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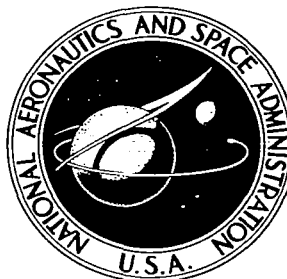
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# ON THE INVERSE OPTIMAL CONTROL PROBLEM IN MANUAL CONTROL SYSTEMS

## by R. W. Obermayer and F. A. Muckler

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IN MANUAL CONTROL SYSTEMS

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## TABLE OF CONTENTS

# ON THE INVERSE OPTIMAL CONTROL PROBLEM
## IN MANUAL CONTROL SYSTEMS[1]

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## SUMMARY

Optimal control theory is briefly reviewed with particular emphasis on the inverse problem of finding the conditions under which a given system is optimum. A specific method for computing the optimal performance weighting coefficients is developed. While the data are inconclusive, application of this technique to some of the mathematical models of manual control systems existing in the literature reveal some intractability with theory, but with the suggestion that some observed trends in the data are consistent with a hypothesis of optimalizing human operator behavior. Some implications to manual control theory and experimental methodology are derived.

## INTRODUCTION

Within recent years developments in modern control theory have given new insights into many tenacious control problems. In particular, modern optimal control theory has made inroads into the problems of control synthesis, allowing the determination of a control law which will optimize on some predetermined basis.

With regard to manual control problems and theory, the ability to synthesize optimal control requirements gives a specification of the functions for optimal performance which may be allocated between man and machine, new and different display and control tasks are suggested, the insights into the manual control tasks provided suggest more comprehensive performance measurement, and theoretical implications are made with regard to appropriate mathematical models and strategies effective to the control task (cf. Obermayer

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and Muckler, 1964). Another option provided by modern optimal control theory is to work the optimization problem backwards: assume that a given control is optimum and attempt to compute the manner in which it may be optimum.

The latter approach, that of the inverse optimal control problem, is the topic of this report. Based on the assumption that the human operator attempts to optimize during manual control, it is believed that applications of the inverse optimal control techniques may shed some light on the strategies and techniques employed. In the following, therefore, the inverse optimal control problem, and conditions necessary for optimality, are explored and a technique developed to compute the nature of a performance index which is optimized by human control functions. Some of the mathematical models existing in the literature are used to compute the nature of performance indices optimized, and the results provide the basis for critical discussion of manual control theory and experimental methodology.

## THE INVERSE OPTIMAL CONTROL PROBLEM

Much of modern optimal control theory takes as its starting point that an index of performance is specified so that optimality can be defined as minimizing the given performance index. Herein lies a fundamental problem, since quite frequently -- if not always -- defining what one means by optimal performance is very difficult. Given a method for achieving rapid solutions, such as the Automatic Synthesis Program (Kalman and Englar, 1963) which provides the optimal control law and transient response once certain performance index matrices are specified, a number of system designers have used a cut-and-try procedure, trying different performance indices until something judged "good" results. As Reynolds and Rynaski (1963) report, "Thus the performance index is used as a performance index -- that is, we choose elements of the H and Q matrices to minimize what we would like to minimize from physical considerations -- and it is used as a 'cut-and-try' parameter. The real criterion of performance is judgment applied during the 'cut-and-try' procedure." In short, an obligation is transferred to the system engineer to mathematically define optimality, an obligation he can only imperfectly fulfill.

Further, the required form for the performance index is that of a scalar, a one-dimensional entity (Zadeh, 1958; Zadeh, 1963). This hardly seems appropriate to express the usual complex multi-facetted descriptions of performance related to even quite simple systems. It is therefore argued that the choice of the performance index to be optimized is arbitrary and subjective, and that

it may be pointless to devote too much effort to finding a control law which is the best in some narrow, individualistic sense.[2]

All this suggests that it may be worthwhile to change methodology. Instead of asking for the control law corresponding to a given performance index, it may be better to seek the performance criteria for which a given control law is- optimal. This problem has come to be called the inverse optimal control problem; it is analogous to the older problem of the inverse problem of the calculus of variations.

## Conditions for Optimality

The scope of the inverse optimal control problem requires some restriction to avoid trivial cases. For example, it is possible to define loss functions under which any control system may be optimal; in particular through the proper choice of loss functions as unstable system may be termed "optimum". Therefore, if we are to seek out the ways a given system may be optimal, it will be expeditious to exclude definitions of optimality which would be universally considered undesirable or impractical by control engineers.

For the purpose of narrowing the allowable definitions of optimality, three control system attributes should be considered: controllability, observability, and stability. Stability, of course, is a long-recognized desirable system property and is generally the first system consideration; controllability and observability are properties first defined by Kalman (1960) and which are required as necessary conditions for the proof of a number of critical control system theorems.

Controllability. The literature distinguishes between various types of controllability, and offers a number of convenient tests for controllability (Kreindler and Sarachik, 1964; Weiss and Kalman, 1964; Stubberud, 1963; Ho, 1962). However, for present purposes, it will suffice to define a plant as completely controllable if for any given initial state a control input exists which will transfer the plant to any other final state in a finite length of time.

A simple example of an uncontrollable plant is shown in Figure 1. It may be seen that in state space the plant can only be controlled along the line $x_1 = x_2$. Kreindler and Sarachik (1964) point out that the lack of controllability in this case may not be critical if one is only interested in the

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[2]    In a recent paper by Kalman (1964) the above objections are pointed out, but from a scientific point of view, study of the inverse optimal control problem is considered of value since: "We might thereby discover general properties shared by all optimal control laws. We might be able to separate control laws which are optimal in any sense".

controllability of the output, and distinguish between state-controllability and output-controllability. These are independent properties with neither implying the other. As a further example, the given figure would demonstrate output-uncontrollability if the output were defined as the difference between $x_1$ and $x_2$; in this case no output variation of any kind would be possible.

