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AN ANALYSIS OF CONFINED VORTEX FLOWS

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by

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ABSTRACT*

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The vortex flow of an incompressible fluid between two finite flat plates is considered. Special attention is given to the case for which the radius of the plates is larger than their separation distance. A momentum integral solution gives the variation of the important parameters β, the modified boundary layer thickness, and g_1, the radial velocity, with the radius for various values of A, a measure of the imposed radial mass flow.

AUTHOR

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LIST OF SYMBOLS

a_o, a_1, a_2	=	constants
A	=	$2kM^2(C_{13} - C_1 + \bar{k}_1 C_{23} - \bar{k}_1 C_2)/(C_1 + \bar{k}_1 C_2 - 1)^2 h_3'(0)$
b	=	a characteristic dimension in the axial direction
b_o, b_1, b_2, b_3	=	constants
c_o, c_1, c_2, c_3	=	constants
C	=	a constant
$C_1, C_{11}, C_{12}, C_{13}$	=	constants; integrals of h_1, h_2, h_3
$C_2, C_{22}, C_{23}, C_{33}$	=	constants; integrals of h_1, h_2, h_3
f, f_1, f_2, f_3	=	functions of the radius
g_1, g_2, g_3	=	functions of the radius
G_1, G_2, G_3	=	functions of x, the similarity variable
h	=	function of the radius
h_1, h_2, h_3	=	functions of ζ ; velocity profiles
i	=	summation index
k	=	the radial velocity at $r = 1$

$k_{1...7}$ = constants

$\bar{k}_1$ = a constant

M = $1/2 \sqrt{Re\ b^2/r_o^2}$; the midplane

n = a constant

O( ) = order of

P = the dimensional pressure

$\bar{P}$ = the dimensionless pressure

Q = the net radial mass flow; 2kM

$\bar{Q}$ = $Q/2\pi \rho r_o b V_o$

r = the radial coordinate

$\hat{r}$ = the radial unit vector

$\bar{r}$ = the dimensionless radial coordinate

r_o = a characteristic dimension in the radial direction

Re = the Reynolds number; $V_o r_o/\nu$

u = the radial velocity component

$\bar{u}$ = the dimensionless radial velocity component

u_i = ordered radial velocity component

$\underline{V}$ = the velocity vector

V_o = a characteristic velocity

v = the tangential velocity component

$\bar{v}$ = the dimensionless tangential velocity component

v_i = ordered tangential velocity component

w = the axial velocity component

$\bar{w}$ = the dimensionless axial velocity component

$\bar{\bar{w}}$ = the transformed axial velocity component

w_i = ordered axial velocity component

x = a similarity variable

z = the axial coordinate

$\hat{z}$ = the axial unit vector

$\bar{z}$ = the dimensionless axial coordinate

$\bar{\bar{z}}$ = the transformed axial coordinate

β = $(C_1 + C_2\bar{K}_1 - 1)\,\delta/M$

δ = the boundary layer thickness; a function of the radius

ϵ = a small positive number

ζ = a similarity variable

θ = the tangential coordinate

$\hat{\theta}$ = the tangential unit vector

ν = the kinematic viscosity

ρ = the density

τ_r = the radial shear stress at the plate

$$\bar{\tau}_r = \tau_r \Big/ \frac{2\mu V_o k}{b} \left[h_1'(0) + \bar{k}_1 h_2'(0)\right] \left[C_1 + \bar{k}_1 C_2 - 1\right]$$

τ_θ = the tangential shear stress at the plate

$$\bar{\tau}_\theta = \tau_\theta \Big/ \frac{2\mu V_o}{b} h_3'(0) \left[C_1 + \bar{k}_1 C_2 - 1\right]$$

' = differentiation

* = dimensional quantity

I.

INTRODUCTION

Recent attention has been focused upon rotating flows within confined regions. In particular, vortex motion of a fluid confined by stationary boundaries has received considerable study.

The swirling flow of an electrically-conducting fluid between two coaxial cylinders in the presence of a magnetic field has been extensively studied in an attempt to predict the performance of a magnetohydrodynamic vortex power generator (1, 2). Such analyses, however, neglected the effect of any confining end walls on the flow.

In an attempt to calculate the effect of the end walls upon the vortex motion, several investigators have studied the swirling motion of a fluid over a single finite flat plate in the absence of magnetic fields. This work has been well summarized by Mack (3) and King (4). More recently Lewellen and King (5) investigated the flow over a single flat plate in the presence of an applied axial magnetic field.

In a typical analysis (3) of the non magnetic flow over a finite stationary plate it is assumed that the flow outside the boundary layer is a vortex (tangential velocity $\sim r^n$) with negligible radial velocity. The flow within the boundary layer is solved by a conventional momentum-integral technique.

The growth of a boundary layer on a finite flat plate in a vortex flow has two effects on the flow. The first is a retardation of the tangential velocity due to the action of viscosity. If, in a confined flow, the action of viscosity is so great as to cause the boundary layers on the two end plates to occupy an appreciable fraction of the volume of the confined region, the vortex motion may be greatly affected. This effect is referred to as boundary-layer blockage.

The second effect concerns the radial mass flow. The radial pressure gradient caused by the vortex motion outside the boundary layer cannot be balanced by such a motion within the boundary layer because of the slowing action of viscosity. This imbalance results in a net force that drives the fluid near the wall radially inward. This radial flow must be compensated by a mass flow into the boundary layer from the outer flow. In a single plate problem such as that analyzed by Mack, the radial mass flow within the boundary layer is compensated by an axial flow toward the plate; it is assumed that the radial velocity is zero outside the boundary layer.

In a problem with the vortex motion occurring in a confined region, the flow picture is different. The slowing of the tangential velocity, causing the boundary layer blockage occurs as described above. However, there is a new condition imposed upon the flow; the radial mass flow must be conserved. For motion occurring between two impermeable plates, the net mass flow at any radial station is a constant independent of the radius since mass is neither added nor removed. In a vortex flow, the radial mass flow within the boundary layer induced by the pressure imbalance is not a constant; it increases as the radius decreases. Therefore, the radial mass flow outside the boundary layers cannot, in general, be a constant. This is equivalent to stating that the radial velocity outside the boundary layer cannot, in general, be a constant independent of the radius (in particular, it cannot be zero). In order to physically maintain the vortex flow this radial velocity must be directed toward the axis of rotation. Thus the above mentioned analysis of the single plate problem is not valid in general for the two plate problem since that analysis does not satisfy the condition that the radial mass flow be conserved.

If the fraction of the radial mass flow diverted into the boundary layer is greater than unity, the radial velocity outside the boundary layer must change direction from toward the vortex axis to away from the axis. In this case the vortex motion breaks down and the flow picture becomes more complicated.

The two plate configuration was initially treated by Vogelpohl (6) in 1944. In this analysis the axial velocity was assumed to be identically zero. The relevance of this work will be discussed subsequently.

Recently, Rosenzweig, Lewellen and Ross (7) analyzed a two plate configuration but the analysis was limited to the case for which the separation distance between the two plates was greater than the radius of the plates.

In the present analysis, the Navier-Stokes equations will be carefully ordered to determine the proper governing equations for flow in a confined region for various values of the Reynolds number and the shape parameter, b/r_o where r_o is the radius of the plates and b is the separation distance between them. Particular attention will be focused on the case b/r_o is smaller than one, i.e., for which the spacing between the plates is smaller than the radius of the plates.

One of the most powerful tools used to solve the non-linear partial differential equations of fluid mechanics is a similarity transformation which reduces the partial differential equations to ordinary differential equations. This device is used by investigators dealing with the problem of one infinite flat plate in a rotating flow.

It will be shown why this method of attack is successful for the case of one plate but cannot be applied to the two plate problem.

Since a true similarity does not exist for the two plate problem, an approximate solution to the problem is carried out. In particular, a momentum integral method is used to calculate the variation of the boundary layer thickness with the radius. Also the variation of the outer radial velocity and the shear stress at the plates with the radius are calculated. An equation relating the net radial mass flow to the pressure gradient is given.

II.

STATEMENT OF THE PROBLEM

The problem is to describe accurately the behavior of a rotating viscous incompressible fluid between two finite flat plates. The analysis will consist of three parts: I) Derivation of the equations governing this flow; II) Demonstration that no similarity transformation exists; III) Solution of the momentum integral equations for the boundary layer thickness, the radial velocity, and shear stress at the wall as functions of the radius.

The plates are situated parallel to each other and their common axis is coincident with the vortex axis of the rotating fluid as shown in figure 1. With the plates separated by a distance b and with the coordinates shown in the figure, the boundary conditions are, in part:

$$\text{At } z = 0 \qquad u = v = w = 0$$

$$\text{At } z = b \qquad u = v = w = 0$$

Let the region of interest be bounded radially by two cylinders at radial stations $r = \epsilon r_o$ and $r = r_o$ where ϵ is a positive number less than one. It is assumed that there is a net radial mass flow inward (toward the vortex axis) between the two plates. It is further assumed that this radial mass flow is sufficiently strong such that the radial velocity is everywhere toward the axis in the region of interest; that is, such that the fraction of radial mass flow in the boundary layer is less than unity. This assumption does not allow the vortex motion to break down due to a reversal of the direction of the radial velocity outside the boundary layer.

The fluid may enter the region of interest at the outer cylinder by being injected tangentially by slot jets, by being blown through a rotating porous cylinder or by some equivalent method such that the boundary conditions at the outer cylinder are:

$$\text{At} \quad r = r_o \qquad v = V_o \qquad u = kV_o \qquad w = 0$$

where k is a negative constant. These boundary conditions prescribe the driving force of the problem.

It is assumed that the cylinder at ϵr_o in no way obstructs the flow of the fluid but only marks the boundary of the region of interest. This assumption does not allow the prescription of velocities at the inner cylinder and therefore precludes consideration of a radial boundary layer on this cylinder. The analysis will not be concerned with the manner of exit of the fluid from the central core.

In practice, the radial mass flow is maintained by an imposed pressure difference between $r = r_o$ and $r = \epsilon r_o$. One result of this analysis will be a relation between the radial mass flow and the imposed pressure difference.

III.

ASSUMPTIONS

The following assumptions will hold for this analysis:

The working fluid is

1. viscous
2. incompressible

The flow is

3. steady
4. laminar
5. axially symmetric

and

6. body forces are absent
7. the properties of the fluid are constant.

IV.

ANALYSIS

I. Derivation of the Governing Equations

The Navier-Stokes equations for a viscous incompressible fluid in cylindrical coordinates are, with the assumptions of axial symmetry, steady flow, and no body force:

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}(u) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right] \qquad a$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right] \qquad b \qquad (1)$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] \qquad c$$

where the velocity is denoted by

$$\underline{V} = u\hat{r} + v\hat{\theta} + w\hat{z}$$

The continuity equation is, with the above assumptions:

$$\frac{1}{r}\frac{\partial}{\partial r}\cdot(ru) + \frac{\partial w}{\partial z} = 0 \qquad d$$

The governing equations will be derived by ordering the terms of equations (1) and ignoring the terms which are shown to be small in the region of interest. In order to facilitate the ordering procedure, the above four equations will first be rendered dimensionless.

The characteristic dimension in the radial direction is the radius of the plates, r_o . The separation distance between the two plates, b , is the characteristic dimension in the axial direction. The characteristic velocity, V_o , will be taken as the tangential free stream velocity at r_o . The characteristic pressure is ρV_o^2 .

Thus

$$
\begin{aligned}
z &= b\bar{z} \qquad & u &= V_o\bar{u} \\
r &= r_o\bar{r} \qquad & v &= V_o\bar{v} \\
P &= \rho V_o^2\bar{P} \qquad & w &= V_o\bar{w}
\end{aligned}
\tag{2}
$$

where the bar denotes a dimensionless quantity.

