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A GUIDE OF THE APPLICATION OF THE LIAPUNOV DIRECT METHOD TO FLIGHT CONTROL SYSTEMS

by Eugene J. Lefferts

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MARTIN MARIETTA CORPORATION
Baltimore, Md.
for

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INTRODUCTION

This report will present the results of a years study in the area of the application of the Liapunov direct method to flight control systems. The primary intent at the initiation of this study was to collect a set or practical applications of the direct method and to prepare a table of appropriate "V" functions suitable for general classes of problems. Unfortunately this aim was not realized and is probably incapable of being realized for many years to come. The number of control problems solved by the direct method and documented in the literature are very few. Instead one finds a wealth of material on procedure for general constructions and the inter-relations between the second method and many of the theoretical areas of modern research such as optimal control theory, stochastic control, the theory of functional equations, etc.

The major body of this report is devoted to a presentation without proofs of the main concepts of the direct method. Chapter I introduces the required matrix and background, and introduces the major transformations by which vector systems may be put into the standard state vector form for subsequent analysis. Chapter II is devoted to the definitions of stability and the second method of Liapunov. In Chapter III is presented the main methods for the construction of Liapunov functions. This presentation is by no means complete but the procedures presented are representative of the major approaches. Chapter IV is devoted to the classical problem of Lur'e. In Chapter V, a discussion of Lagrange stability is given along with a construction procedure for locating bounds on limit sets. In Chapter VI the results of the preceding sections are recast in a form applicable to discrete systems. In Chapter VII an attempt is made to inter-relate the second method to some of the concepts of optimal and adaptive control theory.

It is hoped that this presentation will bring to a larger audience some insight into the importance and general usefulness of the point of view as represented by the second method. It is felt that only by the efforts of a large segment of the practicing engineers in the control field will new results leading to practical synthesis procedures be developed.

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CHAPTER I

NOTATION AND PRELIMINARY CONCEPTS

In this chapter we will review some properties of matrices and vectors and state without proof the main results needed for an understanding of the remaining text. Since the state vectors concept plays such an important role in much of modern control theory, this will be defined and methods of transforming differential equations and block diagrams into the state vector notation will be explained and illustrated by examples.

A. Vectors and Matrices: The capital letters X, Y, Z, will be used to represent variable column matrices or vectors, while small letters will represent scalars. Subscripts will indicate components. Thus

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

The capital letters F, G, H will be used to represent vector functions, while small letters will be used for scalar functions. Thus

$$F(X) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix}.$$

The capital letters A, B, C, P, Q, R will be used to represent matrices, while small letters with double subscripts will represent the elements of a matrix.

A matrix A is an m + n array of elements written as

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix}$$

where the first subscript indicates the row position of the element while the second subscript indicates the column position. The numbers m and n give the row and column dimensions. If A is square, that is, the row dimension is the same as the column dimension, then we refer to the dimension of A as this number. Otherwise we say that A is of dimension m by n.

Two matrices A and B are said to be equal if

$$a_{ij} = b_{ij}$$

for all i and j. Equality can only be defined for matrices of the same dimension.

<u>Addition</u>. The sum or difference of two matrices written as

$$A \pm B = C$$

is defined by

$$c_{ij} = a_{ij} + b_{ij}.$$

Matrix addition satisfies the following

$$A + (B + C) = (A + B) + C \qquad \text{(associativity)}$$
$$A + B = B + A. \qquad \text{(commutativity)}$$

Scalar multiplication: The product of a matrix $A = (a_{ij})$ by a scalar λ is a matrix $B = (b_{ij})$ defined as

$$b_{ij} = \lambda a_{ij} \qquad \text{for all } i \text{ and } j.$$

Scalar multiplication satisfies the following properties

$$\lambda A = A\lambda$$
$$\lambda_1\lambda_2 A = \lambda_1 A \lambda_2 = \lambda_2\lambda_1 A$$
$$\lambda(A + B) = \lambda A + \lambda B$$
$$(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A.$$

Matrix multiplication: Let $A = (a_{ij})$ be a matrix with dimension m by n and let $B = (b_{ij})$ be a matrix with dimension p by q. If $q = m$, then the matrices A, B are said to be conformable in the order BA. If A, B are conformable in the order AB, the product $C = (c_{ij})$ of AB is defined

$$AB = C$$
$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

The matrix C obtained from this multiplication is of dimension m by q. If A, B are conformable in the order BA, then the product BA is given by

$$BA = D = (d_{ij} = \sum_{k=1}^{m} b_{ik}a_{kj})$$

where D is a matrix of dimension p by n. The dimension of C is in general not the same as the dimension of D. Thus matrix multiplication is is not commutative except in special cases. Matrix multiplication satisfies

$$A(BC) = (AB)C \qquad \text{(associative)}$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA.$$

If A is a square matrix we may define powers of A,

$$AA = A^2$$

$$AAA = A(AA) = (AA)A = AA^2 = A^2A = A^3.$$

Powers of A commute

Transposition: Associated with each matrix A is another matrix called the transpose of A and written as A^T. The transpose is obtained by interchanging the rows and columns of A. Thus if A is an m by n matrix, then A^T is an n by m matrix. The transpose satisfies the following properties

$$(A + B)^T = A^T + B^T$$

$$(ABC)^T = C^T B^T A^T$$

$$(A^T)^T = A.$$

Determinants: Associated with each square matrix is a scalar called the determinant of A and written as $|A|$. The determinant has the following properties

$$|A| = |A^T|$$

$$|ABC| = |A|\ |B|\ |C|$$

if $|A| = 0$, A is said to be singular.

Special matrices: There are a number of particular square matrices which we now define. A matrix $A = (a_{ij})$ is said to be diagonal and written as $\mathrm{diag}(\lambda_i)$ if

$$a_{ij} = 0 \qquad i \neq j$$
$$a_{ij} = \lambda_i \qquad i = j.$$

Diagonal matrices commute with each other, thus

$$\mathrm{diag}(\lambda_i)\ \mathrm{diag}(\mu_i) = \mathrm{diag}\ \mu_i\ \mathrm{diag}(\lambda_i) = \mathrm{diag}(\mu_i\lambda_i)$$
$$(\mathrm{diag}\ \lambda_i)^n = \mathrm{diag}(\lambda_i^n).$$

If all the elements of a diagonal matrix are equal to one, then it is called an identity matrix and written as I. The identity matrix has the following property.

$$AI = IA = A$$

for A arbitrary.

If A is nonsingular, there exists a unique matrix called the inverse of A and written as A^{-1}. The inverse has the following properties

$$A^{-1}A = AA^{-1} = I$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$
$$(A^T)^{-1} = (A^{-1})^T$$
$$(A^{-1})^{-1} = A.$$

A matrix A is said to be symmetrical if

$$A = A^T.$$

A matrix A is said to be skew symmetric if

$$A = -A^T.$$

Any arbitrary matrix A can be decomposed into the sum of a symmetric and a skew symmetric matrix

$$A = \left(\frac{A + A^T}{2}\right) + \left(\frac{A - A^T}{2}\right).$$

Characteristic equation: To each square matrix A is associated a scalar equation called the characteristic equation and given by

$$|A - \lambda I| = P_n(\lambda) = (-1)^n(\lambda^n - a_1\lambda^{n-1} + \ldots + (-1)^n a_n) = 0.$$

The roots λ_i of this equation are called the characteristic roots or eigenvalues of A. If all the eigenvalues of A have negative real part, then A is said to be stable. To each eigenvalue λ_i there corresponds a vector Q_i called the eigenvector which has the following property

$$(A - \lambda_i I)Q_i = 0.$$

If the λ_i's of a matrix A are distinct then the corresponding eigenvectors are linearly independent and the matrix Q, called the modal matrix, whose columns are the eigenvectors has the property

$$Q^{-1}AQ = \text{diag}(\lambda_i).$$

Differentiation of matrices: Given any matrix or vector X whose elements depend upon a parameter τ, then the derivative of X with respect to τ is defined as

$$\frac{dX}{d\tau} = \begin{pmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \\ \vdots \\ \frac{dx_n}{d\tau} \end{pmatrix}.$$

The derivative of a product obeys the normal chain law but order of the products must be maintained. Thus

$$\frac{d}{d\tau}(ABC) = \frac{dA}{d\tau}(BC) + A\frac{dB}{d\tau}C + AB\frac{dC}{d\tau}.$$

Since

$$A^{-1} = A^{-1}AA^{-1}$$

we have

$$\frac{dA^{-1}}{d\tau} = \frac{dA^{-1}}{d\tau}AA^{-1} + A^{-1}\frac{dA}{d\tau} + A^{-1}A\frac{dA^{-1}}{d\tau}$$

$$= 2\frac{dA^{-1}}{d\tau} + A^{-1}\frac{dA}{d\tau}A^{-1}$$

$$= -A^{-1}\frac{dA}{d\tau}A^{-1}.$$

Quadratic forms: The scalar

$$X^TQX = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j q_{ij}$$

is called a quadratic form. Without any loss in generality we may always assume that the matrix Q is symmetric since if Q is not symmetric, then

$$Q = R + P$$

where R is symmetric and P is skew symmetric. It follows then

$$X^TQX = X^T(R + P)X = X^TRX + X^TPX.$$

But since P is skew symmetric

$$X^TPX \equiv 0$$

Norms: Associated with every vector X is a scalar called the norm of X and written as $\|X\|$. A norm must satisfy the following postulates

i) $\|X\| > 0$ for all $X \neq 0$

ii) $\|X\| = 0$ $X = 0$

iii) $\|X + Y\| \leq \|X\| + \|Y\|$

iv) $\|\lambda X\| = |\lambda| \; \|X\|$ λ a scalar.

Some specific norms which may be used are the following

(1) $\|X\| = (X^TX)^{1/2} = \sqrt{\Sigma x_i^2}$

(2) $\|X\| = \Sigma |x_i|$

(3) $\|X\| = \max_i |x_i|$.

The first of the above norms is the standard concept of length of a vector. If we examine the space given by the inequality

$$\|X\| \leq 1$$

then (1) gives the interior of the sphere of radius 1

(2) gives a tetrahedron inscriber inside the unit sphere.

(3) gives the cube which circumscribes the unit sphere with its edges parallel to the coordinate axes.

B. State Vector Representation: Throughout this work we will be concerned with the properties of systems of differential equations. We will always assume that the system under study takes the form

$$\text{(I-1)} \qquad \dot{X} = F(X, t)$$

where X is an n-vector and F is a vector function of the vector X and the scalar t. The system(I-1) is equivalent to the set of scalar equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, t) \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n, t) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, t). \end{aligned}$$

If F is autonomous, then system (I-1) takes the form

$$\text{(I-2)} \qquad \dot{X} = F(X).$$

If F is linear in X, then we obtain the form

(I-3) $$\dot{X} = A(t)X + G(t)$$

where A is an n by n constant matrix and G is an n by 1 vector function of time. If G(t) is identically zero then the system is said to be homogeneous and (I-3) takes the form

(I-4) $$\dot{X} = A(t)X.$$

If F is linear, autonomous and homogeneous we obtain the simple form

(I-5) $$\dot{X} = Ax.$$

Every system of differential equations may be transformed into this representation by a suitable change of variables.

Example 1.

$$\ddot{y} + 4\dot{y} + 3y = 0.$$

Let

$$x_1 = y \quad \text{and} \quad x_2 = \dot{y}$$

then

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = -3y - 4\dot{y} = -3x_1 - 4x_2$$

or we obtain

$$\dot{X} = Ax$$

with X and A given by

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}.$$

Example 2.

$$\ddot{y} - y = 0$$
$$\ddot{x} + 3\dot{x} + x = 2\dot{y} - y.$$

Let

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = y, \quad x_4 = \dot{y}.$$

Then we have

$$\dot{x}_1 = \dot{x} = x_2$$
$$\dot{x}_2 = \ddot{x} = -x - 3\dot{x} + 2\dot{y} - y = -x_1 - 3x_2 - x_3 + 2x_4$$
$$\dot{x}_3 = \dot{y} = x_4$$
$$\dot{x}_4 = \ddot{y} = y = x_3.$$

n matrix notation this becomes

$$\dot{X} = Ax$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -3 & -1 & +2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Example 3.

$$\ddot{x} + (1 - x^2)\dot{x} + x = 0.$$

As before with $x_1 = x$ and $x_2 = \dot{x}$ we obtain the representation

$$\dot{X} = F(X)$$

$$X = \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} \qquad\qquad F(X) = \begin{pmatrix} x_2 \\ \\ (x_1^2 - 1)x_2 - x_1 \end{pmatrix}.$$

In general there will be more than one way in which a system can be represented as a vector differential equation. In the previous example we could have made the transformation

$$y_1 = x \quad \text{and} \quad y_2 = \int [\ddot{x} + (1 - \dot{x}^2)\dot{x}]dt = \dot{x} + x - \frac{x^3}{3}$$

thus obtaining the representation

$$\dot{y}_1 = \frac{y_1^3}{3} - y_1 + y_2$$

$$\dot{y}_2 = - y_1.$$

As a first step in the reduction of a system of differential equations into the state vector form, one often encounters an equation of the form

$$x^{(n)} + a_1 x^{(n-1)} + \dots + a_n x = b_o y^{(n)} + b_1 y^{(n-1)} + \dots + b_n y$$

where by the notation $x^{(n)}$ is meant the n^{th} derivative of x. In this representation the coefficients a_i and b_i may depend explicitly upon the independent variable t. This system with some effort may be placed into the form

$$\dot{X} = A(t)X + C(t)y$$
$$x = D^T(t)X + ry$$

providing the coefficients $a_i(t)$ and $b_i(t)$ are sufficiently often differentiable. This transformation may be achieved by the following procedure

$$\dot{x}_1 = x_2 + c_1(t)y$$
$$\dot{x}_2 = x_3 + c_2(t)y$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + c_{n-1}(t)y$$
$$\dot{x}_n = - a_n(t)x_1 - a_{n-1}(t)x_2 - \dots - a_1(t)x_n(t) + c_n(t)y$$
$$x = x_1 + r(t)y$$

where the coefficients $c_i(t)$ are to be determined by substitution into the original scalar differential equation. In this transformation observe that $A(T)$ has the form

$$A(T) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & \dots & -a_1 \end{pmatrix}$$

$$D^T(t) = (1 \quad 0 \quad 0 \quad \dots \quad 0).$$

The elements of the vector $C(t)$ must be determined. To illustrate this procedure consider the following example.