Ho (1962) gives necessary and sufficient conditions for controllability which are helpful in gleaning some insight into the meaning of controllability.
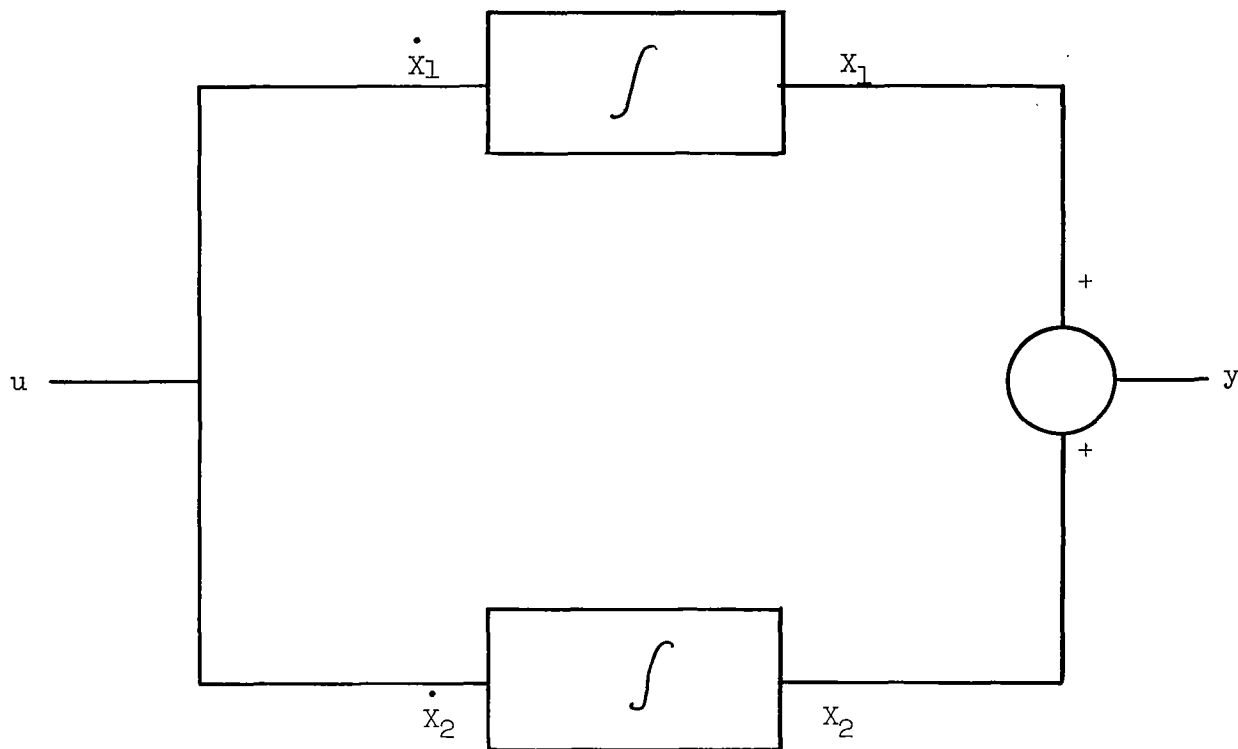


Figure 1. Plant Not Completely Controllable

Restricting attention to single-input time-invariant linear systems, he points out that controllability is independent of coordinate transformations, allowing consideration of the Jordan Canonical Form of the linear system (Figure 2).

The necessary and sufficient conditions for controllability stated by Ho are that (1) $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_q \neq \lambda_{q+1} \neq \ldots \neq \lambda_{q+s}$, and (2) $d_1, d_2, \ldots, d_q, d_{q+1}r_1, d_{q+1}r_2, \ldots, d_{q+s}r_s \neq 0$.

$$
\begin{Bmatrix}
\dot{x}_1 \\
x_2 \\
\vdots \\
x_q \\
\dot{x}_{q+1,1} \\
\dot{x}_{q+1,2} \\
\vdots \\
\dot{x}_{q+1,r_1} \\
\vdots \\
\dot{x}_{q+s,1} \\
\vdots \\
\dot{x}_{q+s,r_s}
\end{Bmatrix}
=
\begin{Bmatrix}
\lambda_1 & & & & & & & & & \\
 & \lambda_2 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & \lambda_q & & & & & & \\
 & & & & \lambda_{q+1} & 1 & & & & \\
 & & & & & \lambda_{q+1} & 1 & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & 1 & & \\
 & & & & & & & \lambda_{q+1} & & \\
 & & & & & & & & \lambda_{q+s} & 1 \\
 & & & & & & & & & \ddots \\
 & & & & & & & & & 1 \\
 & & & & & & & & & \lambda_{q+s}
\end{Bmatrix}
\begin{Bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_q \\
x_{q+1,1} \\
x_{q+1,2} \\
\vdots \\
x_{q+1,r_1} \\
\vdots \\
x_{q+s,1} \\
\vdots \\
x_{q+s,r_s}
\end{Bmatrix}
+
\begin{Bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_q \\
d_{q+1,1} \\
d_{q+1,2} \\
\vdots \\
d_{q+1,r_1} \\
\vdots \\
d_{q+s,1} \\
\vdots \\
d_{q+s,r_s}
\end{Bmatrix} u .
$$

Figure 2. Jordan Canonical form and canonical block diagram of a dynamical system

$d_1$   $\dfrac{1}{s-\lambda_1}$   $x_1$

$d_q$   $\dfrac{1}{s-\lambda_q}$   $x_q$

$d_{q+1,r_1}$   $\dfrac{1}{s-\lambda_{q+1}}$   $x_{q+1,r_1}$   $\dfrac{1}{s-\lambda_{q+1}}$   $x_{q+1,r_1-1}$   $\dfrac{1}{s-\lambda_{q+1}}$   $x_{q+1,1}$

