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EFFECTS OF RADIATION FORCES UPON THE ATTITUDE OF AN ARTIFICIAL EARTH SATELLITE

by

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ABSTRACT

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The motion of an artifical earth satellite about its center of mass as a result of radiation forces is investigated. The satellite is assumed to be symmetrical, both geometrically and dynamically. The sources of radiation considered are direct solar radiation, solar radiation reflected by the earth and its atmosphere, and direct radiation from the earth. For these three cases the forces and torques acting on an arbitrarily shaped satellite are derived. The results are in integral form and are dependent upon the satellite's surface geometry as well as upon its orientation relative to the earth and sun and its mass distribution.

The effects of the radiation torques upon the otherwise unperturbed motion of the satellite about its center of mass can be divided into two motions; motion of the axis of symmetry about the satellite's angular momentum vector (relative to its center of mass), and motion of the angular momentum vector. For the sources considered the motion about the angular momentum vector is an unperturbed Euler-Poinsot motion and the angular momentum vector itself, while remaining practically constant in magnitude, precesses and nutates relative to inertial space. This latter motion is described relative to the earth-sun line for direct solar radiation and relative to a perigee coordinate system for reflected solar and direct earth radiation.

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NOTATION

A. Coordinate Systems

1. Equatorial Reference (X_o, Y_o, Z_o) - An inertial coordinate system with origin located at the center of the earth; the Z_oX_o plane the equatorial plane with Z_o directed towards the Vernal Equinox. Unit vectors $\underline{I}_o$, $\underline{J}_o$, $\underline{K}_o$.
2. Ecliptic Reference (X_o', Y_o', Z_o') - An inertial coordinate system with origin at the center of the earth; the $Z_o'X_o'$ plane the ecliptic plane with Z_o' directed towards the Vernal Equinox. Unit vectors $\underline{I}_o'$, $\underline{J}_o'$, $\underline{K}_o'$.
3. Perigee Reference (X,Y,Z) - A non-inertial coordinate system with origin at the satellite's center of mass; the ZX plane the orbit plane with Z parallel to the position vector at perigee, X parallel to the tangent to the orbit at perigee. Unit vectors $\underline{I}$, $\underline{J}$, $\underline{K}$.
4. Body-Fixed Reference (x,y,z) - A rotating coordinate system with origin at the satellite's center of mass and directed along the satellite's Principal Moments of Inertia directions. Unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$.

$\overline{I}$	Inertia dyadic of the satellite about its CM
I	Radiation intensity
J	Radiation flux
k	R_e/r
L	Lyapunov function
$\underline{\ell}$, $\underline{\ell}'$	Unit vectors, refer to Figure 5
$\underline{M}$	Moment of radiation forces
$\underline{M}_1,\underline{M}_2,\underline{M}_3$	Components of $\underline{M}$ due to direct solar, reflected solar, and direct earth radiation
$\underline{n}$	Unit normal to element of surface area
P	Period of satellite in its orbit
$\underline{p}$	Position vector of CP relative to CM
$\underline{r}$	Position vector of satellite's center of mass relative to center of earth
r	$\lvert\underline{r}\rvert$
$\underline{r}'$	Position vector of element of satellite's surface area relative to CM
$\overline{R}$	Radius of the sphere taken as model for the earth and its atmosphere
R_e	Average radius of the earth
R_{e-S}	Mean earth-sun distance
R_{s-S}	Satellite-sun distance
R_s	Radius of sun
S_s	Solar constant at the mean earth-sun distance
S_s'	Solar constant at the satellite
s	Satellite spin

T	Absolute temperature
t	Time
V	Potential of the moments about the CM
γ, θ, χ	Euler angles between (x,y,z) and (X_o', Y_o', Z_o')
δ	Shadow factor; unity when satellite is in sunlight, zero when it is in the earth's shadow
Δ	Inclination of the ecliptic plane to the equatorial plane
ϵ	Emissivity
φ	Angle between $\underline{H}$ and $\underline{\omega}_e$
φ'	Angle between $\underline{H}$ and $\underline{\omega}_o$
η_1	Angle between surface normal and line from dA_e to the sun
η_2	Angle between surface normal and line from dA_s to the sun
θ	Angle between z-axis and $\underline{e}_s$
θ'	Angle between z-axis and $\underline{e}_r$
λ	Angle between projection of $\underline{H}$ on $\underline{e}_1\underline{e}_2$ plane and $\underline{e}_2$
λ'	Angle between projection of $\underline{H}$ on YZ plane and $\underline{K}$
Λ	Longitude of the sun in the ecliptic plane from Vernal Equinox
μ	Angle between $\underline{H}$ and $\underline{e}_s$
μ'	Angle between $\underline{H}$ and $\underline{I}$
ξ_1, ξ_2	Angles between surface normals and the line connecting dA_e and dA_s

ρ', α', τ'	Reflectivity, absorptivity, and transmissibility of satellite's surface to the earth's (low temperature) radiation
ρ, α, τ	Reflectivity, absorptivity, and transmissibility of satellite's surface to the sun's (high temperature) radiation
$\bar{\rho}$	Average reflectivity of the earth and its atmosphere
σ	Stefan-Boltzmann constant
ϕ	Angle between $\underline{H}$ and z-axis of satellite
Φ	Angle between $\underline{e}_r$ and $\underline{e}_s$
ψ	Angle between the plane of $\underline{H}$ and $\underline{e}_s$ and the plane of $\underline{H}$ and $\underline{k}$
ψ'	Angle between the plane of $\underline{H}$ and $\underline{I}$ and the plane of $\underline{H}$ and $\underline{k}$
Ω	Longitude of Ascending Node of satellite's orbit from Vernal Equinox
ω	Longitude of Perigee from Ascending Node
$\underline{\omega}_e$	Angular velocity of $\underline{e}_s$ relative to inertial space
$\underline{\omega}_o$	Angular velocity of CM about the earth
$\underline{\omega}_p$	Angular velocity of (XYZ) relative to inertial space
$\underline{\omega}_s$	Angular velocity of satellite about its center of mass relative to inertial space

Special attention is called to Figures 1, 5, 7, 8, and 9 in conjunction with the above symbols.

C. Derivatives

The time derivatives of scalar quantities will be denoted by d/dt or $(\dot{\ })$ while the derivatives of vector quantities will carry a subscript (f or r) to denote a derivative in inertial space or a derivative in the body-fixed rotating coordinate system. For example $d\underline{H}/dt)_f = d\underline{H}/dt)_r + \underline{\omega}_s \times \underline{H}$.

Any other necessary notation will be defined in the text.

I. INTRODUCTION

The many recent uses of artificial earth satellites which require a particular attitude of the vehicle (such as the Tiros weather satellites, the Orbiting Astronomical Observatory satellite, and those making use of solar energy) have stimulated interest in the satellite attitude problem. In order to design an effective attitude control system it is first necessary to determine the perturbing torques acting on a particular satellite configuration and then to analyze the effect of these torques upon the satellite's attitude.

The most important perturbing torques in any given case will depend on the particular satellite configuration, the satellite's orbital elements, and the mission of the vehicle. The sources of attitude perturbing torques presently being considered are:

1. Earth's magnetic and electric fields
2. Gravitational fields of the earth and of other celestial bodies
3. Moving parts on or in the satellite
4. Atmospheric drag
5. Meteoroid and Cosmic-ray bombardment
6. Electromagnetic radiation from the sun, earth, and satellite
7. Non-uniform rotation of reference coordinates
8. Gyroscopic effects.

Roberson (1)* has noted the first seven sources and has listed some qualitative order-of-magnitude results. Kershner and Newton (2) qualitatively discuss sources 1-6 and 8 and include a simplified analysis of the effects of magnetic tourques upon satellite spin.

Beletskii (3) and (4) considers the effects of gravity due to a spherical earth, atmospheric drag, and coordinate rotation and by a perturbation method he describes the resulting satellite motion. Colombo (5), in a similar analysis, adds the effects of the earth's magnetic field and the gyroscopic motion resulting from energy dissipation due to internal vibrations. In a different approach, Hagihara (6) linearizes the equations of motion and investigates earth's gravity, atmospheric drag, and magnetic field effects. Roberson (7) formulates the torques arising from certain types of internal moving parts but makes no attempt to solve the resulting equations of motion. The effects of dissipation of energy by elastic vibrations induced by gyroscopic moments has recently been studied by Thomson and Reiter (8) and Meirovitch (9).

The radiation effect has been noted by Roberson (1) but he merely lists some order-of-magnitude estimates. Holl (10), considering direct solar radiation only, assumes the resulting force acting on the satellite can be expressed as $F = p_o A C_f$

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* Numbers in paranthesises and underlined refer to references listed in the Bibliography.

(A = projected area, p_o = radiation pressure in the vicinity of the earth) where the "radiation force coefficient" C_f is found to be within the limits $0 < C_f \leq 2$ for several convex shapes. McElvain (11) appears to be the first one to derive analytical expressions for the force and torque acting on an arbitrarily shaped body due to direct solar radiation. He then determines, for two particular satellite shapes, the change in the satellite's angular momentum necessary to maintain a specified vehicle orientation. There has also been some analysis of the use of radiation forces for satellite stabilization, such as that given by Sohn (12) and Newton (13). A related problem of thermal radiation incident upon an earth satellite has received considerable attention. A few of those who have studied the problem are Cunningham (14) and (15), Katz (16), Altshuler (17), Mark and Ostrach (18), and Wood and Carter (19).

With the exception of the paper by McElvain on direct solar radiation, there appears to be no complete analytical formulation of the forces and torques acting on an artificial earth satellite as a result of incident radiation and there appears to be, without exception, no analytical description of the effects of such torques upon the attitude of the satellite.

The purpose of this analysis, therefore, is to attempt to fill this gap by:

1. Deriving the forces and torques acting on an arbitrarily shaped artificial earth satellite as a result of: (i) direct solar radiation, (ii) solar radiation reflected by the earth

and its atmosphere, and (iii) direct earth radiation.

2. Determining the effects of such radiation torques upon the motion of the satellite about its center of mass, specific configurations being used as examples.

II. FORMULATION OF RADIATION FORCES AND TORQUES

Since Maxwell's formulation of the theory of electromagnetic radiation it has been well known and often quoted that a black surface (one which completely absorbs all radiation impinging on it) whose normal makes an angle β with the incoming radiation experiences a pressure given by

$$p = (J'/c) \cos^2\beta \tag{1}$$

where $J' = J/\cos\beta$ (J = radiation flux) and c is the speed of electromagnetic radiation.* But this effect is not a pressure in the usual sense of the word, rather only the normal component of the resultant force per unit area acting on the surface. In general the force caused by radiation is not normal to the receiving area on which it acts. The resulting force will depend on the radiation flux and the orientation of the receiving area but it will also depend on the radiation properties of the surface (i.e. ρ, α, and τ).

The chief sources of radiation that will cause forces and possibly torques on an artificial earth satellite are:

1. Direct solar radiation
2. Solar radiation reflected by the earth and its atmosphere

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* This effect was first demonstrated experimentally by Lebedew (1901) and later in an extensive series of tests by Nichols and Hull (1903).

3. Direct radiation from the earth and its atmosphere

4. Radiation emitted by the satellite.

The analysis of the last case requires a knowledge of the temperature in the satellite's shell and, in general, it is a much smaller effect than the first three cases. There are special situations, however, when this effect may be comparable to the other three; for example, the pressure experienced by emission from a "black" flat plate receiving solar radiation, and thus emitting an equal amount of radiation, is equal to 2/3 the pressure due to direct solar radiation. In general the satellite does not act as a "black body" (as it will reflect as well as conduct portions of the incident energy) and furthermore it emits radiation in all directions, thus tending to cancel any resulting force, and for these reasons this effect will be neglected in the following analysis.

The first three sources of radiation are all exterior to the satellite and are covered by the following general example, see Figure 2. The incoming radiation of intensity I (parallel to $\underline{e}_1$) makes an angle β_1 with the normal to the surface element dA which has radiation properties ρ, α, and τ (where $\rho + \alpha + \tau = 1$). The radiation, after striking the surface, is partly transmitted through the material, partly absorbed by the material, and partly reflected by the surface. The reflected radiation (parallel to $\underline{e}_2$) makes an angle β_2 with the surface normal. The unit vector $\underline{e}_3$ is perpendicular to $\underline{n}$ and coplanar with $\underline{n}$, $\underline{e}_1$, and $\underline{e}_2$ and can be defined by the equation

$$\underline{e}_3 = \underline{n} \times (\underline{e}_1 \times \underline{n})/\sin \beta_1. \tag{2}$$

If the surface is assumed to be a specular reflector geometric optics yields

$$\beta_1 = \beta_2 = \beta. \tag{3}$$

The incident radiation produces a force

$$d\underline{F}_i = -(\alpha + \rho)\ (J'/c)\ \cos\beta\ dA\ \underline{e}_1 \tag{4}$$

while the reflected radiation produces a force

$$d\underline{F}_r = -\rho\,(J'/c)\ \cos\beta\ dA\ \underline{e}_2. \tag{5}$$

Thus the total force on the element of area dA is

$$d\underline{F} = -(J'/c)\ \cos\beta\ dA\ \left[(\alpha + \rho)\ \underline{e}_1 + \rho\ \underline{e}_2\right] \tag{6}$$

or, in terms of the unit vectors $\underline{n}$ and $\underline{e}_3$

$$d\underline{F} = -(J'/c)\ \cos\beta\ dA\ \left[(\alpha + 2\rho)\ \cos\beta\ \underline{n} + \alpha\underline{n}\times\ (\underline{e}_1 \times \underline{n})\right]. \tag{7}$$

Note the two special cases:

i. Perfect Reflector ($\rho = 1,\ \alpha = \tau = 0$)

From equation (7)

$$d\underline{F} = -2(J'/c)\ \cos^2\beta\ dA\ \underline{n}$$

and the total force acts normal to the surface.

ii. Perfect Absorber ($\alpha = 1,\ \rho = \tau = 0$)

From equation (6)

$$d\underline{F} = -(J'/c)\ \cos\beta\ dA\ \underline{e}_1$$

4. The solar radiation reflected by the earth and its atmosphere can be represented by reflection from a sphere of radius $\bar{R}$ with reflectivity $\bar{\rho}$.
5. Direct radiation from the earth is not effected by the earth's atmosphere.
6. The satellite's surface is everywhere convex.
7. The satellite's surface acts as a specular reflector.
8. The satellite has both dynamic and geometric symmetry (z-axis is the axis for both).
9. Radiation flux is independent of frequency.
10. Earth's orbit around the sun is circular.

Note that assumptions 4 and 5 are definitely intended for circumventing the problem of scattering and absorption of radiation by the earth's atmosphere. This is a separate problem in itself and there is no intention of considering it in this analysis.

The radiation flux at dA_2 due to uniform, diffuse emission from the surface element dA_1, see Figure 3, is

$$dJ_{2-1} = (\epsilon_1 \sigma T_1^4/\pi L^2) \cos\beta_1 \cos\beta_2 \, dA_1. \tag{8}$$

The total radiation flux at dA_2 due to the body No. 1 is

$$J_{2-1} = \int_{A_1^*} (\epsilon_1 \sigma T_1^4/\pi L^2) \cos\beta_1 \cos\beta_2 \, dA_1 \tag{9}$$

where the integration is carried over the surface area of body No. 1 that is "seen" by the element of area dA_2. That is, over all elements of area of body No. 1 such that $0 \leq \beta_1 \leq \pi/2$, and $0 \leq \beta_2 \leq \pi/2$.

As an example consider the radiation flux due to the sun on a flat plate whose normal is parallel to the line joining the centers of mass of the two bodies. Referring to Figure 4,

$$J_{2-1} \cong J_s = \int_{A_s^*} (\sigma T_s^4/\pi L^2) \cos\beta_1 \cos\beta_2 \, dA_s.$$

If R is of the order R_{e-S} then $\beta_2 \cong 0$, and $L \cong R$ so that

$$J_s \cong (\sigma T_s^4/\pi R^2) \int_{A_s^*} \cos \beta_1 \, dA_s.$$

and $\int_{A_s^*} \cos\beta_1 \, dA_s$ is the projected area of the sun as seen by the plate, approximately πR_s^2, thus

$$J_s = (\sigma T_s^4 R_s^2/R^2). \tag{10}$$

The total surface area of the sun is $A_s = 4\pi R_s^2$ so that equation (10) may be rewritten

$$J_s = (\sigma T_s^4 A_s/4\pi R^2). \tag{11}$$

This equation gives the flux of solar radiation of normal incidence on a flat plate at a distance R from the sun (when $R \gg R_s$). When $R = R_{e-S}$ this is usually referred to as the solar constant (S_s) and its value is 2.00 gm cal/cm^2 min (21).

On the basis of equations (7) and (8) and the stated assumptions it is now possible to write the total force acting on an element of surface area of an earth satellite as a result of the three sources of radiation. From Figure 5, where the spherical model of the earth and its atmosphere is shown as the earth itself

$$d\underline{F} = d\underline{F}_1 + d\underline{F}_2 + d\underline{F}_3$$

where

$$d\underline{F}_1 = -\delta\left(\frac{S_s'}{c}\right)\left[\cos^2\eta_2(1+\rho)\underline{n} + \cos\eta_2(1-\rho)\,\underline{n} \times (\underline{e}_s'' \times \underline{n})\right] dA_s \tag{12}$$

and the total force acts parallel to the incident radiation.

If only the normal component of equation (7) is considered and the surface is assumed to be a perfect absorber the "Radiation Pressure" of equation (1) is obtained.

In general the radiation flux J depends on the shape, temperature, and emissivity of the radiation source as well as the position of dA relative to it. Similarly ρ, α, and τ for the receiving surface depend on the temperature of dA, the wavelength of the incident radiation and the angle of incidence. It is also possible that the surface properties (including ρ, α, and τ) of a particular satellite may change with time as a result of external factors. Such a change in surface composition has been noted in the first Soviet satellite by Yatsunskii and Gurko (20).