Substituting equations (2) into equations (1) gives

$$
\bar{u}\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{r_o}{b}\bar{w}\frac{\partial\bar{u}}{\partial\bar{z}} - \frac{\bar{v}^2}{\bar{r}} = -\frac{\partial\bar{P}}{\partial\bar{r}} + \frac{\nu}{V_o r_o}\left[\frac{\partial^2\bar{u}}{\partial\bar{r}^2} + \frac{\partial}{\partial\bar{r}}\left(\frac{\bar{u}}{\bar{r}}\right) + \frac{r_o^2}{b^2}\frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right] \qquad a
$$

$$
\bar{u}\frac{\partial\bar{v}}{\partial\bar{r}} + \frac{r_o}{b}\bar{w}\frac{\partial\bar{v}}{\partial\bar{z}} + \frac{\bar{u}\bar{v}}{\bar{r}} = \frac{\nu}{V_o r_o}\left[\frac{\partial^2\bar{v}}{\partial\bar{r}^2} + \frac{\partial}{\partial\bar{r}}\left(\frac{\bar{v}}{\bar{r}}\right) + \frac{r_o^2}{b^2}\frac{\partial^2\bar{v}}{\partial\bar{z}^2}\right] \qquad b
$$

$$
\bar{u}\frac{\partial\bar{w}}{\partial\bar{r}} + \frac{r_o}{b}\bar{w}\frac{\partial\bar{w}}{\partial\bar{z}} = -\frac{r_o}{b}\frac{\partial\bar{P}}{\partial\bar{z}} + \frac{\nu}{V_o r_o}\left[\frac{\partial^2\bar{w}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{w}}{\partial\bar{r}} + \frac{r_o^2}{b^2}\frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right] \qquad c \tag{3}
$$

$$
\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}(\bar{r}\,\bar{u}) + \frac{r_o}{b}\frac{\partial\bar{w}}{\partial\bar{z}} = 0 \qquad d
$$

The Reynolds number is defined as $\frac{V_o r_o}{\nu}$. If the rotating flow is of a vortex nature, which is the most interesting case, $(V_o r_o)$ is a constant of the flow irrespective of the choice of r_o.

The boundary conditions now are:

At $\bar{Z} = 0$ $\qquad \bar{u} = \bar{v} = \bar{w} = 0$

At $\bar{Z} = 1$ $\qquad \bar{u} = \bar{v} = \bar{w} = 0$

The flow pattern is governed by the values of two dimensionless groups--the Reynolds number and the shape ratio, r_o/b . The effect of one plate on the other is most pronounced for the ratio r_o/b larger than unity; therefore consideration will now be given to this case.

Now it is possible to expand the three velocity components and the pressure in power series in b/r_o .

$$\bar{u} = \sum_{i=o}^{\infty} u_i (b/r_o)^i \qquad a$$

$$\bar{v} = \sum_{i=o}^{\infty} v_i (b/r_o)^i \qquad b$$

$$\bar{w} = \sum_{i=o}^{\infty} w_i (b/r_o)^i \qquad c \qquad (5)$$

$$\bar{P} = \sum_{i=o}^{\infty} P_i (b/r_o)^i \qquad d$$

where the u_i, v_i, w_i, and P_i are independent of b/r_o but still are functions of the Reynolds number.

Substituting equations (5) into equations (3) and grouping terms with equal powers of b/r_o gives

$$w_o \frac{\partial u_o}{\partial \bar{z}} + \frac{b}{r_o}\left[u_o \frac{\partial u_o}{\partial \bar{r}} + w_1 \frac{\partial u_o}{\partial \bar{z}} + w_o \frac{\partial u_1}{\partial \bar{z}} - \frac{v_o^2}{\bar{r}} + \frac{\partial P_o}{\partial \bar{r}}\right] +$$

$$\left(\frac{b}{r_o}\right)^2 \left[u_1 \frac{\partial u_o}{\partial \bar{r}} + u_o \frac{\partial u_1}{\partial \bar{r}} + w_2 \frac{\partial u_o}{\partial \bar{z}} + w_1 \frac{\partial u_1}{\partial \bar{z}} + w_o \frac{\partial u_2}{\partial \bar{z}} - 2\,\frac{v_1 v_o}{\bar{r}} + \frac{\partial P_1}{\partial \bar{r}}\right] \quad \text{a}$$

$$+ \ldots = \frac{1}{Re}\frac{r_o}{b}\frac{\partial^2 u_o}{\partial \bar{z}^2} + \frac{1}{Re}\frac{\partial^2 u_1}{\partial \bar{z}^2} + \frac{1}{Re}\frac{b}{r_o}\left[\frac{\partial^2 u_o}{\partial \bar{r}^2} + \frac{\partial}{\partial \bar{r}}\left(\frac{u_o}{\bar{r}}\right) + \frac{\partial^2 u_2}{\partial \bar{z}^2}\right] + \ldots$$

$$w_o \frac{\partial v_o}{\partial \bar{z}} + \frac{b}{r_o}\left[u_o \frac{\partial v_o}{\partial \bar{r}} + w_1 \frac{\partial v_o}{\partial \bar{z}} + w_o \frac{\partial v_1}{\partial \bar{z}} + \frac{u_o v_o}{\bar{r}}\right] +$$

$$\left(\frac{b}{r_o}\right)^2 \left[u_1 \frac{\partial v_o}{\partial \bar{r}} + u_o \frac{\partial v_1}{\partial \bar{r}} + w_2 \frac{\partial v_o}{\partial \bar{z}} + w_1 \frac{\partial v_1}{\partial \bar{z}} + w_o \frac{\partial v_2}{\partial \bar{z}} + \frac{u_1 v_o}{\bar{r}} + \frac{u_o v_1}{\bar{r}}\right] \quad \text{b}$$

$$+ \ldots = \frac{1}{Re}\frac{r_o}{b}\frac{\partial^2 v_o}{\partial \bar{z}^2} + \frac{1}{Re}\frac{\partial^2 v_1}{\partial \bar{z}^2} + \frac{1}{Re}\frac{b}{r_o}\left[\frac{\partial^2 v_1}{\partial \bar{r}^2} + \frac{\partial}{\partial \bar{r}}\left(\frac{v_o}{\bar{r}}\right) + \frac{\partial^2 v_2}{\partial \bar{z}^2}\right] + \ldots$$

(6)

$$\left[w_o \frac{\partial w_o}{\partial \bar{z}} + \frac{\partial P_o}{\partial \bar{z}}\right] + \frac{b}{r_o}\left[u_o \frac{\partial w_o}{\partial \bar{r}} + w_1 \frac{\partial w_o}{\partial \bar{z}} + w_o \frac{\partial w_1}{\partial \bar{z}} + \frac{\partial P_1}{\partial \bar{z}}\right] +$$

$$\left(\frac{b}{r_o}\right)^2 \left[u_1 \frac{\partial w_o}{\partial \bar{r}} + u_o \frac{\partial w_1}{\partial \bar{r}} + w_2 \frac{\partial w_o}{\partial \bar{z}} + w_1 \frac{\partial w_1}{\partial \bar{z}} + w_o \frac{\partial w_2}{\partial \bar{z}} + \frac{\partial P_2}{\partial \bar{z}}\right] + \ldots \quad \text{c}$$

$$= \frac{1}{Re}\frac{r_o}{b}\frac{\partial^2 w_o}{\partial \bar{z}^2} + \frac{1}{Re}\frac{\partial^2 w_1}{\partial \bar{z}^2} + \frac{1}{Re}\frac{b}{r_o}\left[\frac{\partial^2 w_o}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial w_o}{\partial \bar{r}} + \frac{\partial^2 w_2}{\partial \bar{z}^2}\right] + \ldots$$

$$\frac{\partial w_o}{\partial \bar{z}} + \frac{b}{r_o}\left[\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}u_o) + \frac{\partial w_1}{\partial \bar{z}}\right]$$

$$+ \left(\frac{b}{r_o}\right)^2 \left[\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}u_1) + \frac{\partial w_2}{\partial \bar{z}}\right] + \ldots \quad \text{d}$$

Since the ordered velocity components are independent of b/r_o , each group of terms in equation (6d) may be immediately set equal to zero. The zeroth order group is simply

$$\frac{\partial w_o}{\partial \bar{z}} = 0$$

Immediate integration gives

$$w_o = f(r)$$

Applying the boundary conditions (4) the result is

$$w_o = 0 \qquad (7)$$

This is an interesting result. In most analyses, it is assumed a priori that the terms of the continuity equation are of the same order. In the above analysis, this assumption was not made and the result is that the axial velocity is not of the same order as the other two velocity components; that is, while the radial and tangential velocities are of unit order, the axial velocity is of the order b/r_o .

The first order continuity equation is, from equation (6d),

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} u_o) + \frac{\partial w_1}{\partial \bar{z}} = 0$$

Note that now both terms of the continuity equation are of the same order.

It is physically reasonable that the axial velocity be small since it is prescribed zero on all boundaries where velocities are stipulated and since it has no primary driving force such as an imposed axial pressure gradient. This fact undoubtedly led Vogelpohl to ignore all terms involving the axial velocity or its derivatives. Such an assumption does not allow the buildup of radial flow within the boundary layer due to the imbalanced pressure gradient and is therefore overly restrictive. The present analysis will retain terms containing the axial velocity where they are the same order as other terms. In this way the redistribution of radial flow may be considered.

Equation (7) can now be used to simplify equations (6a, b, c). Note that the zeroth order inertia terms of equations (6a, b) disappear with the introduction of equation (7). For simplicity the higher order terms on each side of equations (6a, b, c) will be dropped. The equations now become

$$u_o \frac{\partial u_o}{\partial \bar{r}} + w_1 \frac{\partial u_o}{\partial \bar{z}} - \frac{v_o^2}{\bar{r}} + \frac{\partial P_o}{\partial \bar{r}} = \frac{1}{Re}\left(\frac{r_o}{b}\right)^2 \frac{\partial^2 u_o}{\partial \bar{z}^2} \qquad \text{a}$$

$$u_o \frac{\partial v_o}{\partial \bar{r}} + w_1 \frac{\partial v_o}{\partial \bar{z}} + \frac{u_o v_o}{\bar{r}} = \frac{1}{Re}\left(\frac{r_o}{b}\right)^2 \frac{\partial^2 v_o}{\partial \bar{z}^2} \qquad \text{b}$$

$$\frac{\partial P_o}{\partial \bar{z}} = \frac{1}{Re}\frac{\partial^2 w_1}{\partial \bar{z}^2} \qquad \text{c} \qquad (8)$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r} u_o) + \frac{\partial w_1}{\partial \bar{z}} = 0 \qquad \text{d}$$

The fact that equations (8) do not contain the second order radial derivative terms is consistent with the neglect of the radial boundary layers discussed on page 6.

The set of equations (8) would be identical with those derived by Vogelpohl if the terms containing w_1 were omitted. However, the ordering carried out above indicates that these terms should not be omitted; therefore, the Vogelpohl equations do not apply to the case of b/r_o small.

The case of b/r_o large will now be briefly investigated to see if this ordering yields the Vogelpohl equations. Assuming r_o/b small, the velocities and pressure may be expanded in powers of r_o/b and substituted into equations (3). Upon ordering with respect to r_o/b, the zeroth order equations are

$$u_o \frac{\partial u_o}{\partial \bar{r}} - \frac{v_o^2}{\bar{r}} = -\frac{\partial P_o}{\partial \bar{r}} + \frac{1}{Re}\left[\frac{\partial^2 u_o}{\partial \bar{r}^2} + \frac{\partial}{\partial \bar{r}}\left(\frac{u_o}{\bar{r}}\right)\right] \qquad a$$

$$u_o \frac{\partial v_o}{\partial \bar{r}} + \frac{u_o v_o}{\bar{r}} = \frac{1}{Re}\left[\frac{\partial^2 v_o}{\partial \bar{r}^2} + \frac{\partial}{\partial \bar{r}}\left(\frac{v_o}{\bar{r}}\right)\right] \qquad b$$

$$u_o \frac{\partial w_o}{\partial \bar{r}} = \frac{1}{Re}\left[\frac{\partial^2 w_o}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial w_o}{\partial \bar{r}}\right] \qquad c \qquad (9)$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r} u_o) = 0 \qquad d$$

These equations are identical to those which Rosenzweig, Lewellen and Ross (7) apply to their central region (Region I of figure 1 in Ref. 7).