$d_{q+1,r_1-1}$

u

$d_{q+1,1}$

$d_{q+s,r_s}$   $\dfrac{1}{s-\lambda_{q+s}}$   $x_{q+s,r_s}$   $\dfrac{1}{s-\lambda_{q+s}}$   $x_{q+s,1}$
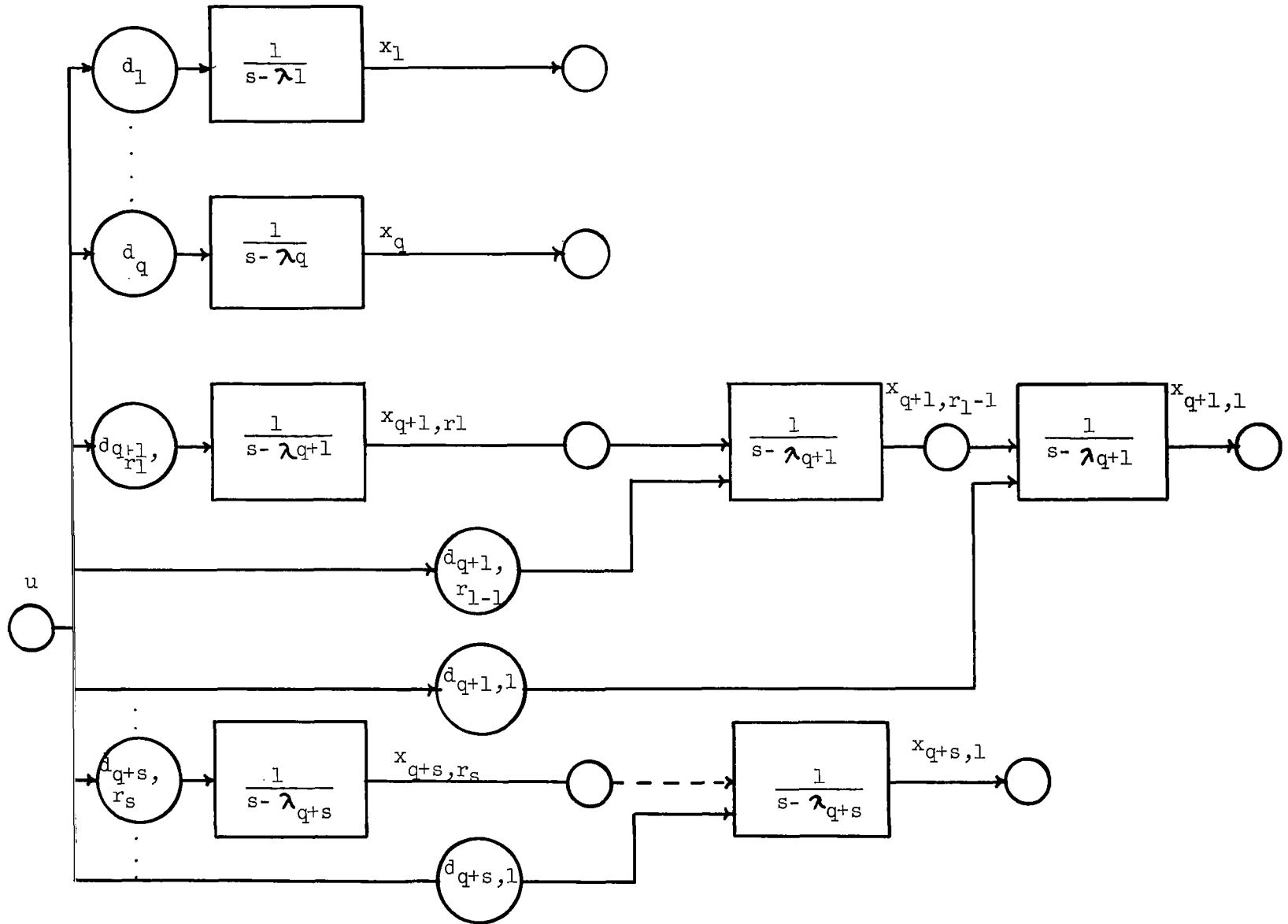
$d_{q+s,1}$

Figure 2.   Jordan Canonical form and canonical block diagram of a dynamical system

Referring to the block diagram (Figure 2), condition 1 points out that if, for example, $\lambda_1 = \lambda_2$ a situation like the previous examples of uncontrollability would result, and if condition 2 is not satisfied then we have simply lost direct or indirect control of one or more integrators.

Observability. The concept of observability is associated with the measureability of the state of the plant. In general, our knowledge of the state of a system is based on observations of the output, and if all state variables affect the output (i.e., there is no motion in state space which leaves the output unaffected) the output is completely observable. Similarly, if control feedback is affected by any change in system state, the control law may be called completely observable.

Incomplete observability implies that current and past states may be only known statistically, and occurs as a result of inaccurate measuring instruments or restricted access to measuring points.

Observability is therefore an ideal and can never be attained in practice. To the extent that probability distributions of past and present states can be constructed, optimal control may be possible with partial observability (cf., Florentin, 1962) as one may combine sequential observations and decisions according to Wald's statistical decision theory. Ostensibly inaccuracies of measurement may be compensated through such procedures, but the total ignorance of some system states is bound to be more serious. If the control law is not completely observable, degenerate cases of optimal control may result.

Stability. A very basic attribute of a control system is the concept stability: If the system is perturbed from its equilibrium, all resulting motions will remain in a small neighborhood of the equilibrium point. A more refined form of this motion is asymptotic stability which requires that the resulting motion converge to the equilibrium point. Clearly, if a control system had neither of these attributes (i.e. was unstable), the system motions would become increasingly large and hence disastrous.

A most powerful tool for the determination of system stability is provided by the second method of Lyapunov (LaSalle and Lefschetz, 1961). Stability can be verified without solving the system equations if one can find a suitable Lyapunov function. $V(x)$ is a Lyapunov function if $V(x)$ is positive definite;[3] if $\dot{V}(x)$ is negative definite one may assert that the equilibrium point is asymptotically stable.

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[3]     A scalar function $V(x)$ is said to be positive definite if $V(0) = 0$, and $V(x) \neq 0$ for $x \neq 0$. If $-V(x)$ is pos. definite, $V(x)$ is then said to be negative definite.

Lyapunov stability theory is of interest in considering the relation between optimal control systems and stable control systems, since the performance index defining optimal control may be a Lyapunov function. Under the condition that the performance index is a Lyapunov function it is guaranteed that the optimal control will be asymptotically stable. If the performance index for a free, linear, stationary system is defined as the integrated error criterion:

$$V(x) = \int p(x)\, dt$$

such that V (x) is finite in a neighborhood of the origin, and p (x) is positive definite, then V (x) is a Lyapunov function and the origin is as asymptotically stable equilibrium point (Kalman, 1960).

Constraints on the inverse optimal control problem. It may be seen from the preceding that if one uses a definition of optimality which insists on complete controllability, complete observability, and asysmptotic stability there is little danger of labelling trivial and degenerate cases as optimal.

To further concentrate attention on a class of problems of great interest in control engineering, it will be well to follow the lead of Kalman (1964) who makes the following assumptions: (1) The plant is described by linear differential equations with constant coefficients, (2) the control law is linear and constant, (3) all state variables are directly measureable, (4) quadratic performance indices are used, and (5) there is only one control variable.

Under the above five conditions, and the additional conditions of (6) complete observability and (7) complete controllability, Kalman (1964) shows that the optimal control law must be stable, and further, a control law is optimal if and only if component variations in the forward loop are diminished by the addition of feedback.

It is evident that systems which are termed optimal in the context of these seven requirements are elements of a set which would be termed excellent by control system engineers. It is believed therefore that these are reasonable constraints on the concept of optimality for the scope of constant coefficient linear systems indicated, and such linear systems which do not satisfy these conditions will be branded inoptimal. These seven requirements shall be assumed in this paper.

The above assumptions and conditions are very restrictive, excluding many interesting problems, but unfortunately current theory does not allow one to consider more sophisticated cases. Certainly performance indices other than quadratic forms are of interest. The condition of complete observability, with all state variables measurable, is a practical problem since

this frequently implies the measurement of many high-order derivatives. If some control variables cannot be measured directly, optimal control theory requires that the missing state variables be estimated from the known ones. This may be done using Wiener filtering techniques and results in the inclusion of dynamical elements as part of the controller.