In this analysis ρ, α, and τ will be assumed constant for solar radiation, both direct and earth-reflected, while for the earth's radiation a different set of values will be taken (ρ', α', τ') but they also will be assumed constant. The other assumptions made in this analysis are:

1. The transmissibility of the satellite's surface is zero ($\rho + \alpha = 1$).
2. The sun and the earth act as diffusely radiating bodies, that is the radiation from these bodies obeys Lambert's Cosine Law.
3. Emissivity ϵ and absolute temperature T are constant for both the sun and the earth ($\epsilon_{sun} = 1$).

is the force due to direct solar radiation,

$$d\underline{F}_2 = -\delta \int_{\bar{A}_e}\left(\frac{\bar{\rho} S_s}{c\pi L^2}\right)\cos\eta \cos\xi_1\left[\cos^2\xi_2(1+\rho)\underline{n} + \cos\xi_2(1-\rho)\underline{n}\times(\underline{\ell}\times\underline{n})\right]d\bar{A}_e\, dA_s \qquad (13)$$

is the force due to solar radiation reflected by the earth and its atmosphere (reflected from a sphere of radius $\bar{R}$), and

$$d\underline{F}_3 = -\int_{A_e}\left(\frac{\epsilon_e \sigma T_e^4}{c\pi L^2}\right)\cos\xi_1\left[\cos^2\xi_2(1+\rho')\underline{n} + \cos\xi_2(1-\rho')\underline{n}\times(\underline{\ell}\times\underline{n})\right]dA_e\, dA_s \qquad (14)$$

is the force due to direct radiation from the earth. The integration in equation (13) is over the surface area of the earth-atmosphere model sphere seen by dA_s and not in the earth's shadow. In equation (14) the integration is over the surface area of the earth seen by dA_s. Note that in general the extent of these surface integrations will depend not only on the relative positions of the earth, sun and satellite but also on the particular element of area dA_s.

The unit vector $\underline{e}_s''$ in equation (12) can be replaced by the unit vector $\underline{e}_s'$ as the dimensions of the satellite are negligible compared to r and R_{s-S}. For the same reason and to simplify the integrations each element of surface area dA_s can be moved parallel to itself to the CM, thus replacing L, $\underline{\ell}$, ξ_1, and ξ_2 with L', $\underline{\ell}'$, ξ_1', ξ_2' in equations (13) and (14), see Figure 5. Also note that S_s' (the solar constant at the satellite) appearing in equation (12) can be related to S_s (the usual solar constant) as follows

$$S_s' = S_s\,(R_{e-S}/R_{s-S})^2$$

or $$S_s' = S_s/\left[1-2(r/R_{e-S})\cos\Phi + (r/R_{e-S})^2\right] \tag{15}$$

thus $$S_s' = S_s\left[1 + 2(r/R_{e-S})\cos\Phi + 0\,(r^2/R_{e-S}^2)\right]. \tag{16}$$

For illustration take $R_{e-S} = 93 \times 10^6$ mi and $r = 46{,}500$ mi so that equation (16) gives

$$S_s' = S_s\left[1 + 10^{-3}\cos\Phi + 0(10^{-6})\right].$$

Because of this small difference between S_s' and S_s the latter value will be used for the remainder of this analysis.

Incorporating the above simplifications into equations (12), (13), and (14)

$$d\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)\left[\cos^2\eta_2(1+\rho)\underline{n} + \cos\eta_2(1-\rho)\underline{n}\times(\underline{e}_s'\times\underline{n})\right]dA_s \tag{17}$$

$$d\underline{F}_2 = -\delta\int_{A_e}\left(\frac{\bar{\rho}S_s}{c\pi L'^2}\right)\cos\eta\cos\xi_1'\left[\cos^2\xi_2'(1+\rho)\underline{n}+\cos\xi_2'(1-\rho)\underline{n}\times(\underline{\ell}'\times\underline{n})\right]d\bar{A}_e\,dA_s \tag{18}$$

$$d\underline{F}_3 = -\int_{A_e}\left(\frac{\epsilon_e\sigma T_e^4}{c\pi L'^2}\right)\cos\xi_1'\left[\cos^2\xi_2'(1+\rho)\underline{n}+\cos\xi_2'(1-\rho)\underline{n}\times(\underline{\ell}'\times\underline{n})\right]dA_e\,dA_s\,. \tag{19}$$

The total force acting on the satellite due to the three sources of radiation is

$$\underline{F} = \int_{A_1^*} d\underline{F}_1 + \int_{A_2^*} d\underline{F}_2 + \int_{A_3^*} d\underline{F}_3 \tag{20}$$

where: A_1^* is the total surface area of the satellite seen by the sun,

A_2^* is the total surface area of the satellite seen by that part of the earth not in the earth's shadow, and

A_3^* is the total surface area of the satellite seen by the earth.

The total external moment about the satellite's center of mass is

$$\underline{M} = \int_{A_1^*} \underline{r}' x d\underline{F}_1 + \int_{A_2^*} \underline{r}' x d\underline{F}_2 + \int_{A_3^*} \underline{r}' x d\underline{F}_3. \qquad (21)$$

If the satellite's surface can be divided into a number of subsections or if the satellite's reflectivity varies from section to section, equation (21) may be replaced by

$$M = \sum_{A_{1_i}^*} \underline{r}_i' x d\underline{F}_{1_i} + \sum_{A_{2_i}^*} \underline{r}_1' x d\underline{F}_{2_i} + \sum_{A_{3_i}^*} \underline{r}_1' x d\underline{F}_{3_i} \qquad (22)$$

where $\underline{F}_{1_i}$ is the force due to direct solar radiation on the surface A_i,

$\underline{F}_{2_i}$ is the force due to reflected solar radiation on the surface A_i,

$\underline{F}_{3_i}$ is the force due to direct earth radiation on the surface A_i, and

$\underline{r}_i'$ is the position vector of the center of pressure of the surface A_i relative to the center of mass of the satellite.

Equations (20) and (21) with equations (17), (18), and (19) formally give the forces and torques acting on an earth satellite, under the stated assumptions, as a result of direct solar radiation, solar radiation reflected by the earth and its atmosphere, and direct radiation from the earth. The surface integrations that are involved for an arbitrarily shaped satellite are very complicated (especially for the determination of $\underline{F}_2$ and $\underline{F}_3$) and except for a few special cases analytical expressions for equations (20) and (21) appear to be unobtainable. The two papers by Cunningham (14) and (15), on a related problem, indicate the nature of this difficulty.

Before proceeding to the analysis of the effects of these torques upon satellite attitude consider the following simple example which will indicate the magnitude of the forces involved. Consider a flat plate of cross-sectional area A situated on the earth-sun line and with its normal directed towards the earth, parallel to the earth-sun line, see Figure 6. As the plate is not in the earth's shadow $\delta = 1$ and equations (17), (18), and (19) become

$$\underline{F}_1 = (S_s/c)\ (1 + \rho) A\ \underline{n} \tag{23}$$

$$\underline{F}_2 = -\int_{\bar{A}_e} \left(\frac{\bar{\rho}\, S_s}{c \pi L'^2}\right) \cos\xi_1' \cos\eta_1 \cos^2\xi_2' (1+\rho)\, d\bar{A}_e\ A\ \underline{n} \tag{24}$$

$$\underline{F}_3 = -\int_{A_e} \left(\frac{\epsilon_e \sigma T_e^4}{c \pi L'^2}\right) \cos\xi_1' \cos^2\xi_2' (1+\rho')\, dA_e\ A\ \underline{n} \tag{25}$$

where it has been noted that $\cos^2\eta_2 = 1$ and that there are no components of force parallel to the plate because of symmetry. These integrals have been worked out in Appendix A and the results are

$$\underline{F}_1 = (S_s/c)\ (1 + \rho) A\ \underline{n} \tag{26}$$

$$\underline{F}_2 = -2(\bar{\rho}\ S_s/c)\ (1 + \rho) I_1 A\ \underline{n} \tag{27}$$

$$\underline{F}_3 = -2(\epsilon_e \sigma T_e^4/c)\ (1 + \rho') I_2 A\ \underline{n} \tag{28}$$

where $$I_1 = (1/15k)\left[2(1-k^5) + 5k^3 - (2 + 3k^2)(1-k^2)^{3/2}\right] \quad (29)$$

$$I_2 = 1/3\left[1 - (1-k^2)^{3/2}\right] \quad (30)$$

$$k = R_e/r$$

and $\bar{R}$ has been taken equal to R_e (that is the spherical model for the earth and its atmosphere has been taken as the earth itself). For numerical results consider the following parameter values:

$R_e = 4000$ mi

$S_s = 2.00$ gm cal/cm^2 min

$c = 3 \times 10^{10}$ cm/sec

$\bar{\rho} = 0.34$

$\epsilon_e = 1.0$

$T_e = 250^\circ K$, $300^\circ K$ (two different values used)

$\sigma = 5.6742 \times 10^{-5}$ erg/cm^2 sec $^\circ K^4$

$\rho = \rho' = 0.$

The results of substituting these values into equations (26), (27), and (28) are listed in Table I, Appendix A for various satellite altitudes.

III. EFFECTS OF TORQUES UPON SATELLITE ATTITUDE

In general there are many different perturbing forces and torques acting on an earth satellite and the resulting motion will depend on all of these forces and torques. All of these perturbations are usually small, relative to the unperturbed motion, and generally they are investigated separately with the total motion being determined by adding the individual results. In this chapter the effects of radiation forces on the motion about its center of mass of an otherwise unperturbed satellite* will be investigated and, assuming no coupling with the other perturbations, the results may be added to the already existing equations for the effects of the earth's gravity, atmospheric drag, etc. which have been analyzed by Beletskii (3), Colombo (5), and others. Direct solar radiation contributes the major component of force and it will be considered first.

A. Direct Solar Radiation

1. Derivation of equations

From equations (20) and (17)

$$\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)\iint_{A_1^*}\left[\cos^2\eta_2\,(1+\rho)\underline{n} + \cos\eta_2\,(1-\rho)\,\underline{n}\times(\underline{e}_s'\times\underline{n})\right]dA_s \qquad (31)$$

where A_1^* is the surface area of the satellite "seen" by the sun.

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* The satellite's unperturbed motion, $\underline{M} = 0$, will be characterized by $\underline{H}$ being constant in inertial space and a regular procession of the axis of symmetry about $\underline{H}$.

From equation (21) the moment about the satellite's center of mass due to $\underline{F}_1$ is

$$\underline{M}_1 = -\delta\left(\frac{S_s}{c}\right)\int_{A_1^*} \underline{r}' \times \left[\cos^2\eta\,(1+\rho)\underline{n} + \cos\eta\,(1-\rho)\underline{n}\times(\underline{e}_s' \times \underline{n})\right] dA_s \quad (32)$$

The unit vector $\underline{e}_s'$ appearing in equations (31) and (32) can be related to more useful vectors by referring to Figure 5.

$$R_{s-S}\,\underline{e}_s' + \underline{r} = R_{e-S}\,\underline{e}_s$$

or

$$\underline{e}_s' = (R_{e-S}/R_{s-S})\underline{e}_s - (r/R_{s-S})\underline{e}_r$$

but

$$R_{e-S}/R_{s-S} = \left[1-2(r/R_{e-S})\cos\Phi + (r/R_{e-S})^2\right]^{-1/2}$$

and

$$r/R_{s-S} = (r/R_{e-S})\,(R_{e-S}/R_{s-S}).$$

Thus

$$\underline{e}_s' = \left[1 + (r/R_{e-S})\cos\Phi + 0(r^2/R_{e-S}^2)\right]\underline{e}_s - \left[(r/R_{e-S}) + 0(r^2/R_{e-S}^2)\right]\underline{e}_r . \quad (33)$$

For $r = 25{,}000$ mi., $r/R_{e-S} = 0.00027$. Thus this difference will be neglected and $\underline{e}_s'$ will be taken equal to $\underline{e}_s$.

Before proceeding to particular satellite shapes consider the following general analysis. As the only configurations under consideration are those with both geometric and dynamic symmetry it would be plausible to assume $\underline{F}_1$ of the form

$$\underline{F}_1 = -\delta\,(F_1'\,\underline{e}_s + F_1''\,\underline{k}) . \quad (34)$$

The first term, $-\delta F_1' \underline{e}_s$, would follow directly from equation (31) if $\rho = 0$, while if $\rho \neq 0$ the combined result would apply as $\underline{e}_s$ and $\underline{k}$ are the only preferential directions. Also because of the symmetry the CP will lie on the axis of symmetry and the moment about the CM due to $\underline{F}_1$ will be

$$\underline{M}_1 = p\underline{k} \times \underline{F}_1 \tag{35}$$

where $p\underline{k} = \underline{p}$, the position vector of the CP relative to the CM. From equations (34) and (35)

$$\underline{M}_1 = -\delta F_1' \, p \, \underline{k} \times \underline{e}_s \tag{36}$$

and finally p and F_1' will be functions only of the angle between $\underline{e}_s$ and $\underline{k}$, that is

$$\underline{M}_1(\theta) = -\delta F_1'(\theta) p(\theta) \, \underline{k} \times \underline{e}_s \tag{37}$$

where $$\cos\theta = \underline{k} \cdot \underline{e}_s \, . \tag{38}$$

The equation of motion about the CM is

$$d\underline{H}/dt)_f = \underline{M}_1 \tag{39}$$

or $$d\underline{H}/dt)_r + \underline{\omega}_s \times \underline{H} = \underline{M}_1 \tag{40}$$

with $$\underline{H} = \bar{I} \cdot \underline{\omega}_s$$

$$\underline{\omega}_s = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$

where $\underline{\omega}_s$ is the angular velocity of (x,y,z) relative to inertial space. Equation (40) may be written out in scalar form to give the usual Euler equations

$$
\begin{aligned}
A(d\omega_x/dt) &+ (C-B)\omega_y\omega_z = \underline{M}_1 \cdot \underline{i} \\
B(d\omega_y/dt) &+ (A-C)\omega_x\omega_z = \underline{M}_1 \cdot \underline{j} \\
C(d\omega_z/dt) &+ (B-A)\omega_x\omega_y = \underline{M}_1 \cdot \underline{k} \ .
\end{aligned}
\tag{41}
$$

From equation (37) $\underline{M}_1 \cdot \underline{k} = 0$ and from the assumed dynamic symmetry $A = B$, thus there is one integral of equation (39) or equations (41)

$$\underline{H} \cdot \underline{k} = h_1 \tag{42}$$

or $$\omega_z = s, \text{ a constant.}$$

But the moment $\underline{M}_1$ has been assumed to depend only on the angle θ thus, if M_1 is derivable from a potential, there is an additional integral of the motion. That is, if

$$M_1(\theta) = -\partial V/\partial\theta \tag{43}$$

then $$\tfrac{1}{2}\underline{H} \cdot \underline{\omega}_s - \underline{H} \cdot \underline{\omega}_e + V(\theta) = h_2 \tag{44}$$

which expresses Conservation of Energy in the rotating coordinate system (x,y,z). Because of the dynamic symmetry and the fact $\underline{\omega}_e$ is constant equation (44) may be rewritten in the scalar form

$$H^2 - 2A\omega_e \cos\gamma\, H + \left[2AH_o\omega_e \cos\gamma_o - H_o^2 + 2A(V-V_o)\right] = 0 \tag{45}$$

where H_o, V_o, and γ_o are the initial (or unperturbed) values of H, V, and γ, and γ is the angle between $\underline{H}$ and $\underline{\omega}_o$. Equation (45) is a quadratic equation for H which yields

$$H = H_o\left\{\frac{A\omega_e\cos\gamma}{H_o} + \sqrt{1 + \left[\left(\frac{A\omega_e\cos\gamma}{H_o}\right)^2 - 2\frac{A\omega_e\cos\gamma_o}{H_o} - 2A\frac{(V-V_o)}{H_o^2}\right]}\right\} \tag{46}$$

or $$H = H_o\left\{1 + \frac{A\omega_e}{H_o}(\cos\mathcal{J} - \cos\mathcal{J}_o) - \frac{A}{H_o^2}(V - V_o) + O\left[\frac{A^2\omega_e}{H_o^2}\right] + O\left[\frac{A^2}{H_o^4}(V - V_o)^2\right]\right\} \tag{47}$$

and the positive sign has been taken for the square root so that H equals H_o when $\mathcal{J}$ and V equal $\mathcal{J}_o$ and V_o respectively. From equation (47) it can be noted that there are no secular changes in H to order H_o^{-1} (M_1 will be periodic in θ but V will not necessarily be periodic). At this point two different problems must be distinguished:

i. $$\frac{A\omega_e}{H_o}(\cos\mathcal{J} - \cos\mathcal{J}_o) - \frac{A}{H_o^2}(V - V_o) << 1 \tag{48}$$

This case corresponds to a satellite whose kinetic energy of rotation about its center of mass is large with respect to the work done by the external force. Note that V is proportional to S_s/c (approximately 9.4×10^{-8} lb/ft^2) and ω_e is about 1°/day. For this case H may be taken equal to its unperturbed value H_o.

ii. $$\frac{A\omega_e}{H_o}(\cos\mathcal{J} - \cos\mathcal{J}_o) - \frac{A}{H_o^2}(V - V_o) = O(1) \tag{49}$$

This case corresponds to a satellite whose kinetic energy of rotation about its center of mass is not large as compared to the work done by the external force; for example, V being a small quantity, a satellite placed in orbit with an angular velocity proportional to ω_e. In such a problem the orbital motion of the CM and the motion about the CM are coupled and a libration-type analysis would be necessary. An analysis of this type was carried out by Beletskii (22) for the case of perturbing torques caused by the earth's gravity.

In the present analysis the conditions of the first case are assumed and for a particular problem with given values of H_o, etc., the extremum values of equation (48) may be computed and compared to unity.

From equation (42)

$$\underline{H} \circ \underline{k} = Cs = H \cos\phi$$

where ϕ is the angle between $\underline{H}$ and $\underline{k}$. Thus for $H = H_o$

$$\cos\phi = Cs/H_o$$

or

$$\phi = \phi_o \tag{50}$$

its unperturbed value. The precession rate of the axis of symmetry about the angular momentum vector, $\dot{\psi}$, is given by

$$\dot{\psi} = H/A$$

when the angular velocity of the plane of $\underline{H}$ and $\underline{e}_s$ is neglected relative to $\dot{\psi}$ (see Figure 7). So if $\underline{H} = H_o$

$$\dot{\psi} = H_o/A \tag{51}$$

its unperturbed value. Thus there are no secular effects in the motion of the axis of symmetry about $\underline{H}$. Any secular effects in the motion must then be confined to the orientation of $\underline{H}$ in inertial space. The angular momentum vector may be expressed as $H_o\underline{h}$ where, from Figure 7,

$$\underline{h} = \cos\mu\ \underline{e}_s + \sin\mu\ (\sin\lambda\ \underline{e}_1 + \cos\lambda\ \underline{e}_2). \tag{52}$$

Then, by equation (39)

$$d\underline{h}/dt)_f = \underline{M}_1/H_o$$

thus $\quad \underline{e}_s \circ (d\underline{h}/dt)_f = 0$

because $\quad \underline{e}_s \circ \underline{M}_1 = 0.$

But $\quad d\underline{h}\circ \underline{e}_s/dt)_f = \underline{h} \circ (d\underline{e}_s/dt)_f + \underline{e}_s \circ (d\underline{h}/dt)_f$

and $\quad d\underline{e}_s/dt)_f = \underline{\omega}_e \times \underline{e}_s = -\omega_e \underline{e}_2$

where $\underline{\omega}_e = \omega_e \underline{e}_1.$ Therefore,

$$d\cos\mu/dt)_f = -\omega_e \underline{h} \circ \underline{e}_2$$

which upon substituting equation (52) becomes

$$d\mu/dt)_f = \omega_e \cos\lambda. \tag{53}$$

That is, except for the effect due to the rotation of $\underline{e}_s$, the angle between $\underline{H}$ and $\underline{e}_s$ remains constant.