The inertia terms in equations (9a, b) now are identical with the inertia terms in the Vogelpohl equations but the viscous terms which appear above are the radial derivatives rather than the axial derivatives. Equations (9) hold in the central region between the two plates but not immediately adjacent to them because the highest order axial derivatives have been omitted. In order to satisfy the boundary conditions on the plates, the second order axial derivatives must be taken into account. Introducing the boundary layer transformation

$$\bar{z} = \frac{r_o}{b} Re^{-1/2} \bar{\bar{z}} \qquad \bar{w} = Re^{-1/2} \bar{\bar{w}}$$

into equations (3), together with the ordering $b/r_o \gg 1$, gives the familiar boundary layer equations with axial rather than radial highest derivatives. But in this case, the inertia terms containing the axial velocity reappear. Thus, the equations derived by Vogelpohl do not accurately govern any phase of rotating flow between two flat plates.

By ordering the Reynolds number with respect to r_o/b for $r_o/b \gg 1$ in equations (8), various types of flow are found in the region of interest.

CASE A) Assume

$$Re \ll (r_o/b)^2 \gg 1$$

In this case, the viscous terms dominate and a Stokes type of flow results from equations (8). The inertia terms may be neglected but the pressure term must be retained since it is the driving force. Equations (8) reduce to

$$\frac{\partial P_o}{\partial \bar{r}} = \frac{\partial^2 u_o}{\partial \bar{z}^2} \qquad \text{a}$$

$$0 = \frac{\partial^2 v_o}{\partial \bar{z}^2} \qquad \text{b}$$

$$(10)$$

$$\frac{\partial P}{\partial \bar{z}} = 0 \qquad \text{c}$$

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} u_o) + \frac{\partial w_1}{\partial \bar{z}} = 0 \qquad \text{d}$$

Employing the boundary conditions on the two plates, the solutions for the velocities are

$$u_o = \frac{C}{\bar{r}} (\bar{z} - \bar{z}^2) \qquad \text{a}$$

$$v_o = 0 \qquad \text{b}$$

$$(11)$$

$$w_1 = 0 \qquad \text{c}$$

where the constant C must be adjusted so that the mass flow at any radial station equals that at the outer cylinder $(\bar{V} = 1)$. That is,

$$C = 6k$$

Equations (11a, b) do not satisfy the boundary conditions at $r = 1$. There exists a radial boundary layer at $r = 1$. Introducing the transformation

$$\bar{r} = b/r_o \bar{\bar{r}}$$

into equations (3) and realizing that the inertia terms are negligible and that the pressure term is of order $\left(\frac{r_o}{b}\right)^2 \frac{1}{Re}$ gives the radial boundary layer equations

$$\frac{\partial^2 u}{\partial \bar{\bar{r}}^2} + \frac{\partial}{\partial \bar{\bar{r}}}\left(\frac{u}{\bar{\bar{r}}}\right) + \frac{\partial^2 u}{\partial \bar{\bar{z}}^2} = 0 \qquad \text{a}$$

(12)

$$\frac{\partial^2 v}{\partial \bar{\bar{r}}^2} + \frac{\partial}{\partial \bar{\bar{r}}}\left(\frac{v}{\bar{\bar{r}}}\right) + \frac{\partial^2 v}{\partial \bar{\bar{z}}^2} = 0 \qquad \text{b}$$

The solution of these equations will complete the solution for the case of $Re \ll \left(\frac{r_o}{b}\right)^2 \gg 1$. This solution will not be pursued further since it is not related to the study of the boundary layer interactions.

CASE B) Assume

$$Re = (r_o/b)^2 \gg 1$$

In this case the viscous terms just balance the inertia terms and viscous flow fills the region of interest; that is, there is tangential flow throughout the region but there is no central core of inviscid flow. The equations (8) become

$$u_o \frac{\partial u_o}{\partial \bar{r}} + w_1 \frac{\partial u_o}{\partial \bar{z}} - \frac{v_o^2}{\bar{r}} + \frac{\partial P_o}{\partial \bar{r}} = \frac{\partial^2 u_o}{\partial \bar{z}^2} \qquad \text{a}$$

(13)

$$u_o \frac{\partial v_o}{\partial \bar{r}} + w_1 \frac{\partial v_o}{\partial \bar{z}} + \frac{u_o v_o}{\bar{r}} = \frac{\partial^2 v_o}{\partial \bar{z}^2} \qquad \text{b}$$

$$\frac{\partial P_o}{\partial \bar{z}} = 0 \qquad \text{c}$$

(13)

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} u_o) + \frac{\partial w_1}{\partial \bar{z}} = 0 \qquad \text{d}$$

and the boundary conditions remain (4).

CASE C) Assume

$$Re \gg \left(\frac{r_o}{b} \right)^2 \gg 1$$

In this case the inertia terms dominate the viscous terms except in the boundary layer regions near the plates. That is, equations (8), with the second derivatives with respect to z omitted, govern the flow in the central region between the two plates. The flow in the two boundary layer regions lying between this central region and the two plates is governed by the full equations (8) since the viscous terms must be employed to satisfy the boundary conditions on the plates.

The conventional method of solving such a set of equations is to calculate the inviscid solution in the entire region of interest; then expand the coordinate normal to the boundary to arrive at the boundary layer equations which are applied to a thin region near the boundary. The transformed normal coordinate is allowed to tend to infinity. The inviscid solution, evaluated at the boundary is then matched with the boundary layer solution evaluated at infinity.

In such an analysis it is implicitly assumed that the boundary layer does not have a first order (i.e., appreciable) effect on the inviscid solution.

This classical analysis works well for flow over a single plate since there is no conservation of radial mass flow to be satisfied. Any radial flow induced in the boundary layer by the imbalanced pressure gradient will be compensated by a second order (i.e., small) axial flow in the inviscid region since it is assumed that there is no radial velocity outside the boundary layer. However, in the case of flow between two plates, any mass flow induced in the boundary layer must be compensated by a corresponding decrease in the radial velocity in the inviscid region since an axial flow is blocked by the presence of the other plate. Therefore, the boundary layer can have a first order effect on the inviscid flow even though the boundary layer remains relatively thin. Since, in the two plate analysis, the effect of the boundary layer on the inviscid solution must be taken into account, the classical boundary layer analysis cannot be accurately used.

It may be noted here that the presence of the second plate introduces a modified Reynolds number; where Re appeared in the one plate analysis now $Re\left(\frac{b}{r_o}\right)^2$ appears. The boundary layer thickness is now of order $\sqrt{\frac{r_o^2}{Reb^2}}$ rather than $\sqrt{\frac{1}{Re}}$.

As the problem stands at the moment, equations (8) with boundary conditions (4) govern the first order (in b/r_o) flow between two flat plates for $Re \geq \left(\frac{r_o}{b}\right)^2 \gg 1$. A transformation is possible which will remove the dimensionless group from the equations and introduce it into the boundary conditions. This transformation will be performed since the equations are more amenable to analysis and solution in this form.

Introducing the transformation

$$\bar{z} = \sqrt{\frac{r_o^2}{Reb^2}}\ \bar{\bar{z}} \qquad \text{a}$$

(14)

$$w_1 = \sqrt{\frac{r_o^2}{Reb^2}}\ \bar{\bar{w}} \qquad \text{b}$$

into equations (8) gives

$$u_o \frac{\partial u_o}{\partial \bar{r}} + \bar{\bar{w}} \frac{\partial u_o}{\partial \bar{\bar{z}}} - \frac{v_o^2}{\bar{r}} + \frac{\partial P_o}{\partial \bar{r}} = \frac{\partial^2 u_o}{\partial \bar{\bar{z}}^2}$$

$$u_o \frac{\partial v_o}{\partial \bar{r}} + \bar{\bar{w}} \frac{\partial v_o}{\partial \bar{\bar{z}}} + \frac{u_o v_o}{\bar{r}} = \frac{\partial^2 v_o}{\partial \bar{\bar{z}}^2}$$

$$\frac{\partial P_o}{\partial \bar{\bar{z}}} = 0$$

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} u_o) + \frac{\partial \bar{\bar{w}}}{\partial \bar{\bar{z}}} = 0$$

or, dropping the bars and subscripts,

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = - \frac{dP}{dr} + \frac{\partial^2 u}{\partial z^2} \qquad \text{a}$$

$$u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{\partial^2 v}{\partial z^2} \qquad \text{b} \qquad (15)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0 \qquad \text{c}$$

The boundary conditions (4) now are

$$\text{At} \quad z = 0 \qquad u = v = w = 0$$

$$\text{At} \quad z = \sqrt{\frac{Reb^2}{r_o^2}} \qquad u = v = w = 0 \qquad (16)$$

Returning to equations (6) it is easily seen that the second order equations are

$$u_1 \frac{\partial u_o}{\partial \bar{r}} + u_o \frac{\partial u_1}{\partial \bar{r}} + w_2 \frac{\partial u_o}{\partial \bar{z}} + w_1 \frac{\partial u_1}{\partial \bar{z}} - 2 \frac{v_o v_1}{\bar{r}} = - \frac{\partial P_1}{\partial \bar{r}} + \frac{r_o^2}{b^2 Re} \frac{\partial^2 u_1}{\partial \bar{z}^2} \qquad \text{a}$$

$$u_1 \frac{\partial v_o}{\partial \bar{r}} + u_o \frac{\partial v_1}{\partial \bar{r}} + w_2 \frac{\partial v_o}{\partial \bar{z}} + w_1 \frac{\partial v_1}{\partial \bar{z}} + \frac{u_1 v_o}{\bar{r}} + \frac{u_o v_1}{\bar{r}} = \frac{r_o^2}{b^2 Re} \frac{\partial^2 v_1}{\partial \bar{z}^2} \qquad \text{b}$$

$$(17)$$

$$\frac{\partial P_1}{\partial \bar{z}} = 0 \qquad \text{c}$$

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} u_1) + \frac{\partial w_2}{\partial \bar{z}} = 0 \qquad \text{d}$$

All the higher order equations are linear and once the first order non-linear equations are solved, all higher order velocity terms can be evaluated.

Equations (15) are identical to the equations governing the flow in the single plate problem. However, there is a fundamental difference between the boundary conditions in the single plate problem and those in the two plate problem. In the single plate problem, u and v must asymptotically approach the inviscid values while the two plate problem u, v and w become exactly zero at a given value of z due to the presence of the second plate.

It is convenient to apply equations (15) to the entire flow field rather than to divide up the flow field as is done in standard boundary layer theory. This formulation avoids the need to match solutions at the edge of the boundary layer. Also it will be shown later that the presence of the boundary layer has a first order effect on the inviscid solution.

II. A Search for a Similarity

To determine whether there exists a similarity transformation for the problem, a fairly general transformation is used.

Assume a similarity of the following form

$$u = f_1(r)G_1(x) \quad \text{a}$$

$$v = f_2(r)G_2(x) \quad \text{b}$$

$$w = f_3(r)G_3(x) \quad \text{c} \qquad (18)$$

$$x = \frac{z}{h(r)} \quad \text{d}$$

Introducing equations (18) into equations (15) gives

$$f_1 f_1' G_1^2 - f_1^2 \frac{h'}{h} G_1 G_1' x + f_3 \frac{f_1}{h} G_3 G_1' - \frac{f_2^2}{r} G_2^2 = -\frac{dP}{dr} + \frac{f_1}{h^2} G_1'' \qquad \text{a}$$

$$f_1 f_2' G_1 G_2 - f_1 f_2 \frac{h'}{h} G_1 G_2' x + f_2 \frac{f_3}{h} G_3 G_2' + f_1 \frac{f_2}{r} G_1 G_2 = \frac{f_2}{h^2} G_2'' \qquad \text{b} \qquad (19)$$

$$\left(f_1' + \frac{f_1}{r} \right) G_1 - f_1 \frac{h'}{h} G_1' x + \frac{f_3}{h} G_3' = 0 \qquad \text{c}$$

where the prime denotes differentiation.