With regard to the restriction of quadratic performance indices, it should be pointed out that if a quadratic performance index is minimized by a particular control, performance indices of other forms may also be minimized. For example, Sherman (1958) showed that with Gaussian signals, and some nongaussian signals, that a Wiener predictor satisfying a mean square error criterion also satisfied any even monotonically increasing error criteria. Brown (1962) extended these results to asymmetric non-mean-square error criteria, as well as to the case of nonstationary Gaussian inputs.

## Application to Manual Control Systems

Much has been said about the human controller tending to perform in an "optimal" fashion and in an adaptive manner (i.e., perform optimally in a number of different control environments). For example, McRuer and Krendel (1957) comment, "Although we would be hard put to specify the precise optimum toward which the subject strives, we can assert that the human operator is both "adaptive" (within a relatively fixed form), and "optimalizing" (to some internal criterion). In fact, the human operator is the very prototype of an adaptive, optimalizing servo system."

It is interesting to pose the question: If the human operator is performing optimally, what performance criteria are the basis for his optimization? In terms of the inverse optimal control problem, this is equivalent to stating: Given a manual control system, under what performance criteria is it optimal?

While extensive considerations have been given to optimal manual control systems (e.g., Birmingham and Taylor, 1954; Frost, 1962), little study has been given to the mode of human operator optimization.

Roig's investigation. One approach to the study of human optimizing behavior is to compare human performance in a given task against a device which is optimal in some known manner. Roig (1962) used this approach in comparing the performance of a human operator against a linear controller which minimized rms error. The task was one-dimension compensatory tracking, with two types of stochastic nongaussian inputs, and with controlled element dynamics of approximately a rate control with large delay. The optimal linear control was known for various amounts of constraint on the controller output.

In comparison to these it appeared that the human operator performed about as well as a highly constrained optimal linear controller. However, while the results were suggestive, no definitive conclusions could be made about the mode of human optimalizing behavior. In particular while similarities between human and optimal controller overall performance were noted, differences were apparent in the time history records.

Leonard's study. Another approach to the study of optimalizing behavior is to vary the parameters of a mathematical description of the human operator to determine if other combinations of parameters could produce superior performance. Using a brute force computer technique, Leonard (1960) evaluated two cases of human operator mathematical models against a minimum mean square error criterion. One case was the mathematical models fitted by Elkind using rectangular spectra of various cut-off frequencies and no controlled element dynamics, and the other case was the mathematical model fitted by the Franklin Institute using the dynamics of the F-80 aircraft in simulated tail-chase conditions. In each case the parameters of the math model were varied and the mean square error score was computed until the minimum mean square error condition was found. In comparing against the published experimental results a similarity was noted between experimental and calculated scores except for the model corresponding to aileron control of the F-80, however, it was observed that the subject's technique in this task was to use loose control of the ailerons and to stress pitch control. Leonard concluded that "the trained human often adopts dynamics that nearly minimize the mean square tracking error (subject to the human's inherent limitations)."

Potential for gaining insight into human behavior. Instructions aside, it may be observed that the subjects of tracking experimentation bring with them a set of strategies and techniques which they apply to the task. In some cases these may be highly individualistic traits, in other cases, there may be a small number of techniques being employed by different subjects. It is possible that there are different methods of achieving the same goals, but on the other hand, different strategies may indicate attempts at achieving different goals. If it were possible to compute the performance indices optimized in a given manual control system, it may then be possible to make some inferences about the task and the strategies employed. Clearly information of this sort is essential to an understanding of the manual control task and the related human operator behavior. It is this that makes attractive the potential modern optimal control theory offers for direct solution of the inverse control problem.

Use of math models. In order to apply existing modern control theory to manual control, it is necessary to have a complete mathematical description of the manual control system. Fortunately, some mathematical description of manual control exist, known variously as human transfer functions, describing

functions, and mimicks. Of course, the other portions of the control system are usually mathematically described. Taking the available data for mathematical human operator models, one may form a mathematical description of a manual control system in precisely the same form as might be applied to some automatic control system. Available tools of the inverse optimal control problem might, therefore, be applied to this situation as well as any other.

While some elegant and complex models have been developed incorporating nonlinear aspects of human response, the only models for which a significant amount of data exists are in terms of linear differential equations with constant coefficients. The following form by McRuer and Krendel (1957) is by far the most tested:

$$\frac{C(S)}{G(S)} = K_p e^{-s\tau} \frac{1 + T_L S}{(1 + T_n S)(1 + T_I S)}$$

It will be noted that this model contains a pure time delay, presumably to account for the reaction time lag of the human operator. The form shown is simplified, more frequently than not, by reducing the number of constants in the numerator or the number of terms in the denominator whenever these terms are not deemed necessary to obtain a good fit to the original human operator responses. A variation of the linear model used by Adams (1963) is:

$$\frac{C(S)}{G(S)} = \frac{C_1 (S + C_2)}{(S + A)^2}$$

The mathematical models of the human operator are ordinarily either measured with no system dynamics at all, or with simple linear dynamics. Except for the time-delay term, then, the composite system of system dynamics and human operator model is describable as a simple linear system. In addition, the models of the human operator are measured with the operator only displayed error information; in such a case, the human operator is presumed to be responding to the instantaneous error and without knowledge of the future nature of the forcing function input. This situation is similar to that defined as "regulator" control where control is applied to nullify the immediate input; and without the qualifications for "servomechanism" control, where control is applied to match the system output to some desired time-history (e.g., that of a pursued target).

A Technique for Calculation

Kalman's linear solution. While modern control theory is deeply involved with nonlinear techniques, it still remains that nonlinear techniques

are not easily generalized and that the most powerful, general statements can be made with linear systems. It is not surprising then that Kalman's solution for optimal linear systems is quite easily stated and is very specific about the nature of the optimal control law for a linear system.

Kalman's solution applies to linear systems of any order (with possibly time-variable coefficients) where the performance index is expressed in terms of quadratic forms (quadratic loss functions). With a quadratic form, the terms are weighted cross-products and squares of the state variables; e.g.:

$$X^T Q X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} Q11 & Q12 \\ Q12 & Q22 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} (Q11 \ X_1 + Q12 \ X_2) \\ (Q12 \ X_1 + Q22 \ X_2) \end{bmatrix}$$

$$= Q11 \ (X_1)^2 + 2Q12 \ (X_1 \ X_2) + Q22 \ (X_2)^2$$

Expressed in matrix notation the system equations and the performance index take the following form:

$$\dot{X} = A(t) X + G(t) u$$

$$J = \frac{1}{2} X^T(t_f) S X(t_f) + \frac{1}{2} \int_o^{tf} \left[ X^T Q X + U^T R U \right] dt$$

Here, $\underline{x}$ is the state vector, $\underline{u}$ the control vector, $\underline{J}$ the scalar performance index, and the others are matrices of constants (possibly time-variable). Kalman requires thematrices S, Q, R to be symmetric, R must have an inverse, and the quantity in brackets must be positive definite. The performance index is composed of the weighting of the state at terminal time ($t_f$), the time-history of the state variables during the intermediate trajectory, and the time-history of the use of control; the relative weighting of each of these factors is determined by the matrices, S, Q and R respectively.