From equations (39), (37), and (52)

$$-\delta \left[F_1'(\theta)p(\theta)/H_o\right] \underline{k}\times\underline{e}_s = d\underline{h}/dt)_f$$

and $-\delta \left[F_1'(\theta)p(\theta)/H_o\right] \underline{k}\times\underline{e}_s = \Big\{-\sin\mu \frac{d\mu}{dt}\underline{e}_s + \cos\mu\left(\frac{d\underline{e}_s}{dt}\right)_f$

$+\cos\mu\frac{d\mu}{dt}(\cos\lambda\,\underline{e}_2 + \sin\lambda\,\underline{e}_1) + \sin\mu(-\sin\lambda\,\underline{e}_2 + \cos\lambda\,\underline{e}_1)\frac{d\lambda}{dt}$

$+\sin\mu\left[\cos\lambda\left(\frac{d\underline{e}_2}{dt}\right)_f + \sin\lambda\left(\frac{d\underline{e}_1}{dt}\right)_f\right]\Big\}. \quad (54)$

But $\quad d\underline{e}_s/dt)_f = \underline{\omega}_e \times \underline{e}_s = -\omega_e \underline{e}_2$

$\quad d\underline{e}_1/dt)_f = 0$

$\quad d\underline{e}_2/dt)_f = \underline{\omega}_e \times \underline{e}_2 = \omega_e \underline{e}_s.$

Thus equation (54) leads to three scalar equations

$$-\sin\mu\frac{d\mu}{dt} + \omega_e \sin\mu\cos\lambda = -\delta\left[\frac{F_1'(\theta)p(\theta)}{H_o}\right](\underline{k}\times\underline{e}_s)\cdot\underline{e}_s \tag{55}$$

$$\sin\lambda\cos\mu\frac{d\mu}{dt} + \sin\mu\cos\lambda\frac{d\lambda}{dt} = -\delta\left[\frac{F_1'(\theta)p(\theta)}{H_0}\right](\underline{k}\times\underline{e}_s)\cdot\underline{e}_1 \quad (56)$$

$$-\omega_e\cos\mu + \cos\mu\cos\lambda\frac{d\mu}{dt} + \sin\mu\sin\lambda\frac{d\lambda}{dt} = -\delta\left[\frac{F_1'(\theta)p(\theta)}{H_0}\right](\underline{k}\times\underline{e}_s)\cdot\underline{e}_2 . \quad (57)$$

Equation (55) is identical to the result already obtained, that is equation (53). Substituting equation (53) into equations (56) and (57)

$$\cos\lambda\frac{d\lambda}{dt} = -\delta\left[\frac{F_1'(\theta)p(\theta)}{H_0\sin\mu}\right](\underline{k}\times\underline{e}_s)\cdot\underline{e}_1 - \omega_e\cot\mu\cos\lambda\sin\lambda \quad (58)$$

$$\sin\lambda\frac{d\lambda}{dt} = \delta\left[\frac{F_1'(\theta)p(\theta)}{H_0\sin\mu}\right](\underline{k}\times\underline{e}_s)\cdot\underline{e}_2 - \omega_e\cot\mu\sin^2\lambda . \quad (59)$$

Note that these are two equations in the one unknown $\dot{\lambda}$. This arises because of the assumption that $\underline{H} = H_o\underline{h}$, when in fact the magnitude of $\underline{H}$ is not constant but is given by equation (47). If $\underline{h}$ is to remain a unit vector the velocity of $\underline{H}$ must be perpendicular to $\underline{H}$ and the axis of symmetry must therefore lie in the plane of $\underline{H}$ and $\underline{e}_s$ (except for the special case $\mu = 0$), this is easily seen from equations (58) and (59) by solving for $\dot{\lambda}$ and equating the two results. But it has already been shown that the axis of symmetry precesses about $\underline{H}$ with a rate $\dot{\psi}$, which is equal to H_o/A, while $\underline{H}$ precesses about $\underline{e}_s$ with a rate which, from equations (58) and (59), is proportional to ω_e and $F_1'p/H_o$. The potential V is generally a very small quantity and $\omega_e \ll H_o/A$ from equation (48) thus $\dot{\lambda} \ll \dot{\psi}$ so that an average value for $\dot{\lambda}$ can be obtained from the above equations by averaging over a complete rotation of the z-axis about the angular momentum vector. Such an averaging process will eliminate the inconsistency in the equations for $\mathring{\lambda}$. In similar analyses for determining the effects of other types of perturbations upon satellite attitude this averaging is performed at the beginning of the formulation of the perturbing moments. This, in effect, replaces the instantaneous moments by an average moment. This method is employed by both Beletskii (3) and Colombo (5). With this in mind consider $\cos\lambda$ times equation (58) added to $\sin\lambda$ times equation (59), thus

$$\frac{d\lambda}{dt} = \delta \frac{F_1'(\theta)p(\theta)}{H_0 \sin\mu}\left[(\sin\lambda \underline{e}_2 - \cos\lambda \underline{e}_1)\cdot(\underline{k}\times\underline{e}_s)\right] - \omega_e \cot\mu \sin\lambda \,. \tag{60}$$

The average precession rate is

$$\overline{\frac{d\lambda}{dt}} = \frac{\delta}{2\pi H_0 \sin\mu}\int_0^{2\pi} F_1'(\theta)p(\theta)\left[(\sin\lambda \underline{e}_2 - \cos\lambda \underline{e}_1)\cdot(\underline{k}\times\underline{e}_s)\right]d\psi - \omega_e \cot\mu \sin\lambda \tag{61}$$

where $\qquad F_1'(\theta) = F_1'[\theta(\psi)]$

and $\qquad p(\theta) = p[\theta(\psi)]$.

But $\underline{e}_2\cdot(\underline{k}\times\underline{e}_s) = \underline{k}\cdot(\underline{e}_s\times\underline{e}_2)$ and $\underline{e}_1\cdot(\underline{k}\times\underline{e}_s) = \underline{k}\cdot(\underline{e}_s\times\underline{e}_1)$ so that equation (61) may be rewritten

$$\overline{\frac{d\lambda}{dt}} = -\frac{\delta}{2\pi H_0 \sin\mu}\int_0^{2\pi} F_1'(\theta)p(\theta)\left[(\sin\lambda \underline{e}_1 + \cos\lambda \underline{e}_2)\cdot\underline{k}\right]d\psi - \omega_e \cot\mu \sin\lambda \tag{62}$$

and from Figure 7

$$\begin{aligned}\underline{k} = {} & (\cos\phi\cos\mu + \sin\phi\sin\mu\cos\psi)\,\underline{e}_s + (\cos\phi\sin\mu\sin\lambda\, + \\ & + \sin\phi\cos\lambda\sin\psi - \sin\phi\cos\mu\sin\lambda\cos\psi)\,\underline{e}_1\, + \\ & + (\cos\phi\sin\mu\cos\lambda - \sin\phi\sin\lambda\sin\psi - \sin\phi\cos\mu\cos\lambda\cos\psi)\,\underline{e}_2\,. \end{aligned} \tag{63}$$

Since $\phi = \phi_o$, from equation (50), equation (62) is

$$\frac{\overline{d\lambda}}{dt} = - \frac{\delta}{2\pi H_o} \int_0^{2\pi} F_1'(\theta) p(\theta) \left[\cos\phi_o - \sin\phi_o \cot\mu \cos\psi \right] d\psi - \omega_e \cot\mu \sin\lambda \quad (64)$$

where μ is considered constant during the integration. Thus the motion of the satellite under the influence of direct solar radiation can be described as an unperturbed motion (ie. no secular effects) of the axis of symmetry about the angular momentum vector which itself remains practically constant in magnitude but precesses and nutates relative to $\underline{e}_s$. The average precession and nutation rates are given by equations (64) and (53) respectively. If the rotation of $\underline{e}_s$ can be neglected $\underline{H}$ performs a regular precession ($\mu = \mu_o$) about $\underline{e}_s$ with an average rate $\overline{d\lambda}/dt)_o$ which is given by

$$\left.\frac{\overline{d\lambda}}{dt}\right)_o = - \frac{\delta}{2\pi H_o} \int_0^{2\pi} F_1'(\theta) p(\theta) \left[\cos\phi_o - \sin\phi_o \cot\mu \cos\psi \right] d\psi. \quad (65)$$

Note that if the satellite enters the earth's shadow ($\delta = 0$) the motion becomes the unperturbed motion ($\underline{M}_1 = 0$) and $\underline{H}$ is constant in inertial space at its value upon entering the shadow.

2. Particular Satellite Configurations

a. Spherical Satellite - Radius R

The force due to direct solar radiation on a spherical satellite of radius R and with surface reflectivity ρ is worked out in Appendix B, the results are

$$\underline{F}_1 = - \delta \, (S_s/c) \, \bar{A}_s \underline{e}_s \quad (66)$$

where $\bar{A}_s = \pi R^2$, the projected area of the satellite as seen by the sun (note the force is independent of ρ).

Thus
$$F_1'(\theta) = (S_s/c)\,\bar{A}_s\,. \tag{67}$$

Consider the following possible locations of the satellite's center of mass relative to the geometric center of the sphere (CP for this case).

i. CM at CP:

$$p(\theta) = 0$$

Therefore $\underline{M}_1 = 0$, and the motion about the CM is the unperturbed motion.

ii. CM displaced from CP along the z-axis:

1. $p = p_o$, a constant

thus
$$\underline{M}_1 = -\delta\,(S_s/c)\bar{A}_s p_o\,\underline{k}\times\underline{e}_s \tag{68}$$

and
$$V(\theta) = -\delta\,(S_s/c)\bar{A}_s p_o \cos\theta \tag{69}$$

and equation (43) is satisfied. From equations (53) and (64)

$$\frac{d\mu}{dt} = \omega_e \cos\lambda \tag{70}$$

$$\overline{\frac{d\lambda}{dt}} = -\frac{\delta}{2\pi H_o}\int_0^{2\pi}\left(\frac{S_s}{c}\right)\bar{A}_s p_o\left[\cos\phi_o - \sin\phi_o \cot\mu \cos\psi\right]d\psi - \omega_e \cot\mu \sin\lambda$$

or, averaging over ψ

$$\overline{\frac{d\lambda}{dt}} = -\delta\left(\frac{S_s}{c}\right)\frac{\bar{A}_s p_o}{H_o}\cos\phi_o - \omega_e \cot\mu \sin\lambda\,.$$

And, as $H_o \cos\phi_o = Cs$, this may be written

$$\overline{\frac{d\lambda}{dt}} = -\delta\left(\frac{S_s}{c}\right)\frac{\bar{A}_s p_o}{Cs}\cos^2\phi_o - \omega_e \cot\mu \sin\lambda\,. \tag{71}$$

Note that the first term in equation (71) is a secular term while the second term (due to the rotation of $\underline{e}_s$) is periodic in μ and λ.

2. $p = p(\theta)$

Thus $$\underline{M}_1(\theta) = -\delta (S_s/c)\bar{A}_s p(\theta)\underline{k}\times\underline{e}_s \tag{72}$$

and $$V(\theta) = (S_s/c)\bar{A}_s \int p(\theta)\sin\theta \, d\theta \tag{73}$$

Therefore

$$\frac{d\mu}{dt} = \omega_e \cos\lambda$$

$$\frac{\overline{d\lambda}}{dt} = -\delta\left(\frac{S_s}{c}\right)\frac{\bar{A}_s\cos\phi_0}{2\pi C_s}\int_0^{2\pi} p[\theta(\psi)][\cos\phi_0 - \sin\phi_0 \cot\mu\cos\psi]d\psi - \omega_e \cot\mu\sin\lambda$$

As an example to show that $\frac{\overline{d\lambda}}{dt}$ does not always have a secular term let $p = p_o\cos\theta$, then from equation (63)

$$\cos\theta = \underline{k}\cdot\underline{e}_s = \cos\phi\cos\mu + \sin\phi\sin\mu\cos\psi$$

thus $$V(\theta) = \delta\left(\frac{S_s}{c}\right)\bar{A}_s \frac{\sin^2\theta}{2} \tag{74}$$

and $$\frac{\overline{d\lambda}}{dt} = -\delta\left(\frac{S_s}{c}\right)\frac{\bar{A}_s p_o\cos\phi_0}{2C_s}\cos\mu(3\cos^2\phi_0 - 1) - \omega_e\cot\mu\sin\lambda. \tag{75}$$

Now the average precession rate is completely periodic in μ as well as having a term also periodic in λ.

b. Cylindrical Satellite - Radius R and height h

The force due to direct solar radiation on a right circular cylinder of radius R and height h with surface reflectivity ρ for both its sides and end pieces has been worked out in Appendix C, the results are

$$\cos\theta = \underline{k}\cdot\underline{e}_s = \cos\phi\,\cos\mu + \sin\phi\,\sin\mu\,\cos\psi$$

and $$\sin\theta = (a\cos^2\psi + b\cos\psi + c)^{1/2}$$

where
$$a = -\sin^2\phi\,\sin^2\mu$$
$$b = -1/2\,\sin 2\phi\,\sin 2\mu$$
$$c = 1 - \cos^2\phi\,\cos^2\mu\,.$$

Thus the average precession rate is

$$\frac{d\lambda}{dt} = -\delta\left(\frac{S_s}{c}\right)\frac{p_o}{2\pi H_o}\left[2Rh\left(1+\tfrac{1}{3}\rho\right)I_1 + \pi R^2(1-\rho)I_2\right] - \omega_e\cot\mu\,\sin\lambda \tag{80}$$

where $$I_1 = \int_0^{2\pi}(a\cos^2\psi + b\cos\psi + c)^{1/2}\left[\cos\phi_o - \sin\phi_o\cot\mu\cos\psi\right]d\psi$$

and $$I_2 = \int_0^{2\pi}\left\{H[\theta(\psi)] - 2H\left[\theta(\psi) - \tfrac{\pi}{2}\right]\right\}\left[\cos\phi_o\cos\mu + \right.$$
$$\left. + \sin\phi_o\sin\mu\cos\psi\right]\left[\cos\phi_o - \sin\phi_o\cot\mu\cos\psi\right]d\psi.$$

c. Paddle Vanes

The forces and torques acting on a symmetric arrangement of flat plates due to direct solar radiation is worked out in Appendix D. The results are exactly the same as for the end pieces of the cylinder in the above section.

B. Reflected Solar and Direct Earth Radiation

1. Derivation of Equations

From equations (18), (19), and (20)

$$\underline{F}_2 = -\delta\int_{A_2^*}\left\{\int_{A_e}\left(\frac{\bar{\rho}S_s}{c\pi L'^2}\right)\cos\eta_1\cos\xi_1'\left[\cos^2\xi_2'(1+\rho)\underline{n} + \cos\xi_2'(1-\rho)\underline{n}\times(\underline{\ell}'\times\underline{n})\right]d\bar{A}_e\right\}dA_s$$

$$\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)\Big\{2Rh\sin\theta\left[\left(1+\tfrac{1}{3}\rho\right)\underline{\varepsilon}_s - \tfrac{4}{3}\rho\cos\theta\,\underline{k}\right] +$$

$$+\pi R^2\cos\theta\left[H(\theta)-2H\left(\theta-\tfrac{\pi}{2}\right)\right]\left[(1-\rho)\underline{\varepsilon}_s + 2\rho\cos\theta\,\underline{k}\right]\Big\} \qquad (76)$$

thus $F_1'(\theta) = \left(\frac{S_s}{c}\right)\left\{2Rh\sin\theta\left(1+\tfrac{1}{3}\rho\right) + \pi R^2\cos\theta(1-\rho)\left[H(\theta)-2H\left(\theta-\tfrac{\pi}{2}\right)\right]\right\}.$ (77)

Again consider various possible locations of the CM relative to the geometric center of the cylinder (which is the CP).

i. CM at CP:

$$p(\theta) = 0$$

Thus $\underline{M}_1 = 0$, and again the motion is unperturbed about the CM.

ii. $p = p_o$, a constant

Then

$$\underline{M}_1(\theta) = -\delta\left(\frac{S_s}{c}\right)p_o\left\{2Rh\sin\theta\left(1+\tfrac{1}{3}\rho\right)+\pi R^2\cos\theta(1-\rho)\left[H(\theta)-2H\left(\theta-\tfrac{\pi}{2}\right)\right]\right\}\underline{k}\times\underline{\varepsilon}_s \qquad (78)$$

$$V(\theta) = \delta\left(\frac{S_s}{c}\right)p_o\Big\{Rh\left(1+\tfrac{1}{3}\rho\right)\left[\theta-\tfrac{1}{2}\sin 2\theta\right] +$$

$$+\pi R^2(1-\rho)\left[\tfrac{1}{2}\sin^2\theta\,H(\theta)+\cos^2\theta\,H\left(\theta-\tfrac{\pi}{2}\right)\right]\Big\} \qquad (79)$$

and equation (43) is still satisfied. Therefore

$$\frac{d\mu}{dt} = \omega_e\cos\lambda$$

and $$\frac{d\lambda}{dt} = -\delta\frac{p_o}{2\pi H_o}\int_0^{2\pi}F_1'[\theta(\psi)][\cos\phi_o - \sin\phi_o\cot\mu\cos\psi]\,d\psi - \omega_e\cot\mu\sin\lambda$$

where $F_1'(\theta)$ is given by equation (77). From equation (63)

$$\underline{F}_3 = -\int_{A_3^*}\left\{\int_{A_e}\left(\frac{\epsilon_e \sigma T_e^4}{c\pi L'^2}\right)\cos\xi_1'\left[\cos^2\xi_2'(1+\rho')\underline{n} + \cos\xi_2'(1-\rho')\underline{n}\times(\underline{\ell}'\times\underline{n})\right]dA_e\right\}dA_s$$

where $\bar{A}_e$ is the surface area of the earth-atmosphere model sphere seen by dA_s and not in the earth's shadow, A_2^* is the total surface area of the satellite seen by that part of the model sphere not in the shadow, A_e is the surface area of the earth seen by dA_s, and A_3^* is the total surface area of the satellite seen by the earth.