Example 4.

$$\ddot{x} + a_1(t)\dot{x} + a_2(t)x = b_o(t)\ddot{y} + b_1(t)\dot{y} + b_i(t)y.$$

We have that this system takes the form

$$\dot{X} = A(t)X + C(t)y$$
$$x = x_1 + r(t)y$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}.$$

Thus we have

$$x = x_1 + r(t)y$$
$$\dot{x}_1 = x_2 + c_1(t)y$$
$$\dot{x}_2 = - a_2x_1 - a_1x_2 + c_2(t)y.$$

Solving the first of these for x_1 and differentiating we have

$$x_1 = x - ry$$
$$\dot{x}_1 = \dot{x} - \dot{r}y - r\dot{y} = x_2 + c_1y.$$

Now solving for x_2 and differentiating we obtain

$$x_2 = \dot{x} - (\dot{r} + c_1)y - r\dot{y}$$

$$\dot{x}_2 = \ddot{x} - (\ddot{r} + \dot{c}_1)y - (2\dot{r} + c_1)\dot{y} - r\ddot{y} = -a_2x_1 - a_1x_2 + c_2y$$

thus we have

$$\ddot{x} - (\ddot{r} + c_1)y - (2\dot{r} + c_1)\dot{y} - r\ddot{y} = -a_2(x - ry) - a_1(\dot{x} - (r + c_1)y - r)) + c_2y.$$

Collecting terms we obtain

$$\ddot{x} + a_1\dot{x} + a_2x = r\ddot{y} + (2\dot{r} + c_1 + a_1r)\dot{y} + (\ddot{r} + \dot{c}_1 + a_2r + a_1(\dot{r} + c_1) + c_2))y.$$

Equating the above coefficients on the right to the original equation we obtain

$$r = b_o(t)$$

$$2\dot{r} + c_1 + a_1r = b_1(t)$$

$$\ddot{r} + \dot{c}_1 + a_2r + a_1\dot{r} + a_1c_1 + c_2 = b_2(t).$$

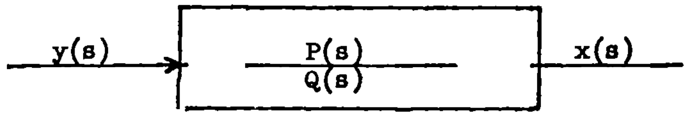
Thus we have the solutions

$$r = b_o(t)$$

$$c_1 = b_1(t) - a_1b_o(t) - 2\dot{b}_o(t)$$

$$c_2 = b_2(t) - a_1(b_1 - 2\dot{b}_o - a_1b_o) - a_2(b_o) + \ddot{b}_o - \dot{b}_1 + \dot{a}_1b_o.$$

In many cases, engineering problems appear in terms of a block diagram where each block of this representation gives a relationship between the input and output in terms of their Laplace transforms. Thus a typical block has the form



where $P(s)/Q(s)$ is a rational polynomial in s where the numerator order is equal to or less than the order of the denominator. Thus it is sufficient to show the transformation of such a block into the state vector representation. We will assume that $P(s)$ and $Q(s)$ have the form

$$Q(s) = s^n + a_1 s^{n-1} + \dots + a_n$$

$$P(s) = b_o s^n + b_1 s^{n-1} + \dots + b_n.$$

Since the transfer function of the block diagram representation is given by

$$\frac{x(s)}{y(s)} = \frac{P(s)}{Q(s)}$$

then we have

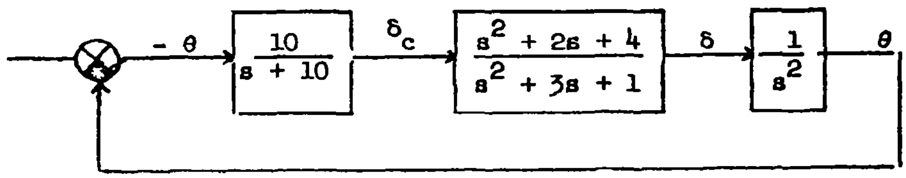
$$Q(s)x(s) = P(s)y(s).$$

If s is given the interpretation of the operators $\frac{d}{dt}$ this equation becomes identical to what we obtained before, namely

$$x^{(n)} + a_1 x^{(n-1)} + \dots + a_n x = b_o y^{(n)} + b_1 y^{(n-1)} + \dots + b_n y.$$

Thus the preceeding method gives directly the desired representation.

Example 5. Consider the following system



In transfer notation we have

$$\frac{\theta}{\delta} = \frac{1}{s^2}$$

$$\frac{\delta}{\delta_c} = \frac{s^2 + 2s + 4}{s^2 + 3s + 1}$$

$$\frac{\delta_c}{\theta} = \frac{-10}{s + 10}.$$

Representing these in terms of differential equations we have

$$\theta = \delta$$

$$\ddot{\delta} + 3\dot{\delta} + \delta = \ddot{\delta}_c + 2\delta_c + 4\delta_c$$

$$\delta_c + 10\delta_c = - 10\theta.$$

For the first of these equations we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \delta$$

$$\theta = x_1.$$

For the second we have

$$\dot{x}_3 = x_4 - \delta_c$$
$$\dot{x}_4 = - x_3 - 3x_4 + 6\delta_c$$
$$\delta = x_3 + \delta_c.$$

For the third equation we obtain

$$\dot{x}_5 = - 10x_5 - 10\theta$$
$$\delta_c = x_5.$$

Combining these we have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3 + x_5$$
$$\dot{x}_3 = + x_4 - x_5$$
$$\dot{x}_4 = - x_3 - 3x_4 + 6x_5$$
$$\dot{x}_5 = - 10x_5 - 10x_1$$

or

$$X = AX,$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -3 & 6 \\ -10 & 0 & 0 & 0 & -10 \end{pmatrix}.$$

For systems represented by a block diagram, there is an alternate way of obtaining a state vector representation which has many advantages providing the denominator has no repeated roots. The advantage of this procedure is that the matrix A can be obtained directly as a diagonal matrix. Consider the block diagram of transfer function

$$\frac{x}{y} = \frac{p(s)}{q(s)} = \frac{p(s)}{\prod_{i=1}^{n} (s - \lambda_i)}$$

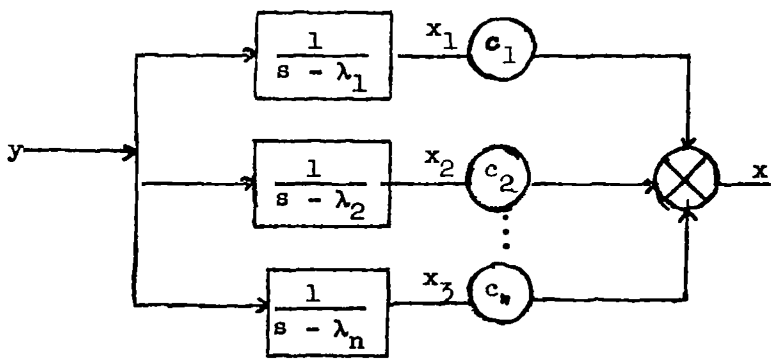
where λ_i's are the roots of $q(s)$. If none of these are repeated, then we know that we can represent this in a partial fraction form. Thus

$$\frac{x}{y} = \frac{p(s)}{\prod_{i=1}^{n}(s - \lambda_i)} = \Sigma \frac{c_i}{s - \lambda_i}$$

where the c_i are the residues given by

$$c_i = \lim_{s \to \lambda_i} (s - \lambda_i) \frac{p(s)}{q(s)} = \frac{p(\lambda_i)}{q'(\lambda_i)}.$$

With this representation we have the following diagram.



If we define the equations for each block we have

$$\dot{x}_1 = \lambda_1 x_1 + y$$

$$\dot{x}_2 = \lambda_2 x_2 + y$$

$$\vdots$$

$$\dot{x}_n = \lambda_n x_n + y$$

$$x = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

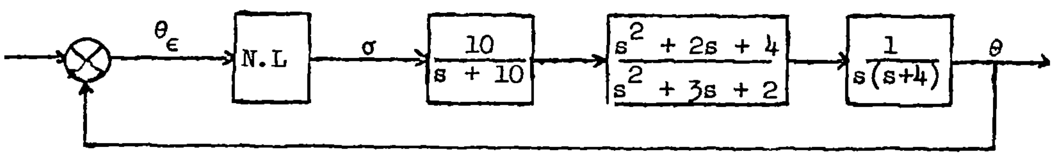
or

$$\dot{X} = \text{diag}(\lambda_i)X = 1y$$

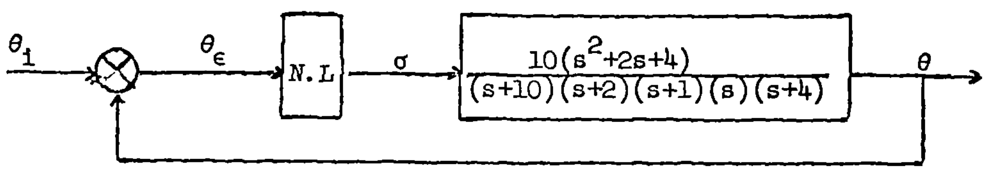
$$x = C^T X$$

where the vector 1 is a vector with ones in every element.

Example 6. As an example of this procedure consider the system



This system can be represented by the form



Therefore we have

$$\frac{\theta}{\sigma} = \frac{10(s^2+2s+4)}{(s+10)(s+2)(s+1)(s+4)(s)} = \frac{c_1}{s+10} \quad \frac{c_2}{s+2} \quad \frac{c_3}{s+1} \quad \frac{c_4}{s+4} \quad \frac{c_5}{s}$$

where

$$c_1 = \lim_{s \to -10} \frac{10(s^2+2s+4)}{(s+2)(s+1)(s+4)(s)} = 7/36 \qquad \lambda_1 = -10$$

$$c_2 = \lim_{s \to -2} \frac{10(s^2+2s+4)}{(s+10)(s+1)(s+4)(s)} = 5/4 \qquad \lambda_2 = -2$$

$$c_3 = \lim_{s \to -1} \frac{10(s^2+2s+4)}{(s+10)(s+2)(s+4)(s)} = 10/9 \qquad \lambda_3 = -1$$

$$c_4 = \lim_{s \to -4} \frac{10(s^2+2s+4)}{(s+10)(s+2)(s+4)(s)} = -5/6 \qquad \lambda_4 = 4$$

$$c_5 = \lim_{s \to 0} \frac{10(s^2+2s+4)}{(s+10)(s+4)(s+2)(s+1)} = 1/2 \qquad \lambda_5 = 0.$$

Thus the above system has the form

$$\dot{X} = \mathrm{diag}(\lambda_i)X + 1\sigma$$

$$\theta = C^T X$$

$$\sigma = F(\theta_c)$$

$$\theta_\epsilon = \theta_i - \theta = \theta_i - C^T X.$$

C. Scalar Functions: In most of the applications of the direct method of Liapunov we will need to consider certain scalar functions and some of their properties. In this section we will bring together these properties with examples to illustrate them.

Def. 1. We say that the scalar function $V(x) = V(x_1, x_2, \ldots, x_n)$ is definite for $\|X\| < k$, if for all choices of X with $\|X\| < k$, it assumes values on one sign only and vanishes only when $\|X\| = 0$.

Def. 2. We say that the function $V(X) = V(x_1, x_2, \ldots, x_n)$ is semidefinite for $\|X\| < k$, if for all choices of X with $\|X\| < k$ it assumes values of one sign, but it may vanish for values of X other than $\|X\| = 0$.

Def. 3. We say that the function $V(X) = V(x_1, x_2, \ldots, x_n)$ is indefinite if it is neither definite nor semidefinite.

Example 7.

$$V(x_1, x_2) = x_1^2 + x_2^2.$$

V is positive definite for all values of X.

Example 8.

$$V(x_1, x_2) = x_1^2 + x_2^2 - x_1^3.$$

V is positive definite for all X with $\|X\| < 1$.

Example 9.

$$V(x_1, x_2) = (x_1 - x_2)^2.$$

V is positive semidefinite since $V = 0$ whenever $x_1 = x_2$.

Example 10.

$$V(x_1, x_2, x_3) = x_1^2 + x_2^2.$$

V is positive semidefinite since $V = 0$ whenever $x_1 = x_2 = 0$ and x_3 is arbitrary.

$$V(X, t) \leq U(X).$$

Example 12.

$$V(X, t) = \frac{x_1^2 + x_2^2}{1 + t}.$$

V is positive semi-definite for $t \geq 0$ but not definite since V approaches zero for large t.

Example 13.

$$V(X, t) = (x_1^2 + x_2^2)(1 + \frac{1}{1 + t}).$$

V is positive definite.

Example 14.

$$V = x_1^2 + tx_2^2.$$

V is positive definite for $t > 0$ but it does not have an infinitesimal small upper bound.

Example 15.

$$V = x_1^2 + x_2^2 \sin^2 t.$$

V is positive semidefinite and it does possess an infinitesimal upper bound.

Example 16.

$$V = (x_1^2 + x_2^2)(1 + \sin^2 t).$$

$$Q_{n \times n} = \begin{pmatrix} P_{m \times m} & R_{m \times n-m} \\ R^T_{m-m \times m} & S_{n-m \times n-m} \end{pmatrix}$$

where P is positive definite, then Q is positive definite providing the $n-m$ by $n-m$ matrix

$$T = S - R^T P^{-1} R$$

is positive definite.

If A is stable then corresponding to any given positive definite matrix P, there exists a positive definite matrix Q such that

$$A^T Q + QA = - P.$$

When V depends explicitly upon the scalar, time, then the definitions of definiteness must be modified.

Def. 4. The scalar function $V(X, t)$ is positive definite for $\|X\| < k$, if there exists a positive definite function $W(X)$ such that

$$V(X, t) > W(X) \quad \text{for} \quad \|X\| < k$$

and

$$V(0, t) = 0.$$

Def. 5. The scalar function $V(X, t)$ admits an infinitesimal small upper bound if there exists a positive definite function $U(X)$ such that

Example 11.

$$V(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2.$$

V is indefinite since in every neighborhood of $\|X\| = 0$, it assumes values which are negative and values which are positive.

Example 12.

$$V(x_1, x_2) = \int_0^{x_1} f(s)ds + x_2^2$$

where we are given $x_1 f(x_1) > 0$, V is positive definite.

In many applications we must consider a scalar function V of the form

$$V = X^T Q X$$

where Q is a symmetric matrix. This quadratic form is positive definite if the associated matrix Q is positive definite. A symmetric matrix Q is positive definite if the following relations are satisfied.

$$q_{11} > 0 \quad \begin{vmatrix} q_{11} & q_{11} \\ q_{21} & q_{22} \end{vmatrix} > 0 \quad \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} > 0 \ldots (Q) > 0.$$

In some cases the computation to determine the definiteness of Q can be simplified if part of the above inequalities are known to be satisfied. Thus consider the matrix Q which is partitioned as follows

V is positive definite and it possesses an infinitesimal upper bound. In fact V has the bounds

$$x_1^2 + x_2^2 < V < 2(x_1^2 + x_2^2).$$

CHAPTER II

STABILITY AND THE DIRECT METHOD OF LIAPUNOV

The object of all stability criteria is to determine the stability of a system of differential equations without knowledge of the form of the solutions. In general it is not sufficient to know merely the existence or nonexistence of stability, but it is required to have some reasonable estimate of the size of the region of stability. For linear systems this poses no problem since if stability exists, it is global, whereas for non-linear systems stability is a local property.

In this section we will clarify the various types of stability and introduce the direct or so-called second method of Liapunov. This approach will be illustrated by some elementary examples.

A. <u>Stability</u>: Throughout this work we will be concerned with the stability of an equilibrium point of the system of differential equations

$$\dot{X} = F(X, t). \tag{II-1}$$

By an equilibrium point we mean those particular values of

$$X = X_{\epsilon}$$

such that

$$F(X_{\epsilon}, t) = 0.$$

It is assumed that the equilibrium point under discussion has been transformed to the origin. Therefore

$$F(0, t) = 0$$

and $X = 0$ is a solution of (II-1). It is further assumed that $F(X, t)$ is sufficiently smooth to ensure the existence of a unique solution.

Definition 1. The solution, $X = 0$, is said to be stable if given any $\epsilon > 0$ and t_o, there exists a $\delta(\epsilon, t_o) > 0$ such that for $\|X(t_o)\| > \delta$ implies that

$$\|X(t)\| < \epsilon \quad \text{for all} \quad t \geq t_o.$$

Definition 2. The solution, $X = 0$, is said to be asymptotically stable if 1) it is stable and 2) if

$$\lim_{t \to \infty} \|X(t)\| \to 0.$$

Definition 3. The solution, $X = 0$, is said to be unstable if given an $\epsilon > 0$, then for any δ regardless of how small, $\|X(t_o)\| < \delta$ implies $\|X(t)\| > \epsilon$ eventually. If δ can be chosen independent of t_o, then the stability is said to be uniform.