For any system of this quite general form, Kalman asserts that the optimal control law is a linear feedback of the state vector.

$$U = (-R^{-1} G^T P) X$$

Here the matrix $P(t)$ satisfies a matrix Riccati differential equation:

$$\dot{P} = PGR^{-1}G^TP - PA - A^TP - Q$$

$$P(t_f) = S$$

Inverting Kalman's technique. There are a number of characteristics of the manual control tasks for which mathematical models are available, which permit working Kalman's technique backwards.

1. The human operators are tracking continuously throughout an experimental trial without giving any particular consideration for conditions terminating the trial. They are not trying to achieve any particular state at the end of the trial (at least they are given no instructions to this effect). This permits matrix S to be set to zero.

2. The control law is specified as a constant relation (since the human operator model has constant coefficients): $U = -KX$. Under these conditions, the result is given by the steady state solution of the Riccati matrix differential equation. Under this condition $\dot{P} = 0$.

Kalman's solution, in the form shown here, is only applicable if the manual control task corresponds to the regulator problem. An explicit non-trivial result for the servomechanism problem is not currently possible.

With the above provisions, one is left with only the task of solving for performance matrices R and Q, and the Ricatti differential equations becomes an algebraic equation ($\dot{P} = 0$). For a given constant control system, the control law is known; if the feedback gains are inserted in the above equations one may then solve a system of simultaneous algebraic equations for the unknown elements of the performance matrices. The details of this calculation procedure are given in the Appendix; however it should be pointed out here that it is not possible to write a sufficient number of equations to solve for all unknown elements of the performance matrices. It is necessary to normalize with respect to the weighting on the control input ($R = 1$), and even then is only possible to solve for $\underline{n}$ elements of the Q matrix (where n is the order of the total system). In the following this means that the Q matrix weighting the state variables takes on the following form:

$$Q = \begin{bmatrix} Q_{11} & \cdot & \cdot & 0 \\ 0 & Q_{22} & & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & & \cdot & \cdot & Q_{nn} \end{bmatrix}$$

13

While other variations may be reasonable, this selection weights only the squares of the state variables, assuming no weighting of cross-products of the state variables (i.e., no requirement that the state variables are correlated).

An example. While the calculation procedure is explained fully in the Appendix, the following example will serve to suggest the general procedure and to point up the assumptions involved in applying the technique.

One of the simpler models used to describe human tracking behavior is the following:

$$\frac{Y_p \ \text{(Pilot response)}}{\varepsilon \ \text{(Displayed system error)}} = \frac{K_1}{T_1 + 1} \ e^{-s\tau}$$

To apply the calculation procedure the lag term, $e^{-s\tau}$, is ignored, assuming that this portion of the response is inadvertent and not a part of an attempt of the human operator to track in an optimal fashion.

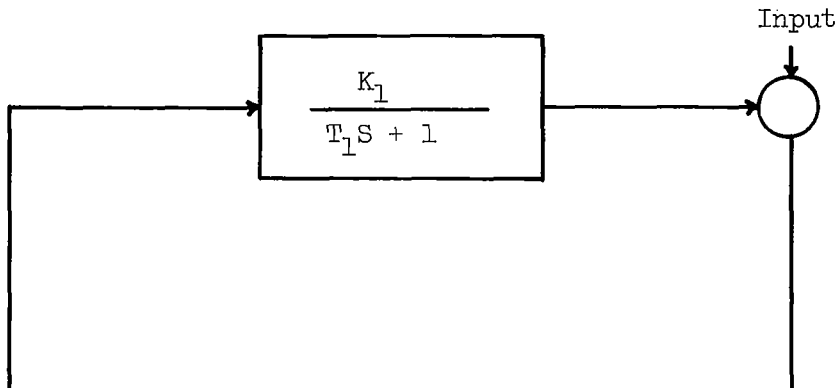1.  For position control tracking (no system dynamics) the following block diagram results:



Figure 3. Manual Control System block diagram

2.  Equivalently, this diagram may be shown in two parts (after utilizing block diagram algebra), corresponding to the "control" and to the "plant".
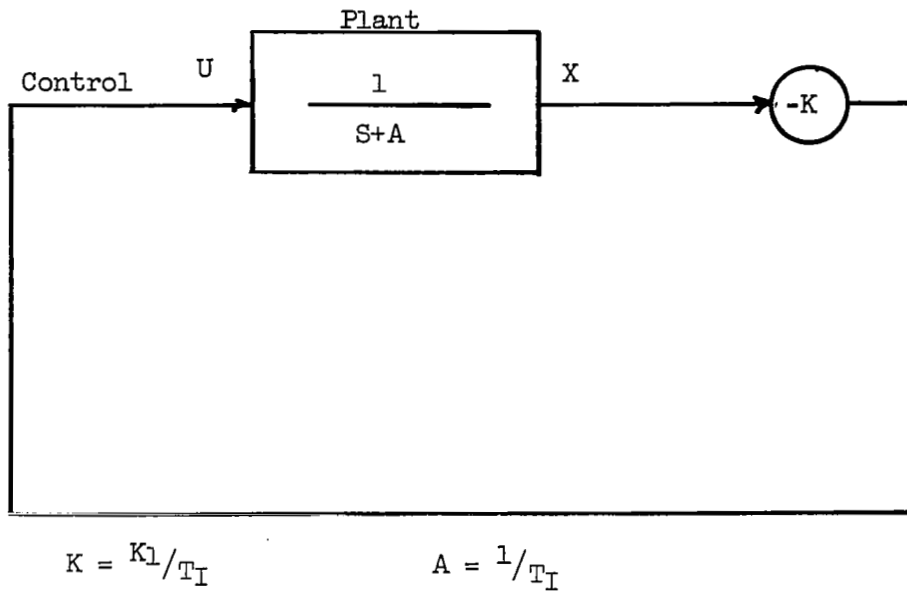
14

Figure 4.  Manipulation of Manual Control System
block diagram

For a higher order system, a partial fraction expansion is found, allowing a
similar block diagram with a number of first-order systems in parallel.