It has already been noted that these surface integrations are very complicated for an arbitrary position of an arbitrarily shaped satellite (even with the assumed geometric symmetry). The analytical description of $\underline{F}_2$ is especially complicated by the earth's shadow for even if the satellite does not pass through the shadow the reflected solar radiation which the satellite receives would be influenced by it. In order to make some statements about the effects of $\underline{F}_2$ and $\underline{F}_3$ upon the satellite's attitude the following form will be assumed for both forces

$$\underline{F}_{2,3} = F'_{2,3}(\theta')\underline{e}_r + F''_{2,3}(\theta')\underline{k} \tag{81}$$

where the shadow factor δ in $\underline{F}_2$ has been dropped for the time being and θ' is the angle between $\underline{e}_r$ and $\underline{k}$. Because of the assumed satellite symmetry this form will be correct for $\underline{F}_3$ when the satellite's orbit about the earth is circular (otherwise it would also depend on r). It is correct for $\underline{F}_2$ only when the satellite is on the earth-sun line but it will serve as an approximation for other positions of the satellite (in general it would also depend on r, θ, and Φ).

The moment about the satellite's center of mass will then be

$$\underline{M}_{2,3} = p_{2,3}(\theta')F'_{2,3}(\theta')\ \underline{k} \times \underline{e}_r \tag{82}$$

where the separation distance p is also assumed to depend only on θ'. The subscripts will now be dropped on $\underline{F}$, $\underline{M}$, etc., where it is to be understood that the results apply to reflected solar as well as direct earth radiation. Again, as in section A, the moment will be assumed derivable from a potential so that

$$M(\theta') = -\partial V'/\partial\theta'. \tag{83}$$

If the analysis is restricted to circular orbits there are two integrals of the motion

$$\underline{H}\cdot\underline{k} = h'_1 \tag{84}$$

or $\qquad \omega_z = s$, a constant

and $$\frac{1}{2}\underline{H}\cdot\underline{\omega}_s - \underline{H}\cdot\underline{\omega}_o + V'(\theta') = h'_2 \tag{85}$$

which are exactly the same as equations (42) and (44) with $\underline{\omega}_e$ (angular velocity of the earth about the sun) replaced by $\underline{\omega}_o$ (angular velocity of the satellite about the earth, constant for a circular orbit). Equation (85) leads to the following relation for H

$$H = H_o\left\{1 + \frac{A\omega_o}{H_o}(\cos\gamma' - \cos\gamma_o') - \frac{A}{H_o^2}(V' - V_o') + O\left[\frac{A^2\omega_o}{H_o^2}\right] + O\left[\frac{A^2}{H_o^4}(V' - V_o')^2\right]\right\} \tag{86}$$

where H_o, V_o', and $\mathcal{S}_o'$ are the initial or unperturbed values of H, V', and $\mathcal{S}'$, and $\mathcal{S}'$ is the angle between $\underline{H}$ and $\underline{\omega}_o$. Similar to equation (48), the case of interest is

$$\frac{A\omega_o}{H_o}\left(\cos\mathcal{S}' - \cos\mathcal{S}_o'\right) - \frac{A}{H_o^2}\left(V' - V_o'\right) << 1 \tag{87}$$

so that H may be replaced by its unperturbed value H_o. Note that if equation (87) is satisfied equation (48) is also satisfied as $\omega_e << \omega_o$. From equation (84) $\underline{H}\cdot\underline{k} = H\cos\phi = h_1'$ so if $H = H_o$

$$\phi = \phi_o. \tag{88}$$

The precession rate of $\underline{k}$ about $\underline{H}$, see Figure 8, is $\dot{\psi}' = H/A$ thus for $H = H_o$

$$\dot{\psi}' = H_o/A \tag{89}$$

its unperturbed precession rate. Thus, as in the case of direct solar radiation, there are no secular effects in the motion of the axis of symmetry about the angular momentum vector. From Figure 8, $\underline{H} = H_o\underline{h}'$ where

$$\underline{h}' = \cos\mu'\ \underline{I} + \sin\mu'\ (\sin\lambda'\ \underline{J} + \cos\lambda'\underline{K}) \tag{90}$$

and the equation of motion about the satellite's center of mass is

$$d\underline{h}'/dt)_f = \underline{M}(\theta')/H_o$$

or

$$-\sin\mu'\frac{d\mu'}{dt}\underline{I} + \cos\mu'(\sin\lambda'\underline{J} + \cos\lambda'\underline{K})\frac{d\mu'}{dt} + \sin\mu'(\cos\lambda'\underline{J} - \sin\lambda'\underline{K})\frac{d\lambda'}{dt} + \cos\mu'(\underline{\omega}_p\times\underline{I}) + \sin\mu'[\sin\lambda'(\underline{\omega}_p\times\underline{J}) + \cos\lambda'(\underline{\omega}_p\times\underline{K})]$$

$$= \frac{\underline{M}(\theta')}{H_o} \tag{91}$$

where $\underline{\omega}_p$ is the angular velocity of (XYZ) relative to inertial space (caused by changes in the orbital elements as a result of orbit perturbations). That is

$$\underline{\omega}_p = \omega_{p_1} \underline{I} + \omega_{p_2} \underline{J} + \omega_{p_3} \underline{K} \tag{92}$$

with

$$\omega_{p_1} = \sin i \cos\omega \frac{d\Omega}{dt} - \sin\omega \frac{di}{dt}$$

$$\omega_{p_2} = \cos i \frac{d\Omega}{dt} + \frac{d\omega}{dt} \tag{93}$$

$$\omega_{p_3} = \sin i \sin\omega \frac{d\Omega}{dt} + \cos\omega \frac{di}{dt}$$

Equation (91) thus leads to three scalar equations for the two unknowns $\dot{\mu}'$ and $\dot{\lambda}'$:

$$\frac{d\mu'}{dt} = -\frac{\underline{M}\cdot\underline{I}}{H_0} - \omega_{p_3}\sin\lambda' + \omega_{p_2}\cos\lambda' \tag{94}$$

$$\cos\mu'\sin\lambda'\frac{d\mu'}{dt} + \sin\mu'\cos\lambda'\frac{d\lambda'}{dt} = \frac{\underline{M}\cdot\underline{J}}{H_0} + \omega_{p_1}\sin\mu'\cos\lambda' - \omega_{p_3}\cos\mu' \tag{95}$$

$$\cos\mu'\cos\lambda'\frac{d\mu'}{dt} - \sin\mu'\sin\lambda'\frac{d\lambda'}{dt} = \frac{\underline{M}\cdot\underline{K}}{H_0} - \omega_{p_1}\sin\mu'\sin\lambda' + \omega_{p_2}\cos\mu'. \tag{96}$$

The reason for the inconsistency in these equations (three equations in two unknowns) has already been explained. To determine average rates of change, equations (95) and (96) may be combined to yield

$$\frac{d\lambda'}{dt} = \frac{\underline{M}}{H_0\sin\mu'}\cdot\left[\cos\lambda'\underline{J} - \sin\lambda'\underline{K}\right] + \omega_{p_1} - \cot\mu'\left[\omega_{p_3}\cos\lambda' + \omega_{p_2}\sin\lambda'\right]. \tag{97}$$

Note that in both equations (94) and (97) the effects of the perturbing moments contribute only the first term on the right hand side of each equation, the remaining terms are a result of the rotation of the (XYZ) coordinate system due to perturbations in Ω, ω, and i.

From equation (82), $\underline{M} = F'(\theta')p(\theta')\ \underline{k} \times \underline{e}_r$ thus

$$\begin{aligned} \underline{M} \cdot \underline{I} &= F'(\theta')p(\theta') \cos f\ \underline{k} \cdot \underline{J} \\ \underline{M} \cdot \underline{J} &= F'(\theta')p(\theta')(\sin f\ \underline{K} - \cos f\ \underline{I}) \cdot \underline{k} \\ \underline{M} \cdot \underline{K} &= -F'(\theta')p(\theta') \sin f\ \underline{k} \cdot \underline{J} \ . \end{aligned} \tag{98}$$

Thus equations (94) and (97) may be written

$$\frac{d\mu'}{dt} = -\frac{F'(\theta')p(\theta')}{H_0 \sin\mu'} \cos f\ \underline{k} \cdot \underline{J} - \omega_{p_3} \sin\lambda' + \omega_{p_2} \cos\lambda' \tag{99}$$

$$\frac{d\lambda'}{dt} = \frac{F'(\theta')p(\theta')}{H_0 \sin\mu'} \left[\cos\lambda'(\sin f\ \underline{K} - \cos f\ \underline{I}) + \sin\lambda' \sin f\ \underline{J}\right] \cdot \underline{k} + $$

$$+ \omega_{p_1} - \cot\mu' \left[\omega_{p_3} \cos\lambda' + \omega_{p_2} \sin\lambda'\right] \tag{100}$$

where ω_{p_1}, ω_{p_2}, ω_{p_3} are given by equation (93), while from Figure 8

$$\begin{aligned} \underline{k} = {} & (\cos\phi \cos\mu' + \sin\phi \sin\mu' \cos\psi')\underline{I} + (\cos\phi \sin\mu' \sin\lambda' + \\ & + \sin\phi \cos\lambda' \sin\psi' - \sin\phi \cos\mu' \sin\lambda' \cos\psi')\ \underline{J} + \\ & + (\cos\phi \sin\mu' \cos\lambda' - \sin\phi \sin\lambda' \sin\psi' - \sin\phi \cos\mu' \cos\lambda' \cos\psi')\underline{K} \end{aligned} \tag{101}$$

In order to determine the secular effects in equations (99) and (100) (and to clear up the inconsistency in these equations) it will be necessary to replace the instantaneous moments by the average moments. This is achieved by averaging the perturbing moments over a complete rotation of $\underline{k}$ about $\underline{H}$ (ψ' ranging from 0 to 2π) and over a complete revolution of the satellite about the earth (f ranging from 0 to 2π). The values of $d\omega/dt$, di/dt, etc. to be substituted in equations (99) and (100) are the average values and thus will not be averaged here. The average rates of change of μ' and λ' are

$$\frac{\overline{d\mu'}}{dt} = -\frac{1}{4\pi^2}\int_0^{2\pi}\left\{\int_f \frac{F'[\theta'(\psi')]\,p[\theta'(\psi')]}{H_0 \sin\mu'}\cos f\,(\underline{k}\cdot\underline{J})\,df\right\}d\psi' - \omega_{p_3}\sin\lambda' + \omega_{p_2}\cos\lambda' \tag{102}$$

$$\frac{\overline{d\lambda'}}{dt} = \frac{1}{4\pi^2}\int_0^{2\pi}\left\{\int_f \frac{F'[\theta'(\psi')]\,p[\theta'(\psi')]}{H_0 \sin\mu'}\left[\cos\lambda'(\sin f\,\underline{K} - \cos f\,\underline{J}) + \sin\lambda'\sin f\,\underline{I}\right]\cdot\underline{k}\,df\right\}d\psi'$$

$$+\,\omega_{p_1} - \cot\mu'\left[\omega_{p_3}\cos\lambda' + \omega_{p_2}\sin\lambda'\right]. \tag{103}$$

At this point the effects of each force, $\underline{F}_2$ and $\underline{F}_3$, can be analyzed separately.

i. <u>Direct Earth Radiation</u>

In this case the average on f is from 0 to 2π because the earth's shadow has no effect. From the above equations

$$\frac{\overline{d\mu'}}{dt} = -\omega_{p_3}\sin\lambda' + \omega_{p_2}\cos\lambda' \tag{104}$$

$$\frac{\overline{d\lambda'}}{dt} = \omega_{p_1} - \cot\mu'\left[\omega_{p_3}\cos\lambda' + \omega_{p_2}\sin\lambda'\right] \tag{105}$$

as the effects of the perturbing moment $\underline{M}_3$ are completely periodic in the true anomaly. The satellite's motion for direct earth radiation, under the stated assumptions, can be described, therefore, as an unperturbed motion of the axis of symmetry about the angular momentum vector which itself remains practically constant in magnitude but changes its orientation relative to (XYZ) as a result of orbit perturbations only.

ii. Reflected Solar Radiation

This falls into two classes: a) satellite never enters the earth's shadow, and b) satellite enters earth's shadow at $f = f_1$ and leaves shadow at $f = f_2$.

a) No Shadow Effect - The effect of the perturbing moment $\underline{M}_2$ is completely periodic in f and averaging over f from 0 to 2π gives

$$\overline{\frac{d\mu'}{dt}} = -\omega_{p_3}\sin\lambda' + \omega_{p_2}\cos\lambda' \tag{106}$$

$$\overline{\frac{d\lambda'}{dt}} = \omega_{p_1} - \cot\mu'[\omega_{p_3}\cos\lambda' + \omega_{p_2}\sin\lambda'] \tag{107}$$

the same result as equations (104) and (105).

b) Satellite Eclipsed by Earth's Shadow - When the satellite enters the shadow $\underline{F}_2 = 0$ so that the average over f is not zero even though the effect is still periodic in f. From equations (102) and (103)

$$\overline{\frac{d\mu'}{dt}} = -\frac{(\sin f_1 - \sin f_2)}{4\pi^2 H_0 \sin\mu'}\int_0^{2\pi} F_3'[\theta'(\psi')]p[\theta'(\psi')](\underline{k}\cdot\underline{J})d\psi' - \omega_{p_3}\sin\lambda' + \omega_{p_2}\cos\lambda' \tag{108}$$

$$\overline{\frac{d\lambda'}{dt}} = -\frac{1}{4\pi^2 H_0 \sin\mu'}\int_0^{2\pi} F_3'[\theta'(\psi')]p[\theta'(\psi')]\{\cos\lambda'[(\cos f_1 - \cos f_2)\underline{K} + (\sin f_1 - \sin f_2)\underline{I}] + \sin\lambda'(\cos f_1 - \cos f_2)\underline{J}\}\cdot\underline{k}\,d\psi' + \omega_{p_1} - \cot\mu'[\omega_{p_3}\cos\lambda' + \omega_{p_2}\sin\lambda'] \tag{109}$$

where μ' is considered constant during the integration. These integrals can not be worked out completely until particular forms for $F_3'(\theta')$ and $p(\theta')$ are specified. The satellite's motion due to reflected solar radiation is, therefore, an unperturbed motion of the axis of symmetry about the angular momentum vector which remains approximately constant in magnitude and whose orientation relative to (XYZ) is given by equations (106) and (107) if the satellite does not pass through the earth's shadow, or by equations (108) and (109) for satellite eclipses.

The angular velocity of (XYZ) relative to inertial space, $\underline{\omega}_p$, depends on orbit perturbations and for any particular satellite orbit the important perturbations will have to be worked out by some appropriate technique in order to determine $d\Omega/dt$, $d\omega/dt$, and di/dt. As an example, it is well known that the oblateness of the earth produces, among other effects, a secular increase in both Ω and ω (see Kozai (23) as an example)

$$d\Omega/dt = -(2\pi\varepsilon \bar{R}_e^2/p^2P)\cos i \tag{110}$$

$$d\omega/dt = (\pi\varepsilon \bar{R}_e^2/p^2P)(5\cos^2 i - 1) \tag{111}$$

where p is the focal parameter of the orbit, $\bar{R}_e$ the equatorial radius of the earth, P the orbital period, i the inclination of the orbit to the equatorial plane, and ε a dimensionless parameter which characterizes the oblateness of the earth ($\varepsilon = 1.6331 \times 10^{-3}$).

The derivations of eclipse conditions, duration, and geometry are listed in Appendix E.

2. Particular Satellite Configurations

As a result of the above analysis the only cases of interest are satellites receiving reflected solar radiation and passing through the earth's shadow. But no shapes have been found where analytical results for $\underline{F}_2$ and $\underline{M}_2$ can be computed (by analytical results it is meant that $\underline{F}_2$ and $\underline{M}_2$ can be determined as functions of the satellite's attitude and position relative to the earth and sun). For the sake of an example, therefore, consider $F_2'(\theta') = F_o$ and $p(\theta') = p_o$, both constant. Thus if the satellite enters the earth's shadow at $f = f_1$ and leaves it at $f = f_2$

$$\overline{\frac{d\mu'}{dt}} = -\frac{F_o p_o}{2\pi H_o}(\sin f_1 - \sin f_2)\cos\phi_o \sin\lambda' - \omega_{p_3}\sin\lambda' + \omega_{p_2}\cos\lambda' \qquad (112)$$

$$\overline{\frac{d\lambda'}{dt}} = -\frac{F_o p_o}{2\pi H_o}\left[(\cos f_1 - \cos f_2)\cos\phi_o + (\sin f_1 - \sin f_2)\cos\phi_o \cot\mu' \cos\lambda'\right] +$$

$$+ \omega_{p_1} - \cot\mu'\left[\omega_{p_3}\cos\lambda' + \omega_{p_2}\sin\lambda'\right]. \qquad (113)$$

Note that the first term in equation (113) is a secular term.

IV. PARTICULAR SOLUTION

In the previous chapter the effects of radiation torques upon the attitude of a satellite were investigated by a perturbation-type analysis. In that analysis it was necessary to make certain assumptions about the initial conditions, see equation (48), as well as to average over certain of the variables. The reason for this was because of the non-linear character of the equations of motion about the center of mass which do not, in general, admit analytical solutions for an arbitrary external moment.

In this chapter the possibility of a regular precession of the axis of symmetry about the vector $\underline{e}_s$ will be investigated for the case of direct solar radiation. In order to determine if this motion is a particular solution to the equations of motion the external moment necessary to maintain it will be derived and compared to the results of the previous chapters.