To illustrate these definitions, consider a pendulum at rest. If the pendulum is initially at rest and then it is disturbed slightly, it will oscillate about its rest position. If the initial displacement is small, then the amplitude of oscillation will be small. The equilibrium position is said to be stable. If a small amount of damping is present, not only will the amplitude of oscillation be small, but in time it will damp to zero. The equilibrium position is said to be asymptotically stable. Now consider the inverted pendulum at rest. Regardless of how small an initial displacement is given, the pendulum will move far from the rest position. In this case the rest position is unstable.

When the system is linear and autonomous, then (II-1) takes the form

$$\dot{X} = AX. \tag{II-2}$$

The only equilibrium points of this equation is $X = 0$. The question of stability can be answered in terms of the roots of the characteristic equation

$$|A - \lambda I| = 0.$$

If all these roots have negative real part, then the origin is asymptotically stable. If any roots have positive real part or if any roots are repeated with zero real part, then the system is unstable. Thus, to determine the stability of a linear autonomous system, one needs only to determine the nature of the real part of all characteristic roots. The roots themselves do not need to be determined.

Various methods have been developed to determine the stability of linear systems. One of the easiest of such methods to apply is the Routh-Hurwitz criteria. This criteria examines an array formed from the coefficients of the characteristic equation. The number of changes of sign of the first column of this array is equal to the number of roots with positive real part. This array may be determined by the following procedure. Let the characteristic equation be given as

$$\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0. \tag{II-3}$$

Consider the array defined as

$$
\begin{array}{ccccc}
1 & a_2 & a_4 & a_6 & \cdots \\
a_1 & a_3 & a_5 & a_7 & \cdots \\
b_1 & b_2 & b_3 & b_4 & \\
c_1 & c_2 & c_3 & & \\
d_1 & d_2 & & &
\end{array}
$$

where the elements b_i, c_i, d_i, etc., are formed as follows

$$b_1 = a_2 - \frac{a_3}{a_1}, \quad b_2 = a_4 - \frac{a_5}{a_1}, \quad \ldots, \quad b_i = a_{2i} - \frac{a_{2i+1}}{a_1}$$

$$c_1 = a_3 - \frac{a_1 b_2}{b_1}, \quad c_2 = a_5 - \frac{a_1 b_3}{b_1}, \quad \ldots, \quad c_i = a_{2i+1} - \frac{a_1 b_{i+1}}{b_1}$$

$$d_1 = b_2 - \frac{b_1 c_2}{c_1}, \quad d_2 = b_3 - \frac{b_1 c_3}{c_1}, \quad \ldots, \quad d_i = b_{i+1} - \frac{b_1 c_{i+1}}{c_1}$$

$$e_1 = c_2 - \frac{c_1 d_2}{d_1}, \quad e_2 = c_3 - \frac{c_1 d_3}{d_1}, \quad \ldots, \quad e_i = c_{i+1} - \frac{c_1 d_{i+1}}{d_1} .$$

Example 1. Consider the characteristic equation

$$\lambda^3 + \lambda^2 + 4\lambda + 30 = (\lambda + 3)(\lambda - 1 + 3_i)(\lambda - 1 - 3_i) = 0.$$

The Routh array becomes

1	4	0
1	30	0
-26	0	0
30	0	0

Thus there are two changes of sign and there exist two roots with positive real part.

Since criteria for the stability of linear systems are easy to apply, one naturally desires to apply the same procedures when a nonlinear system is encountered. This often leads to a ruthless linearization of the system equations. The problem then becomes: under what conditions does the stability of the nonlinear approximation represent the stability of the nonlinear system? This question may be answered by the following.

Consider the nonlinear autonomous system

$$\dot{X} = F(X) \qquad \text{(II-4)}$$

where $F(X)$ has the representation

$$F(X) = AX + G(X)$$

with

$$\lim_{X \to 0} \frac{G(X)}{\|X\|} \to 0$$

and A given by

$$A = \left(a_{ij} = \frac{\partial f_i}{\partial x_j}\right)_{X = 0}.$$

System (II-4) thus may be approximated locally by

$$\dot{X} = AX. \qquad \text{(II-2)}$$

<u>Theorem</u> 1: If all of the eigenvalues of A have negative real part, then the origin of (II-4) is asymptotically stable.

Theorem 2: If any eigenvalue of A has positive real part, then the origin of (II-4) is unstable. Thus the stability of the linear approximation carries over to the nonlinear system providing the linear system is asymptotically stable or unstable. In the case where the linear system is only stable, no conclusion can be obtained about the nonlinear system. When this occurs the stability is determined by the higher order terms.

Example 2. Consider the nonlinear system

$$\ddot{x} + a\dot{x} + bx + x^2 = 0.$$

In state notation this takes the form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -bx_1 - x_1^2 - ax_2.$$

This system possesses two equilibrium points

(1) $X_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, (2) $X_2 = \begin{pmatrix} -b \\ 0 \end{pmatrix}$.

The linear approximation is given by the Jacobian matrix

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b - 2x_1 & -a \end{pmatrix}.$$

At the equilibrium position X_1, the linear approximation becomes

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = bx_1 - ax_2.$$

For $a > 0$, $b > 0$, this equilibrium position is asymptotically stable.

For the equilibrium position X_2, the linear approximation becomes

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = bx_1 - ax_2.$$

For $a > 0$, $b > 0$ at least one root has positive real part and this equilibrium position is unstable.

Example 2: Consider the motion of a rotating projectile given by the equations

$$\ddot{\alpha} \cos\beta - 2\dot{\alpha}\dot{\beta} \sin\beta + a\dot{\beta} = b \sin\alpha$$

$$\ddot{\beta} + \dot{\alpha}^2 \sin\beta \cos\beta - a\dot{\alpha} \cos\beta = b \sin\beta \cos\alpha.$$

If we make the usual transformations

$$x_1 = \alpha, \quad x_2 = \dot{\alpha}, \quad x_3 = \beta, \quad x_4 = \dot{\beta}$$

we obtain the state vector representation

$$\dot{X} = F(X)$$

where F takes the form

$$F = \begin{pmatrix} x_2 \\ \dfrac{b \sin x_1 + 2x_2x_4 \sin x_3 - ax_4}{\cos x_3} \\ x_4 \\ b \sin x_3 \cos x_1 - x_2^2 \sin x_3 \cos x_3 + ax_2 \cos x_3 \end{pmatrix}.$$

The origin represents an equilibrium point. The matrix A for the linear approximation takes the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ b & 0 & 0 & -a \\ 0 & 0 & 0 & 1 \\ 0 & a & b & 0 \end{pmatrix}.$$

The characteristic equation for this system is

$$\lambda^4 + (a^2 - 2b)\lambda^2 + b^2 = 0$$

with the four roots given by

$$\lambda_i = \pm \sqrt{- \frac{(a^2 - 2b) + a\sqrt{a^2 - 4b}}{2}} .$$

For $a^2 - 4b < 0$, at least two roots have positive real parts and thus the equilibrium position is unstable. If $a^2 - 4b \geq 0$, all four roots are imaginary, thus the linear system is stable, but we obtain no information as to the stability of the nonlinear system. This result is still useful since if we desire stability of the above system it is necessary that

$$a^2 - 4b \geq 0.$$

This result is not sufficient for stability though. We will return to this example later and determine conditions sufficient for stability.

B. The Direct Method of Liapunov: The preceeding procedure suffers from two main disadvantages: (1) it is restricted to functions F(X) which are analytic and (2) even when it gives the stability, this result is local. We must turn to other procedures in order to obtain an estimate of the size of the region of asymptotic stability. If the linear approximation has one or more roots with zero real part, then the question of stability must be answered by examining the nonlinear terms of the equation.

The only tool which is sufficiently powerful to enable one to treat existing nonlinearities is the Liapunov Second or Direct Method. The term method is actually a misnomer since no method as such really exists. In actuality the set of theorems which make up the direct method are existence theorems and they offer an approach or point of view rather than a precise method. To illustrate this approach consider the following example.

Example 3: Let a system be described by the equations

$$\dot{x}_1 = x_2 - ax_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = - x_1 - ax_2(x_1^2 - x_2^2).$$

The linear approximation has eigenvalues which are imaginary. Thus no information is given as to the stability of the nonlinear system. Let us consider the distance from the origin to an arbitrary point on a solution and differentiate this distance.

$$r(x_1, x_2) = x_1^2 + x_2^2$$

$$\dot{r}(x_1, x_2) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1[x_2 - ax_1(x_1^2 + x_2^2)] + 2x_2[- x_1 - ax_2(x_1^2 + x_2^2)]$$

$$= - 2a(x_1^2 + x^2)^2 = - 2ar.$$

Thus integrating $\dot{r}$ from some initial t_o to t we obtain

$$\int_{t_o}^{t} \dot{r}d\tau = \int_{t_o}^{t} - 2a(x_1^2 + x_2^2)^2 = r(x_1(t), x_2(t)) - \dot{r}(x_1(t_o), x_2(t_o))$$

or finally we have

$$r(x_1(t), x_2(t)) = r(x_1(t_o), x_2(t_o)) - 2a \int_{t_o}^{t} (x_1^2 + x_2^2)^2 dt.$$

Observe that if $a > 0$, then $r(x_1(t), x_2(t))$ is a steadily decreasing function and must go to zero as t increases without bound. Thus the solution that starts at the point $X(t_o)$ must return to the origin and we have asymptotic stability. If $a < 0$, the converse is true and the solutions grow without bound and we have instability. If $a = 0$ r remains constant and we have stability.

The curves given by the equation

$$r(x_1, x_2) = k$$

represent circles in the (x_1, x_2) plane. The fact that the directional derivative $\frac{dr}{dt}$ evaluated along the solution of the system were negative, implied that the solutions must cross the curve $r = k$ from outside to inside. Instead of circles we could have considered any other nonintersecting family of closed curves surrounding the origin.

The Direct Method of Liapunov embodies this point of view in investigating the stability of nonlinear systems. It formalizes the above intuitions and geometric approach into a set of theorems. The method depends upon a scalar function or "V" function which represent a contracting family of closed surfaces surrounding the origin and such that its derivative possesses desired properties. These theorems are as follows.

<u>Theorem</u> 3. If there exists a function $V(x_1, x_2, \ldots, x_n, t)$ which is definite while its total derivative given by

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} x_i + \frac{\partial V}{\partial t} = \nabla V^T \cdot \dot{X} + \frac{\partial V}{\partial t} = \nabla V^T \cdot F(X, t) + \frac{\partial V}{\partial \cdot}$$

is semidefinite of opposite sign, then the equilibrium solution $X = 0$ is stable.

<u>Theorem</u> 4. If there exists a function $V(X, t)$ which is definite while its total derivatives

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V^T \cdot F(X, t)$$

is definite of the opposite sign, then the solution $X = 0$ is asymptotically stable.

If the function $V(X, t)$ used in the two above theorems in addition possesses an infinitesimal upper bound, then the obtained stability is uniform.

<u>Theorem</u> 5. If there exists a function $V(X, t)$ which is indefinite while its derivative

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V^T \cdot F(X, t)$$

is definite, then the solution $X = 0$ is unstable.

The above theorems have stated conditions upon a scalar function $V(X)$ which are sufficient to determine the stability. In actuality these theorems are also necessary, that is, for example if we are given a system

$$\text{(II-1)} \qquad \dot{X} = F(X, t) \qquad F(0, t) = 0$$

which is asymptotically stable, then there exist a $V(X, t)$ which satisfies the conditions given by Theorem 4. Unfortunately this does not aid us in the determination of an appropriate V function. For any given problem there is not a unique V function, but in many cases one can obtain many choices of V each of which will give more or less information. What one really desires is a "V" function which gives the strongest kind of stability in the largest possible space. Unfortunately this is demanding quite a lot, and we must settle for much less. For some problems a complete answer can only be determined from several constructions.

In general "V" functions which insure stability are easier to construct than those which insure asymptotic stability. A natural choice for a "V" function would be the total energy of the system. To illustrate this consider the mechanical system with a nonlinear spring.

<u>Example</u> 4.

$$m\ddot{x} + a\dot{x} + k(x - \frac{x^3}{6}) = 0.$$

As a choice of a "V" function consider the sum of the kinetic energy given by

$$\text{K.E.} = \frac{1}{2} m\dot{x}^2$$

and the potential energy stored in the spring

$$\text{P.E.} = \int_0^x k(u - \frac{u^3}{6})du = k(\frac{x^2}{2} - \frac{x^4}{24})$$

then V is given by

$$V(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 + \frac{k}{2} x^2(1 - \frac{x^2}{12}).$$

V is positive definite providing $|x| < \sqrt{12}$. Differentiating we have

$$\begin{aligned}\dot{V} &= m\dot{x}(\ddot{x}) + k\dot{x}(x - \frac{x^3}{6}) \\ &= m\dot{x}[- \frac{a}{m} \dot{x} - \frac{k}{m} (x - \frac{x^3}{6})] + k\dot{x}(x - \frac{x^3}{6}) \\ &= - a\dot{x}^2\end{aligned}$$

which is negative semidefinite. Thus we have that the rest position is stable for all X such that $V(X) = C$ is contained in the region $\|x\| \leq \sqrt{12}$. $V = C$ represents a closed curve for

$$C \leq \frac{3}{8}.$$

Thus we have stability for all x and $\dot{x}$ such that

$$\frac{1}{2} m \dot{x}^2(0) + \frac{k}{2} x^2(0)(1 - \frac{x^2(0)}{12}) \leq \frac{3k}{2}$$

For values of c greater than this, V does not represent closed curves. This is illustrated in Figure 1.

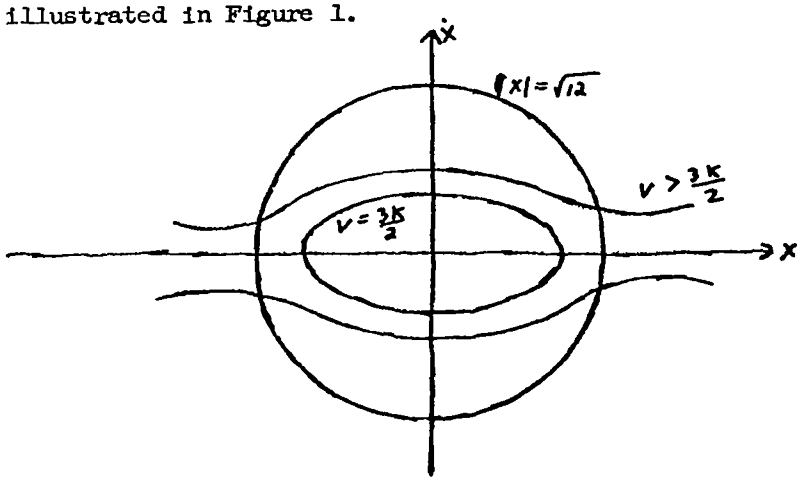


Fig. 1.

The results obtained from this construction are not exceptional. Due to the presence of the damping term we would expect to have asymptotic stability. Another choice of V could be sought which would give this stronger type of stability, but this proves to be unnecessary. LaSalle[1] has given an extension of Theorem 4 which permits these stronger results to be obtained directly. This extension is based upon an examination of the set of values for which $\dot{V}$ vanishes. This set may be divided into two subsets, the first of which are transition points, that is, the solutions just pass through these points, and the invariant points. The latter subset has the property that any solution or initial condition that enters or starts in this set remains there for all t. In practical cases it is hoped that the invariant set consists only of the equilibrium position under investigation. This is the case in the preceeding example.

$$\dot{V} = -a\dot{x}^2.$$

Thus $V = 0$ whenever $\dot{x} = 0$ and x is arbitrary. If the point $(0, \lambda)$ is a transition point, then $\dot{V}$ is only momentarily zero and becomes negative. For $\dot{V}$ to remain zero that $\dot{\lambda}$ must remain zero but this implies that x is a constant $x = c$. If these values are placed in the original differential equation we obtain

$$k(c - \frac{c^3}{6}) = 0$$

or

$$c = 0, \quad x\sqrt{6}, \quad -\sqrt{6}.$$

Thus the invariant set consists of three point only. For the set x and $\dot{x}$ given by

$$V(x, \dot{x}) < \frac{3k}{2}$$

$x = 0$ is the only invariant subset, thus we conclude that the origin is asymptotically stable.