    3.  Here, the system equations are:

$$\dot{X} = -aX + U = Ax + GU$$

where $A = -a$; $G = 1$

The control law is a constant relation:

$$U = -Kx$$

Also, from Kalman's result:

$$U = -R^{-1}G^{T}Px; \text{ here } R = R^{-1} = 1; G = G^{T} = 1$$

$$\therefore \dot{u} = -PX; P = K$$

15

4. These results may be substituted in the Riccati equation (setting $\dot{P} = 0$)

$$\dot{P} = 0 = PG(R^{-1}G^T P) - PA - A^T P - Q$$

$$0 = K^2 + Ka + Ka - Q$$

5. One may then solve for the only remaining variable, the performance weighting Q:

$$Q = K\,(K+2a)$$

$$= (K_1^2 + 2K_1)\,/\,(T_I)^2$$

In this case it may be seen that constant Q corresponds approximately (for $K_1 >> 2$) to a constant gain-bandwidth criterion. If the human operator were to track with a consistent basis for optimization, we would then expect that mathematical models corresponding to consistent optimalizing behavior would yield a constant gain-bandwidth product. This is precisely the observation made by Elkind and Forgie (1959) for mathematical models with a variety of rectangular input spectra.

A number of assumptions must be made in order to apply this technique to the calculation of optimal performance indices using existing human operator models. For convenience these may be listed as follows:

1. Only quadratic performance indices are considered.

2. R = 1, i.e., the results are normalized with respect to the weighting of the use of control.

3. The off-diagonal terms of the quadratic performance matrix are all zero.

4. The mathematical model of human tracking must be linear, the delay term is ignored, and a partial-fraction expansion must exist (i.e. no multiple roots, a condition imposed by the requirement for complete controllability).

5. Control is defined in terms of the optimal regulator problem.

# A COMPUTER INVESTIGATION

## Procedure

To investigate the suitability of inverse optimal control techniques to the study of human tracking behavior, the inverse technique suggested by Kalman's solution was programmed for a high-speed digital computer. The basic approach was to use existing mathematical descriptions of the human operator to achieve a mathematical description of a given total manual control system; through digital computer computation an optimal performance index corresponding to each manual control system was derived. The form of the performance index thus achieved was the performance index which would be minimized by the given manual control system.

The technique outlined in the previous section produces two computational problems: first, for a given control system, a system of simultaneous linear algebraic equations must be set up, and then the system of equations must be solved for the coefficients of the loss functions. Correspondingly two basic programs were written in the FORTRAN computing language, with slight changes necessary for different system dynamics. The linear algebraic equation solved is (see Appendix):

$$0 = PGK - PA - A^T P - T^T qT$$

Here, P and q are the solutions of the Riccati equations and the loss function coefficients, respectively, and the remaining terms of the above equations are constants determined by the system parameters. The solution then is in terms of the symmetrical n×n P matrix and the diagonal n×n q matrix; for present purposes the P matrix is of no direct interest. The first digital computer program then consisted of the straightforward task of calculating the constants of the above set of linear equations, and the second program was a routine for computing the matrix inverse and solving simultaneous linear algebraic equations.

The data were taken from McRuer and Krendel (1957) (also in Senders, 1959, pp 3-4) and Adams (1963). However, in each case it was not possible to use the data exactly as presented. In the case of the McRuer-Krendel data (i.e. Russell, Franklin Institute and Elkind data) a pure time delay is included in the human operator model (an exponential term in Laplace transform notation). The time delay term is not consistent with the finite state model assumed by the inverse optimal computational technique. The lag term was therefore ignored for computer computation (another approach would be to use a Padé approximation for the lag term).

In the case of Adams' data, the model incorporates equal roots in the denominator which yields an ambiguous partial fraction expansion and which

corresponds to a plant which is not completely controllable. The course of action taken here was to approximate Adams' model with a controllable form with distinct roots. The computer was set up using Adams' parameters, but instead of using the double root $a$, distinct roots of $a \pm d$ were used; a number of runs were made with decreasing $d$ until $d = 0.001$ to assure that the solutions were well-behaved. In all cases a well-behaved convergence was observed with variation occurring only in the high order significant digits.

## Results

The results of the digital computer solutions are shown in Tables 1, 2 and 3. The number of state variables and hence the order of Q depends upon the order of the total man-machine system (human operator dynamical model + controlled element dynamics).[4]

It will be noted in some cases Q contains negative terms, and there is little consistency in these data.

## Comments

Small-sample results. Before any extensive discussion based on the results presented here, it should be pointed out that relatively little data are presented here. Very few data points are available for each condition, and only the data of a few total subjects are considered - generally only one subject for each condition. There is then little one can say about trends, or lack of trends, and about apparent variability. This investigation is quite exploratory.

The transfer function data used here are derived by several investigators. A transfer function form was adopted by each investigator which in his judgment produced a good fit to the empirical data. It would be understandable if differences in form of fitted functions and procedures varied with investigator.

Inoptimal results. A number of the manual control system conditions considered lead to a calculation of negative performance indices. Since this indicates a weighting of state variables errors so that increased error is taken as something desirable, one might therefore conclude that these manual control systems represent inoptimal conditions. However, there are various possible interpretations.

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[4]    Therefore, $Q = (Q11)$, or $Q = (Q11\ Q22)$, or $Q = (Q11\ Q22\ Q33)$, for total system dynamics of first, second and third order, respectively.

TABLE 1

POSITION CONTROL:   FIRST ORDER SYSTEM ELKIND'S DATA

| COND. | K | $1/T_I$ | Q |
|---|---|---|---|
| R .96 | 2.113 | 3.65 | 115.7 |
| R1.6 | .9333 | 3.77 | 38.9 |
| R2.4 | .7079 | 1.885 | 6.82 |
| F1 | 3.350 | 1.13 | 22.9 |
| F2 | 17.78 | 0.314 | 346. |
| F3 | 44.67 | 0.1885 | 74.3 |
| B1 | 2.818 | 4.78 | 311. |
| B2 | 1.189 | 5.03 | 96.1 |
| B3 | 0.8912 | 12.6 | 409. |
| B4 | 0.9660 | 12.6 | 456. |
| B5 | 3.589 | 1.88 | 71.0 |
| B6 | 7.674 | 1.00 | 74.1 |
| B9 | 1.047 | 12.6 | 507. |
| B10 | ----- | 2.82 | ----- |

TABLE 2

SECOND AND THIRD ORDER SYSTEMS

| INVEST-IGATOR | K | $1/T_I$ | $1/T_N$ | $1/T_L$ | Q11 | | Q22 | | Q33 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Franklin: | 100 | .04 | 1.5 | .5 | + | 36.720 | + | 168.96 | | |
| | 40 | .11 | 4.55 | 2.0 | + | 420.84 | + | 153.45 | | |
| | 1.5 | .2 | 11.0 | 3.0 | + | 25.410 | + | 19.250 | | |
| Elkind: | 53.09 | .22 | 1.885 | | + | 503.14 | − | 44.040 | | |
| | 37.58 | .314 | 6.22 | | + | 5674.3 | − | 146.80 | | |
| | 13.34 | .785 | 12.3 | | + | 19076.7 | − | 257.60 | | |
| | 5.623 | 1.73 | 30.3 | | | 117813. | − | 589.60 | | |
| | 14.45 | .88 | 17.8 | | + | 58301. | − | 452.60 | | |
| | 2.818 | 3.14 | 6.28 | | + | 5279. | − | 111.14 | | |
| Russell: | 17.7 | 0.1 | | 0.68 | + | 3.5049 | + | 9.0045 | | |
| | 7.55 | 0.13 | 25 | 0.9 | + | 3053.6 | + | 6158.1 | − | 109.2 |
| | 10. | 3 | 14 | 2 | + | 8467.2 | + | 2688.0 | − | 84.000 |