For a similarity to exist, the function of r in any term of a given equation must be the same, within a multiplicative constant, as the r function in any other term of that equation. This requirement gives rise to the so-called compatibility equations.

First, the compatibility equations for f_1 and h will be analyzed. From equations (19a, c) it is easily seen that the compatibility equations for f_1 and h are

$$f_1 f_1' = k_1 f_1^2 \frac{h'}{h} \qquad \text{a}$$

$$f_1 f_1' = k_2 \frac{f_1}{h^2} \qquad \text{b} \qquad (20)$$

$$f_1' + \frac{f_1}{r} = k_3 f_1 \frac{h'}{h} \qquad \text{c}$$

These are three equations for two unknowns. In general such a set of equations does not yield a non-zero solution. However, this set may be solved if a condition is placed on the constants k_1 and k_3. First equations (20a, c) will be solved for f_1 and h. Then these solutions will be substituted back into equation (20b) in an attempt to satisfy that equation also. The most general functions which will solve (20a, c) are

$$f_1 = k_4 r^{k_1/k_3 - k_1} \qquad \text{a}$$

$$h = k_5 r^{1/k_3 - k_1} \qquad \text{b} \qquad (21)$$

If $k_1 = k_3$, the exponents of the radius become infinity in equations (21). Since these equations only have meaning for $r < 1$ and since $r^{\infty} = 0$ for $r < 1$, the functions f_1 and h become zero for $k_1 = k_3$.

Substitution of equations (21) into equation (19b) shows that equation (19) is satisfied if

$$k_3 - k_1 = k_1 + 2$$

$$k_5 = \left[\frac{k_2(k_3 - k_1)}{k_4 k_1}\right]^{1/2}$$

Substituting equations (21) into (19) and solving for f_2 and f_3 gives

$$h = k_5 r^{1/k_1 + 2} \qquad \text{a}$$

$$f_1 = k_4 r^{k_1/k_1 + 2} \qquad \text{b}$$

$$f_2 = k_6 r^{k_1/k_1 + 2} \qquad \text{c} \qquad (22)$$

$$f_3 = k_7 r^{-1/k_1 + 2} \qquad \text{d}$$

The similarity still holds if k_4, k_5, k_6, and k_7 are set equal to unity for convenience.

Changing notation $(k_1/k_1 + 2 = n)$, equations (18) now are

$$u = r^n G_1(z/r^{1 - n/2}) \qquad \text{a}$$

$$v = r^n G_2(z/r^{1 - n/2}) \qquad \text{b} \qquad (23)$$

$$w = r^{n - 1/2} G_3(z/r^{1 - n/2}) \qquad \text{c}$$

This is a known result; it is just the similarity used successfully by Lewellen and King (5) to reduce a similar set of equations for a single plate. The above analysis was performed to show that this similarity is the most general one that will transform the equations (15) into ordinary differential equations. Before a similarity transformation is successful, it must also transform the boundary conditions of a given problem. That is, the boundary conditions in the transformed problem must be applied at constant values of the similarity variable. There is no trouble with the boundary conditions at $z = 0$, it is transformed into one at $x = 0$. The trouble lies with the upper boundary condition.

In the case of the single plate, the upper limit is removed to $z = \infty$ which transforms into $x = \infty$. If the same technique is tried in the two plate problem, namely removing the upper limit to infinity, the effect of the upper plate is lost. This defeats the purpose of the analysis. Therefore the upper boundary condition must be applied at a finite value of z. Since $x = \frac{z}{r^{1-n/2}}$, z = constant transforms into x = constant only for a particular value of n, $n = 1$.

The particular flow pattern for $n = 1$ is modified after wheel flow. The velocities u and v have z-dependent profiles which are magnified by a multiplicative factor r while the profile of w is independent of the radius. This is the same similarity used by von Kármán to reduce the equations in his classic rotating plate problem. This flow pattern occurs only if there is no net radial mass flow. This can be shown in the following way. The net radial mass flow, Q, is given by the equation

$$Q = 2\pi \rho r_o b V_o \int_0^{\sqrt{Re\, b^2/r_o^2}} r u dz \qquad (24)$$

Since mass is neither added to nor subtracted from the flow at a general radius, r, the net mass flow cannot be a function of r; Q is a constant. With $n = 1$, equation (23a) becomes

$$u = r\, G(z)$$

and equation (24) can be written

$$r^2 \int_0^{\sqrt{Re\ b^2/r_o^2}} G(z)dz = \text{const.}$$

The only way this relation can hold for any radius is for both the value of the integral and the value of the constant to be zero. But this means that the net radial mass flow is zero.

Ruling out this unique case, which does not produce the desired vortex motion, it is seen that the similarity fails to transform the boundary conditions properly for the two plate problem. Therefore, no similarity exists for the two plate problem.

III. A Momentum Integral Solution

It is of interest to calculate some of the effects which the two flat plates produce on the flow. These include the boundary layer blockage caused by the buildup of the boundary layers on the plates and the variation of the radial velocity with the radius.

These effects may be calculated by a momentum integral solution of the problem posed by equations (15) and (16).

The differential equations to be approximated are

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{dP}{dr} + \frac{\partial^2 u}{\partial z^2} \qquad \text{a}$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{\partial^2 v}{\partial z^2} \qquad \text{b} \qquad (15)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \qquad \text{c}$$

The complete boundary conditions are

$$\begin{aligned}
&\text{At} \quad z = 0 && u = v = w = 0 && \text{a} \\
&\text{At} \quad z = \sqrt{Re\, b^2/r_o^2} && u = v = w = 0 && \text{b} \\
&\text{At} \quad z = 1/2\sqrt{Re\, b^2/r_o^2} && w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 && \text{c} \\
&\text{At} \quad r = 1 && u = k < 0 \qquad v = 1 \qquad w = 0 && \text{d}
\end{aligned} \tag{25}$$

The differential equations are valid for $Re \geqslant \left(\frac{r_o}{b}\right)^2 \gg 1$. The conditions (25a, b) are a reiteration of conditions (16) where $z = \sqrt{Re\, b^2/r_o^2}$ describes the upper plate. Condition (25c) is applied at the midplane between the two plates and expresses the conditions of symmetry existing between the two plates. Condition (25d) is the velocity state prescribed at the outer edge of the plates. It is assumed that the fluid is injected at $r = 1$ such that the axial velocity is zero. The tangential velocity is chosen equal to unity at $r = 1$ in agreement with the original nondimensionalization. The radial velocity is chosen as some negative constant since it is assumed to be negative everywhere in the region of interest. This negative constant is related to the net radial mass flow, Q by equation (24); that is,

$$Q = 2\pi\rho r_o b V_o \sqrt{Re\, b^2/r_o^2}\; k < 0$$

or, introducing a dimensionless mass flow,

$$\overline{Q} = Q/2\pi\rho r_o b V_o = k\sqrt{Re\ b^2/r_o^{\ 2}} < 0 \qquad (26)$$

In the following analysis, only the region from $z = 0$ to $z = 1/2\sqrt{Re\ b^2/r_o^{\ 2}}$ will be considered and the symmetry conditions at the latter boundary will be employed. Because of the symmetry, this is equivalent to considering the entire region between the two plates. For brevity, let $M = 1/2\sqrt{Re\ b^2/r_o^{\ 2}}$, thus $z = M$ is the midplane between the two plates.

Now to derive the integral equations: As a preliminary step, multiply the continuity equation (15c) by the radial velocity component u and integrate the result with respect to z from zero to M. Applying the boundary conditions

$$w(0) = w(M) = 0$$

and realizing that

$$\frac{\partial M}{\partial r} = 0$$

gives the result

$$\int_0^M \left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} \right) dz = \frac{1}{r}\frac{d}{dr}\int_0^M ru^2 dz \qquad (27a)$$

Now multiply the continuity equation by the tangential velocity v and again integrate from zero to M. This gives

$$\int_0^M \left(u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} \right) dz = \frac{1}{r}\frac{d}{dr}\int_0^M ruv dz \qquad (27b)$$

These equations (27) will be used to simplify the integrals of the momentum equations.

Integrate the radial momentum equation (15a) with respect to z from zero to M. Simplifying the result by using equation (27a) and the symmetry condition $\frac{\partial u(M)}{\partial z} = 0$ gives

$$\frac{1}{r}\frac{d}{dr}\int_0^M ru^2 dz + \int_0^M \left(\frac{dP}{dr} - \frac{v^2}{r}\right) dz = -\left.\frac{\partial u}{\partial z}\right|_{z=0} \tag{28}$$

Integrate the tangential momentum equation (15b) from zero to M. Simplifying the result with equation (27b) and the symmetry condition $\frac{\partial v(M)}{\partial z} = 0$ gives

$$\frac{1}{r^2}\frac{d}{dr}\int_0^M r^2 uv dz = -\left.\frac{\partial v}{\partial z}\right|_{z=0} \tag{29}$$

An additional condition appearing in the two-plate problem which does not appear in the single plate problem is the conservation of radial mass flow. A combination of equations (24) and (26) gives

$$r\int_0^M u dz = kM < 0 \tag{30}$$

Equations (28), (29) and (30) are equations for the radial variations of the velocities u and v and the pressure p.

It is assumed that the Reynolds number is sufficiently large such that the viscous effects of the plates do not reach the midplane between the two plates. That is, the boundary layer thickness, $\delta(r)$, is less than the value of the midplane M everywhere in the region of interest.

It was shown in section II that a true similarity does not exist for the problem. However, it is known from single plate analyses (3) that the profiles of u and v are similar with respect to the parameter z/δ (r) for $z < \delta(r)$. It is expected that the presence of the second plate will not affect this local similarity, therefore it is assumed that the profiles of u and v are similar with respect to the parameter $z/\delta(r)$ for $z < \delta(r)$. Also it is assumed that u and v are independent of z for $z > \delta$.

The value of δ is assumed to be zero at $r = 1$. This means that any boundary layer on the outer cylinder is ignored.

It is convenient to introduce the transformation

$$\zeta = z/\delta \qquad\qquad dz = \delta\, d\zeta$$

The upper limit of integration becomes M/δ and equations (28), (29), and (30) become

$$\frac{1}{r}\frac{d}{dr}\left[r\delta\int_0^{M/\delta} u^2 d\zeta\right] + \int_0^{M/\delta}\left(\frac{dP}{dr} - \frac{v^2}{r}\right)d\zeta = -\frac{1}{\delta}\frac{\partial u}{\partial\zeta}\bigg|_0 \tag{31}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\delta\int_0^{M/\delta} uv\,d\zeta\right] = -\frac{1}{\delta}\frac{\partial v}{\partial\zeta}\bigg|_0 \tag{32}$$

$$r\delta\int_0^{M/\delta} u\,d\zeta = kM \tag{33}$$

At this point in a momentum integral analysis of the single plate problem it is usually assumed that (3)

$$u = g_1(r)h_1(\zeta)$$

$$v = g_2(r)h_2(\zeta)$$

Such a form for the radial velocity u is acceptable in the single plate analysis since it is assumed that the radial velocity is identically zero outside the boundary layer; the continuity equation is satisfied by an axial flow from infinity. This cannot be done in the two plate problem. The radial velocity must have a non-zero value outside the boundary layer. The radial velocity also may overshoot in the boundary layer. This overshoot is caused by the fact that radial mass flow is drawn into the boundary layer by the imbalanced pressure gradient in the boundary layer. There is no guarantee that the ratio of the maximum value of the radial velocity in the boundary layer to the value outside the boundary layer is a constant independent of the radius. Therefore, the radial velocity at a given cross-section may be divided into two parts; the first part consists of the portion of u that is non-zero outside of the boundary layer and does not have an overshoot in the boundary layer, the second part consists of the remainder of u which is zero outside of the boundary layer and is a measure of the overshoot.