The formal statement of the appropriate extension theorem is as follows.

<u>Theorem</u> 6. Assume that there exists a "V" function for the system (II-3) which is positive definite and such that its derivative is negative semi-definite. Let S be the set such that X is in S if $\dot{V}(X) = 0$. Let S_I be the invariant subset of S, then all solutions of (II-3) approach S_I.

For linear systems the determination of stability by the direct method must given identical results as any of the procedures in common usage. Consider the linear system (II-2)

$$\dot{X} = AX.$$

As a Liapunov function we will consider a generalization of the energy namely a quadratic form in the state variables. Thus consider V as

$$V = X^T Q X \tag{II-5}$$

where Q is positive definite. $\dot{V}$ is given by

$$\begin{aligned} \dot{V} &= \dot{X}^T Q X + X^T Q \dot{X} \\ &= X^T A^T Q X + X^T Q A X \\ &= X^T [A^T Q + QA] X. \end{aligned} \tag{II-6}$$

It is desired that $\dot{V}$ be negative definite so we assume $\dot{V}$ takes the form

$$\dot{V} = - X^T P X \tag{II-7}$$

where P is positive definite. This requires a solution to the equation

$$A^T Q + QA = -P \tag{II-8}$$

for a matrix Q which is positive definite. We have stated before that such a solution can always be determined providing the matrix A is stable. If P is chosen as the identity matrix this relation becomes

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

This become equivalent to the system of equations

$$\begin{pmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & a_{11} + a_{22} & a_{21} \\ 0 & 2a_{12} & 2a_{22} \end{pmatrix} \begin{pmatrix} q_{11} \\ q_{12} \\ q_{22} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \tag{II-9}$$

In order to have a solution for the elements of Q it is necessary and sufficient for the three by three determinant of the coefficient matrix to be nonvanishing. This gives the requirement

$$4(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21}) \neq 0.$$

Under the assumption this condition is satisfied, the matrix Q may be determined.

(II-10)

$$Q = \frac{1}{4\,\mathrm{tr}\,A\,|A|}\begin{pmatrix} -2(|A| + a_{22}^2 + a_{21}^2) & 2(a_{11}a_{21} + a_{12}a_{22}) \\ 2(a_{11}a_{21} + a_{12}a_{22}) & -2(a_{12}^2 + a_{11}^2 + |A|) \end{pmatrix}$$

where we have defined

$$\mathrm{tr}\,A = (a_{11} + a_{22})$$

$$|A| = a_{11}a_{22} - a_{12}a_{21}.$$

For Q to be positive definite we require the Hurwitz conditions to be satisfied, thus

$$q_{11}q_{22} - q_{12}^2 > 0 \qquad q_{11} > 0.$$

The first of these conditions implies

$$\frac{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}{2(\mathrm{tr}\,A)^2\,|A|} > 0$$

which implies that

$$|A| > 0.$$

If this condition is used in the second inequality we obtain

$$\frac{-\,(|A| + a_{22}^2 + a_{21}^2)}{2\,\mathrm{tr}\,A\,|A|} > 0$$

To satisfy this we must require

$$\text{tr } A < 0.$$

Thus for a two dimensional system the application of the direct method gives rise to the two conditions on the elements of the matrix A to insure asymptotic stability, namely

$$\begin{aligned} &a_{11} + a_{22} < 0 \\ &a_{11}a_{22} - a_{21}a_{12} > 0. \end{aligned} \qquad \text{(II-11)}$$

For this simple problem it would appear that the amount of computation required was excessive as compared to the usual linear procedures. In general this is true, but the aim in this example was to indicate the large amount of freedom one has in using this method. In a later section the procedure used above will be used as a starting point for the analysis of nonlinear systems.

CHAPTER III

CONSTRUCTION OF LIAPUNOV FUNCTIONS

The theorems which form the basis of the direct method are in the nature of existence theorems in that they give conditions for stability based upon the properties of the scalar "V" functions, but they do not indicate how such functions are to be constructed. This limitation has prompted many investigators to develop general methods of construction. For some problems it was observed that a suitable choice of a Liapunov function was given by the total energy or momentum of the system. This consideration has led to a construction from the integrals of motion if some of these can be determined. Other investigators have started with the quadratic form which is the basis for linear systems and generalized this to quadratic forms in which the elements of the matrix are assumed to be functions of the state variables.

In most cases these general procedures of construction still require considerable ingenuity and as such have not been developed to the point that they may be considered as an algorithm which leads directly to the determination of the stability. Two such general construction procedures approach this state of development namely the Zubov[2] construction and the construction due to Ingwerson[3], both of which can be implemented upon a digital computer. Unfortunately all such procedures suffer from one basic limitation, namely, the inability to determine whether the result of the construction process possesses the desired definiteness properties.

This limitation forces the problem of construction of Liapunov functions to be divided into two problems. These problems are (1) determine if a given homogeneous function of degree $2n$ is definite, and (2) assuming that problem (1) is solved construct a suitable Liapunov function.

For some particular function solutions to problem 1 can be determined but no general results are available.

A. Construction from the Integrals of Motion. For many problems primarily of a gyroscopic nature "V" functions can be constructed from a consideration of one or more of the integrals of motion. Such V functions actually represent solution curves and as such their derivations vanish identically. Thus one only obtains stability. To illustrate this procedure consider the motion of a vehicle rotating about its center of mass.

Example 1. Assuming principle axes the equations of a rotating vehicle with no applied torques become

$$I_x\dot{p} + (I_z - I_y)rq = 0$$

$$I_y\dot{q} + (I_x - I_z)pr = 0$$

$$I_z\dot{r} + (I_y - I_x)pq = 0.$$

This system possesses the four equilibrium positions

$$\epsilon_1 = (0, 0, 0) \quad \epsilon_2 = (p_o, 0, 0) \quad \epsilon_3 = (0, q_o, 0) \quad \epsilon_4 = (0, 0, r_o).$$

To investigate their stability we will assume that the inertias are ordered as follows

$$I_z > I_y > I_x.$$

This implies

$$\frac{I_z - I_y}{I_x} = c_1 > \frac{I_y - I_x}{I_z} = c_3 > \frac{I_x - I_z}{I_y} = - c_2 \quad \text{with } c_3, c_2, c_1 \text{ positive.}$$

The stability of ϵ_1 is somewhat trivial and may be determined by considering a Liapunov function of the form

$$V = \frac{I_x\, p^2 + I_y\, q^2 + I_z\, r^2}{2}.$$

Then $\dot{V}$ becomes

$$\begin{aligned} \dot{V} &= I_x\, p\dot{p} + I_y q\dot{q} + I_z r\dot{r} \\ &= p[(I_y - I_z)ry] + q[(I_z - I_x)pr] + r[I_y - I_z]pq \\ &= pqr[I_y - I_z + I_z - I_x + I_x - I_y] = 0. \end{aligned}$$

Thus the equilibrium position (0, 0, 0) is stable. We now investigate the equilibrium position ϵ_2. To facilitate this investigation we transfer the equilibrium position to the origin by means of the following

$$x_1 = p - p_o \qquad x_2 = q \qquad x_3 = r$$

$$\begin{aligned} \dot{x}_1 &= - c_1 x_2 x_3 \\ \dot{x}_2 &= c_2 x_1 x_3 + c_2 p_o x_3 \\ \dot{x}_3 &= - c_3 x_1 x_2 - c_3 p_o x_2. \end{aligned}$$

Let us divide the second and third of the above equations by the first to obtain

$$\frac{dx_2}{dx_1} = - \frac{c_2}{c_1}\, \frac{x_1}{x_2} - \frac{c_2}{c_1}\, \frac{p_o}{x_2}$$

$$\frac{dx_3}{dx_1} = \frac{c_3}{c_1}\, \frac{x_1}{x_3} + \frac{c_3}{c_1}\, \frac{p_o}{x_3}.$$

Integrating these two equations we obtain

$$c_1x_2^2 + c_2x_1^2 + 2c_2p_ox_1 = k_o$$

$$c_1x_3^2 - c_3x_1^2 - 2c_3p_ox_1 = k_1.$$

Let us define the two scalar functions V_o and V_1 as

$$V_o = c_1x_2^2 + c_2x_1^2 + 2c_2p_ox_1$$

$$V_1 = c_1x_3^2 - c_3x_1^2 - 2c_3p_ox_1.$$

Neither V_o nor V_1 are Liapunov functions but both V_o and V_1 are such that

$$\frac{dV_o}{dt} \equiv 0 \qquad \frac{dV_1}{dt} \equiv 0.$$

Therefore it follows that any function $V = k_oV_o + k_1V_1$ also has the property that $\frac{dV}{dt} \equiv 0$. Consider as a Liapunov function

$$\frac{c_2}{c_1} V_1 + \frac{c_3}{c_1} V_o = c_3x_2^2 + c_2x_3^2 = V_2.$$

This is only semi-definite since it vanishes for $x_2 = x_3 = 0$ and x_1 arbitrary. To complete the construction we need a dependence upon x_1. To achieve this let us add V_o^2 to V_2. Thus

$$V = c_3x_2^2 + c_2x_3^2 + (c_1x_2^2 + c_2x_1^2 + 2c_2p_ox_1)^2.$$

Now V is positive definite and its derivative $\dot{V}$ is identically zero. Thus the equilibrium position ϵ_2 is stable. If the same procedure is applied to ϵ_4 it is also found to be stable. To investigate ϵ_3 we once again transfer the equilibrium position to the origin. This gives the system

$$\dot{x}_1 = - c_1x_2x_3 - c_1q_ox_3$$

$$\dot{x}_2 = c_2x_1x_3$$

$$\dot{x}_3 = - c_3x_1x_2 - c_3q_ox_3.$$

Examining the linear approximation, we have one eigenvalue with positive real part. Therefore this equilibrium is unstable. Thus we conclude that rotations about the axes of maximum and minimum moments of inertia are stable whereas rotations about the axis of intermediate inertia is unstable.

We now wish to return to the problem discussed in Example 2 of the the preceding section. The equations of motion are

$$\dot{x} = x_2$$

$$\dot{x}_2 = \frac{b \sin x_1 + 2x_2x_4 \sin x_3 - ax_4}{\cos x_3}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = b \sin x_3 \cos x_1 - x_2^2 \sin x_3 \cos x_3 + ax_2 \cos x_3.$$

We had obtained as a result of the linear analysis that for stability it is necessary that

$$a^2 - 4b \geq 0.$$

We now assume this inequality is satisfied. If the equation for $\dot{x}_2$ is multiplied by $\cos^2 x_3x_2$ and the equation for x_4 is multiplied by x_4 and the results added together, to give

$$x_2\dot{x}_2 \cos^2 x_3 + x_4\dot{x}_4 = b \cos x_3 \sin x_1x_2 + bx_4 \sin x_3 \cos x_1 + x_2^2x_4 \sin x_3 \cos x_3.$$

Now if equations for $\dot{x}_1$ and $\dot{x}_3$ are substituted into the above we obtain

$$x_2\dot{x}_2 \cos^2 x_3 - x_2^2 \sin x_3 \cos x_3\dot{x}_3 + x_4\dot{x}_4 = b \cos x_3 \sin x_1\dot{x}_2 + b \sin x_3 \cos x_1\dot{x}_3$$

or

$$\frac{d}{dt} \{\cos^2 x_3 \frac{x_2^2}{2} + \frac{x_4^2}{2}\} = - \frac{d}{dt} \{b \cos x_3 \cos x_1\}.$$

Thus we have one integral of motion given by

$$V_o = \frac{x_4^2}{2} + \frac{x_2^2}{2} \cos^2 x_3 + b \cos x_3 \cos x_1.$$

For a Liapunov function consider

$$V = V_o - b.$$

With this choice of V we have $\dot{V} \equiv 0$. If $\bar{V}$ is positive definite then we would have stability, but for V to be positive definite we must have $b < 0$. If $b = - d$, then V becomes

$$V = \frac{x_4^2}{2} + \frac{x_2^2}{2} \cos^2 x_3 + d[1 - \cos x_3 \cos x_1].$$

For the norm of x small V is dominated by the first terms in the series expansion of the trigometric functions. Thus it behaves like

$$\frac{x_1^2}{2} + \frac{x_2^2}{2} + d \frac{x_3^2}{2} + d\frac{x_1^2}{2}$$

and is positive definite. Thus for $b > a^2$ the equilibrium position is unstable, for $b < 0$, the equilibrium is stable and for $0 < b < a^2$ we have no information. By constructing another integral of motion the region of stability could possibly be improved.

B. Construction by Extension of Quadratic Form. The most useful approach to the construction of a Liapunov function is to use a quadratic form in all or in some of the state variables. If the nonlinearities are odd functions of their argument at least in a neighborhood of the equilibrium position, then their integrals become additional choices for the construction of a Liapunov function. This procedure was used in Example (II-4), and will be discussed in more detail when the problem of Lur'e is encountered.

The advantage of a quadratic form is obvious when one considers a linear system since conditions for asymptotic stability are replaced by finding positive definite solutions of the system of algebraic equations

$$A^TQ + QA = -P. \tag{III-1}$$

If the matrix P is specified, then the elements of the matrix Q can be determined as a solution of a system of algebraic equations whose coefficients are the elements of the matrix A. For a second order system with $P = I$, this was computed (II-10) in the previous section. For various choices of P the matrix Q could be tabulated for arbitrary choices of A. The amount of work of course would be prohibitive unless the form of A was suitably restricted.

Starting from this point of view, several apparent options are available. For the nonlinear autonomous system

$$\dot{X} = F(X) \tag{III-2}$$

one could assume that F can be written in the form $F(X) = A(X)X$ where $A(Y)$ is a matrix whose elements depend upon the state. (III-2) would then take the form

$$\dot{X} = F(X) = A(X)X. \tag{III-3}$$

Consider as a V function the quadratic form

$$V = X^T QX. \tag{III-4}$$

Its derivative becomes for Q constant

$$\dot{V} = X^T[A^T(X)Q + QA(X)]X.$$

Equating the matrices

$$A^T(X)Q + QA(X) = - P. \tag{III-5}$$

Q could be determined, but unfortunately its elements are not constant but depend upon the vector X. This causes a revision in the form of $\dot{V}$. If Q is initially assumed to depend upon the state vector then $\dot{V}$ takes the form

$$\dot{V} = X^T[A^TQ + QA + \dot{Q}]X \tag{III-6}$$

with the resulting equation

$$A^T(X)Q(X) + Q(X)A(X) + \dot{Q}(X) = - P \tag{III-7}$$

to be solved for the elements of Q. With various restrictions upon the matrix Q and restrictions as to the type of the nonlinear function F, various approaches analogous to this have been proposed by Szegö[4], Ku and Puri[5]. In the above treatment it must be observed that in obtaining equation (III-3) the matrix $A(X)$ is not unique.