## TABLE 3

## $2/_S$ DYNAMICS-ADAM'S DATA

| $C_1$ | $C_2$ | $A=B$ | $Q_{11}$ | $Q_{22}$ | $Q_{33}$ |
|-------|-------|-------|----------|----------|----------|
| 32.2  | 2.27  | 4.54  | +5345.7  | +1037.1  | -64.401  |
| 23.1  | 3.03  | 5.0   | +4902.5  | + 288.88 | -46.203  |
| 8.61  | 1.492 | 3.03  | + 165.26 | + 76.564 | -17.220  |
| 10.22 | 2.324 | 3.45  | + 564.69 | + 20.004 | -20.441  |
| 14.02 | 0.571 | 2.0   | + 64.151 | + 244.70 | -28.040  |
| 24.6  | 2.324 | 3.45  | +3269.8  | + 401.90 | -49.201  |
| 16.0  | 1.492 | 3.03  | + 570.31 | + 260.52 | -32.000  |
| 5.93  | 2.70  | 4.0   | + 256.86 | - 31.226 | -11.860  |

Perhaps the clearest statement of the troublesome results is to say that the calculation procedure used here was unable in some cases to point up the manner in which certain systems are optimal. Some restrictions are placed on the nature of optimal systems which may be at variance with the manual control systems. For example, the performance index is assumed to be a quadratic form since this was consistent with existing theoretical developments, but a system which optimized on the absolute value of system error may appear inoptimal in the light of these assumptions.

Additional state variables. The required number of state variables for system description is equal to the order of the total dynamical system. It will be noted that the manual control systems yielding negative results incorporate feedback of less than all state variables (determined by the number of constants in the numerator of the transfer function).[5] These systems do not satisfy Kalman's requirement for complete observability, and represent systems in which control is not based on the full state vector. By Kalman's definition, such systems are inoptimal.

There are two possibilities where such systems may be optimal, even by Kalman's criteria: (1) the dynamical portion of the transfer function may include a prediction of the seemingly missing state variables, or (2) the

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[5]     Since $u = Kx$ and $R = \begin{bmatrix} 1 \end{bmatrix}$, $X^TQX + u^TRu = X^TQX + X^TK^TK X = X^T(Q+K^TK)X$. When the negative elements of $Q$ correspond to the zero elements of $K$ (unobserved states), there is question of the positive definiteness of the performance index, and hence asymptotic stability is not assured by Lyapunov's theorem.

precision of transfer function measurement was insufficient to determine human operator response to all state variables (i.e., high-order derivatives of system error). Of course, both explanations may simultaneously hold.

With regard to the first possible explanation, a quote from Kalman (1964) should suffice: "These assumptions are of course highly restrictive. One obtains a hierarchy of problems depending on the number of control variables and the number of state variables which can be measured directly.

"If all state variables can be measured, the optimal controller does not contain dynamical elements because the best control action at any instant depends only on the value of the state variables at that instant. But if some control variables cannot be measured directly -- which happens very often in practical problems -- optimal control theory requires that the missing state variables be estimated from the known ones using Wiener filtering techniques. The Wiener filter will contain dynamical elements which are to be regarded as a part of the controller."

The technique for calculating optimal performance indices used in this investigation makes no allowance for Wiener prediction of missing state variables, nor is it apparent at this time how this could be accomplished.

The other possibility is that the human operators did depend on high-order state variables for control, but that this was not apparent in deriving a fit to his responses. To demonstrate this, an additional state variable was considered for one transfer function (see Table 4). The weighting of the additional state variable in the control law was varied until an optimal performance index (in the sense of the assumed form) could be calculated. In the case shown, a moderate weighting of the missing state variable could yield the desired result without changing other transfer function constants. If the altered transfer function form were used in fitting to the empirical data, all constants would change, with possibly even a smaller weighting to the high-order state variable sufficing.

TABLE 4

ADDITION OF NEW STATE VARIABLES

| K | $1/T_I$ | $1/T_N$ | $1/T_L$ | $Q_{11}$ | $Q_{22}$ |
|---|---|---|---|---|---|
| 53.09 | .22 | 1.885 | 4.40 | 505.1 | 2.01 |

## Inverse Optimal Control Theory and Mathematical Models of the Human Operator

The inverse optimal control theory is aimed at determining the manner in which a given control system is best. To the extent that there is insight into the goals applied to the manual control task, better descriptions of manual control behavior will result.

In a more abstract sense, the theory of the inverse optimal control problem allows a set of numbers to be assigned to a given tracking trial -- the coefficients of the loss functions which are optimized. Other sets of numbers can be assigned to the same tracking trials which correspond to the coefficients of mathematical models which fit the data. These sets of numbers are transformations of each other, and to the extent that the transformations are 1:1, they are all equivalent. However, in general, the differing models are not entirely equivalent to each other, and the loss functions optimized may correspond to many variations of the measured models. That is, the differing models may fit the data in different ways and in differing degrees, and it is possible that a variety of model coefficients may correspond to optimization on the basis of the same criteria in a variety of different circumstances.

As an example, it has been observed that the human operator may adapt so that the measured linear model of his performance is different with different controlled element dynamics and gains. It is reasonable to hypothesize that over a range of circumstances he may be attempting to optimize performance on the _same_ basis, necessitating that his behavior, and the corresponding linear model, be _different_. It is therefore possible that a practice of correlating mathematical models to the conditions under which they represent optimal performance may form a basis for consolidating a range of models representing similar behavior. Although the evidence is not conclusive, Elkind's result that over a variety of conditions performance tended to maintain an approximately-constant gain-bandwidth product, is an encouraging sign of constant optimalizing behavior.

## Manual Control Experimental Methodology

While it may be argued that differences in human operator response under different conditions may indicate optimalizing behavior, it may also be argued that differences in human operator response, particularly between subjects given the same task, or sudden changes in a specific subject's response, may represent optimalizing behavior, but with different bases for optimization.

For the most part, there is little reason to suppose that existing data are representative of consistent optimalizing behavior, since we do very little to constrain the subject's basis for optimization. The instructions generally imply some vague minimization of error, but as we have seen to adequately specify optimal performance means indicating a weighting of the use of control, the error and appropriate derivatives, i.e., the control and some complete set of state variables. The appropriate method for conveying the desired optimization process is obscure; however, it is clear that until our experimental methodology is improved, we will be collecting data from various different subjects doing various different tasks at their own whimsy.