Thus it is assumed that

$$u = g_1(r)h_1(\zeta) + g_2(r)h_2(\zeta) \quad \text{a}$$

$$(34)$$

$$v = g_3(r)h_3(\zeta) \quad \text{b}$$

where $\delta < M$ throughout the region of interest. The profiles h_1, h_2, and h_3 are assumed to be known; their forms are determined in the appendix. Equations (31), (32), and (33) become

$$\frac{1}{r}\frac{d}{dr}\left[r\,\delta g_1^2\int_0^{M/\delta} h_1^2\,d\zeta + 2r\,\delta g_1 g_2\int_0^{M/\delta} h_1 h_2\,d\zeta + r\,\delta g_2^2\int_0^{M/\delta} h_2^2\,d\zeta\right]$$

$$+ M\frac{dP}{dr} - \frac{\delta {g_3}^2}{r}\int_0^{M/\delta} h_3^2\,d\zeta = -\frac{1}{\delta}\left[h_1'(0)g_1 + h_2'(0)g_2\right] \qquad (35)$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\delta g_1 g_3\int_0^{M/\delta} h_1 h_3\,d\zeta + r^2\delta g_2 g_3\int_0^{M/\delta} h_2 h_3\,d\zeta\right] = -\frac{g_3}{\delta}h_3'(0) \qquad (36)$$

$$r\,\delta g_1\int_0^{M/\delta} h_1\,d\zeta + r\,\delta g_2\int_0^{M/\delta} h_2\,d\zeta = kM \qquad (37)$$

It should be noted here that the upper limit of integration is a function of r through $\delta(r)$. It is just this type of radial dependence that foiled the attempt to find a proper similarity transformation for the two plate problem in section II. Since the upper limit of integration is a function of the radius, the integrals are not independent of the radius. This difficulty can be circumvented by splitting the integral into two parts. The first part is an integration from the lower plate to the edge of the boundary layer (from $\zeta = 0$ to $\zeta = 1$); this integral is independent of the radius. The second part is an integration from $\zeta = 1$ to $\zeta = M/\delta$. In this region it is assumed that $h_1 = h_3 = 1$

and $h_2 = 0$. This means that u and v are functions only of the radius in this region. Thus the second integral is a linear function of M/δ. Equations (35), (36), and (37) become

$$\frac{d}{dr}\left\{rg_1^2\left[M + (C_{11} - 1)\delta\right] + 2C_{12}r\delta g_1g_2 + r\delta g_2^2 C_{22}\right\} + Mr\frac{dP}{dr}$$
$$- g_3^2\left[M + (C_{33} - 1)\delta\right] = -\frac{r}{\delta}\left[h_1'(0)g_1 + h_2'(0)g_2\right] \quad (38)$$

$$\frac{d}{dr}\left\{r^2g_1g_2\left[M + (C_{13} - 1)\delta\right] + C_{23}r^2\delta g_2g_3\right\} = -\frac{g_3r^2}{\delta}h_3'(0) \quad (39)$$

$$rg_1\left[M + (C_1 - 1)\delta\right] + C_2 r\delta g_2 = kM \quad (40)$$

where

$$C_{11} = \int_0^1 h_1^2\, d\zeta \quad \text{a} \qquad C_{12} = \int_0^1 h_1h_2\, d\zeta \quad \text{e}$$

$$C_{22} = \int_0^1 h_2^2\, d\zeta \quad \text{b} \qquad C_{33} = \int_0^1 h_3^2\, d\zeta \quad \text{f}$$

(41)

$$C_{13} = \int_0^1 h_1h_3\, d\zeta \quad \text{c} \qquad C_{23} = \int_0^1 h_2h_3\, d\zeta \quad \text{g}$$

$$C_1 = \int_0^1 h_1\, d\zeta \quad \text{d} \qquad C_2 = \int_0^1 h_2\, d\zeta \quad \text{h}$$

The constants C are evaluated in the appendix.

The set of equations (38), (39), and (40) are three equations for five unknowns g_1, g_2, g_3, $\frac{dP}{dr}$, and δ. Another two relationships between these unknowns must be found. A number of equations can be found by evaluation of equations (15) at the lower plate $(z = 0)$ and at the midplane $(z = M)$. Expressing u and v by equations (34), evaluation of equations (15a, b) at $z = 0$ gives

$$\frac{dP}{dr} = \frac{1}{\delta^2}\left(g_1 h_1''(0) + g_2 h_2''(0) \right) \qquad \text{a}$$

(42)

$$0 = g_3 h_3''(0)/\delta^2 \qquad \text{b}$$

and evaluation of equations (15a, b, c) at $z = M$ gives

$$g_1 g_1' - g_3^2/r = -\frac{dP}{dr} \qquad \text{c}$$

$$g_1 \frac{d}{dr}(r g_3) = 0 \qquad \text{d}$$

(42)

$$\frac{d}{dr}(r g_1) = 0 \qquad \text{e}$$

The evaluation of equation (15c) at $z = 0$ was omitted since nothing is gained by the reintroduction of w. Equations (42) are written with the assumption that $\delta < M$, therefore the viscous terms are absent from the right hand side of equations (42c, d). Equation (42b) is merely a condition upon the assumed form of h_3. Equations (42d, e) may be directly integrated to yield

$$g_3 = \frac{1}{r} \qquad g_1 = \frac{k}{r} \qquad (43a,b)$$

where the boundary condition (25d) has been applied. Note that g_3 represents the radial variation of the tangential velocity and g_1 represents the radial variation of the radial velocity outside the boundary layer. It must now be decided whether equations (43a, b) can be used as additional equations or must be rejected. Both of these equations are written ignoring the effect of the boundary layers. It must be decided whether the velocities at the midplane are affected by the presence of the boundary layers or not.

The tangential velocity together with the prescribed radial velocity toward the axis of rotation is the driving force of the problem. The only way that the tangential velocity can be reduced from its free stream value is by the action of viscous forces. By definition, δ marks the upper limit of the action of the viscous forces. Since it is assumed that $\delta < M$, the tangential velocity at the midplane is not significantly affected by the presence of the boundary layers. Therefore equation (43a) which represents the radial variation of the tangential velocity can be used as an additional relationship.

On the other hand, it was stated in the introduction that there are strong radial velocities in the boundary layer. By the conservation of radial mass flow, this requires a corresponding decrease in the radial velocity outside the boundary layer. Thus the presence of the boundary layer affects the radial velocity even in the region outside the boundary layer. Therefore, equation (43b) which gives the radial variation of the radial velocity external to the boundary layer is inaccurate with boundary layers present and will be ignored.

In order to make the set of equations (38), (39), (40) and (43a) complete, one more relationship must be found between the unknowns. Either equation (42a), (42c) or a combination of the two will give the additional relationship. However, the resulting set of equations is quite cumbersome and not amenable to solution. Since the momentum integral solution is an approximate solution, it is not out of order to introduce a simplifying assumption which will make the equations much more amenable to solution. This assumption is

$$g_2 = \overline{k}_1 g_1 \tag{44}$$

where $\overline{k}_1$ is independent of the radius but depends on M and k. This is equivalent to assuming that the ratio of the maximum value of the radial velocity in the boundary layer to the value outside the boundary layer is a constant independent of the radius. This assumption restricts the form of the velocity overshoot in the boundary layer but still allows one to gain a measure of the influence of the boundary layer on the outer flow. The assumption allows an exact closed-form solution to be obtained.

Substitution of equations (43a) and (44) into equations (38), (39), and (40) gives

$$\frac{d}{dr}\left\{rg_1^2\left[M + (C_{11} + 2\overline{k}_1 C_{12} + \overline{k}_1^{\,2} C_{22} - 1)\delta\right]\right\} + Mr\frac{dP}{dr} \tag{45}$$

$$-\frac{1}{r^2}\left[M + (C_{33} - 1)\delta\right] = -\frac{r}{\delta} g_1\left[h_1'(0) + \overline{k}_1 h_2'(0)\right]$$

$$\frac{d}{dr}\left\{rg_1\left[M + (C_{13} + \bar{k}_1\, C_{23} - 1)\,\delta\right]\right\} = -\frac{r}{\delta}\, h_3'(0) \tag{46}$$

$$rg_1\left[M + (C_1 + \bar{k}_1\, C_2 - 1)\,\delta\right] = kM \tag{47}$$

Equation (47) may be solved for g_1 in terms of δ. Substitution of that result into equation (46) gives

$$\frac{\delta\delta'}{\left[M + (C_1 + \bar{k}_1\, C_2 - 1)\,\delta\right]^2} = -\frac{h_3'(0)r}{kM^2\,(C_{13} - C_1 + \bar{k}_1\, C_{23} - \bar{k}_1\, C_2)} \tag{48}$$

where the prime dentoes differentiation.

Direct integration of this equation yields the expression for the boundary layer thickness,

$$\log\left[1 + \frac{(C_1 + \bar{k}_1\, C_2 - 1)}{M}\,\delta\right] - \frac{(C_1 + \bar{k}_1\, C_2 - 1)\delta}{M + (C_1 + \bar{k}_1\, C_2 - 1)\delta}$$

$$= \frac{(C_1 + \bar{k}_1\, C_2 - 1)^2\, h_3'(0)\,(1 - r^2)}{2kM^2(C_{13} - C_1 + \bar{k}_1\, C_{23} - \bar{k}_1\, C_2)} \tag{49}$$

where the boundary condition $\delta(1) = 0$ has been applied.

To study the character of this equation note that k is a measure of the mass flow and is negative. If it is zero the solution is meaningless since the analysis leading to the solution is invalid. It is shown in the appendix that

$$C_1 + \bar{k}_1\, C_2 - 1 > 0 \qquad \text{and} \qquad C_{13} - C_1 + \bar{k}_1\, C_{23} - \bar{k}_1\, C_2 < 0$$

Therefore the solution (49) does not exhibit singularities.

It is possible to determine the nature of $\delta(r)$ without an explicit solution. Assume that δ has a maximum at some $r \neq 0$, ignoring for the moment the fact that the equations are not valid for $r < \epsilon$. If equation (48) is to hold then the denominator of the left hand side, $M + (C_1 + \bar{k}_1 C_2 - 1)\delta$, must be equal to zero. Solving this for δ gives $\delta_{max} = - M/C_1 + \bar{k}_1 C_2 - 1$. This states that $\delta_{max} < 0$, but δ is a physical quantity always greater than zero. Therefore the leading assumption is invalid and δ has its maximum at $r = 0$.

Equation (48) asserts that δ' is zero at $r = 0$, infinite for $r = 1$ (because $\delta = 0$) and negative but finite for $0 < r < 1$. Therefore it follows that $\delta(0)$ is finite.

The maximum value of δ within the region of interest occurs at $r = \epsilon$. For ϵ small, δ_{max} may be approximated by $\delta(0)$.

For convenience introduce the notation

$$\beta = (C_1 + \bar{k}_1 C_2 - 1)\, \delta/M \qquad \text{a}$$

$$A = \frac{2kM^2(C_{13} - C_1 + \bar{k}_1 C_{23} - \bar{k}_1 C_2)}{(C_1 + \bar{k}_1 C_2 - 1)^2\, h_3'(0)} \qquad \text{b} \qquad (50)$$

Note that β is a modified boundary layer thickness and A is a measure of the imposed radial mass flow.