The most fruitful of the approaches based upon the extension of the quadratic form is due to Ingwerson[3] whose method will be described in more detail. Instead of starting with a variable matrix $Q(X)$ and obtaining equation (III-7) to solve one can observe that for Q a constant the elements q_{ij} are given by

$$q_{ij} = \frac{\partial 2V}{\partial x_i \partial x_j}.$$

The natural question to ask is the following: Could one start with a matrix $Q(X)$ which is a solution to the equation

$$A^T(X)Q(X) + Q(X)A(X) = - P(X) \qquad \text{(III-8)}$$

and then integrate the resulting matrix, Q, twice to obtain a scalar V function? For the matrix Q to be the second derivative of a scalar V, the elements of Q must satisfy two relations, namely

$$\text{(III-9)} \qquad \begin{array}{ll} \text{(a)} & q_{ij} = q_{ji} \\ \text{(b)} & \dfrac{\partial q_{ij}}{\partial x_k} = \dfrac{\partial q_{ik}}{\partial x_j} \qquad j, k \neq i. \end{array}$$

In general a matrix R satisfying an equation of the form (III-8) will always have property (a) but in general it will not satisfy the condition (b). If the elements of Q are formed from the matrix R by the following relation

$$(q_{ij} = r_{ij}(0, \ldots, 0, \ldots, x_i, 0 \ldots, x_j, 0, \ldots, 0)$$

then Q will always satisfy both relations (a) and (b). Observe that each element q_{ij} only depends upon the two components of X namely x_i and x_j.

With Q now determined and satisfying the conditions (III-9), it may be integrated to give the gradient vectors ∇V.

(III-10) $$\nabla V = \int_0^x Q dx$$

where each component ∇V_i of ∇V is defined as

(III-11)
$$\nabla V_i = \sum_{j=1} \int_0^{x_j} q_{ij} dx_j = \int_0^{x_1} q_{i1} dx_1 + \int_0^{x_2} q_{i2} dx_2 + \dots + \int_0^{x_n} q_{in}\, dx_n$$

Once ∇V is obtained both V and $\dot{V}$ can be determined. If the original system is in the form (III-2), then $\dot{V}$ is given by:

(III-12) $$\dot{V} = \nabla V^T F(X).$$

In order to obtain V one must integrate ∇V along some path to the point X. In general such line integrals depend upon the path of integration. To have this integral to be independent of the path it is sufficient that the matrix of partial derivatives of the vector ∇V have a vanishing skew symmetric part. In three dimensional space, this is equivalent to the vanishing of the curl of ∇V. From our construction of Q, it represents this matrix of partials and is symmetric. Thus ∇V can be integrated independently of the path. This gives for V the equation

(III-13) $$V = \int_c \nabla V^T dx$$

where c is any path connecting the origin to the point X. A convenient path to use is to integrate along paths parallel to the coordinate axis. For this path the integral for V becomes

$$V = \int^{(x_1,0,0,\ldots,0)} \nabla V_1^T dx_1 + \int^{(x_1,x_2,0,\ldots,0)} \nabla V_2^T + \ldots + \int_0^X \nabla V_n^T dx_n.$$

(III-14)

In this development we have not indicated how the matrix A, which is used in equation (III-8) is obtained. Ingwerson does not represent the vector $F(X)$ as $A(X)X$, instead he starts with the original equation (III-2) and differentiates to obtain

$$\dot{X} = F(X)$$

$$\ddot{X} = A(X)\dot{X},$$

where $A(X)$ is the matrix defined as

(III-15) $$A(X) = (a_{ij}(X) = \frac{\partial f_i}{\partial x_j}).$$

To facilitate the computation F is assumed to be a vector of the form

(III-16) $$F^T = (x_2, x_3, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$$

which is the form obtained from the vector representation of a single scalar equation of degree n. Thus A will be in the so-called companion form.

In the construction of a V by this approach, it is not necessary to demand that the matrix P from equation (III-8) be positive definite, but it is sufficient for P to be semi-definite and then apply LaSalle's extension.

At the completion of this construction, the determination of the associated stability still depends upon V being positive definite. In general V will be complicated and thus the testing for definiteness is very difficult. If the matrix P is chosen such that $\dot{V}$ is definite, then complete information will be determined since if V is indefinite, then by Theorem II-5 instability would result.

Before attacking a sample problem it is worthwhile tabulating the steps in the above construction. We assume a system if for (III-2) given

$$\dot{X} = F(X).$$

Step 1. Determine the matrix

$$A = (a_{ij} = \frac{\partial f_i}{\partial x_j}).$$

Step 2. Choose a matrix P which is either definite or semi-definite.

Step 3. Construct a matrix R such that

$$A^T R + RA = - P.$$

Step 4. Construct a matrix Q where Q is obtained from R by setting in each element r_{ij} all variables to zero except x_i and x_j.

Step 5. Construct ∇V by integrating Q

$$V = \int_0^X Q dX = (\nabla V_i = \sum_{j=1}^{n} \int_0^{x_j} q_{ij} dx_j).$$

Step 6. Construct V by performing the line integral of ∇V

$$V = \int_0^{x_1, 0, \dots, 0} \nabla V_1 dx_1 + \int_0^{x_1, x_2, 0, \dots, 0} \nabla V_2 dx_2 + \dots + \int_0^{x_1, x_2, \dots, x_n} \nabla V_n dx_n.$$

Step 7. Determine if V is definite or indefinite. If P was chosen to be semi-definite and V is indefinite no information is obtained and steps 2 through 7 must be repeated. If P is definite or if P was semi-definite with V definite the solution is complete.

Example 2. As an example consider the construction of a Liapunov function for a phase-locked communication loop. The describing equations are given by

$$\ddot{x} + (a + b \cos x)\dot{x} + k \sin x = 0.$$

In vector notation this becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -k \sin x_1 - (a + b \cos x_1)x_2 \end{pmatrix}.$$

Step 1. The matrix A is

$$A = \begin{pmatrix} 0 & 1 \\ -k \cos x_1 + b \sin x_1 x_2 & -(a + b \cos x_1) \end{pmatrix}.$$

Step 2. If $b < a$ then for P consider the matrix

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 2(a + b \cos x_1) \end{pmatrix}.$$

P is positive semi-definite.

Step 3. The matrix R for this choice of P is

$$R = \begin{pmatrix} k \cos x_1 - bx_2 \sin x_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Step 4. The desired matrix Q is

$$Q = \begin{pmatrix} k \cos x_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Step 5. Integrating Q we obtain for ∇V

$$\nabla V = \begin{pmatrix} \int_0^{x_1} k \cos x_1 dx_1 \\ \int_0^{x_2} dx_2 \end{pmatrix} = \begin{pmatrix} k \sin x_1 \\ x_2 \end{pmatrix}.$$

Step 6. Integrating ∇V we obtain for V

$$V = \int_0^{x_1} k \sin x_1 dx_1 + \int_0^{x_2} x_2 dx_2$$

$$= k(1 - \cos x_1) + \frac{x_2^2}{2}.$$

Thus for $\|x\| < \pi$ V is positive definite. $\dot{V}$ is given by $\nabla V^T \cdot F(x)$

$$(k \sin x_1 \quad x_2) \begin{pmatrix} x_2 \\ -k \sin x_1 - x_2(a + b \cos x_1) \end{pmatrix} = \left(-x_2^2(a + b \cos x_1) \right)$$

for $b < a$ $\dot{V}$ is negative semi-definite. Thus we can conclude that the origin is stable. Applying the extension due to LaSalle we have

$$\dot{V} = 0 \quad \text{for} \quad x_2 = 0 \quad \text{and} \quad x_1 \quad \text{arbitrary}$$

$$\text{and} \quad x_2 \quad \text{arbitrary} \quad x_1 = \cos^{-1} - a/b$$

for $b < a$ the second set does not exist. For $x_2 = 0$, the system equations imply that $x_1 = c$ where c is given by

$$k \sin c = 0 \quad \text{thus} \quad c = \pm n\,\pi.$$

Interior to the set $\|X\| < \pi$ the invariant set consists of the origin, thus we conclude the phase-lock system is asymptotically stable for all x_1 and x_2 interior to the curve $V = c$ where c is chosen such that V is inscribed in the curve $\|X\| < \pi$.

An alternate construction procedure called the variable gradient method has been proposed by Gibson and Schultz[6]. This procedure is in many ways analogous to the one above. Its main departure is in that it does not start with the matrix Q and integrate twice, but rather it starts with the vector ∇V. If ∇V is assumed to be known, then both V and $\dot{V}$ are given in terms of it.

(III-17) $$\dot{V} = \nabla V^T F(X)$$

(III-18) $$V = \int_L \nabla V^T dX.$$

In the approach advocated by the above two authors, an arbitrary form for ∇V is assumed. With many free positions $\dot{V}$ is formed from the assumed form of ∇V. $\dot{V}$ is then constrained to be at least semi-definite. This constraint fixes some of the free parameters in ∇V. Conditions of symmetry on the Jacobian matrix of ∇V are enforced to insure that ∇V

is the gradient of a scalar. These symmetry restrictions give $\frac{n(n-1)}{2}$ constraints on the elements of ∇V. The remaining free parameters are chosen to satisfy these constraints. In practice one is forced to iterate between these last two steps. Once these constraints are satisfied V may be obtained in the same manner as for the Ingwerson method with the resulting problem of the determination of the definiteness of V. Before we try to make comparison between this procedure and the previous one we wish to consider an example.

Example 3. Consider the system given by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = - kx_1^2 x_2 - rx_1^3 \qquad k > 0 \qquad r > 0.$$

For V we assume the form

$$\nabla V = \begin{pmatrix} a_{11}(x_1)x_1 + a_{12}(x_1, x_2)x_2 \\ a_{21}(x_1, x_2)x_1 x_2 \end{pmatrix}$$

where as yet a_{11}, a_{12}, and a_{21} are arbitrary functions to be determined. In terms of ∇V we can solve for $\dot{V}$ to obtain

$$\begin{aligned} \dot{V} &= \nabla V^T F(x) \\ &= - ra_{21}x_1^4 + x_2^2(a_{12} - kx_1^2) + x_1 x_2(a_{11} - ka_{21}x_1^2 - rx_1^2). \end{aligned}$$

We now wish to choose some of the functions a_{ij} so that $\dot{V}$ is at least negative semi-definite. This can be achieved if we make the following hold

a) $ra_{21} \geq 0$

b) $a_{12} - kx_1^2 \leq 0$

c) $a_{11} - ka_{21}x_1^2 - rx_1^2 = 0.$

Condition b is satisfied providing

$$a_{12} = kx_1^2 - c \quad \text{with} \quad c > 0 \text{ but arbitrary.}$$

With this choice of a_{12} we can now impose the symmetry conditions upon the Jacobian matrix. This condition requires

$$\frac{\partial}{\partial x_2}(a_{12}(x_1, x_2)x_2) = \frac{\partial}{\partial x_1}(a_{21}(x_1, x_2)x_1)$$

$$kx_1^2 - c = a_{21} + x_1 \frac{\partial a_{21}}{\partial x_1}.$$

Solving this equation we obtain

$$a_{21}(x_1, x_2)x_1 = \frac{kx_1^3}{3} - cx_1$$

or

$$a_{21} = \frac{kx_1^2}{3} - c.$$

Applying condition (a) we have the requirement that $a_{21} > 0$. To satisfy this we would require $c \leq 0$. Thus the only choice remaining is for $c = 0$. The third condition requires

$$a_{11} = ka_{21}x_1^2 + rx_1^2 = k^2 \frac{x_1^4}{3} + rx_1^2.$$

With these choices we have for ∇V the vector

$$\nabla V = \begin{bmatrix} \frac{k^2 x_1^5}{3} + r x_1^3 + k x_1^2 x_2 \\ \frac{k x_1^3}{3} + x_2 \end{bmatrix} .$$

Integrating we obtain for V

$$V = k^2 \frac{x_1^6}{18} + r \frac{x_1^4}{4} + k \frac{x_1^3 x_2}{3} + \frac{x_2^2}{2}$$

$$= \frac{1}{18} [(k x_1^3 + 3x_2)^2] + r \frac{x_1^4}{4}$$

which is positive definite. $\dot{V}$ becomes

$$\dot{V} = - r \frac{k}{3} x_1^4$$

which is negative semi-definite. The invariant set consists only of the origin. Since V represents closed curves for all $\|X\|$, then the conclusion is that this system is asymptotically stable throughout the whole finite plane.

The primary difference in the variable gradient procedure and the modified Liapunov construction appears to be in the procedure for obtaining ∇V. Ingwerson's procedure gives immediately a choice for ∇V based upon a selection of the matrix P, whereas the variable gradient procedure requires considerable ingenuity to apply. It has been the author's experience on many sample problems that comparable results were obtained with the two methods. If a choice was to be made it would have to be based upon the ease of construction. The Ingwerson procedure has one solid advantage, namely it could be formalized sufficiently to be placed upon a computer.

C. Non-Quadratic Constructions: Numerous other procedures have been advocated for the construction of Liapunov functions which are not based upon any starting quadratic form. It is recognized that the variable gradient procedure is not based upon properties of quadratic forms, but due to its similarity to the construction due to Ingwerson it was classified in the same section.

The most general process for a Liapunov function construction is the one proposed by Zubov[2]. This procedure has been discussed in considerable detail by Margolis and Vogt[7]. Szegö[8] has also proposed a similar procedure.

Basically the Zubov construction is based upon examining the partial differential equation for $\dot{V}$ and obtaining solutions to this equation in terms of a power series expansion. This restricts the construction to non-linear functions which are analytic. The utility of the Zubov construction does not lie in the determination of the stability or instability of a system, but rather in obtaining the complete domain of stability. The procedure for construction is as follows. We assume a system of the form (III-2)

$$\dot{X} = F(X).$$

If we have a Liapunov function $V(X)$ its derivative is given by

$$\dot{V} = \nabla V^T F(X).$$

If the equilibrium of (III-2) is asymptotically stable then we know there exists a V function such that $\dot{V}$ takes the form

$$\dot{V} = \nabla V^T F(X) = - W(X) \qquad \text{(III-19)}$$

where $W(X)$ is positive definite. Zubov assumes $W(X)$ to be of the form

(III-20) (1) $W(X) = U(X)[1 - V]$

(III-21) (2) $W(X) = U(X)[1 + F^T F][1 - V]$

where $U(X)$ is assumed to be a positive definite quadratic form. Thus the problem of constructing a Liapunov function is equivalent to solving the partial differential equation

(III-22) $$\nabla V^T \cdot F(X) = U(X)[1 - V]$$

or

(III-23) $$\nabla V^T \cdot F(X) = U(X)[1 + F^T F][1 - V].$$

This partial differential equation can be solved in terms of an infinite series of functions which are homogeneous. Thus V is assumed to take the form

(III-24) $$\sum_{k=2}^{\infty} V_k(X) = \sum_{k=2}^{\infty} V_k(x_1, x_2, \ldots, x_n)$$

where the functions $V_k(X)$ are homogeneous of degree k, that is

(III-25) $$V_k(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k V(x_1, x_2, \ldots, x_3).$$

Every first partial derivative of a homogeneous function of degree k is homogeneous of degree $k-1$. Under the assumption that F is analytic, then F has an expansion of the form

$$F = \sum_{i=1}^{\infty} F_i(X)$$

where each of the vector functions are homogeneous of degree i. If these two series are substituted into the equation (III-20),

$$\text{(III-26)} \qquad \nabla^T \{\sum_{k=2}^{\infty} V_k(X)\}\{\sum_{i=1}^{\infty} F_i(X)\} = U(X)[1 - \sum_{k=2}^{\infty} V_k(X)].$$

If the homogeneous forms of the same degree appearing on each side of equation (III-26) are equated one obtains the set of equations

$$\begin{aligned} &\nabla V_2^T F_1 = U(X) \\ &\nabla V_3^T F_1 + \nabla V_2^T F_2 = 0 \\ \text{(III-27)} \quad &\nabla V_4^T F_1 + \nabla V_3^T F_2 + \nabla V_2^T F_3 = - U(X) V_2 \\ &\vdots \\ &\nabla V_k^T F_1 + \nabla V_{k+1}^T F_2 + \ldots + \nabla V_2^T F_{k-1} = - U(X) V'_{k-2}. \end{aligned}$$

This system may be solved recursively to give each term in V in terms of the previous terms. Each homogeneous term of degree k in the expansion of V has the form

$$V_k = \Sigma\, a_{i_1 i_2}, \ldots, {}_{i_3}, x_1^{i_1} x_2^{i_2}, \ldots, x_n^{i_n}$$

where the summation is over all combinations of the indices with the sum of the indices equal to k. As an example for a second order system

$$V_4 = a_{40} x_1^4 + a_{31} x_1^3 x_2 + a_{22} x_1^2 x_2^2 + a_{13} x_1 x_2^3 + a_{04} x_2^4.$$

For large order system (third or higher) the work in such a solution would be prohibitive.