The angular velocity of (x,y,z) relative to inertial space can be written as

$$\underline{\omega}_s = \omega_z \underline{k} + \underline{\omega} \tag{114}$$

where $\underline{\omega}$ is perpendicular to $\underline{k}$. Then

$$d\underline{k}/dt)_f = \underline{\omega}_s \times \underline{k} = \underline{\omega} \times \underline{k}$$

but $$\underline{k} \times (d\underline{k}/dt)_f = \underline{k} \times (\underline{\omega} \times \underline{k}) = \underline{\omega}$$

thus $$\underline{\omega}_s = \omega_z \underline{k} + \underline{k} \times (d\underline{k}/dt)_f \, . \tag{115}$$

The angular momentum is

$$\underline{H} = C\,\omega_z \underline{k} + A\,\underline{k} \times (d\underline{k}/dt)_f . \tag{116}$$

The equation of motion about the satellite's center of mass as a result of direct solar radiation is

$$d/dt\left[A\,\underline{k}\times(d\underline{k}/dt)_f + C\omega_z\underline{k}\right]_f = \underline{M}_1 . \tag{117}$$

The particular solution sought to this equation, see Figure 9, is

$$\begin{aligned} \theta &= \theta_o \\ \omega_z &= s \\ d\underline{k}/dt)_f &= \Omega_o\,\underline{e}_s \times \underline{k} + \underline{\omega}_e \times \underline{k} \end{aligned} \tag{118}$$

where $\Omega_o = \dot{\gamma}$ (a constant) is the precession rate of $\underline{k}$ about $\underline{e}_s$ and $\underline{\omega}_e$ is the angular velocity of $\underline{e}_s$. Substituting these conditions into equation (117) leads to

$$\begin{aligned} Cs\left[\Omega_o\,\underline{e}_s\times\underline{k} + \underline{\omega}_e\times\underline{k}\right]_{\theta=\theta_o} &+ A\,\underline{k}\times\Big\{\Omega_o\big[(\underline{\omega}_e\times\underline{e}_s)\times\underline{k} + \underline{e}_s\times(\Omega_o\,\underline{e}_s\times\underline{k} + \\ &+ \underline{\omega}_e\times\underline{k})\big] + \underline{\omega}_e\times(\Omega_o\,\underline{e}_s\times\underline{k} + \underline{\omega}_e\times\underline{k})\Big\}_{\theta=\theta_o} = \underline{M}_1(\theta_o) \end{aligned} \tag{119}$$

where $d\underline{e}_s/dt)_f = \underline{\omega}_e \times \underline{e}_s$ and $d\underline{\omega}_e/dt)_f = 0$. Expanding the triple cross products in equation (119) and collecting terms one finds

$$\begin{aligned} \Big[\Omega_o\big\{Cs - A\left[2(\underline{\omega}_e\cdot\underline{k}) + \Omega_o\cos\theta\right]\big\}\,\underline{e}_s\times\underline{k} &+ \\ + \left[Cs - A(\underline{\omega}_e\cdot\underline{k})\right]\underline{\omega}_e\times\underline{k}\Big]_{\theta=\theta_o} &= \underline{M}_1(\theta_o). \end{aligned} \tag{120}$$

Thus, for a regular precession of $\underline{k}$ about $\underline{e}_s$ while $\underline{e}_s$ rotates with a constant angular velocity $\underline{\omega}_e$, $\underline{M}_1(\theta_o)$ must satisfy equation (120). But for direct solar radiation, under the previously stated assumptions, $\underline{M}_1$ is of the form

$$\underline{M}_1 = F_1'(\theta)p(\theta)\, \underline{e}_s \times \underline{k} \quad . \tag{121}$$

So that $$\underline{M}_1(\theta_o) = F_1'(\theta_o)p(\theta_o)(\underline{e}_s \times \underline{k})_{\theta=\theta_o} \tag{122}$$

which does not have a component $\underline{\omega}_e x\underline{k}$ and thus does not satisfy equation (120). Note that the term $\left[Cs - A(\underline{\omega}_e \cdot \underline{k})\right] \underline{\omega}_e x\underline{k}$ in equation (120) is zero only when $\underline{\omega}_e \cdot \underline{k} = Cs/A$ and $|Cs/A\omega_e| \leq 1$, while $\underline{\omega}_e x\underline{k}$ is never zero, unless $\theta_o = \pi/2$, as $\underline{\omega}_e$ is perpendicular to $\underline{e}_s$.

A. Regular Precession

From equations (120) and (122) it is observed that a regular precession is a particular solution for the case of direct solar radiation if $\underline{\omega}_e = 0$. That is over short intervals of time, say a day or less, when $\underline{e}_s$ may be considered constant in inertial space equation (122) satisfies equation (120). If $\underline{\omega}_e$ is set equal to zero and equation (122) is substituted into equation (120) one finds

$$\left[Cs\,\Omega_o - A\cos\theta_o\,\Omega_o^2 - F_1'(\theta_o)p(\theta_o)\right](\underline{e}_s x\underline{k})_{\theta=\theta_o} = 0. \tag{123}$$

So if $\underline{k}$ is neither parallel nor perpendicular to $\underline{e}_s$ equation (123) is a quadratic expression for the precession rate Ω_o

$$\Omega_o = \frac{Cs \pm \sqrt{C^2s^2 - 4A\cos\theta_o F_1'p}}{2A\cos\theta_o} \tag{124}$$

or

$$\Omega_o = \frac{Cs}{2A\cos\theta_o}\left[1 \pm \sqrt{1 - \frac{4A\cos\theta_o}{C^2s^2}F_1'(\theta_o)p(\theta_o)}\right]. \tag{125}$$

The regular precession rates given by this equation are exactly the same as those for the Heavy Symmetric Top with one point fixed in which case $F_1' = Mg$ (M = mass of the top and g = acceleration of gravity) and p is the distance of the center of gravity from the fixed point.

The quantity under the radical in equation (125) is approximately equal to unity($F_1' \sim S_s$) and thus can be expanded to give a "fast" and a "slow" precession rate

$$\Omega_{o_1} = \frac{Cs}{A\cos\theta_o} + O\left[\frac{F_1'(\theta_o)p(\theta_o)}{Cs}\right] \tag{126}$$

$$\Omega_{o_2} = \frac{F_1'(\theta_o)p(\theta_o)}{Cs} + O\left[\frac{A\cos\theta_o}{C^3s^3}F_1'^2(\theta_o)p^2(\theta_o)\right] \tag{127}$$

For this particular case ($\underline{\omega}_e = 0$) there are three integrals of the motion

$$\tfrac{1}{2}\underline{H} \cdot \underline{\omega}_s + V = h_1 \tag{128}$$

$$\underline{H} \cdot \underline{e}_s = h_2 \tag{129}$$

$$\underline{H} \cdot \underline{k} = h_3. \tag{130}$$

Equation (128) is Conservation of Energy and follows from equation (44), equations (129) and (130) are Conservation of Angular Momentum. These integrals may be written in terms of Euler angles, see Figure 9, as follows

$$\tfrac{1}{2}A\,(\dot{\theta}^2 + \dot{\gamma}^2\sin^2\theta) + \tfrac{1}{2}C\,\omega_z^2 + V(\theta) = h_1 \tag{131}$$

$$A\sin^2\theta\,\dot{\gamma} + C\cos\theta\,\omega_z = h_2 \tag{132}$$

$$C\,(\dot{\chi} + \dot{\gamma}\cos\theta) = C\,\omega_z = h_3 \tag{133}$$

The particular solution of a regular precession of $\underline{k}$ about $\underline{e}_s$ is

$$\begin{aligned} \theta &= \theta_0 \\ \dot{\gamma} &= \Omega_0 \\ \omega_z &= s \end{aligned} \tag{134}$$

where Ω_0 is given by equation (125).

B. <u>Stability of Regular Precession</u>

In the above section the conditions for a regular precession, equation (125), have been determined for the case of direct solar radiation when $\underline{e}_s$ can be considered constant in inertial space. It is now possible to determine the stability of this motion, in the sense of Lyapunov, because of the existence of the integrals of the motion. Consider as the unperturbed motion equations (134) which will now be perturbed by instantaneous perturbations x_i

$$\begin{aligned} \theta &= \theta_0 + x_1 \\ \dot{\gamma} &= \Omega_0 + x_2 \\ \omega_z &= s + x_3 \\ \dot{\theta} &= x_4 \end{aligned} \tag{135}$$

and the stability will be investigated with respect to x_1, x_2, x_3, and x_4.

Consider the Lyapunov function L

$$L(x_1,x_2,x_3,x_4) = K_1L_1 + K_2L_2 + K_3L_3 + K_4L_3^2 - L_o \tag{136}$$

where $L_o = K_1\left\{\frac{1}{2}A(\Omega_o^2 \sin^2\theta_o) + \frac{1}{2}Cs^2 + V(\theta_o)\right\} +$

$$+K_2\left\{A\sin^2\theta_o\,\Omega_o + Cs\cos\theta_o\right\} + K_3 s + K_4 s^2$$

$$L_1 = h_1 = \frac{1}{2}A\left[x_4^2 + (\Omega_o + x_2)^2\sin^2(\theta_o + x_1)\right] + $$

$$+\frac{1}{2}C(s + x_3)^2 + V(\theta_o + x_1) \tag{137}$$

$$L_2 = h_2 = A\sin^2(\theta_o + x_1)(\Omega_o + x_2) + C\cos(\theta_o + x_1)(s + x_3)$$

$$L_3 = s + x_3$$

and K_1, K_2, K_3, K_4 are constants, yet to be determined. Assume that $V(\theta_o + x_1)$ can be expanded in a power series in x_1 with the coefficients depending on θ_o

$$V(\theta_o + x_1) = a_o(\theta_o) + a_1(\theta_o)x_1 + a_2(\theta_o)x_1^2 + \ldots \quad . \tag{138}$$

Since $L(0,0,0,0) = 0$ and $dL/dt = 0$ (L_1, L_2, L_3 being integrals of the motion) the sufficient condition for stability according to Lyapunov is whether L is positive definite or not. This can be determined by expanding L in terms of x_1, x_2, x_3, x_4 and retaining only up to second order terms, (24). That is

$$L(x_1,x_2,x_3,x_4) = x_1^2\left[K_2A\Omega_o(\cos^2\theta_o - \sin^2\theta_o) - \frac{1}{2}K_2Cs\cos\theta_o + \right.$$

$$\left. +\frac{1}{2}K_1A\Omega_o^2(\cos^2\theta_o - \sin^2\theta_o) + K_1a_2\right] + x_2^2\left(\frac{1}{2}K_1A\sin^2\theta_o\right) + x_3^2\left(K_4 + \frac{1}{2}K_1C\right) +$$

$$+ x_4^2\left(\frac{1}{2}K_1A\right) + x_3\left[K_1Cs + K_2C\cos\theta_o + 2K_4s + K_3\right] + x_2\left[K_1A\Omega_o\sin^2\theta_o + \right.$$

$$\left. +K_2A\sin^2\theta_o\right] + x_1\left[K_1A\Omega_o^2\sin\theta_o\cos\theta_o + \right. \tag{139}$$

$$+2K_2 A\Omega_o \sin\theta_o\cos\theta_o - K_2 Cs\sin\theta_o + K_1 a_1] + x_1 x_2[2K_2 A\sin\theta_o\cos\theta_o +$$

$$+ 2K_1 A\Omega_o \sin\theta_o \cos\theta_o] - x_1 x_3 (K_2 C\sin\theta_o) + O(x_i^2 x_j) .$$

Let $K_1 = 1$, $K_2 = -\Omega_o$, $K_3 = -Cs + C\Omega_o\cos\theta_o - 2sK_4$ and note from equation (123) that $A\Omega_o^2\cos\theta_o\sin\theta_o - Cs\Omega_o\sin\theta_o + M(\theta_o) = 0$. If these relations are substituted into equation (139) one obtains

$$L = [\tfrac{1}{2}A\Omega_o^2\sin^2\theta_o + \tfrac{1}{2}\cot\theta_o M(\theta_o) + a_2]x_1^2 + \tfrac{1}{2}A\sin^2\theta_o x_2^2 +$$
$$+ (K_4 + \tfrac{1}{2}C)x_3^2 + \tfrac{1}{2}Ax_4^2 + [a_1 + M(\theta_o)]x_1 +$$
$$+ x_1 x_3 C\Omega_o\sin\theta_o + O(x_i^2 x_j). \tag{140}$$

From equation (138) $\partial V/\partial x_1 = (\partial V/\partial \theta)(\partial\theta/\partial x_1) = a_1 + 2a_2x_1 + \ldots$, but $\partial\theta/\partial x_1 = 1$ so that $\partial V/\partial\theta = \partial V/\partial x_1 = -M(\theta)$ and $a_1(\theta_o) = -M(\theta_o)$. Thus the term in x_1 drops out of the above equation and the condition for L to be positive definite is supplied by Sylvester's Theorem, namely

(i) $\tfrac{1}{2}A\Omega_o^2\sin^2\theta_o + \tfrac{1}{2}\cot\theta_o M(\theta_o) + a_2(\theta_o) > 0$

(ii) $\sin\theta_o \neq 0$

(iii) $\tfrac{1}{4}A\sin^2\theta_o[\tfrac{1}{2}A\Omega_o^2\sin^2\theta_o + \tfrac{1}{2}\cot\theta_o M(\theta_o) + a_2(\theta_o)](\tfrac{1}{2}C + K_4) -$
$- \tfrac{1}{8}AC^2\Omega_o^2\sin^4\theta_o > 0.$

But if $\sin\theta_o \neq 0$ the third condition is easily satified as soon as the first condition is satisfied, K_4 still being arbitrary. Stability, therefore, reduces to the two conditions

$$\sin\theta_o \neq 0 \tag{141}$$

$$\frac{1}{2}A\Omega_o^2\sin^2\theta_o + \frac{1}{2}\cot\theta_o M(\theta_o) + a_2(\theta_o) > 0 \tag{142}$$

where $M(\theta) = -\partial V/\partial\theta$

$$V(\theta_o + x_1) = a_o(\theta_o) + a_1(\theta_o)x_1 + a_2(\theta_o)x_1^2 + \ldots$$

and $a_1(\theta_o) = -M(\theta_o)$.

In order to illustrate these equations consider the case of a spherically shaped satellite with its center of mass displaced a constant distance from the geometric center of the sphere. From equations (66), (68), and (69)

$$\underline{F}_1 = -\delta(S_s/c)\bar{A}_s\underline{e}_s$$

$$\underline{M}_1(\theta) = -\delta(S_s/c)\bar{A}_s p_o \underline{k}\times\underline{e}_s$$

$$V(\theta) = -\delta(S_s/c)\bar{A}_s p_o \cos\theta.$$

Thus $F_1' = (S_s/c)\bar{A}_s$, $M(\theta) = F_1'p_o\sin\theta$, and

$$M(\theta_o) = F_1'p_o\sin\theta_o \tag{143}$$

$$V(\theta_o + x_1) = F_1'p_o\left[\cos\theta_o - x_1\sin\theta_o - \frac{1}{2}x_1^2\cos\theta_o + \ldots\right] \tag{144}$$

and $a_1(\theta_o) = -F_1'p_o\cos\theta_o = -M(\theta_o)$.

Stability, therefore, is determined by equations (141) and (142). But for this case $\frac{1}{2}\cot\theta_o M(\theta_o) + a_2(\theta_o) = \frac{1}{2}F_1'p_o\cos\theta_o - \frac{1}{2}F_1'p_o\cos\theta_o = 0$ and this regular precession is stable in the

Lyapunov sense if

$$\begin{aligned} \sin\theta_o &\neq 0 \\ \tfrac{1}{2}A\,\Omega_o^2 \sin^2\theta_o &> 0 \end{aligned} \qquad (145)$$

where Ω_o is given by equation (125).

REMARKS

The average precession and nutation rates for reflected solar and direct earth radiation, equations (102) and (103), can be transformed to the $(\underline{e}_s, \underline{e}_1, \underline{e}_2)$ coordinate system and added directly to the results for direct solar radiation, equations (53) and (64), to give the total perturbation. This transformation will involve $\overline{d\mu'}/dt$, $\overline{d\lambda'}/dt$, ω_{p_1}, ω_{p_2}, ω_{p_3}, Ω, ω, i, Δ, Λ, $\underline{\omega}_e$ and, in addition, the angles μ', λ', etc. will have to be expressed in terms of the angles μ, λ, etc.. Such a combination may be necessary in order to obtain more information about the motion but for the present analysis it would only obscure the results.

It should be noted that all conclusions based on average precession and nutation rates are applicable only to the average motion. The averaging procedure is intended to illustrate the presence or absence of secular perturbations in the attitude motion and it does not necessarily indicate the same for other types of perturbations. These other perturbations can be found by studying the non-averaged motion, for example equations (60), (99), and (100) as opposed to the averaged equations (64), (102), and (103).

For a particular satellite configuration with given initial conditions equations (53) and (64), as well as equations (102) and (103), can be integrated numerically to give nutation and

APPENDIX A. FORCES ON FLAT PLATE SITUATED ON EARTH-SUN LINE

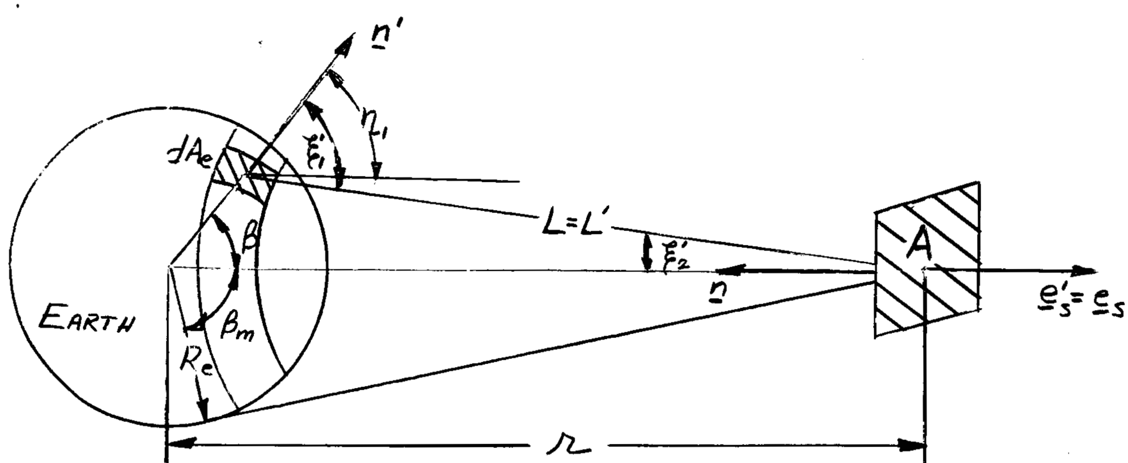
From Figure 6 and equations (23), (24), and (25) the three components of force are

$$\underline{F}_1 = \left(\frac{S_s}{c}\right)(1+\rho)A\underline{n} \tag{A-1}$$

$$\underline{F}_2 = -\int_{\bar{A}_e}\left(\frac{\bar{\rho}\, S_s}{c\pi L'^2}\right)\cos\xi_1' \cos\eta_1 \cos^2\xi_2'(1+\rho)\,d\bar{A}_e\, A\underline{n} \tag{A-2}$$

$$\underline{F}_3 = -\int_{A_e}\left(\frac{\epsilon_e \sigma T_e^4}{c\pi L'^2}\right)\cos\xi_1' \cos^2\xi_2'(1+\rho')\,dA_e\, A\underline{n} \tag{A-3}$$

For this particular example let the spherical model representing the earth and its atmosphere be the earth itself. That is $\bar{R} = R_e$ and then the surface integrations for $\underline{F}_2$ and $\underline{F}_3$ will be over the same sphere.