The expression for β and A may be somewhat simplified by introduction of the calculated values of the constants c from the appendix and by noting that $M = 1/2 \sqrt{Re\ b^2/r_o^2}$ and $kM = \bar{Q}$ where $\bar{Q} = Q^*/2\pi\rho r_o b V_o$. Thus

$$\beta = \frac{\bar{k}_1 - 4}{6} \frac{r_o}{b} \frac{1}{\sqrt{Re}} \delta \qquad a$$

(50')

$$A = - \frac{\bar{Q}\sqrt{Re}}{(\bar{k}_1 - 4)^2} \frac{b}{r_o} \frac{266 + 64\,\bar{k}_1}{35} \qquad b$$

Note that $\bar{Q}$ is negative so that A is always positive.

Now equation (49) becomes

$$\log(1 + \beta) - \frac{\beta}{1 + \beta} = \frac{1 - r^2}{A} \qquad (51)$$

This equation is solved graphically in figure 2. This graphical solution may be used to obtain plots of β versus r for various values of A ; these are plotted in figure 3.

As was stated above, the maximum value of β occurs at $r = 0$; β_{max} versus (A) is plotted in figure 4.

Rewriting equation (47) in terms of β gives

$$\frac{g_1}{k} = \frac{1}{r(1 + \beta)} \qquad (52)$$

This relationship is plotted in figure 5 for various values of A.

The radial shear stress at the plate is given by

$$\tau_r = \mu \left. \frac{\partial u^*}{\partial z^*} \right|_{z = 0}$$

where the asterisks indicate dimensional quantities. In terms of the functions g , h and δ this relation is

$$\tau_r = \frac{\mu V_o}{r_o}\sqrt{Re}\left(h_1'(0) + \overline{k}_1\, h_2'(0)\right)\frac{g_1}{\delta} \tag{53a}$$

This relationship is plotted in figure 6 for various values of A .

The tangential shear stress at the plate is given by

$$\tau_\theta = \mu \left.\frac{\partial v^*}{\partial z^*}\right|_{z=0} ,$$

or in terms of g , h and δ it is

$$\tau_\theta = \frac{\mu V_o}{r_o}\sqrt{Re}\; h_3'(0)\,\frac{1}{r\delta} \tag{53b}$$

This relationship is plotted in figure 6 for various values of A .

With δ and g_1 known as functions of r , equation (45) gives $\frac{dP}{dr}$ as a function of r and the various constants appearing in the problem. This equation may be considered as a relationship between the pressure gradient and the net radial mass flow $\overline{Q}$, which is just $2kM$.

So far $\overline{k}_1$ has been an unknown; it may be determined by use of equations (42a, c) combination to get rid of $\frac{dP}{dr}$ gives, noting that $g_3 = 1/r$ and $g_2 = \overline{k}_1 g_1$,

$$\delta^2\left(g_1 g_1' - \frac{1}{r^3}\right) = -\left(g_1 h_1''(0) + \overline{k}_1 g_1 h_2''(0)\right)$$

The term g_1' may be found from equations (52) and (51). The result is

$$(\overline{k}_1 - 4)^2(2\overline{k}_1 + 1) = \frac{72M^2}{r^2 k}\left[\frac{2r^2\beta k^2}{A} - \beta^2(1+\beta) - \frac{\beta^2 k^2}{1+\beta}\right] \tag{53}$$

If the right hand side of equation (53) were independent of r, equation (44) would be exact instead of approximate. Knowing M, k and A, equation (53) may be averaged over r and solved for $\bar{k}_1$.

If $|k| \ll 1$ and $\bar{k}_1 \gg 4.5$ a good approximation is

$$(\bar{k}_1 - 4)^3 = \frac{36M^2\beta^2(1+\beta)}{r^2(-k)} \qquad (54)$$

An alternative method of evaluating $\bar{k}_1$ is by experimentation.

Since $\bar{k}_1$ depends on A (through its dependence on β) and A is a function of $\bar{k}_1$, $\bar{k}_1$ will have to be evaluated by trial and error, knowing M and k.

V.

DISCUSSION OF RESULTS

Figure 2 is the graphical solution of equation (51). It plots β, the modified boundary layer thickness, as a function of $1 - r^2/A$ where the parameter A is a measure of the strength of the imposed radial mass flow.

Figure 3 is obtained directly from figure 2. Figure 3 is a plot of β, the modified boundary layer thickness as a function of the radius for various values of A, the measure of the radial mass flow. This figure shows the actual form that the boundary layer will have on the plate.

Note that the boundary layer thickness is a strong function of the radial mass flow. This functional dependence is shown in figure 4 which plots β_{max}, the maximum value of the modified boundary layer thickness as a function of A, the measure of the radial mass flow. β_{max} is directly a measure of the boundary layer blockage; this figure shows that the amount of boundary layer blockage depends strongly upon the applied radial mass flow.

It has been repeatedly stated in this analysis that the boundary layer draws radial mass flow from the outer flow. This influence of the boundary layers upon the outer flow is depicted in figure 5. It is a plot of g_1/k, the radial velocity, as a function of the radius for various values of A. $A = \infty$ corresponds to a very large imposed radial mass flow such that the boundary layers are very small and have negligible influence on the outer flow. As the value of the radial mass flow is decreased, the boundary layers at a fixed radius grow larger and exert a larger influence upon the outer flow. Note that the value of the radial velocity may remain relatively small over a significant portion of the radius. It is possible that the influence of the boundary layer may be so great as to cause g_1/k to become negative. In this case the vortex motion would break down (this case was not considered in this analysis).

Figure 6 plots the tangential and radial shear stresses as a function of r for various values of A, the measure of the radial mass flow. The shear stresses are both infinite at $r = 1$, due to the singularity which appears in all boundary layer calculations at a sharp leading edge, and at $r = 0$, because the velocities tend to infinity as r tends to zero. The shear stresses have minimum values in the region $.55 < r < .65$ for a large range of A. As A grows large, indicating a large radial mass flow, the boundary layer becomes thin and velocity gradients become large. Therefore τ increases as A increases.

VI.

CONCLUSIONS

It has been shown that the first order equations governing the flow between two flat plates for $b/r_o \ll 1$ are identical to those governing the flow over one plate.

Although the equations are the same for the two problems, the boundary conditions are fundamentally different. The boundary conditions for the two plate problem preclude the use of a similarity transformation.

The momentum integral solution of the two plate problem is basically different from that of the single plate problem because the conservation of radial mass flow is used as one of the governing integral equations in the former.

The solution shows the strong dependence of the boundary layer thickness and the radial velocity on the imposed radial mass flow. This dependence is not brought out in a single plate analysis since radial mass flow is not a governing parameter.

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APPENDIX

The purpose of this appendix is to assume reasonable profiles for the functions h_1, h_2 and h_3 and to calculate the constants C_1, C_2, C_{11}, C_{12}, C_{13}, C_{22}, C_{23} and C_{33}.

The boundary conditions to be satisfied are

$$\begin{aligned}
&\text{At } \zeta = 0: \quad h_1 = h_2 = h_3 = 0 \qquad h_3''(0) = 0 && \text{a} \\
&\text{At } \zeta = 1: \quad h_1 = h_3 = 1 \qquad h_2 = 0 && \text{b} \quad \text{(A1)} \\
&\text{At } \zeta = 1: \quad h_1' = h_2' = h_3' = 0 && \text{c}
\end{aligned}$$

The requirement that $h_3''(0) = 0$ is a direct result of equation (42b) which is an evaluation of the tangential momentum equation at the plate. It has been previously assumed that h_1 and h_3 are identically one and h_2 is identically zero for greater than one. Condition (A1c) is an expression of the requirement that the functions should join smoothly at ζ equal to one. For greater smoothness, higher derivatives may also be set equal to zero at ζ equal to zero.

The profile for h_3, which is associated with the tangential velocity v, is expected to increase monotonically from zero at ζ equal to zero to one at ζ equal to one. The profile for h_1, which is associated with the regular part of the radial velocity u, is expected to vary in a similar manner. The profile for h_2, which is associated with the overshoot of the radial velocity, is expected to increase from zero at ζ equals zero, to reach a maximum for some $0 < \zeta < 1$ and to return to zero at ζ equals one.

In analyses of this type, the function most commonly assumed is a polynomial in the independent variable with coefficients which are determined from the boundary conditions.

It will be assumed that h_1 is a quadratic polynomial and h_2 and h_3 are cubic polynomials since these are the simplest polynomicals which are able to satisfy the necessary conditions:

$$h_1 = a_o + a_1 \zeta + a_2 \zeta^2 \quad \text{a}$$

$$h_2 = b_o + b_1 \zeta + b_2 \zeta^2 + b_3 \zeta^3 \quad \text{b} \qquad \text{(A2)}$$

$$h_3 = c_o + c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3 \quad \text{c}$$

Since the assumed form for h_1 has three unknown coefficients and h_1 must satisfy three boundary conditions it is completely determined:

$$h_1 = \zeta(2 - \zeta) \qquad \text{(A3)}$$

Similarly, h_3 has four unknown coefficients and four boundary conditions to satisfy:

$$h_3 = \frac{\zeta}{2}(3 - \zeta^2) \qquad \text{(A4)}$$

The case for h_2 is not so clear cut. There are only three boundary conditions to be satisfied, but if it is assumed that h_2 is a quadratic polynomial application of the boundary conditions requires that the constants are all zero. Therefore for a non-zero solution, it must be assumed that h_2 is a cubic polynomial. Now three of the constants b can be found in terms of the fourth. There is no loss of generality if this fourth constant is set equal to unity. The profile for h_2 is:

$$h_2 = \zeta(1 - \zeta)^2 \tag{A5}$$

The profiles h_1, h_2 and h_3 are plotted in figure 7.

Now that the profiles are chosen, the constants C are easily calculated.

$$C_{11} = \int_0^1 h_1^2 \, d\zeta = \frac{7}{12} \qquad C_{12} = \int_0^1 h_1 h_2 \, d\zeta = \frac{1}{20}$$

$$C_{22} = \int_0^1 h_2^2 \, d\zeta = \frac{1}{105} \qquad C_{33} = \int_0^1 h_3^2 \, d\zeta = \frac{29}{70}$$

$$C_{13} = \int_0^1 h_1 h_3 \, d\zeta = \frac{61}{120} \qquad C_{23} = \int_0^1 h_2 h_3 \, d\zeta = \frac{19}{420} \tag{A6}$$

$$C_1 = \int_0^1 h_1 \, d\zeta = \frac{2}{3} \qquad C_2 = \int_0^1 h_2 \, d\zeta = \frac{1}{12}$$

Evaluation of the functions h_1, h_2, and h_3 at the plate gives

$$h_1'(0) = 2 \qquad h_1''(0) = -2$$

$$h_2'(0) = 1 \qquad h_2''(0) = -4 \tag{A7}$$

$$h_3'(0) = \frac{3}{2} \qquad h_3''(0) = 0$$

The groups of constants found in equations (45), (46), and (47) become.

$$C_1 + \overline{k}_1 C_2 - 1 = \frac{\overline{k}_1 - 4}{12}$$

$$C_{13} - C_1 + \overline{k}_1 C_{23} - \overline{k}_1 C_2 = -\left(\frac{19}{120} + \frac{4\overline{k}_1}{105} \right)$$

$$C_{11} + 2\overline{k}_1 C_{12} + \overline{k}_1^{\,2} C_{22} - 1 = -\frac{5}{12} + \frac{\overline{k}_1}{10} + \frac{\overline{k}_1^{\,2}}{105}$$

$$C_{33} - 1 = -\frac{41}{70} \tag{A8}$$

$$C_{13} + \overline{k}_1 C_{23} - 1 = -\frac{59}{120} + \frac{19\overline{k}_1}{420}$$

The expression $(C_1 + \overline{k}_1 C_2 - 1)$ is just the ratio of the average radial velocity in the boundary layer to the radial velocity outside the boundary layer. It has been stated that there exist strong radial velocities in the boundary layer which draw fluid from the region outside the boundary layer. If this is true, then the average radial velocity in the boundary layer must be larger than the value outside of the boundary layer and the above expression must be always positive. By equation (A8), this means that

$$\bar{k}_1 > 4 \qquad \text{and} \qquad C_1 + C_2\bar{k}_1 - 1 > 0$$

By substitution of the values given in equation (A6) into the expression $C_{13} - C_1 + \bar{k}_1(C_{23} - C_2)$ it is easily seen that

$$C_1 + \bar{k}_1 C_2 > C_{13} + \bar{k}_1 C_{23}$$

Therefore the inequalities expressed on page 38 are shown to hold.