As was stated previously, the main advantage of the Zubov construction is that the whole region of asymptotic stability is given. If X is a point which belongs to the region of asymptotic stability, then $V(X) \leq 1$. Thus the boundary of the region of asymptotic stability is given by $V(X) = 1$. To illustrate this method consider the following.

Example 4.

$$\dot{x}_1 = - x_1 + x_2 + x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = - x_1 - x_2 + x_2(x_1^2 + x_2^2).$$

Thus $F = F_1 + F_3$ where

$$F_1 = \begin{pmatrix} - x_1 + x_2 \\ - x_1 - x_2 \end{pmatrix} \quad F_3 = \begin{pmatrix} x_1^3 + x_1 x_2^2 \\ x_1^2 x_2 + x_2^3 \end{pmatrix}.$$

For the function $U(X)$ consider

$$U(X) = 2(x_1^2 + x_2^2).$$

Observe that the origin is asymptotically stable, since the characteristic roots

$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = (\lambda - 1 + i)(\lambda - 1 - i)$$

have negative real part. Therefore the Zubov construction is applicable. For V we assume a series of homogeneous form

$$V = V_2 + V_3 + \dots + V_n.$$

The components of V may be determined from the recursive relations

$$\nabla V_2^T F_1 = - U$$

$$\nabla V_3^T F_1 + \nabla V_2^T \cdot F_2 = 0$$

$$\nabla V_4^T F_1 + \nabla V_3^T F_2 + \nabla V_2^T F_3 = UV_2$$

$$\vdots$$

$$\nabla V_K^T F_1 + \nabla V_{K-1}^T F_2 + \ldots + \nabla V_2^T F_{K-1} = UV_{K-2}.$$

Observe that the first of the above relations is identical to finding a matrix Q such that

$$A^T Q + QA = - P$$

where Q is the matrix of V_2, A is given by $F_1 = AX$ and P is the matrix of the quadratic form U(X). Thus the first equation requires

$$(2q_{11}x_1 + 2q_{12}x_2)(-x_1 + x_2) + (2q_{12}x_1 + 2q_{22}x_2)(-x_1 - x_2) = - 2(x_1^2 + x_2^2),$$

from which we obtain

$$q_{11} = 1 \quad q_{12} = 0 \quad q_{22} = 1.$$

Thus $V_2 = x_1^2 + x_2^2$.

In the second of the above equations $\nabla V_3^T F_1 + \nabla V_2^T F_2 = 0$ we are given $F_2 = 0$ therefore $\nabla V_3^T F_1 = 0$ or $V_3 = 0$. For the next term in the recursive relation we have

$$\nabla V_4^T F_1 + \nabla V_2^T F_3 = UV_2.$$

V_4 must have the form

$$V_4 = a_{40}x_1^4 + a_{31}x_1^3x_2 + a_{22}x_1^2x_2^2 + a_{13}x_1x_2^3 + a_{04}x_2^4.$$

Therefore this relation gives

$$(4a_{40}x_1^3 + 3a_{31}x_1^2x_2 + 2a_{22}x_1x_2^2 + a_{12}x_2^3)(-x_1 + x_2) + (2x_1)(+x_1^3 + x_2^2x_1)$$
$$+ (a_{31}x_1^3 + 2a_{22}x_1^2x_2 + 3a_{13}x_1x_2^2 + 4a_{04}x_2^3)(-x_1 - x_2) + (2x_2(x_2x_1^2 + x_2^3)$$
$$= 2x_1^4 + 4x_1^2x_2^2 + 2x_2^4.$$

Equating coefficients of the same powers we obtain the algebraic relationships

$$-4a_{40} - a_{31} + 2 = 2$$
$$4a_{40} - 4a_{31} - 2a_{22} = 0$$
$$3a_{31} - 4a_{22} - 3a_{13} + 4 = 4$$
$$2a_{22} - 4a_{13} - 4a_{04} = 0$$
$$a_{31} - 4a_{04} + 2 = 2.$$

The solution to the above by inspection since they are homogeneous equations is $a_{40} = a_{31} = a_{22} = a_{13} = a_{04} = 0$. Thus $V_4 = 0$. It follows that all terms V_i are zero for $i > 4$ also. Thus

$$V(X) = x_1^2 + x_2^2.$$

The complete region of asymptotic stability is given by

$$V = x_1^2 + x_2^2 = 1.$$

The Zubov construction in the last example led to a closed form for V. In general this will not be the case so that one has a finite number of terms in a series representation for V. It is known that V will give the complete region of asymptotic stability, but the natural question to ask is to what extent can the region of asymptotic stability be approximated by an approximation to the V function? The answer is in the affirmative since if $(V)^k$ is the approximation up to terms of degree k, let α be the minimum value of $(V)^k$ over the set X where $(\dot{V})^k = 0$. Then the set given by $(V)^k = \alpha$ is contained in the domain of asymptotic stability.

The Zubov construction plays an important role in the theory of optimal control. This relation will be given in more detail in a later section, but to illustrate the interrelation consider the Zubov partial differential equation

(III-22) $$\dot{V} = \nabla^T V \cdot F = - U(X)[1 - V].$$

Dividing through by $(1 - V)$ and multiplying by dt we obtain

(III-28) $$\frac{dV}{1-V} = - U(X)dt.$$

Integrating both sides to obtain

(III-29) $$- \ln(1 - V)\Big|_{V(t_o)}^{V(T)} + - \int_{t_o}^{T} U(X)dt.$$

If we make the transformation $\ln(1 - V) = W$ (III-29) becomes

(III-30) $$W(t_o) - W(T) = - \int_{t_o}^{T} U(X)dt.$$

In the conventional representation we have

$$W(T) = W(t_o) + \int_{t_o}^{t} U(X)dt \tag{III-31}$$

which resembles the form for most performance criteria in optimal control problems.

Numerous other procedures for the construction of Liapunov functions have been proposed. Most of these lack the generality of the three methods thus far discussed. One such method due to Infante and Clark[9], although restricted to second order systems gives an interesting geometrical interpretation of the Liapunov function. The method consists in modifying the system of equations until it becomes an exact differential equation, thus giving an integral of motion. The conditions for exactness coupled with the requirement that the cross product of the original velocity vector with the modified system, insure that trajectories enter the regions given by the solution of the modified equation.

A useful construction for low order systems has been proposed by Reiss and Geiss[10] which is based upon using the differential equation and performing an integration by parts until one arrives at a definite form. This construction often gives useful insight even when the system is of high order. This method is much easier to illustrate with an example than to explain.

Example 5. Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = - f(x_2)x_3 - 2x_2 - x_1.$$

As a tentative choice of $\dot{V}$ consider the semi-definite form

$$\dot{V} = - x_3^2.$$

Then integrating we have

$$\int \dot{v}dt = - \int x_3^2 dt$$

$$\begin{aligned} V &= - \int x_3\dot{x}_2 dt = - \int x_3 dx_2 = - x_3x_2 + \int x_2\dot{x}_3 dt \\ &= - x_3x_2 - \int x_2 f(x_2)x_3 dt - 2 \int x_2^2 dt - \int x_2x_1 dt \\ &= - x_3x_2 - \int x_2 f(x_2)dx_2 - 2 \int x_2 dx_1 - \int x_1 dx_1 \\ &= - x_3x_2 - \int x_2 f(x_2)dx_2 - 2 \int x_2 dx_1 - \frac{x_1^2}{2}. \end{aligned}$$

Let us now examine the integral

$$\int x_2 dx_1 = x_2x_1 - \int x_1\dot{x}_2 dt = x_2x_1 - \int x_1x_3 dt$$

but from the last equation we have

$$- x_1 = \dot{x}_3 + f(x_2)x_3 + 2x_2.$$

Therefore

$$\begin{aligned} \int x_2 dx_1 &= x_2x_1 + \int x_3(\dot{x}_3 + f(x_2)x_3 + 2x_2)dt \\ &= x_2x_1 + \frac{x_3^2}{2} + x_2^2 + \int x_3^2 f(x_2)dt. \end{aligned}$$

Substituting this back into the expression for V we have

$$V = - x_3x_2 - \frac{x_1^2}{2} - \int x_2 f(x_2)dx_2 - 2x_2x_1 - x_3^2 - 2x_2^2 - 2 \int x_3^2 f(x_2)dt.$$

Now define the variable W as

$$W = V + 2\int x_3^2 f(x_2)dt = -x_3x_2 - \frac{x_1^2}{2} - 2x_2x_1 - x_3^2 - 2x_2^2 - \int x_2 f(x_2)dx_2$$

$$= -\frac{1}{2}[2x_2 + x_1]^2 - \frac{1}{4}[2x_3 + x_2]^2 - \int^{x_2} (f(x_2) - \tfrac{1}{2})x_2 dx_2.$$

Thus for $f(x_2) > \frac{1}{2}$, W is negative definite

$$\frac{dW}{dt} = \frac{dV}{dt} + 2x_3^2 f(x_2)$$

$$= -x_3^2 + 2x_3^2 f(x_2) = x_3^2[2f(x_2) - 1]$$

and for $f(x_2) > \frac{1}{2}$, $\dot{W}$ is positive semi-definite. Thus the origin is stable.

CHAPTER IV

THE PROBLEM OF LUR'E

The application of the Liapunov second method has had its most success in the treatment of the classical problem due to Lur'e. To date this has been the only general problem that has been solved. Fortunately many practical problems from the area of control and guidance may be formulated in its form. The equations representing this classical problem are of the form

$$\text{(IV-1)} \qquad \begin{aligned} \dot{Y} &= AY + B\delta \\ \sigma &= C^T X - r\delta \end{aligned}$$

where Y is an n-vector, δ and σ are scalars although these problems may be generalized to the case where they are vectors. To complete the specification of the equations, a relation must be given between the variables σ and δ. This relation is generally given in one of the two forms

(1) $\dot{\delta} = f(\sigma)$ (indirect control)

(2) $\delta = f(\sigma)$ $r = 0$ (direct control).

In general the nonlinearity $f(\sigma)$ is assumed to possess one of the following restrictions

(a) $\sigma f(\sigma) > 0$ $f(0) = 0$

(b) $k_1\sigma^2 \leq \sigma f(\sigma) \leq k_2\sigma^2$ $f(0) = 0$

(c) $\int_0^\sigma f(t)dt > 0$ for all σ.

If the variable δ is eliminated, the system (IV-1) takes the more familiar form

$$\dot{X} = AX + Bf(\sigma)$$

(IV-2)

$$\sigma = C^T X$$

for the problem of direct control, and

$$\dot{X} = AX + Bf(\sigma)$$

(IV-3)

$$\dot{\sigma} = C^T X - rf(\sigma)$$

for the problem of indirect control. This last representation is obtained by making the transformation

$$\dot{Y} = AX + B\delta.$$

The problem of direct control may always be put into the form of the problem of indirect control by differentiating the equation for σ. This gives

$$\dot{\sigma} = C^T \dot{X} = C^T AX + C^T Bf(\sigma).$$

Thus the problem takes the form

$$\dot{X} = AX + Bf(\sigma)$$

(IV-4)

$$\dot{\sigma} = C^T X - rf(\sigma)$$

where $C_1^T = C^T A$ and $r = - C^T B$.

In the original treatment of this problem Lur'e assumed that the problem was in the canonical form

$$\dot{X} = \mathrm{diag}\ (\lambda_i) X + 1\ f(\sigma)$$

(IV-5)

$$\dot{\sigma} = C^T X - rf(\sigma)$$

for the problem of indirect control and in the canonical form

$$\text{(IV-6)} \qquad \begin{aligned} \dot{X} &= \text{diag}\ (\lambda_i)X + 1\ f(\sigma) \\ \sigma &= C^T X \end{aligned}$$

for the problem of direct control. The vector $\underline{1}$ is a vector all of whose elements are unity.

Observe that any block diagram with a single nonlinear gain element with no repeated open loop roots may be placed into the Lur'e canonical form. This will be illustrated by an example later. The problem posed by Lur'e was to find conditions on the vector C^T and the scalar r to insure asymptotic stability throughout the whole plane. To determine this it is first necessary to determine the equilibrium points of the above system. For the problem of indirect control we have the system

$$\text{(IV-7)} \qquad \begin{aligned} \dot{X} &= AX + B\delta \\ \dot{\delta} &= f(\sigma) \\ \sigma &= \mathbb{C}^T X - r\delta. \end{aligned}$$

Thus the equilibrium points are given by the solutions to

$$\text{(IV-8)} \qquad \begin{aligned} AX + B\delta &= 0 \\ C^T X - r\delta &= 0. \end{aligned}$$

In order for the origin to be the only equilibrium point the determinant of the coefficient of the above must be nonvanishing, that is,

$$\text{(IV-9)} \qquad \begin{vmatrix} A & B \\ C^T & -r \end{vmatrix} \neq 0.$$

For the non-critical case, that is for A to be stable which implies A is nonsingular, this becomes equivalent to the relationship

$$r + C^T A^{-1} B \neq 0. \tag{IV-10}$$

Observe that this relation was also required to insure that the transformation to the form (IV-3) was permissible. For the problem of direct control the equilibrium points are given by the solutions of the equation

$$AX + Bf(C^T X) = 0. \tag{IV-11}$$

Lur'e in his construction of a Liapunov function assumes that the eigenvalues λ_i all have negative real part. The critical case, that is, with one or more eigenvalues having zero real part requires special treatment. In the subsequent treatment we will in addition assume all eigenvalues to be real. This is done primarily to simplify the discussion. In addition we place the restriction on $f(\sigma)$ due to LaSalle[12], namely

$$\lim_{\sigma \to \infty} \int_0^\sigma f(\sigma) d\sigma \to \infty.$$

For the problem of indirect control, Lur'e considered a Liapunov function of the form

$$V = \sum_{i=1}^n a_i x_i^2 - \sum_{i=1}^n \sum_{j=1}^n \frac{\alpha_1 \alpha_2 x_i x_j}{\lambda_i + \lambda_j} + \int_0^\sigma f(s) ds \tag{IV-12}$$

where it is required that the numbers a_i are positive and the numbers α_i are arbitrary. The derivative of V takes the form

$$\dot{V} = \sum_{i=1}^n 2\lambda_i a_i x_i^2 - \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j x_i x_j + \sum_{i=1}^n 2a_i x_i f(\sigma)$$
$$- 2 \sum_{i=1}^n x_i \sum_{j=1}^n \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j} f(\sigma) + \Sigma c_i x_i f(\sigma) - rf^2(\sigma).$$

Observe that the terms

$$\sum_{i=1}^{n} 2\lambda_i a_i x_i^2 \qquad \text{is negative definite}$$

$$- \Sigma \Sigma\, \alpha_i \alpha_j x_i x_j = - (\Sigma\, \alpha_i x_i)^2 \quad \text{is negative semi-definite}$$

$$- rf^2(\sigma) \quad \text{is negative providing} \quad r > 0.$$

Collecting terms we have

$$\text{(IV-13)} \qquad \dot{V} = \Sigma\, 2\lambda_i a_i x_i^2 - (\Sigma\, \alpha_i x_i)^2 + \sum_{i=1}^{n} x_i [c_i + 2a_i - 2 \sum_{j=1}^{n} \frac{\alpha_i \alpha_j}{\lambda_i + \lambda}]^2 f(\sigma) - rf^2(\sigma).$$

Thus all terms of $\dot{V}$ are either negative definite or semi-definite except for the coefficient of $f(\sigma)$ which is indefinite. If this term can be chosen to be identically zero, then $\dot{V}$ is negative definite. This gives the requirement

$$\text{(IV-14)} \qquad \Sigma\, x_i [c_i + 2a_i - 2 \sum_{j=1}^{n} \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j}] \equiv 0.$$

But this can only be satisfied if the system of n equations

$$\text{(IV-15)} \qquad c_i + 2a_i - 2 \sum_{j=1}^{n} \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j} = 0 \qquad i = 1, 2, \ldots, n.$$

Thus the problem has been reduced to the algebraic problem of choosing the quantities α_i to be real such that the system (IV-15) is satisfied. The set of equations (IV-15) are called by Lur'e the set of resolving equations.