Note that in both integrations the portion of the earth "seen" by the plate is the spherical cap defined by the angle β_m ($\cos\beta_m$ =

precession angles as a function of time (or some other convenient parameter). Such an analysis would completely determine the satellite's attitude motion as a result of the radiation torques and it would also indicate the dependence of the resulting motion upon the initial conditions.

R_e/r), also note that there is symmetry about the line joining the earth and the plate. From the above figure

$$\cos\eta_1 = \cos\beta$$

$$\cos\xi_2' = \frac{r - R_e\cos\beta}{L'}$$

$$\cos\xi_1' = \left(\frac{r - R_e\cos\beta}{L'}\right)\cos\beta - \left(\frac{R_e\sin\beta}{L'}\right)\sin\beta$$

$$L'^2 = R_e^2 + r^2 - 2rR_e\cos\beta$$

$$dA_e = 2\pi R_e^2\sin\beta\, d\beta$$

Thus the forces become

$$\underline{F}_1 = \left(\frac{S_s}{c}\right)(1+\rho)A\,\underline{n} \qquad \text{(A-4)}$$

$$\underline{F}_2 = -2\left(\frac{\bar{\rho}S_s}{c}\right)(1+\rho)A\,I_1\,\underline{n} \qquad \text{(A-5)}$$

$$\underline{F}_3 = -2\left(\frac{\epsilon_e\sigma T_e^4}{c}\right)(1+\rho')A\,I_2\,\underline{n} \qquad \text{(A-6)}$$

where $$I_1 = k^2\int_0^{\beta_m}\frac{\cos\beta(\cos\beta-k)(1-k\cos\beta)^2}{(1+k^2-2k\cos\beta)^{5/2}}\sin\beta\, d\beta \qquad \text{(A-7)}$$

$$I_2 = k^2\int_0^{\beta_m}\frac{(\cos\beta-k)(1-k\cos\beta)^2}{(1+k^2-2k\cos\beta)^{5/2}}\sin\beta\, d\beta \qquad \text{(A-8)}$$

and $0 \le \beta_m \le \pi/2$ for $R_e \le r < \infty$, $k = R_e/r$.

$$I_2 = k^2\left[I_2^{I} + I_2^{II} + I_2^{III} + I_2^{IV}\right]$$

where $$I_2^{I} = -k\int_0^{\beta_m}\sin\beta\left[1+k^2-2k\cos\beta\right]^{-5/2}d\beta = \frac{1}{3}\left[(1-k^2)^{-3/2} - (1-k)^{-3}\right]$$

$$I_2^{II} = (1+2k^2)\int_0^{\beta_m}\cos\beta\sin\beta\left[1+k^2-2k\cos\beta\right]^{-5/2}d\beta = (1+2k^2)\Big\{-\frac{1}{3k}\left[k(1-k^2)^{-3/2}\right.$$
$$\left.-\,(1-k)^{-3}\right] + \frac{1}{3k^2}\left[(1-k^2)^{-1/2} - (1-k)^{-1}\right]\Big\}$$

$$I_2^{III} = -k(2+k^2)\int_0^{\beta_m}\cos^2\beta\sin\beta[1+k^2-2k\cos\beta]^{-5/2}d\beta = -k(2+k^2)\Big\{-\frac{1}{3k}[k^2(1-k^2)^{-3/2}-$$
$$-(1-k)^{-3}] + \frac{2}{3k^2}[k(1-k^2)^{-1/2}-(1-k)^{-1}] + \frac{2}{3k^3}[(1-k^2)^{1/2}-(1-k)]\Big\}$$

$$I_2^{IV} = k^2\int_0^{\beta_m}\cos^3\beta\sin\beta[1+k^2-2k\cos\beta]^{-5/2}d\beta = k^2\Big\{-\frac{1}{3k}[k^3(1-k^2)^{-3/2}-(1-k)^{-3}]+$$
$$+\frac{1}{k^2}[k^2(1-k^2)^{-1/2}-(1-k)^{-1}] + \frac{2}{k^3}[k(1-k^2)^{1/2}-(1-k)] + \frac{2}{3k^4}[(1-k^2)^{3/2}-(1-k)^3]\Big\}.$$

Thus
$$I_2 = k^2\sum_{i=I}^{IV} I_2^i = \frac{1}{3k^2}[1-(1-k^2)^{3/2}]. \tag{A-9}$$

$$I_1 = k^2[I_1^I + I_1^{II} + I_1^{III} + I_1^{IV}]$$

where

$$I_1^I = -k\int_0^{\beta_m}\cos\beta\sin\beta[1+k^2-2k\cos\beta]^{-5/2}d\beta = -\frac{k}{(1+2k^2)}I_2^{II}$$

$$I_1^{II} = (1+2k^2)\int_0^{\beta_m}\cos^2\beta\sin\beta[1+k^2-2k\cos\beta]^{-5/2}d\beta = -\frac{(1+2k^2)}{k(2+k^2)}I_2^{III}$$

$$I_1^{III} = -k(2+k^2)\int_0^{\beta_m}\cos^3\beta\sin\beta[1+k^2-2k\cos\beta]^{-5/2}d\beta = -\frac{k(2+k^2)}{k^2}I_2^{IV}$$

$$I_1^{IV} = k^2\int_0^{\beta_m}\cos^4\beta\sin\beta[1+k^2-2k\cos\beta]^{-5/2}d\beta = -\frac{k}{3}[k^4(1-k^2)^{-3/2}-(1-k)^{-3}]+$$
$$+\frac{4}{3}[k^3(1-k^2)^{-1/2}-(1-k)^{-1}] + \frac{4}{k}[k^2(1-k^2)^{1/2}-(1-k)] +$$
$$+\frac{8}{3k^2}[k(1-k^2)^{3/2}-(1-k)^3] + \frac{8}{15k^3}[(1-k^2)^{5/2}-(1-k)^5] .$$

Thus

$$I_1 = \sum_{i=I}^{IV} I_1^i = \frac{1}{15k}\left[2(1-k^5) + 5k^3 - (2+3k^2)(1-k^2)^{3/2}\right]. \qquad \text{(A-10)}$$

The results of substituting the parameter values from Chapter II into equations (A-4), (A-5), and (A-6) are listed in Table I where h is the height of the satellite above the surface of the earth ($h = r - R_e$). Note that $\underline{F}_3$ has been worked out for two different values of T_e (250 and 300°K) to show possible extreme values of this force for the current range of temperatures for the earth if it is considered to radiate as a black body ($\epsilon_e = 1$).

Table I - Forces on Flat Plate Situated on Earth-Sun Line

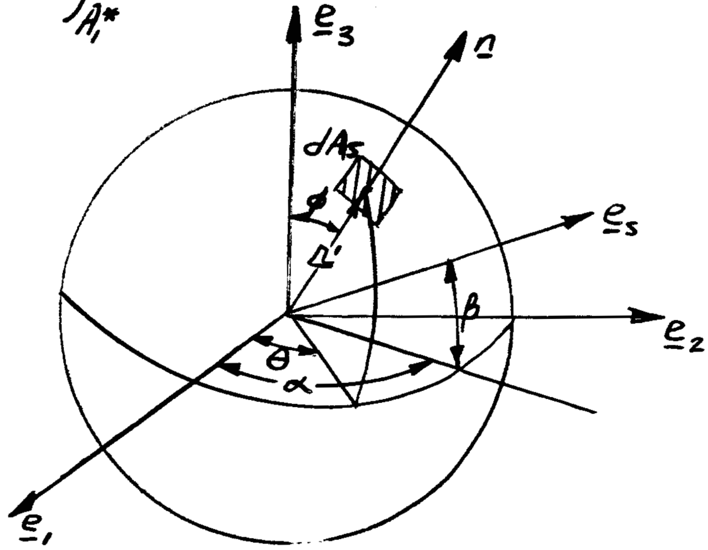
r mi.	h mi.	F_1/A lb/ft^2	F_2/A lb/ft^2	$(F_3/A)_{250°}$ lb/ft^2	$(F_3/A)_{300°}$ lb/ft^2
4000	0	9.4×10^{-8}	2.13×10^{-8}	1.02×10^{-8}	2.12×10^{-8}
4050	50	9.4×10^{-8}	2.12×10^{-8}	1.01×10^{-8}	2.11×10^{-8}
4100	100	9.4×10^{-8}	2.10×10^{-8}	1.00×10^{-8}	2.10×10^{-8}
4200	200	9.4×10^{-8}	2.06×10^{-8}	9.93×10^{-7}	2.06×10^{-8}
4300	300	9.4×10^{-8}	2.02×10^{-8}	9.70×10^{-7}	2.02×10^{-8}
4400	400	9.4×10^{-8}	1.96×10^{-8}	9.48×10^{-7}	1.97×10^{-8}
4500	500	9.4×10^{-8}	1.91×10^{-8}	9.22×10^{-7}	1.92×10^{-8}
4600	600	9.4×10^{-8}	1.85×10^{-8}	8.95×10^{-7}	1.86×10^{-8}
4700	700	9.4×10^{-8}	1.80×10^{-8}	8.74×10^{-7}	1.81×10^{-8}
4800	800	9.4×10^{-8}	1.74×10^{-8}	8.50×10^{-7}	1.76×10^{-8}
4900	900	9.4×10^{-8}	1.68×10^{-8}	8.22×10^{-7}	1.71×10^{-8}
5000	1000	9.4×10^{-8}	1.62×10^{-8}	8.00×10^{-7}	1.66×10^{-8}
5500	1500	9.4×10^{-8}	1.38×10^{-8}	6.91×10^{-7}	1.43×10^{-8}
6000	2000	9.4×10^{-8}	1.17×10^{-8}	5.99×10^{-7}	1.24×10^{-8}
6500	2500	9.4×10^{-8}	1.00×10^{-8}	5.21×10^{-7}	1.08×10^{-8}
7000	3000	9.4×10^{-8}	9.46×10^{-7}	4.58×10^{-7}	9.53×10^{-7}
9000	5000	9.4×10^{-8}	5.14×10^{-7}	2.86×10^{-7}	5.94×10^{-7}
14000	10000	9.4×10^{-8}	1.98×10^{-7}	1.21×10^{-7}	2.54×10^{-7}
24000	20000	9.4×10^{-8}	5.88×10^{-6}	4.08×10^{-6}	8.48×10^{-6}

$\eta_2 = 0$, $\bar{R} = R_e$, $\bar{\rho} = 0.34$, $\rho = \rho' = 0$, $\epsilon_e = 1$, $T_e = 250, 300°K$, $h = r - R_e$.

APPENDIX B. FORCE AND TORQUE ON SPHERICAL SATELLITE AS A RESULT OF DIRECT SOLAR RADIATION

From equation (20), with $\underline{e}'_s$ replaced by $\underline{e}_s$

$$\underline{F} = -\gamma\left(\frac{S_s}{c}\right)\iint_{A_1^*}\left[\cos^2\eta_2(1+\rho)\,\underline{n} + \cos\eta_2(1-\rho)\,\underline{n}\times(\underline{e}_s\times\underline{n})\right]dA_s$$



From the figure above

$$\underline{n} = \cos\phi\,\underline{e}_3 + \sin\phi(\cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2)$$

$$\underline{e}_s = \sin\beta\,\underline{e}_3 + \cos\beta(\cos\alpha\,\underline{e}_1 + \sin\alpha\,\underline{e}_2)$$

$$\cos\eta_2 = \underline{n}\cdot\underline{e}_s = \cos\phi\sin\beta + \sin\phi\sin\beta(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$dA_s = R^2\sin\phi\,d\phi\,d\theta$$

$$\underline{r}' = R\,\underline{n}$$

The coordinate system $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is arbitrary, thus orient it so that $\underline{e}_s = \underline{e}_3$. Then

$$\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\cos^2\phi(1+\rho)\underline{n}+\cos\phi(1-\rho)\underline{n}\times(\underline{e}_s\times\underline{n})\right]R^2\sin\phi\, d\phi\, d\theta \quad \text{(B-1)}$$

or

$$\underline{F}_1 = -2\pi\delta\left(\frac{S_s}{c}\right)R^2\int_0^{\frac{\pi}{2}}\sin\phi\left[\cos^3\phi(1+\rho)+\cos\phi\sin^2\phi(1-\rho)\right]d\phi\ \underline{e}_s$$

$$\underline{F}_1 = -2\pi\delta R^2\left(\frac{S_s}{c}\right)\left[\frac{1}{4}(1+\rho)+\frac{1}{4}(1-\rho)\right]\underline{e}_s$$

$$\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)\bar{A}_s\,\underline{e}_s \quad \text{(B-2)}$$

where $\bar{A}_s = \pi R^2$, the projected area of the satellite as seen by the sun. From equation (21) and (B-1)

$$\underline{M}_1(0) = -\delta\left(\frac{S_s}{c}\right)\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R^3\underline{n}\times\left[\cos^2\phi(1+\rho)\underline{n}+\cos\phi(1-\rho)\underline{n}\times(\underline{e}_s\times\underline{n})\right]\sin\phi\, d\phi\, d\theta \quad \text{(B-3)}$$

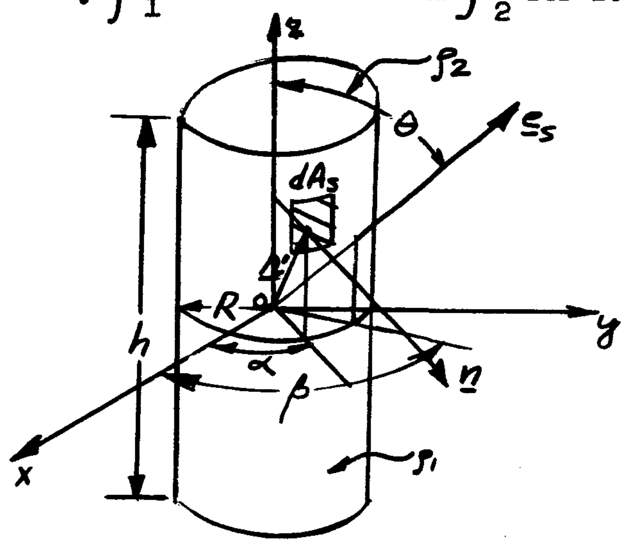
or $$\underline{M}_1(0) = -\delta R^3\left(\frac{S_s}{c}\right)(1-\rho)\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left\{\cos\phi\sin\phi\,\underline{n}\times\left[\underline{n}\times(\underline{e}_s\times\underline{n})\right]\right\}d\phi\, d\theta$$

and $\underline{M}_1(0) = 0.$ (B-4)

Thus point 0, the geometric center of the sphere, is the center of pressure (CP) for this case.

APPENDIX C. FORCE AND TORQUE ON A CYLINDRICAL SATELLITE DUE TO DIRECT SOLAR RADIATION

Consider a right circular cylinder of radius R and height h with reflectivity ρ_1 for its sides and ρ_2 for its end pieces



From equations (20) and (17)

$$\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)\iint_{A_1^*}\left[\cos^2\eta_2(1+\rho)\underline{n} + \cos\eta_2(1-\rho)\underline{n}\times(\underline{e}_s\times\underline{n})\right]dA_s$$

a) Sides - Reflectivity ρ_1.

$$\underline{n} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$$

$$\underline{e}_s = \cos\theta\,\underline{k} + \sin\theta(\cos\beta\,\underline{i} + \sin\beta\,\underline{j})$$

$$\cos\eta_2 = \underline{n}\cdot\underline{e}_s = \sin\theta(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$\underline{n}\times(\underline{e}_s\times\underline{n}) = \underline{e}_s - \cos\eta_2\,\underline{n}$$

$$dA = R\,d\alpha\,dz$$

where $-h/2 \leq z \leq -h/2$, $\beta - \pi/2 \leq \alpha \leq \beta + \pi/2$ are the limits on z and α for the sunlit portion of the cylinder's sides.

Integrating first over z

$$\underline{F}_{1_s} = -\delta\left(\frac{S_s}{c}\right)Rh\int_{\beta-\frac{\pi}{2}}^{\beta+\frac{\pi}{2}}\left[(1-\rho_1)\cos\eta_2\, \underline{e}_s + 2\rho_1\cos^2\eta_2\, \underline{n}\right]d\alpha$$

then over α

$$\underline{F}_{1_s} = -2\delta\left(\frac{S_s}{c}\right)Rh\sin\theta\Big[(1-\rho_1)\underline{e}_s + \rho_1\sin\theta\Big\{\Big[2\cos^3\beta + \frac{2}{3}(\sin^2\beta - \cos^2\beta)\cos^3\beta +$$

$$+ \frac{4}{3}\sin^4\beta\cos\beta\Big]\underline{i} + \Big[-\frac{2}{3}(\sin^2\beta - \cos^2\beta)\sin^3\beta + \frac{4}{3}\sin\beta\cos^4\beta + 2\sin^3\beta\Big]\underline{j}\Big\}\Big]$$

$$\underline{F}_{1_s} = -2\delta\left(\frac{S_s}{c}\right)Rh\sin\theta\left[(1-\rho_1)\underline{e}_s + \frac{4}{3}\rho_1\sin\theta(\cos\beta\,\underline{i} + \sin\beta\,\underline{j})\right].$$

But $\underline{e}_s = \cos\theta\,\underline{k} + \sin\theta(\cos\beta\,\underline{i} + \sin\beta\,\underline{j})$, thus $\underline{F}_{1_s}$ becomes

$$\underline{F}_{1_s} = -\delta\left(\frac{S_s}{c}\right)2Rh\sin\theta\left[\left(1+\frac{1}{3}\rho_1\right)\underline{e}_s - \frac{4}{3}\rho_1\cos\theta\,\underline{k}\right] \qquad \text{(C-1)}$$

b) End Pieces - Reflectivity ρ_2

Only one end piece is seen by the sun and the force acting on it is the same as a flat plate of area πR^2 whose normal makes an angle θ or $\pi - \theta$ (depending on whether the top or bottom piece is seen) with the incoming radiation. Therefore

$$\underline{F}_{1_e} = -\delta\left(\frac{S_s}{c}\right)\pi R^2\cos\theta\left[H(\theta) - 2H\left(\theta-\frac{\pi}{2}\right)\right]\left[(1-\rho_2)\underline{e}_s + 2\rho_2\cos\theta\,\underline{k}\right] \qquad \text{(C-2)}$$

which follows directly from equations (20) and (17) where

$\eta_2 = \theta, \pi - \theta$; $\underline{n} = \underline{k}, -\underline{k}$; and $\int_{A_1^*} dA = \pi R^2$.