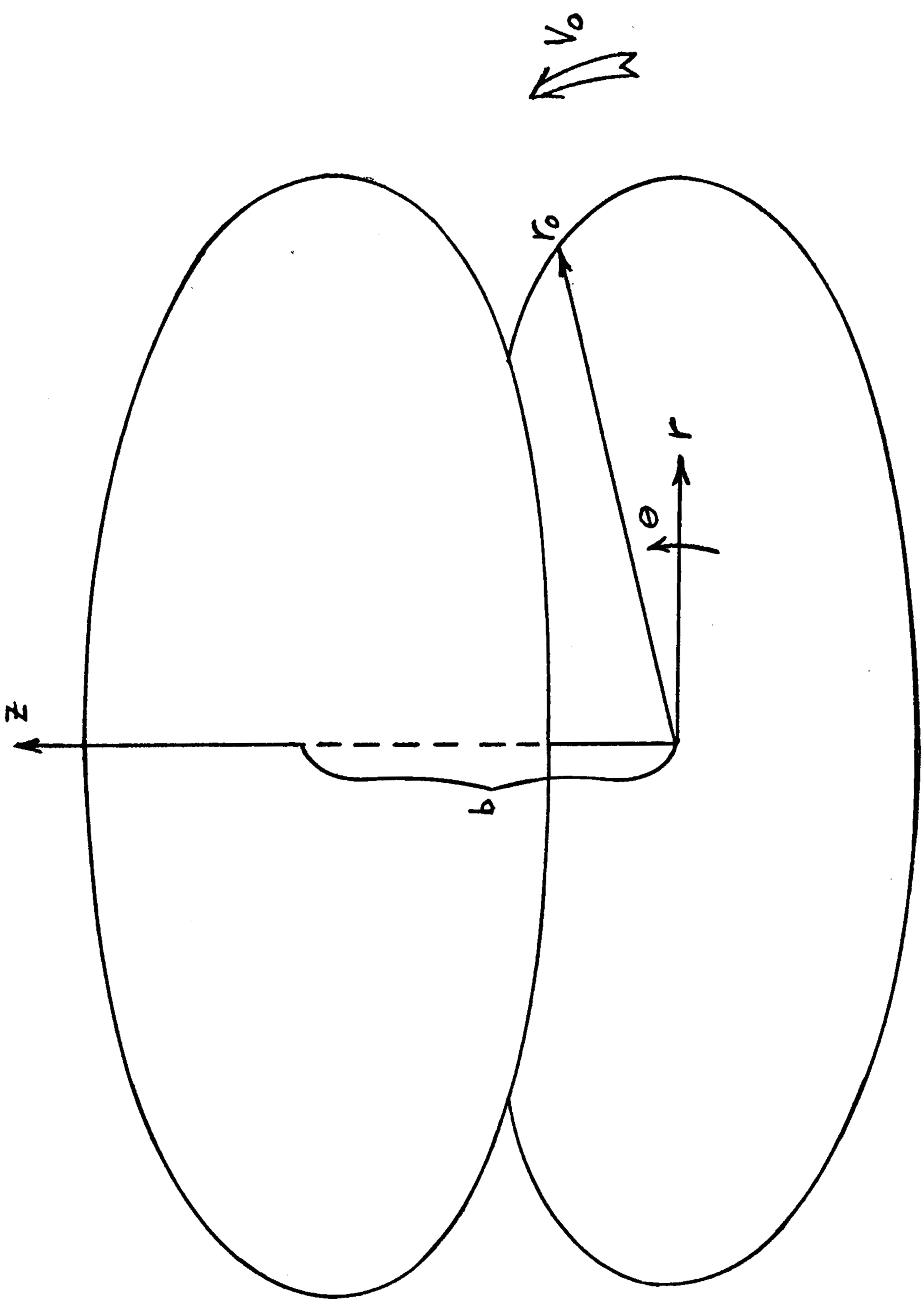
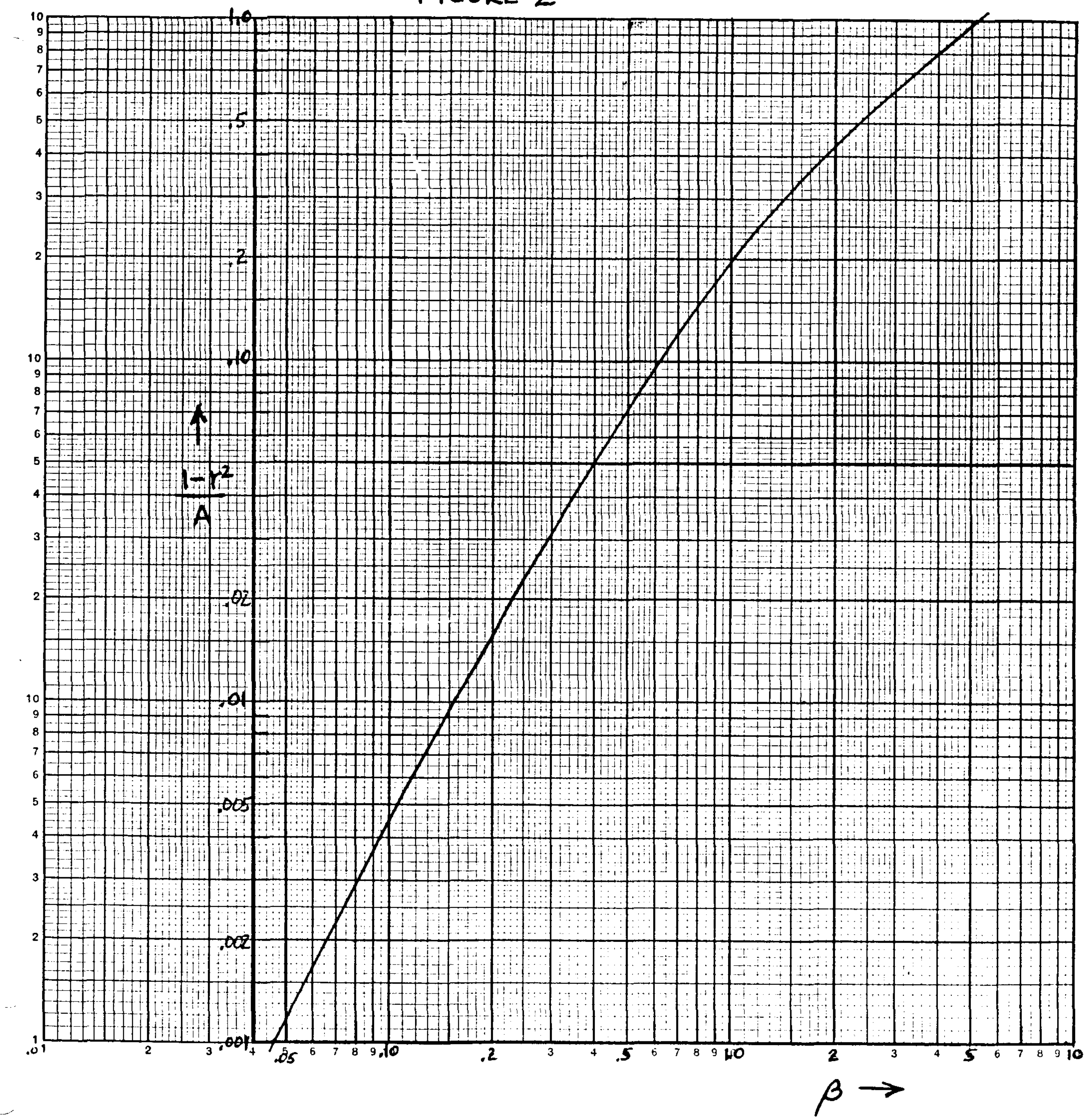


FIGURE 1

FIGURE 2



1.0
.5
.2
.10
.02
.01
.005
.002
.001
$\frac{1-r^2}{A}$
.05
.10
.2
.5
1.0
2
5
10
$\beta \rightarrow$