If in the equation for $\dot{V}$ given by (IV-13) the quantity

$$2\sqrt{r}\, f(\sigma)\Sigma\, \alpha_i x_i$$

is added and substracted then equation (IV-13) takes the modified form

$$\text{(IV-16)} \qquad \dot{V} = \Sigma\, 2\lambda_i a_i x_i^2 - (\Sigma\, \alpha_i x_i + \sqrt{r}\, f(\sigma))^2 + \sum_{i=1}^{n} x_i [c_i + 2a_i + 2\sqrt{r}\, \alpha_i \sum_{j=1}^{n} \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j}]\, f(\sigma)$$

and the resolving equations become

$$\text{(IV-17)} \qquad c_i + 2a_i + 2\sqrt{r}\, \alpha_i - 2\, \Sigma \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j} = 0 \quad i = 1,2, \ldots, n.$$

In practice the quantities a_i are dropped from the resolving equations since they may be chosen arbitrarily small. This gives the two forms which are used in practice.

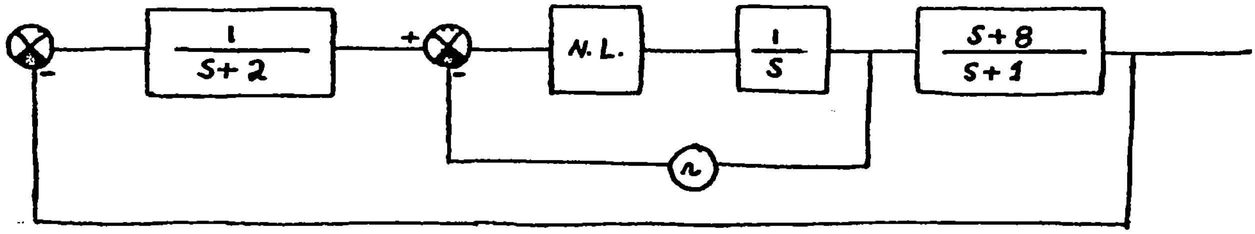
$$\text{(IV-18)} \qquad 2\alpha_i\, \Sigma \frac{\alpha_j}{\lambda_i + \lambda_j} = c_i \qquad i = 1,2, \ldots, n$$

or

$$\text{(IV-19)} \qquad 2\alpha_i [\Sigma \frac{\alpha_j}{\lambda_i + \lambda_j} - \sqrt{r}\,] = c_i \quad i = 1,2, \ldots, n.$$

The solution of these equations for real α_i is sufficient for asymptotic stability. If $f(\sigma)$ is such that $\sigma f(\sigma) > 0$ then the region of asymptotic stability becomes the whole space. To illustrate Lur'e's solution for the problem of indirect control consider the following:

Example 1. Choose the parameter r to guarantee asymptotic stability for all nonlinearities such that $\sigma f(\sigma) > 0$, where the system is given by the block diagram



This system takes the canonical form

$$\dot{x}_1 = -x_1 + f(\sigma)$$
$$\dot{x}_2 = -2x_2 + f(\sigma)$$
$$\dot{\sigma} = -7x_1 + 6x_2 - rf(\sigma).$$

Applying the canonical equations (IV-14) we obtain

(1) $$\alpha_1^2 + \frac{2}{3}\alpha_1\alpha_2 + 2\alpha_1\sqrt{r} = 7$$

(2) $$\frac{2}{3}\alpha_1\alpha_2 + \frac{\alpha_2^2}{2} + 2\alpha_2\sqrt{r} = -6.$$

For asymptotic stability we require real solutions for α_1 and α_2 of these two equations. If equation (2) is subtracted from (1) and if 1/2 of (2) is added to (1) we obtain

(3) $$\alpha_1^2 - \frac{\alpha_2^2}{2} + 2\sqrt{r}\,(\alpha_1 - \alpha_2) = 13$$

(4) $$\alpha_1^2 + \alpha_1\alpha_2 + \frac{\alpha_2^2}{4} + 2(\alpha_1 + \frac{\alpha_2}{4})\sqrt{r} = 4.$$

Observe that equation (3) is a hyperbola while equation (4) is the pair of straight lines

(5) $$\alpha_1 + \frac{\alpha_2}{2} = \pm\sqrt{4 + r} - \sqrt{r}.$$

If equation (5) is solved for α_1 and this result substituted into equation (3) we obtain

$$(6) \qquad -_{4} + (2\sqrt{r} \pm \sqrt{4 + r})\alpha_2 + 9 = 0.$$

α_2 and consequently α_1 will always be real providing the discrimenant of equation (6) is positive

$$(2\sqrt{r} \pm \sqrt{4 + r})^2 - 9 > 0$$

or

$$5r - 5 \pm 4\sqrt{4r + r^2} > 0.$$

Since by hypothesis $r > 0$ we must have

$$5r - 5 > 4\sqrt{4r + r^2}.$$

An approximate solution is given by $r = 13$. Thus for all $r > 13$, the origin is asymptotically stable.

For a solution of the problem of direct control Lur'e considered a V function which is identical to (IV-7) but without the integral term.

$$(\text{IV-20}) \qquad V = \sum_{i=1}^{n} a_i x_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\alpha_i \alpha_2 x_i x_j}{\lambda_i + \lambda_j}.$$

Differentiating V, $\dot{V}$ has the form

$$(\text{IV-21}) \qquad \begin{aligned} \dot{V} = 2 \Sigma \lambda_i a_i x_i^2 - \Sigma \Sigma \alpha_i \alpha_j x_i x_j + 2 \Sigma a_i x_i f(\sigma) \\ - 2 \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j} f(\sigma). \end{aligned}$$

If $\sigma f(\sigma)$ is added and subtracted to (IV-21) and the terms regrouped, then $\dot{V}$ becomes

$$(\text{IV-22}) \qquad \dot{V} = 2\,\Sigma\,\lambda_i a_i x_i^2 - (\Sigma\,\alpha_i x_i)^2 - \sigma f(\sigma) + \Sigma\, x_i [2a_i - 2\Sigma \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j} + c_i].$$

For asymptotic stability we require the set of resolving equations

$$(\text{IV-18}) \qquad 2\,\Sigma \frac{\alpha_i \alpha_j}{\lambda_i + \lambda} = c_i \qquad i = 1,2, \ldots, n$$

which are the same as the first form for the problem of indirect control. In general it appears that better results are obtained for a problem of indirect control then for the corresponding representation as a problem of direct control.

Example 2. Consider the system given by

$$\dot{x}_1 = -2x_1 + f(\sigma)$$
$$\dot{x}_2 = -3x_2 + f(\sigma)$$
$$\dot{x}_3 = -5x_3 + f(\sigma)$$
$$\sigma = \frac{1}{3} x_1 - x_2 + \frac{2}{3} x_3.$$

The Lur'e canonical equations become from equation (IV-18)

$$(1) \qquad \frac{\alpha_1^2}{2} + \frac{2}{5}\alpha_1\alpha_2 + \frac{2}{7}\alpha_1\alpha_3 = -\frac{1}{3}$$

$$(2) \qquad \frac{2}{5}\alpha_1\alpha_2 + \frac{1}{3}\alpha_2^2 + \frac{1}{4}\alpha_2\alpha_3 = 1$$

$$(3) \qquad \alpha_1\alpha_3 + \frac{1}{4}\alpha_2\alpha_3 + \frac{1}{5}\alpha_3^2 = -\frac{2}{3}.$$

For asymptotic stability we require real solutions for the quantities α_1, α_2, and α_3. If equation (1) is multiplied by 2, equation (2) is multiplied by 3 and equation (3) is multiplied by 5 and the results summed we obtain

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_1\alpha_2 + 2\alpha_1\alpha_3 + 2\alpha_2\alpha_3 = -1$$

or

$$(4) \qquad (\alpha_1 + \alpha_2 + \alpha_3)^2 = -1$$

this immediately demonstrates that there exist no real solutions. Therefore we have ascertained no information as to the stability. Let us now recast this problem into the form of an indirect control. The equations take the form

$$\begin{aligned}
\dot{x}_1 &= -2x_1 + f(\sigma) \\
\dot{x}_2 &= -3x_2 + f(\sigma) \\
\dot{x}_3 &= -5x_3 + f(\sigma) \\
\dot{\sigma} &= -\frac{2}{3}x_1 + 3x_2 - \frac{10}{3}x_3 + 0 \cdot f(\sigma).
\end{aligned}$$

Observe that the term r in this example is zero. Applying the Lur'e resolving equation we obtain

$$\frac{\alpha_1^2}{2} + \frac{2}{5}\alpha_1\alpha_2 + \frac{2}{7}\alpha_1\alpha_3 = \frac{2}{3}$$

$$\frac{2}{5}\alpha_1\alpha_2 + \frac{1}{3}\alpha_2^2 + \frac{1}{4}\alpha_2\alpha_3 = -3$$

$$\frac{2}{7}\alpha_1\alpha_3 + \frac{1}{4}\alpha_2\alpha_3 + \frac{1}{5}\alpha_3^2 = \frac{10}{3}.$$

The values

$$\alpha_1 = \frac{10}{3} \quad \alpha_3 = -12 \quad \alpha_3 = \frac{35}{3}$$

satisfy the above equations. Therefore we conclude that the origin is asymptotically stable.

The problem of indirect control has been independently recast into a general matrix formulation independently by Lefschetz[13] and Yakubovich[14]. This representation has an advantage in that it does not require a canonical representation. Consider the system (IV-3)

$$\text{(IV-3)} \qquad \begin{aligned} \dot{X} &= AX + Bf(\sigma) \\ \dot{\sigma} &= C^T X - rf(\sigma). \end{aligned}$$

As a Liapunov function consider the quadratic form plus integral

$$V = X^T QX + \int_0^\sigma f(s)ds.$$

$\dot{V}$ takes the form

$$\dot{V} = X^T[A^T Q + QA]X + f(\sigma)B^T QX + X^T QB\, f(\sigma) + C^T X\, f(\sigma) - rf^2(\sigma).$$

Since A is stable by hypothesis, we have for any positive P

$$A^T Q + QA = -P.$$

Define the vector D as

$$D^T = B^T Q + \frac{C^T}{2}.$$

Then $\dot{V}$ becomes

$$\dot{V} = -X^TPX + D^TX f(\sigma) + X^TD f(\sigma) - rf^2(\sigma). \quad \text{(IV-23)}$$

If a new state vector $Y^T = (X^T\ f(\sigma))$ is defined, then $\dot{V}$ takes the simple form

$$\dot{V} = -Y^TRY \quad \text{(IV-24)}$$

where R is the matrix

$$R = \begin{pmatrix} +P & -D \\ D^T & r \end{pmatrix}$$

For asymptotic stability we require R to be positive definite. Since P is already positive definite this gives as a condition the scalar equation

$$r - D^TP^{-1}D > 0. \quad \text{(IV-25)}$$

Comparing the construction due to Lur'e with that of Lefschetz, many differences become apparent. The Lefschetz construction gives a single scalar equation (IV-25) sufficient for the determination of stability, whereas the Lur'e construction requires a solution of a system of n nonlinear algebraic equations (IV-18) or (IV-19). On the other hand, by the Lefschetz construction, one must choose a suitable matrix P, all such choices do not lead to a solution since for some, inequality (IV-25) may not be satisfied. Ideally one would like to choose P to maximize (IV-25), but this would in all probability require much more effort then the solution of Lur'e resolving equations. Thus the simplification of the Lefschetz conditions may in many cases be an illusion. One would expect for the same choice of Liapunov functions that the Lefschetz construction would lead to a larger domain of stability. This is indeed the case as we will see when it is applied to Example 1. The Lefschetz construction also generalizes naturally to more than one nonlinearity.

Example 1 - continued. We now wish to analyze this construction by means of the Lefschetz construction. The system equations were

$$\dot{x}_1 = - x_1 + f(\sigma)$$
$$\dot{x}_2 = - 2x_2 + f(\sigma)$$
$$\dot{\sigma} = - 7x_1 + 6x_2 - rf(\sigma).$$

For the matrix P, we will choose the diagonal matrix

$$P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with λ_1 positive. The matrix Q becomes

$$\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} -2Q_{11} & -3Q_{12} \\ -3Q_{12} & -4Q_{33} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$$

or

$$Q = \begin{pmatrix} \frac{\lambda_1}{2} & 0 \\ 0 & \frac{\lambda_2}{4} \end{pmatrix},$$

The vector $D^T = (B^T Q + C\frac{T}{2})$ becomes

$$D^T = \left(\frac{\lambda_1 - 3}{2}, \frac{\lambda_2 + 12}{4}\right).$$

Thus for asymptotic stability we require

$$r - \left(\frac{\lambda_1 - 3}{2} \quad \frac{\lambda_2 + 12}{4}\right) \begin{pmatrix} -\frac{1}{\lambda_1} & 0 \\ 0 & -\frac{1}{\lambda_2} \end{pmatrix} \begin{pmatrix} \frac{\lambda_1 + 3}{2} \\ \frac{\lambda_2 + 12}{4} \end{pmatrix} > 0.$$

This gives the condition

$$r > \frac{(\lambda_1 - 3)^2}{4\lambda_1} + \frac{(\lambda_2 + 12)^2}{16\lambda_2}.$$

To minimize the right hand side subject to the constraint that $\lambda_1 > 0$ and $\lambda_2 > 0$ give

$$\lambda_1 = 3 \qquad \text{(by inspection)}$$

$$\lambda_2 = 12.$$

Thus we have asymptotic stability for $r > 3$. By the Lur'e construction we obtained asymptotic stability for $r > 13$.

The chief disadvantage of the Lur'e and Lefschetz construction is that they reject many practical systems which may be stable. This rejection comes about for two reasons: (1) the matrix A may be unstable or (2) the system may be unstable for systems with gains which are too large. Another disadvantage to these constructions is that in many design cases, the range of parameters obtained to insure stability are unrealistic. This is due primarily to the requirement of asymptotic stability for arbitrary nonlinearity.

For a specific nonlinearity one would expect improved results, but there is no way of taking advantage of this knowledge in the above constructions.

