N) e^{\theta N}$$

or

$$M(\theta) = \frac{p^3 e^{3\theta}}{(1 - qe^{\theta})^3}$$

This further implies that

$$\mu = \left. \frac{dM(\theta)}{d\theta} \right|_{\theta=0} = \frac{3}{p} \text{ days}$$

and also that

$$\sigma^2 = \nu_2 - \mu^2 = \left. \frac{d^2 M(\theta)}{d\theta^2} \right|_{\theta=0} - \mu^2$$

or

$$\sigma^2 = \frac{3q}{p^2}$$

REFERENCES

1. Solloway, C. B., *A Simplified Statistical Model for Missile Launching*, Technical Report No. 32-431, Jet Propulsion Laboratory, Pasadena, California, May 1, 1963.
2. Lass, H., and C. B. Solloway, *A Statistical Problem Related to the Launching of a Missile*, Technical Report No. 32-124, Jet Propulsion Laboratory, Pasadena, California, July 20, 1961.