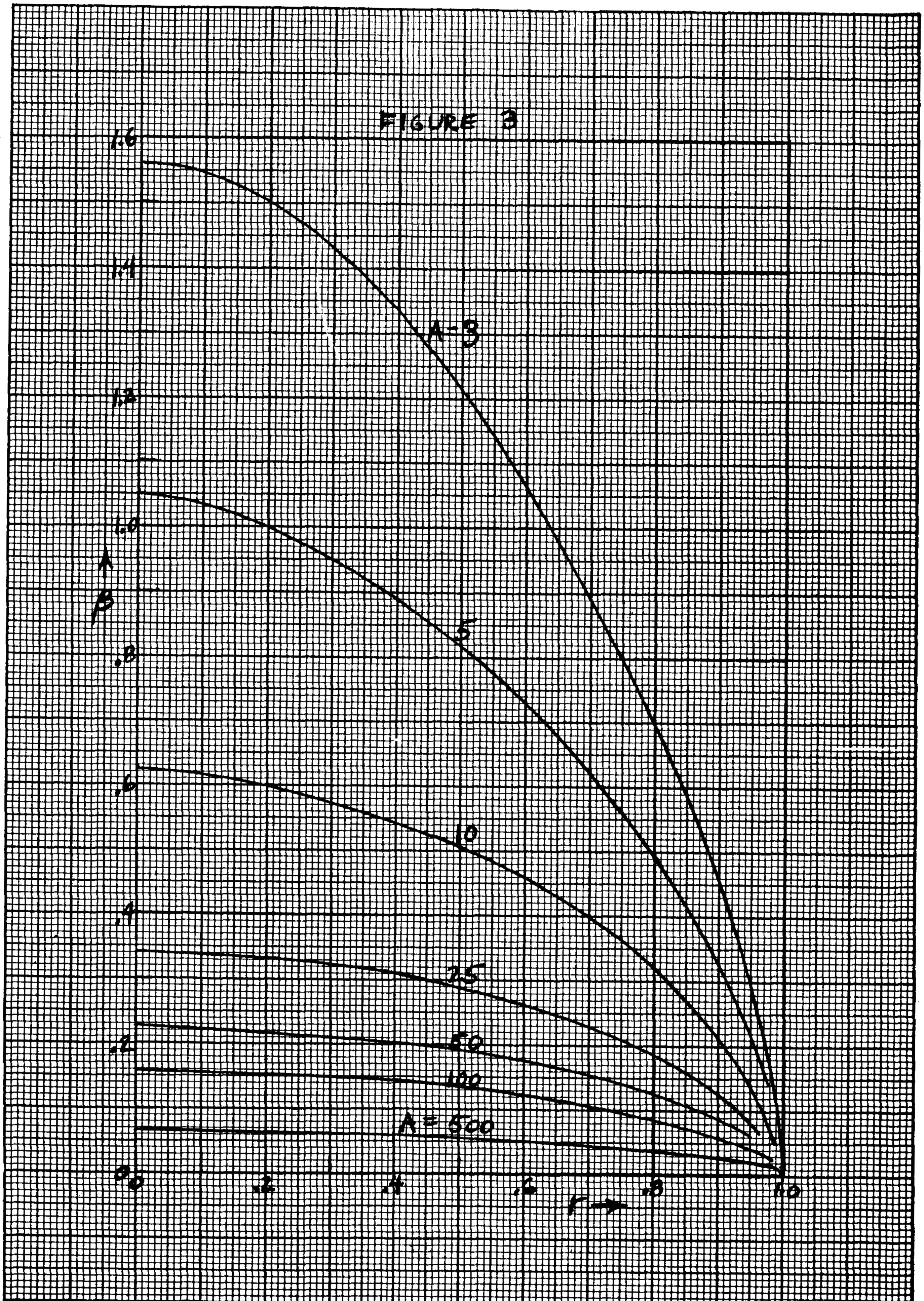
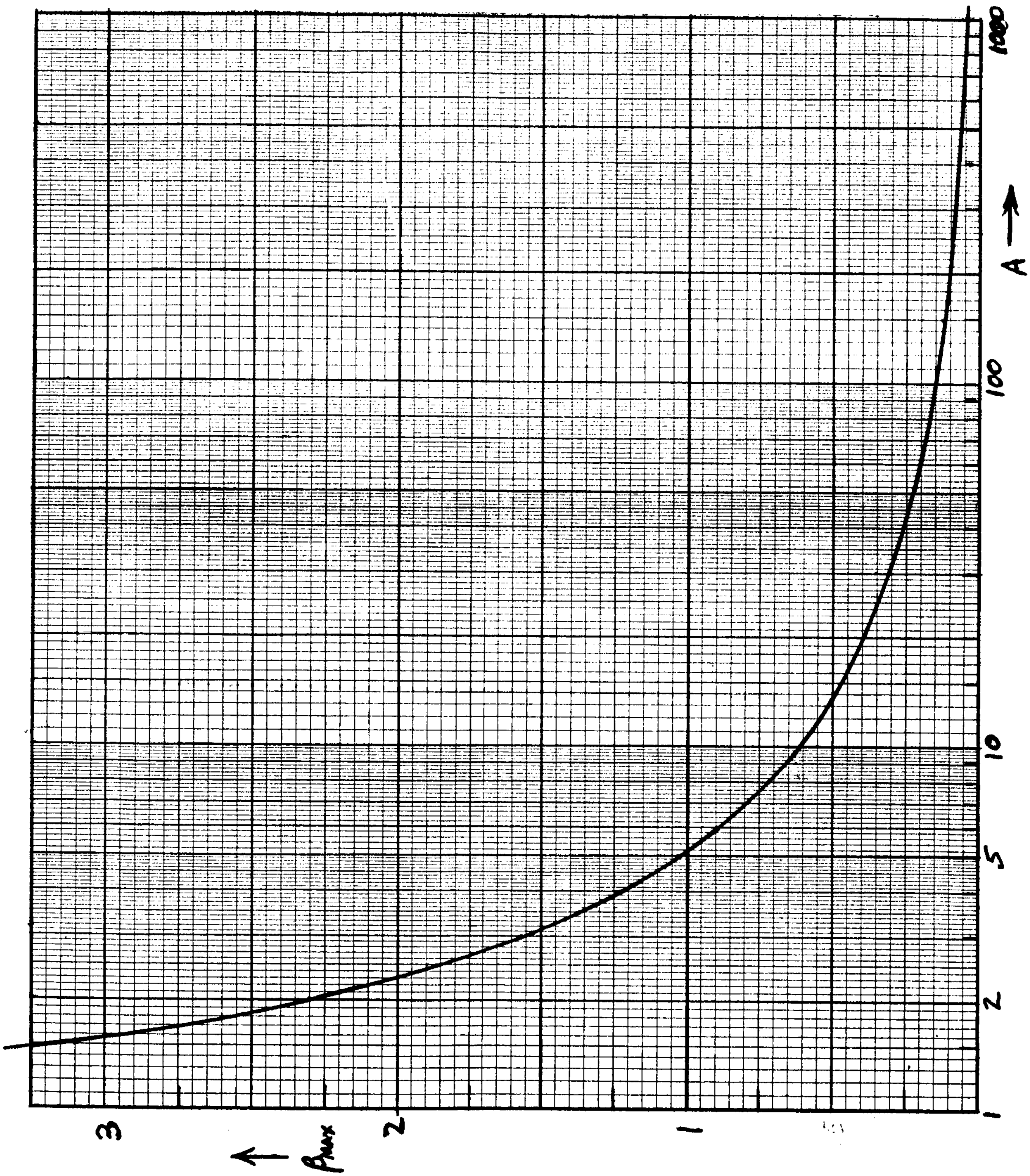


FIGURE 3
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0.0
.2
.4
.6
.8
1.0
β
r
A=3
5
10
25
50
100
A=500

FIGURE 4



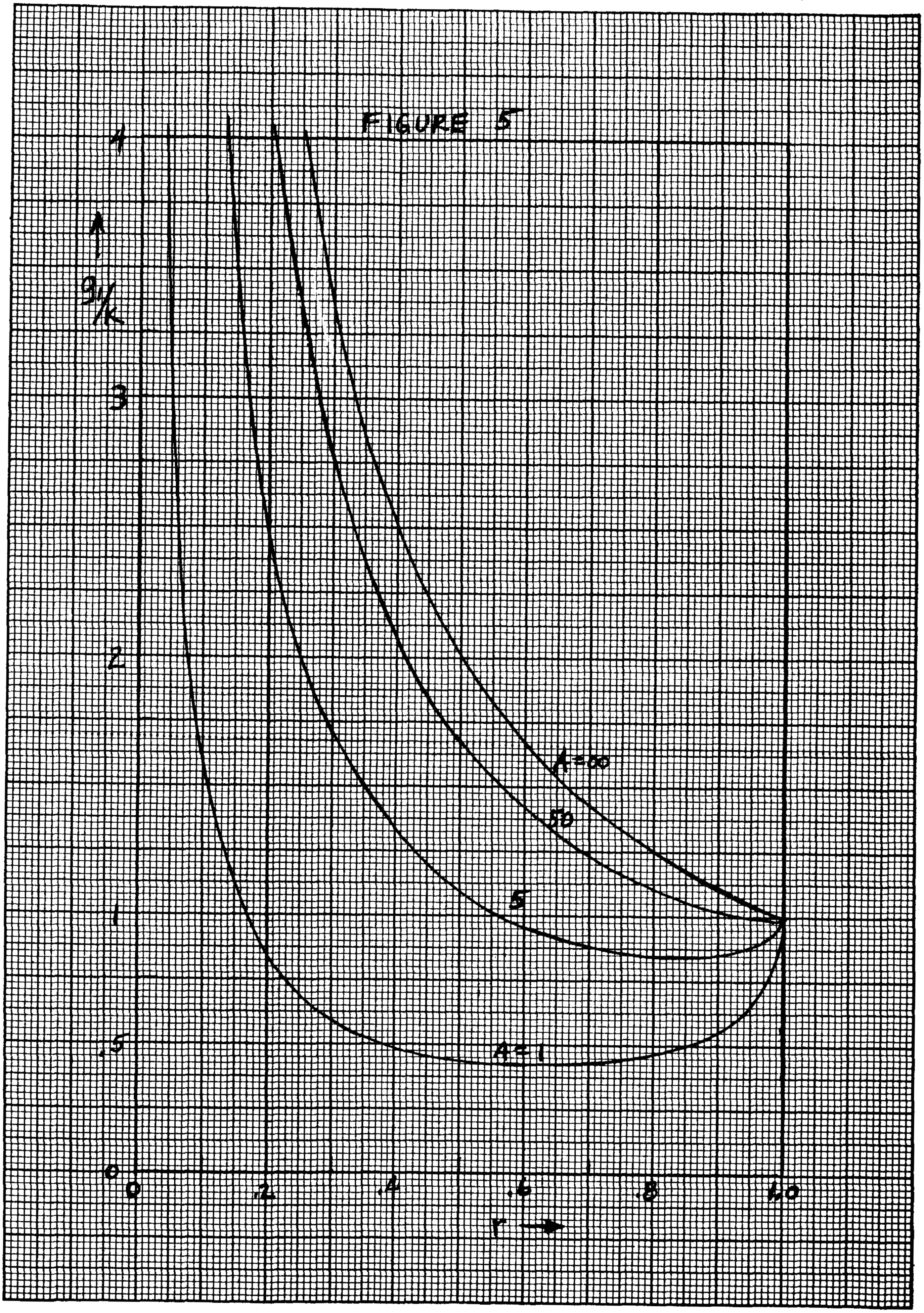


FIGURE 5
q_1/k
4
3
2
1
.5
0
0
.2
.4
.6
.8
1.0
r
A=∞
50
5
A=1

FIGURE 6

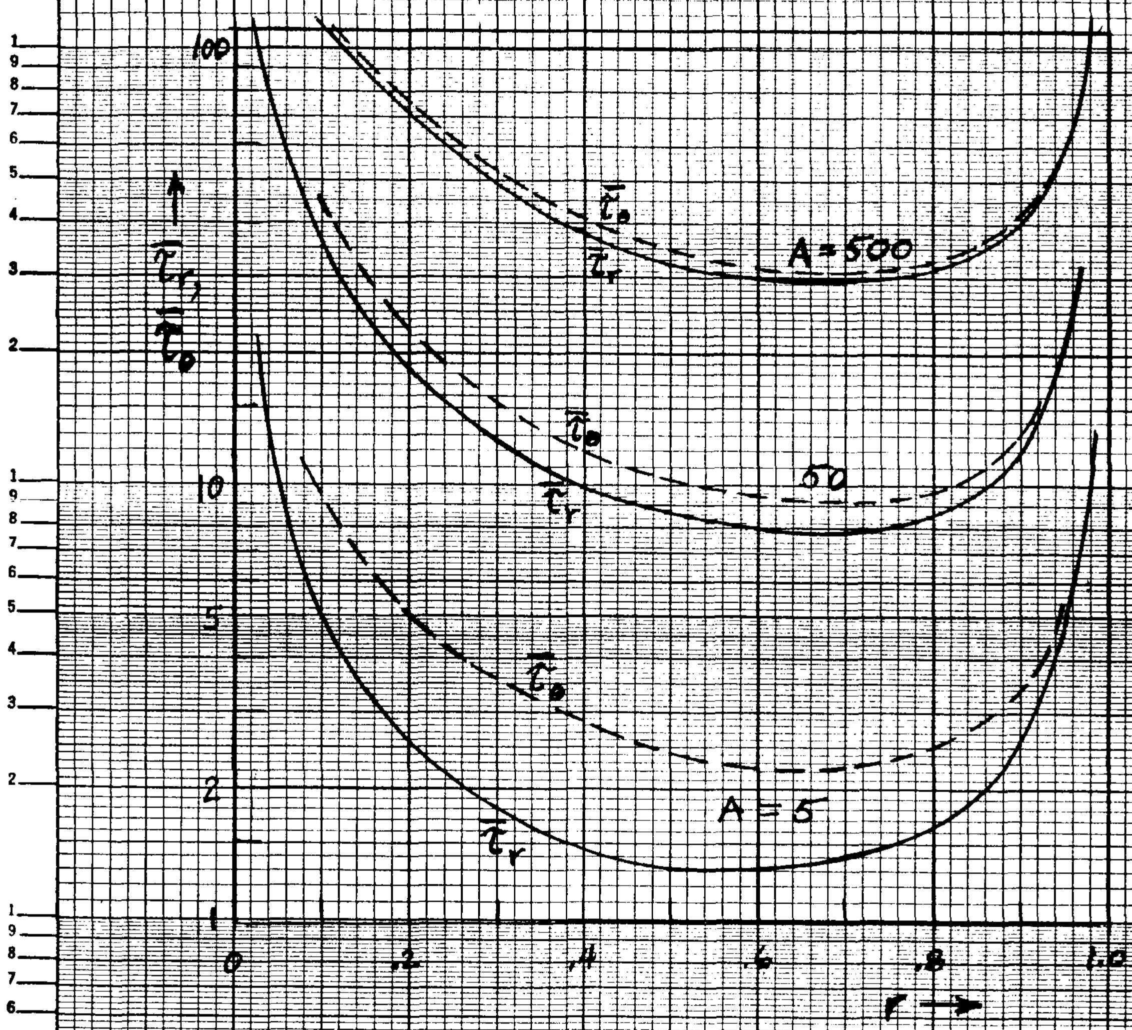


FIGURE 7