The total force acting on the cylinder as a result of direct solar radiation, $\underline{F}_1$, is

$$\underline{F}_1 = \underline{F}_{1_s} + \underline{F}_{1_e} \;. \qquad \text{(C-3)}$$

The moment due to $\underline{F}_1$ about point 0 (geometric center of the cylinder) is

$$\underline{M}_1(0) = \int_{A_1^*} \underline{r}' \times d\underline{F}_{1_s} + \left\{ \frac{h}{2} \left[H(\theta) - 2H\left(\theta - \frac{\pi}{2}\right) \right] \right\} \underline{k} \times \underline{F}_{1_e}$$

and $\underline{r}' = R(\cos\alpha\ \underline{i} + \sin\alpha\ \underline{j}) + z\ \underline{k}$, thus

$$\underline{M}_{1_s}(0) = -\delta\left(\frac{S_s}{c}\right) \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\beta-\frac{\pi}{2}}^{\beta+\frac{\pi}{2}} [R(\cos\alpha\ \underline{i} + \sin\alpha\ \underline{j}) + z\ \underline{k}] \times [\cos^2\eta_2 (1+\rho_1)\underline{n} + (1-\rho_1)\cos\eta_2\, \underline{n} \times (\underline{e}_s \times \underline{n})]\, dA_s$$

Integrating first over z

$$\underline{M}_{1_s}(0) = -\delta\left(\frac{S_s}{c}\right) \int_{\beta-\frac{\pi}{2}}^{\beta+\frac{\pi}{2}} R^2 h (\cos\alpha\ \underline{i} + \sin\alpha\ \underline{j}) \times [\cos^2\eta_2 (1+\rho_1)\underline{n} + \cos\eta_2 (1-\rho_1)\underline{n} \times (\underline{e}_s \times \underline{n})]\, dA_s$$

Note that $\cos\alpha\ \underline{i} + \sin\alpha\ \underline{j} = \underline{n}$ so that

$$\underline{n} \times [\cos^2\eta_2 (1+\rho_1)\underline{n} + \cos\eta_2 (1-\rho_1)\underline{n} \times (\underline{e}_s \times \underline{n})] = (1-\rho_1)\cos\eta_2\ \underline{n} \times \underline{e}_s$$

and $$\underline{M}_{1_s}(0) = -\delta\left(\frac{S_s}{c}\right) R^2 h (1-\rho_1) \int_{\beta-\pi/2}^{\beta+\pi/2} \sin\theta \cos(\alpha-\beta) [\cos\theta \sin\alpha\ \underline{i} - \cos\theta\cos\alpha\ \underline{j} + \sin\theta \sin(\beta-\alpha)\ \underline{k}]\, d\alpha$$

$$= -\delta\left(\frac{S_s}{c}\right) \frac{\pi R^2 h}{2} (1-\rho_1) \sin\theta\cos\theta (\sin\beta\ \underline{i} - \cos\beta\ \underline{j})$$

Thus

$$\underline{M}_1(0) = -\delta\left(\frac{S_s}{c}\right) \pi R^2 h\, \frac{\sin 2\theta}{4} (\sin\beta\ \underline{i} - \cos\beta\ \underline{j})\, (\rho_2 - \rho_1) \tag{C-4}$$

If $\rho_1 = \rho_2$, $\underline{M}_1(0) = 0$ and point 0 is the center of pressure for $\underline{F}_1$ otherwise the CP is displaced from point 0 a distance p such that

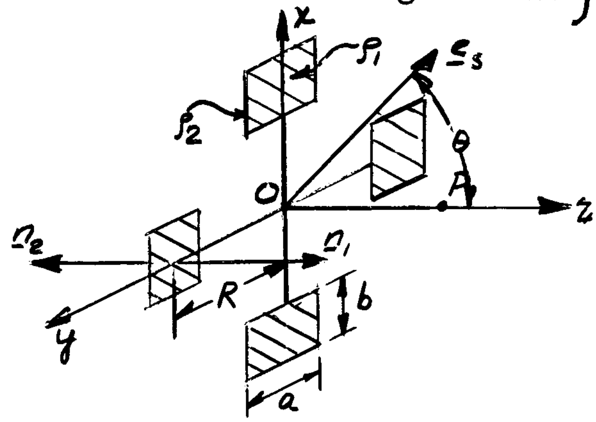
$$\underline{M}_1(0) = \underline{M}(CP) + \underline{p} \times \underline{F}_1 \tag{C-5}$$

but $\underline{M}(CP) = 0$ by definition, so if $\underline{p} = p\ \underline{k}$

$$p = \left| \frac{\underline{M}_1(0)}{\underline{k} \times \underline{F}_1} \right| \tag{C-6}$$

APPENDIX D. FORCE AND TORQUE ON PADDLE VANES DUE TO DIRECT SOLAR RADIATION

Consider a symmetric arrangement of flat, rectangular plates all of the same dimensions. The surface reflectivity will be taken as ρ_1 on the +z side of the arrangement and ρ_2 on the -z side.



$$A_t = 4ab$$

$$\underline{n}_1 = \underline{k}$$

$$\underline{n}_2 = -\underline{k}$$

$$\cos \eta_2 = \underline{n} \cdot \underline{e}_s = \cos\theta \quad .$$

From equation (20)

$$\underline{F}_1 = -\delta A_t\left(\frac{S_s}{c}\right)\Big\{\left[H(\theta)-H(\theta-\tfrac{\pi}{2})\right]\left[(1+\rho_1)\cos^2\theta\,\underline{k}+(1-\rho_1)\cos\theta\,\underline{k}\times(\underline{e}_s\times\underline{k})\right]+$$

$$+H(\theta-\tfrac{\pi}{2})\left[-(1+\rho_2)\cos^2\theta\,\underline{k}-(1-\rho_2)\cos\theta\,\underline{k}\times(\underline{e}_s\times\underline{k})\right]\Big\} \qquad \text{(D-1)}$$

But $\underline{k}\times(\underline{e}_s\times\underline{k}) = \underline{e}_s - \cos\theta\underline{k}$, thus

$$\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)A_t\cos\theta\,\Big\{\left[H(\theta)-H(\theta-\tfrac{\pi}{2})\right]\left[(1-\rho_1)\underline{e}_s+2\rho_1\cos\theta\,\underline{k}\right]-$$

$$-H(\theta-\tfrac{\pi}{2})\left[(1-\rho_2)\underline{e}_s+2\rho_2\cos\theta\,\underline{k}\right]\Big\}. \qquad \text{(D-2)}$$

The moment about point 0 is

$$\underline{M}_1(0) = \sum_{i=1}^{4} \underline{r}_i' \times \underline{F}_i = 0 \qquad \text{(D-3)}$$

because of symmetry. Therefore point 0 is the center of pressure for this arrangement.

If it is necessary to determine the moment about point P as a result of $\underline{F}_1$

$$\underline{M}_1(P) = -\underline{p} \times \underline{F}_1 \qquad \text{(D-4)}$$

where $\underline{p}$ is the position of point P relative to point 0. For $\underline{p} = p\underline{k}$

$$\underline{M}_1(P) = \delta\left(\frac{S_s}{c}\right)A_t\, p\cos\theta\left\{\left[H(\theta) - H\left(\theta - \frac{\pi}{2}\right)\right](1-\rho_1) - H\left(\theta - \frac{\pi}{2}\right)(1-\rho_2)\right\}\underline{k} \times \underline{e}_s \,. \qquad \text{(D-5)}$$

If the reflectivity is the same on both sides of each plate ($\rho_1 = \rho_2 = \rho$)

$$\underline{F}_1 = -\delta\left(\frac{S_s}{c}\right)A_t\cos\theta\left[(1-\rho)\underline{e}_s + 2\rho\cos\theta\,\underline{k}\right]\left[H(\theta) - 2H\left(\theta - \frac{\pi}{2}\right)\right] \qquad \text{(D-6)}$$

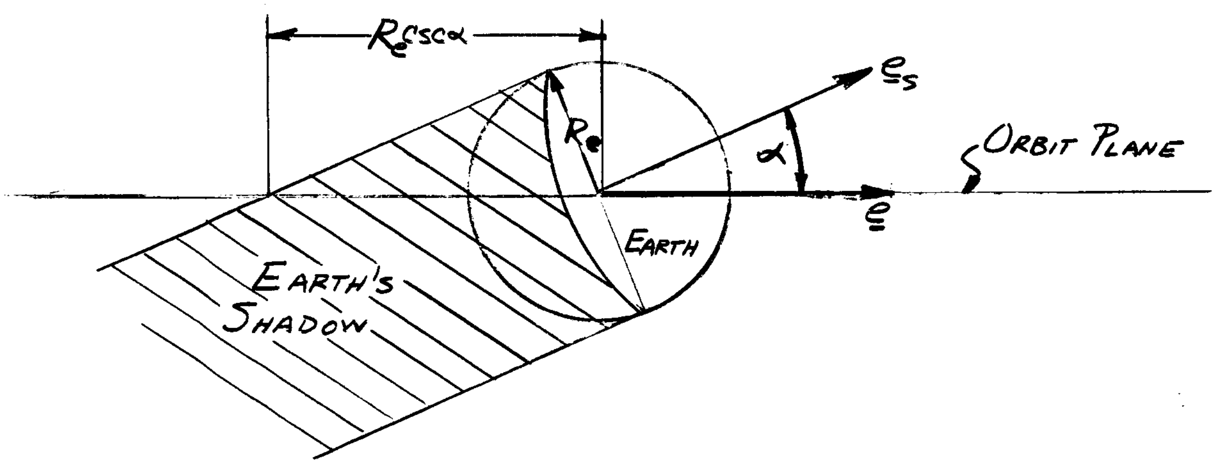
and

$$\underline{M}_1(P) = \delta\left(\frac{S_s}{c}\right)A_t\, p\cos\theta\,(1-\rho)\left[H(\theta) - 2H\left(\theta - \frac{\pi}{2}\right)\right]\underline{k} \times \underline{e}_s \,. \qquad \text{(D-7)}$$

APPENDIX E. ECLIPSES BY THE EARTH'S SHADOW

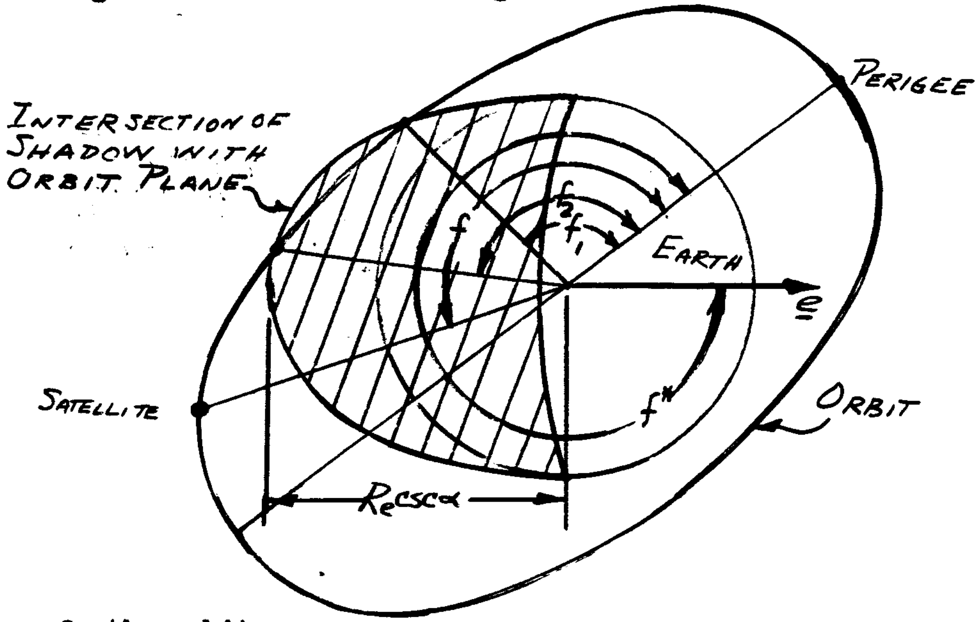
In order to determine the conditions for a satellite to be eclipsed by the earth's shadow, duration of eclipse, geometry, etc. the following assumptions will be made:

1) Earth's shadow is a circular cylinder of radius R
2) The shadow has no penumbra
3) Atmospheric refraction effects are negligible
4) The orbit is unperturbed.



The orbit plane intersects the axis of the earth's shadow (parallel to $\underline{e}_s$) at an angle α and the intersection in the orbit plane will be half an ellipse with semi-major axis $R_e \csc\alpha$ and semi-minor axis R_e, $\underline{e}$ is a unit vector parallel to the projection of $\underline{e}_s$ on the orbit plane. Note the limiting cases; $\alpha = \pi/2$ - intersection is a circle of radius R_e, $\alpha = 0$ - intersection is a semi-infinite rectangle of width $2R_e$.

The orientation of the satellite's orbit relative to the shadow's intersection on the orbit plane may be defined by the angle f^* as shown in the figure below.



On the orbit

$$r = \frac{a(1-e^2)}{1+e\cos f} \tag{E-1}$$

on the intersection ellipse

$$r_1 = \frac{R_e}{[1-\cos^2\alpha\cos^2(f-f^*)]^{1/2}} \tag{E-2}$$

when $\pi/2 \leq (f - f^*) \leq 3\pi/2$.

And the conditions for the satellite to pass through the earth's shadow are

$$\cos \Phi < 0 \tag{E-3}$$

$$r \sin \Phi < R_e \tag{E-4}$$

where Φ is the angle between $\underline{e}_r$ and $\underline{e}_s$. If an eclipse does occur the entering and leaving true anomalies, f_1 and f_2, can be determined by the identity $r = r_1$ or, from equations (E-1) and (E-2)

$$\frac{R_e}{a(1-e^2)}(1+e\cos f) = [1-\cos^2\alpha\cos^2(f-f^*)]^{1/2} \tag{E-5}$$

In general there will be four roots to this transcendental equation but two of the roots will be discarded as not fulfilling the condition of equation (E-3).

The duration of the eclipse can be determined from Kepler's equation

$$E - e\sin E = n(t-t_o) \tag{E-6}$$

where $t = t_o$ corresponds to perigee passage, $n = 2\pi/P$ is the mean motion, P = orbital period, and E is the eccentric anomaly which is related to f by

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2} \tag{E-7}$$

Thus the time spend in the earth's shadow is

$$\Delta t = t_2 - t_1 = \frac{P}{2\pi}\left[(E_2-E_1)+e(\sin E_1-\sin E_2)\right] \tag{E-8}$$

The conditions for and duration of a satellite eclipse, therefore, are completely specified when α and f^* are known. It is easily shown by spherical trigonometry, see Figure 1, that

$$\sin\alpha = \cos i\sin\Lambda\sin\Delta + \sin i(\sin\Lambda\cos\Delta\cos\Omega - \cos\Lambda\sin\Omega) \tag{E-9}$$

$$\sin(\omega+f^*) = \frac{\cos i\sin\alpha + \sin\Lambda\sin\Delta}{\cos\alpha\sin i} \tag{E-10}$$

$$\cos \Phi = \cos\alpha \cos (f-f^*) \tag{E-11}$$

For the special case of a circular orbit ($e = 0$) the conditions become quite simple: $r = r_c$ a constant and the conditions are

$$r_c > R_e \cos\alpha, \text{ no eclipse} \tag{E-12}$$

$$r_c < R_e \cos\alpha, \text{ eclipse when } \pi/2 \leq f-f^* \leq 3\pi/2 \tag{E-13}$$

and from equation (E-5)

$$\left(\frac{R_e}{r_c}\right)^2 = 1 - \cos^2\alpha \cos^2(f-f^*)$$

or

$$(f-f^*) = \cos^{-1}\left\{\pm \frac{\sqrt{1-\left(\frac{R_e}{r_c}\right)^2}}{\cos\alpha}\right\} \tag{E-14}$$

As perigee is undefined for $e = 0$ measure f such that $f^* = \pi/2$, then by symmetry $f_1 + f_2 = 3\pi$, and the duration of the eclipse becomes

$$\Delta t_c = \left(\frac{3\pi - 2f_1}{2\pi}\right) P_c \tag{E-15}$$

where P_c is the period of the satellite in a circular orbit.