## The Tractability of Mathematical Models to Theory

It is of course essential as a first requirement that the input and output of a given model accurately match the measured time histories. The ability to match the given data is limited by the precision of measurement, and, therefore, within the bounds of measurement accuracy some equivocation must exist with regard to the model form and the magnitude of model coefficients. It is also quite reasonable, if one cannot detect the difference at the input and output of two models, one more complex than the other, to use the simpler model. However, it is apparent from this study that one value of these models is to permit theoretical analyses, and that these analyses may be hampered if the model form is inappropriate. We may ask whether the time lag included in many models is entirely justifiable, whether a model must incorporate equal roots, or if the feedback of higher order state variables cannot be included based on our maximum measurement capability. Thus, it is desirable that in addition to providing a good fit to empirical results, that the form of the model be consistent with theoretical requirements. (Another example of this requirement, is that transfer functions fitted to partial frequency response data, may require modification to provide stable response).

## Model Goodness-of-fit

It is clear that even if we restrict attention to the accuracy of reproducing the original input-output time histories, that a good fit should be sought at several levels. If the concept of state is at all reasonable, it should be apparent that a number of aspects of performance must be specified to completely describe a given system. If a given model of human operator control is of higher order than the first degree, then requiring only a fit to the error signal, such as a minimum mean squared error fit, is ignoring many critical aspects of control behavior. While many methods have theoretical foundations which require a given type fit, the efficacy of the model should be checked by comparison against higher order state variables.

## Calculation of the Bases for Optimal Performance

The technique for calculating optimal loss coefficients used in this study requires that the control system be linear, that control is based on a feedback of the entire state variable, that performance is optimized on the basis of quadratic loss functions and assumes a regulator control. Further, as presented here, it is only possible to calculate the diagonal terms of the weighting matrix. Clearly a more general technique would be desirable to delimit the full range of nontrivial performance indices which a given control system may optimize. The only virtue of the present technique is that it may define in at least one way a given system is optimum, and in the case of manual control system theory this may prove invaluable. Further testing with an extensive base of data is required to evaluate the worth of this and other feasible techniques.

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# APPENDIX

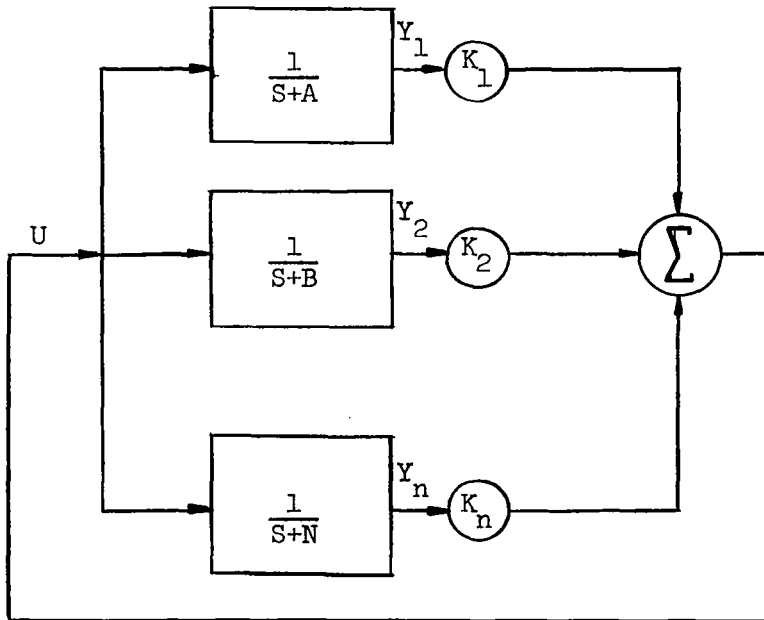## CALCULATION OF OPTIMAL LOSS FUNCTION COEFFICIENTS

1. Given the linear system
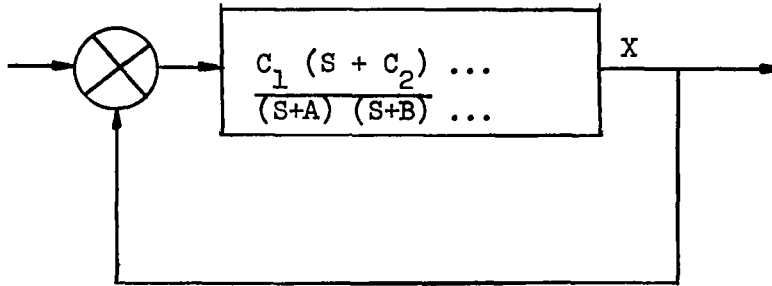


2. Through the partial-fraction expansion,

$$\frac{C_1 (S + C_2) \dots}{(S+A) (S+B) \dots} = \frac{K_1}{S+A} + \frac{K_2}{S+B} + \dots \text{ create the equivalent block diagram:}$$

3. Here the state variables, Y, are apparent, and there exists a relationship $X = TY$.

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{bmatrix} = \begin{bmatrix} X(t) \\ \dot{X}(t) \\ \cdot \\ \cdot \\ \cdot \\ X^{(n)}(t) \end{bmatrix}$$

4. From the last block diagram, we may write directly

$$Y = \begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{bmatrix} = \begin{bmatrix} -A & & \bigcirc \\ & -B & \\ & & \cdot \\ & & \cdot \\ \bigcirc & & \cdot \\ & & -N \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} U$$

$$= AY + GU$$

$$U = - \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{bmatrix} = -KY$$

5.  Since $U = -R^{-1}G^{T}PY$ if the control is optimum, and $U = -KY$: $K = R^{-1}G^{T}P$

6.  With $\dot{P} = 0$, the Riccati equation becomes

$$0 = PGR^{-1}G^{T}P - PA - A^{T}P - Q$$

and substituting $K = R^{-1}G^{T}P$

$$0 = PGK - PA - A^{T}P - Q$$

7.  In this equation, Q corresponds to the Y state variables and the performance index

$$J = \tfrac{1}{2}\int_{0}^{T} (Y^{T}QY + U^{2})\, dt$$

we wish to solve for q of the performance index:

$$J = \tfrac{1}{2}\int_{0}^{T} (X^{T}qX + U^{2})\, dt$$

but since $X = TY$

$$X^{T}qX = (TY)^{T}q\, (TY) = Y^{T}(T^{T}qT)\, Y = Y^{T}QY$$

$$\therefore\quad Q = T^{T}qT$$

8.  It remains to solve the simultaneous linear equations,

$$0 = PGK - PA - A^{T}P - T^{T}qT$$

for the loss coefficients q.