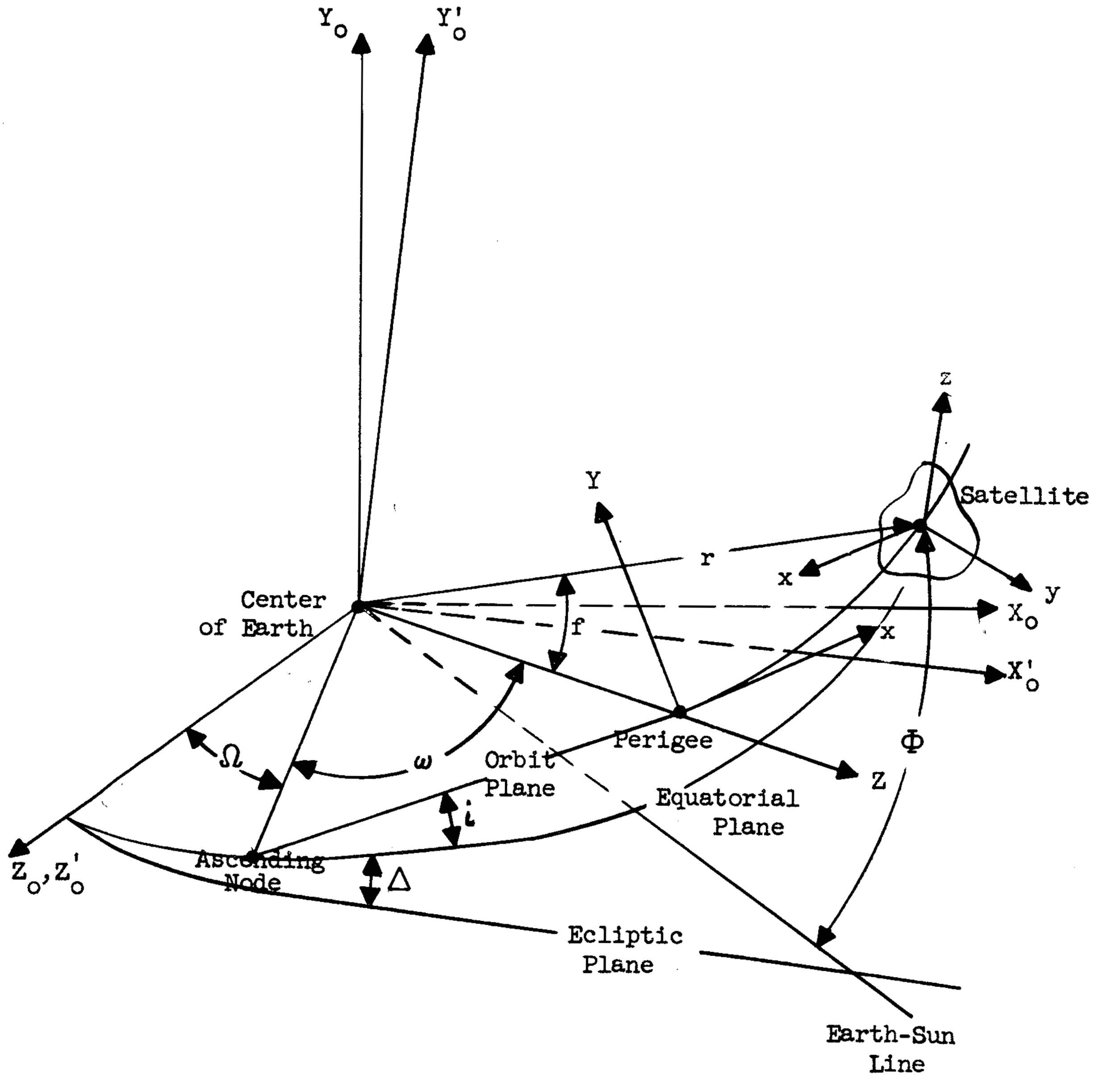


Figure 1. Coordinate Systems and Geometry

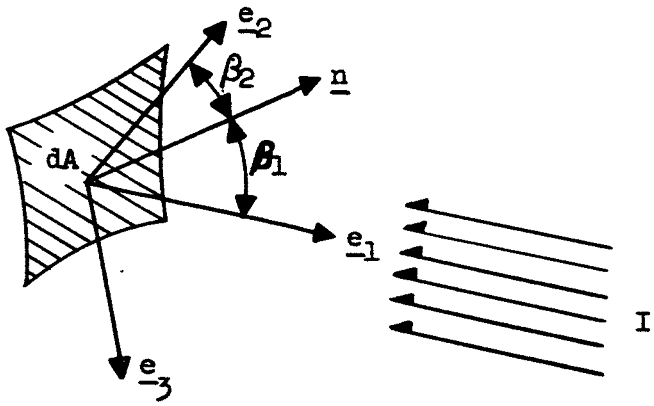


Figure 2. Surface Geometry for Incident and Reflected Radiation

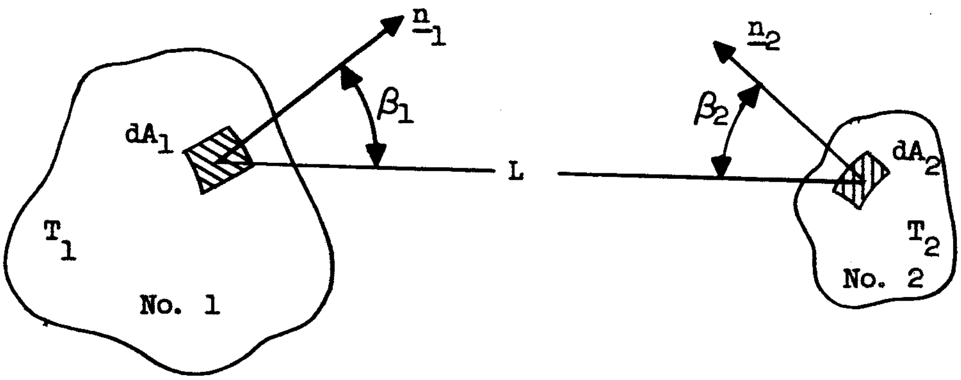


Figure 3. Radiation at dA_2 due to Emission from dA_1

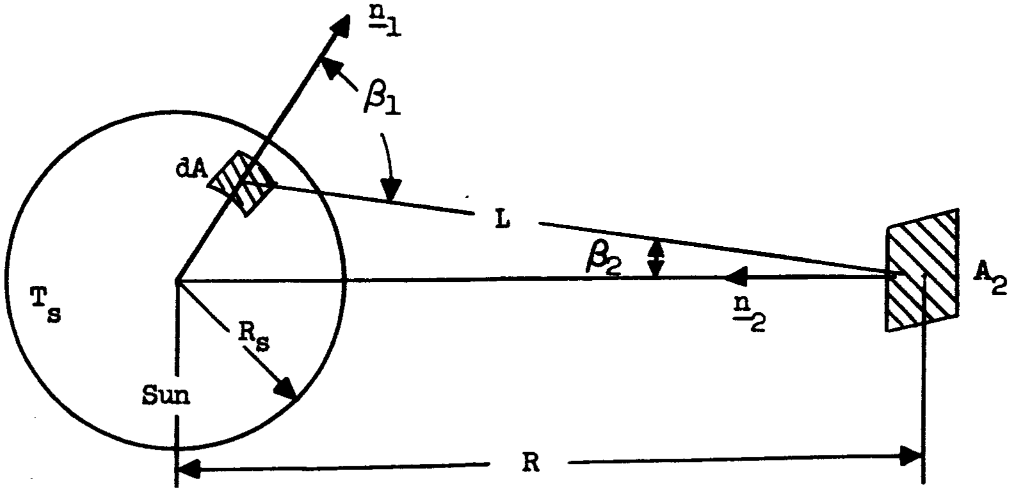


Figure 4. Solar Radiation of Normal Incidence on a Flat Plate

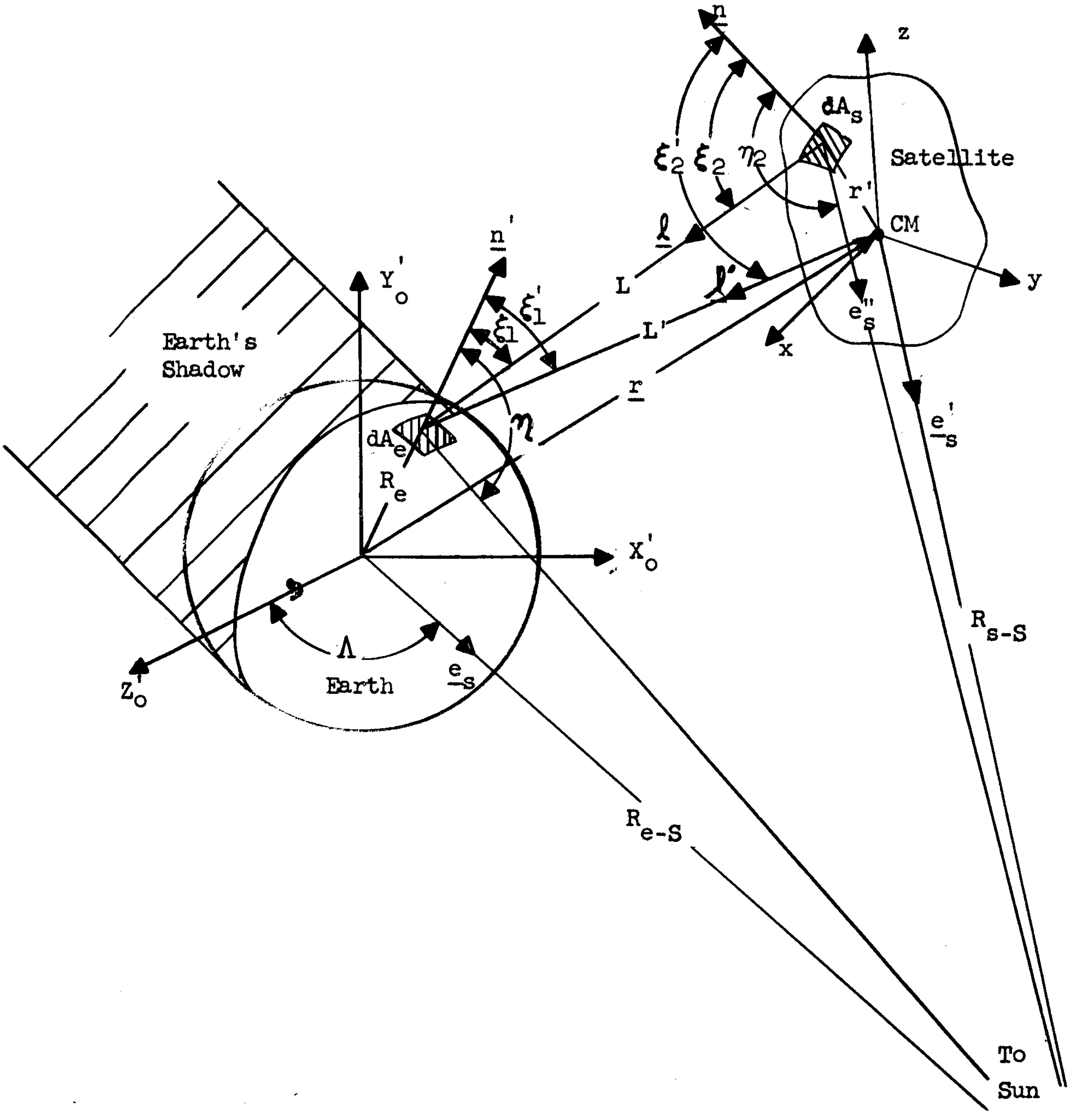


Figure 5. Geometry of Receiving Element dA_s Relative to Emitting area dA_e on the Earth and Relative to the Sun

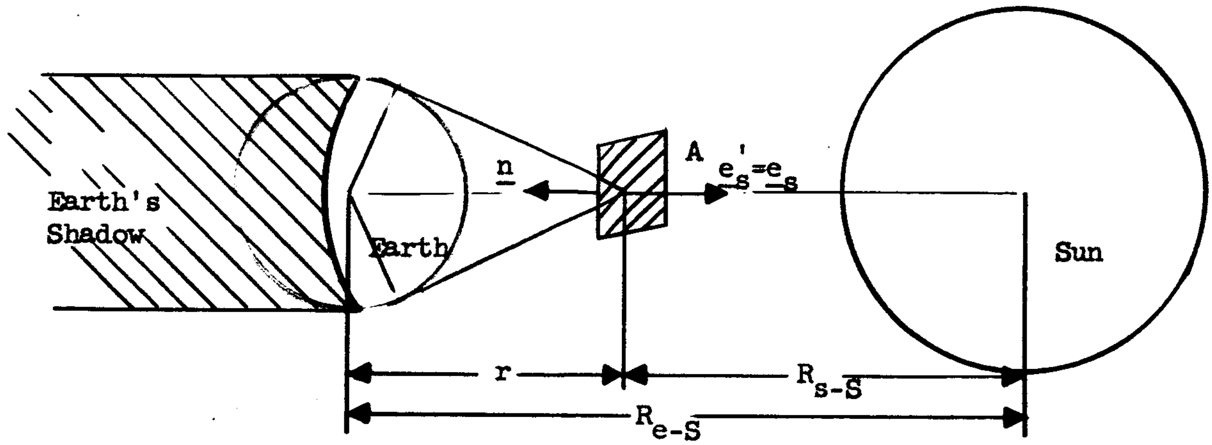


Figure 6. Flat Plate on Earth-Sun Line (not to scale)

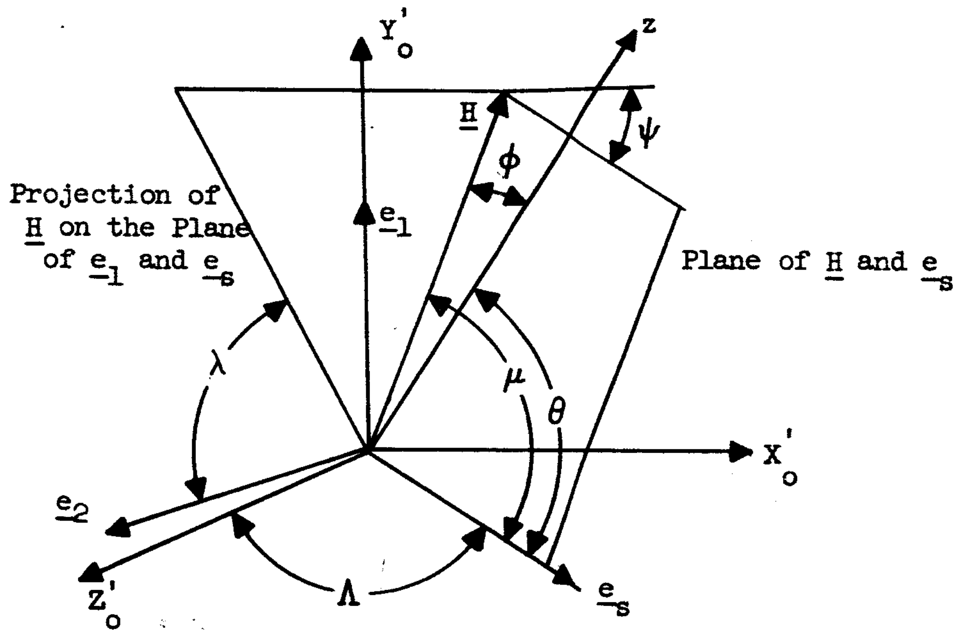


Figure 7. Orientation of H and z-axis Relative to e_s and Inertial Space

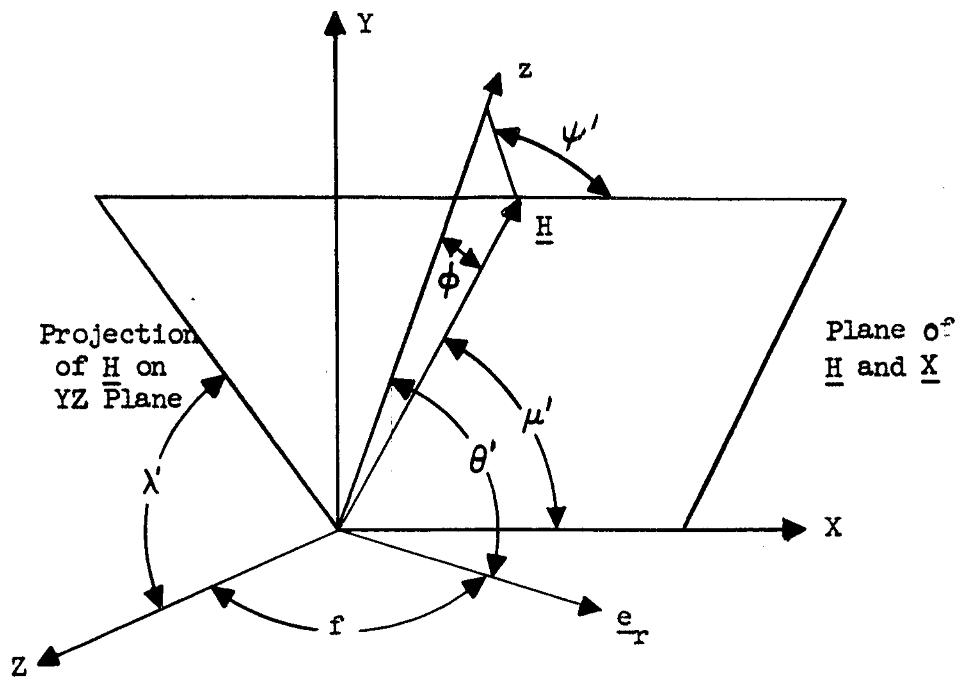


Figure 8. Orientation of H and z-axis Relative to (X,Y,Z)

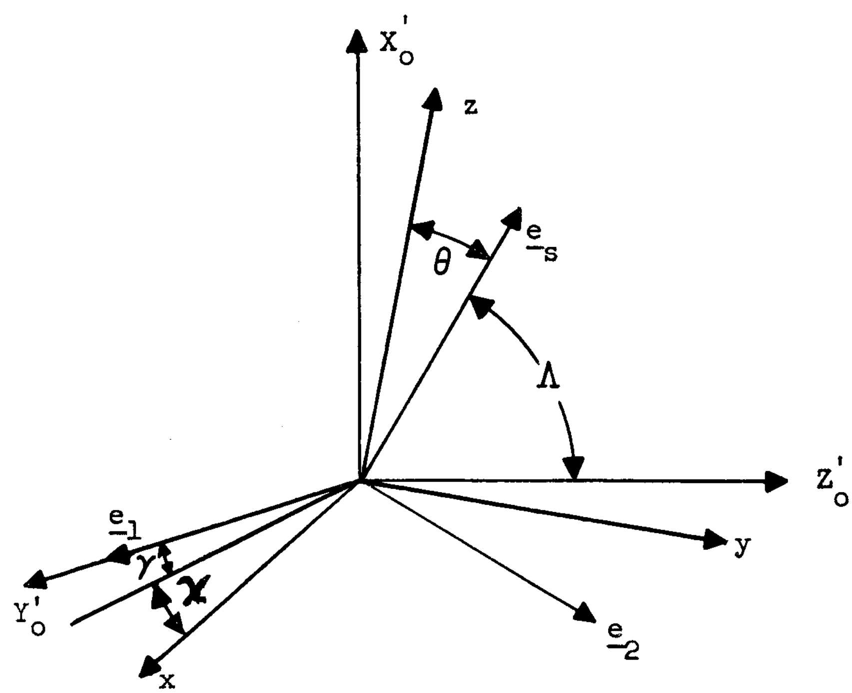


Figure 9. Orientation of Axis of Symmetry Relative to ($\underline{e}_s$, $\underline{e}_1$, $\underline{e}_2$)

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