To bypass the first of the two objections consider the problem of direct control (IV-2)

$$\text{(IV-2)} \qquad \begin{aligned} \dot{X} &= AX + Bf(\sigma) \\ \sigma &= C^T X \end{aligned}$$

where it is assumed that A is not stable. For this system to be stable it will be necessary to restrict $f(\sigma)$ more than was originally specified for the Lur'e problem. This restriction will be of the form

$$\text{(IV-26)} \qquad k_1\sigma^2 < \sigma f(\sigma) < k_2\sigma^2.$$

To obtain these restrictions assume the $f(\sigma)$ is of the form

$$\text{(IV-27)} \qquad f(\sigma) = k_1\sigma + g(\sigma)$$

with $\sigma g(\sigma) > 0$. If (IV-27) is substituted into (IV-2) to give

$$\text{(IV-28)} \qquad \begin{aligned} \dot{X} &= (A + k_1 BC^T)X + Bg(\sigma) = A_1 X + Bg(\sigma) \\ \sigma &= C^T X \end{aligned}$$

where k_1 is chosen large enough to insure that $A_1 = A + k_1 BC^T$ is stable. We now transform (IV-28) into a problem of indirect control by differentiation to obtain

$$\text{(IV-29)} \qquad \begin{aligned} \dot{X} &= A_1 X + Bg(\sigma) \\ \sigma &= C^T X \\ \dot{\sigma} &= C^T AX + C^T Bg(\sigma) - k\sigma + kC^T X \end{aligned}$$

where we have added and subtracted $k\sigma = kC^TX$ to both sides. Consider a Liapunov function of the Lefschetz type

$$V = X^TQX + \int_0^\sigma g(s)ds \tag{IV-30}$$

$$\dot{V} = X^T[A_1^TQ + QA_1]X + g(\sigma)B^TQX + X^TQB\, g(\sigma)$$
$$+ g(\sigma)(C^TA_1X + C^TB\, g(\sigma)) - k\sigma g(\sigma) + g(\sigma)kC^TX.$$

If we add and subtract $k_3g^2(\sigma)$ to $\dot{V}$ and collect terms we obtain

$$\dot{V} = - X^TPX + g(\sigma)[B^TQ + \frac{C^TA_1}{2} + \frac{kC^T}{2}]X + X^T[QB + \frac{A_1^TC}{2} + \frac{kC}{2}]g(\sigma)$$
(IV-31)
$$+ g^2(\sigma)[C^TB - k_3] + g(\sigma)[- k\sigma + k_3g(\sigma)],$$

where as before we have $A_1^TQ + QA_1 = - P$. If we define

$$D^T = [B^TQ + \frac{C^TA_1 + kC^T}{2}]$$
$$Y^T = [X^T, g(\sigma)]$$

then $\dot{V}$ takes the form

$$\dot{V} = - Y^TSY + g(\sigma)[- k\sigma + k_3g(\sigma)] \tag{IV-32}$$

where S is the matrix

$$S = \begin{pmatrix} P & -D \\ -D^T & k_3 - C^TB \end{pmatrix}$$

For

(IV-35) $$k_3 - C^T B = r > 0$$

(IV-36) $$r + D^T P^{-1} D > 0$$

we have asymptotic stability for all $g(\sigma)$ such that

(IV-37) $$\sigma g(\sigma) < \frac{k}{k_3} \sigma^2.$$

In terms of our original system this requires

(IV-38) $$k_1 \sigma^2 < f(\sigma) < (k_1 + \frac{k}{k_3}) \sigma^2.$$

Once again it must be observed that these are only sufficient conditions for asymptotic stability. For a given problem of order higher than the second the computational work to check for any specific choice of P becomes large. Even after this is finished one has no assurance that the particular choice of P gives useful results. Various attempts have been made to develop optimum choices of the matrix P for the Lefschetz problem. Partial results have been obtained by Morozan[15].

For many practical design problems, the broad generality of the stability conditions for such arbitrary nonlinearities does not justify the complex computations. For such systems one often has a known nonlinearity and what is required is assurance of stability throughout a region given by the perturbations of the nonlinearity about a nominal value. Quick answers can be obtained to questions of this type by much simpler means. If the nonlinearity is replaced by its nominal value, then the system is linear and there exists a quadratic form Liapunov function with negative definite derivative. If the nonlinearity is permitted to vary a small amount about its nominal value, then $\dot{V}$ will vary by a corresponding amount. Thus the magnitude of the variation of the nonlinearity can be determined such that $\dot{V}$ remains negative definite. To illustrate this approach consider the following:

Example 3.

$$\dot{x}_1 = f(x_1) + a_{12}x_2$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2.$$

Assume that $f(x_1) = a_{11}x_1 + k(x_1)$ where a_{11} does not necessarily represent the initial slope of $f(x_1)$. Consider the auxilary linear system

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2$$
$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2.$$

For this linear system consider the Liapunov function

$$V = Y^T Q Y$$

with

$$\dot{V} = - Y^T P Y = - W.$$

In this linear system we now vary the coefficient a_{11} to the value $a_{11} + \delta$. The value of $\dot{V}$ will change due to this variation and its new value is

$$\dot{V} = - W + \frac{\partial V}{\partial y_1} \delta y_1 = - W + (2q_{11}y_{11} + 2q_{12}y_2)\delta y_1.$$

If δ remains small, then $\dot{V}$ will remain negative. Assume that this is true for

$$- k_1 < \delta < k_2 \quad k_1 > 0, \quad k_2 > 0.$$

Now if the deviation from the nominal $k(x_1)$ is such that

$$- k_1 < \frac{k(x_1)}{x_1} < k_2$$

then $V = X^T Q X$ is a Liapunov for the nonlinear system with negative definite derivative

$$\dot{V} = - W + k(x_1) \frac{\partial V}{\partial x_1}.$$

Thus the origin is asymptotically stable for all nonlinearities $f(x_1)$ such that

$$(a_{11} - k_1)x_1 \leq f(x_1) \leq a_{11} + k_2 x_1.$$

To determine the numbers k_1 and k_2, one must examine the expression

$$- X^T P X + 2\delta(q_{11}x_1^2 + q_{12}x_1x_2).$$

Obviously the solution will depend upon the choice of P, but for any P some selection of δ will be obtained. Let us assume that $P = 2k^2 I$, then we require

$$\dot{V} = 2x_1^2(\delta q_{11} - k^2) + 2\delta q_{12}x_1x_2 - 2k^2x_2^2$$

to be negative definite. Thus it is sufficient for

$$k^2 - \delta q_{11} > 0$$

$$k^2(k^2 - \delta q_{11}) > \frac{\delta^2 q_{12}^2}{4}.$$

If this last inequality becomes an equality we have solving for δ in $q_{12}^2\delta^2 + 4k^2 q_{11}\delta - 4k^4 = 0$

$$\delta = -\frac{2k^2}{q_{12}^2}\,[\,+q_{11} \pm \sqrt{q_{11}^2 + q_{12}^2}.$$

Thus δ must be in the interval

$$-\frac{2k^2}{q_{12}^2}\,[q_{11} + \sqrt{q_{11}^2 + q_{12}^2}\,] \leq \delta \leq -\frac{2k^2}{q_{12}^2}\,[q_{11} - \sqrt{q_{11}^2 + q_{22}^2}\,].$$

New results pertaining to the Lur'e problem have been obtained using frequency response methods. These results were first reported by Popov[16] with extensions due to Kalman[17] and Rekasius[18]. These results consist in giving conditions upon the existence of a Liapunov function of the type assumed by Lur'e. It is felt that these results should lead to near optimum choices of the matrix P in such constructions, but as yet such constructions have not been obtained.

<u>Theorem</u> IV-1. Consider the system

$$\dot{X} = AX - Bf(C^T X)$$

with

$$0 \leq \sigma f(\sigma) \leq k\sigma^2.$$

A is assumed to be stable and the system is assumed completely controllable and observable, that is

$B,\ AB,\ \ldots,\ A^{n-1}B$ are linearly independent

$C,\ A^T C,\ \ldots,\ (A^T)^{n-1} C$ are linearly independent.

Then there exists a Liapunov function of the form

$$V = X^T Q X + \beta \int_0^{\sigma} f(\sigma)\,d\sigma$$

$$V = X^T QX + \beta \int_0^{\sigma} f(\sigma)d\sigma$$

with

$$\dot{V} \equiv \nabla V^T \dot{X} \leq 0 \quad \text{for all} \quad X$$

if and only if the following conditions are satisfied: there exist two real constants α and β such that

$$\alpha \geq 0 \quad \alpha + |\beta| > 0$$

and

$$g(s) = (\alpha - \beta S)C^T(SI - A)^{-1}B$$

is such that real part of $s > 0$ implies real part of $g(s) \geq 0$.

A problem related to that of Lur'e which has received considerable attention is the problem of Aizerman[19]. This problem is a sort of converse of the problem of Lur'e and may be stated as follows. Consider the system given by (IV-2)

$$\dot{X} = AX + Bf(\sigma)$$

(IV-2)

$$\sigma = C^T X.$$

Assume that the solution $X = 0$ is asymptotically stable for

$$f(\sigma) = k\sigma \quad \text{with} \quad k_1 < k < k_2.$$

The question arises, is the solution of (IV-2) stable for all $f(\sigma)$ such that

$$k_1\sigma^2 < \sigma f(\sigma) < k_2\sigma^2.$$

Aizerman conjectured that this was true. Unfortunately numerous counter-examples have been found. Counterexamples and additional restrictions sufficient for the validity of the conjecture have been given by Cartwright[20], Mufti[21], Bergen and Williams[22].

CHAPTER V

BOUNDEDNESS AND TIME VARYING SYSTEMS

A. Lagrange Stability and Boundedness. The problems considered in the previous sections have been primarily concerned with the determination of the stability of an equilibrium position. In such problems one assumed that the only disturbance was an initial impulse which corresponded to an initial state near the equilibrium and the objective was to determine the extent of this region of stability. By such methods many systems would be rejected as being unstable but, from an engineering point of view, instability can be tolerated if the solutions do not grow too large. For example, in many control systems it is known that given designs will result in small limit cycles or similar types of behavior. This is especially true for many space vehicle control systems. In such systems there is a requirement to be able to obtain limits or bounds on the amplitude of these limit cycles. For problems of higher dimension on encounters so-called limit sets; that is, surfaces in a high dimensional space which all solutions approach. If the solutions close on such surfaces, limit cycles result, but in general the geometric structure of such solutions is extremely complicated.

A second area in which the previous treatment is inadequate is in the treatment of systems which are being continuously excited. Under such constantly acting perturbations the question naturally arises as to the effect on the stability of the equilibrium. For stable linear systems it is known that if the disturbances are small, the response will also be small. One would expect for nonlinear systems that if the system is asymptotically stable then small disturbances should produce bounded outputs.

The concept of boundedness or Lagrange stability as it is referred to by LaSalle[28] was systematically treated by means of the direct method of Liapunov by Yoshizawa[24,25]. As with the concept of stability, many kinds of boundedness can be defined. Only some of the specific definitions will be given here. In what follows the mathematical model of the systems under discussion will take the form

$$\dot{X} = F(X, t) \tag{V-1}$$

or for the autonomous case

$$\dot{X} = F(X). \tag{V-2}$$

Definition 1. The system (V-1) is said to be bounded if for any $\alpha > 0$, and t_o there exists a positive number $\beta(\alpha, t_o)$ such that if $\|X_o\| < \alpha$ then

$$\|X(t, x_o, t_o)\| < \beta \quad \text{for} \quad t \geq t_o.$$

Definition 2. The system (V-1) is said to be ultimately bounded for the bound β if for any $\alpha > 0$ and t_o there exists positive numbers β and $T(\alpha, t_o)$ such that if $\|X(t_o)\| < \alpha$ then

$$\|X(t, X_o, t_o)\| < \beta \quad \text{for} \quad t > t_o + T.$$

If in definition 1, the quantity β can be chosen independent of t_o, then the system (V-1) is said to be uniformly bounded. If T in definition 2 can be chosen independent of t_o, then (V-1) is said to be uniform-ultimately bounded.

For linear homogeneous systems, the concept of stability of the origin and the concept of boundedness are equivalent. If the function $F(X, t)$ in (V-1) is periodic in t, then ultimate boundedness implies uniform ultimate boundedness. Thus for system (V-2) these concepts are equivalent.

The two main theorems relating the concept of boundedness to Liapunov functions are as follows:

Theorem V-1: Let Ω^* be the region defined by $0 \leq t \leq \infty$, $\|X\| > r$. If there exists a function $V(X, t)$ which is positive definite in the region Ω^*, while its derivative

$$\frac{dV}{dt} = \nabla V^T \cdot F + \frac{\partial V}{\partial t} \tag{V-3}$$

is negative semi-definite in the interior of Ω^*, then the solutions of (V-1) are uniformly bounded.

Theorem V-2: If there exists a Liapunov function $V(X, t)$ which is positive definite in Ω^*, while its derivative (V-3) is negative definite in the interior of Ω^*, then the solutions of (V-1) are uniformly ultimately bounded.

Observe that the above two theorems reduce to the theorems on stability and asymptotic stability if r in the definition of the set Ω^* is set equal to zero.

Example 1. Consider the system

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = W(t)$$

where we assume

(1) $f(x, \dot{x}) > 0$ for all x and $\dot{x}$

(2) $k(x) = \int_0^x g(s)ds > 0$

(3) $\lim_{x \to \infty} k(x) \to \infty$

(4) $\int_0^\infty |W(t)|dt < \infty$.

Consider as a Liapunov function

$$V(x, \dot{x}, t) = \frac{\dot{x}^2}{2} + k(x) - \int_0^t |W(s)|ds.$$

For the norm of x sufficient large we have that V is positive definite. The derivative of V becomes

$$\begin{aligned}\dot{V} &= \dot{x}\ddot{x} + \dot{x}g(x) - |W(t)| \\ &= \dot{x}[- \dot{x}f(x, \dot{x}) - g(x) + W(t)] + \dot{x}g(s) - |W(t)| \\ &= - f(x, \dot{x})\dot{x}^2 - |W(t)| + W(t)\dot{x} \\ &= - |W(t)| - \dot{x}[f(x, \dot{x})\dot{x} - W(t)].\end{aligned}$$

Thus for the norm of x sufficiently large, $\dot{V}$ is negative semi-definite and all solutions are uniformly bounded. If in the above example $W(t)$ was identically zero, then the equilibrium position is stable.

Example 2. Consider the autonomous system

$$\dot{X} = AX + G(X)$$

where the nonlinearity is of a saturation type that is $\|G(X)\| < K$ for all X. If A is assumed to be stable, then a Liapunov function is given by

$$V = X^T QX$$

where $\dot{V}$ is given by

$$\begin{aligned}\dot{V} &= X^T[A^TQ + QA]X + 2X^TQG(X) \\ &= - X^TPX + 2X^TQG(X).\end{aligned}$$

For any choice of P positive definite it is obvious that $\dot{V}$ is negative semi-definite for the $\|X\|$ sufficiently large. Thus all solutions are bounded. In particular if A is diagonal and P is the identity matrix then

$$\dot{V} = - r^2 + \Sigma \frac{x_i g_i(x)}{\lambda_i}$$

and an estimate of the bound is given by

$$r = \frac{nK}{\min |\lambda_i|} .$$

The main practical application of the concept of Lagrange stability is to couple it with the concept of instability to obtain bounds on limit sets or limit cycle behavior. If for a system described by (V-2), the origin is unstable and the region of instability is given by $\|X\| \leq r_1$, while at the same time all solutions are ultimately bounded by $\|X\| = r_2$ $r_1 < r_2$ then all solutions must enter the region defined by these two spheres $r_1 < \|X\| < r_2$. Therefore a limit set or cycle must exist in this region. In practice it is required to obtain a better estimate of this limit set than the one given by r_1 and r_2, thus one desires a procedure to construct a Liapunov function to do this.

A construction procedure due to Szegö[4] meets these requirements for certain restricted forms of the system (V-2) Szegö 's construction is based on obtaining a Liapunov function which is positive definite while its derivative is indefinite on a closed curve. The function $\dot{V}$ is constrained to have the form

(V-4) $$\dot{V} = U(X)W(\theta(X))$$

where U, W, and θ have the following properties

(1) $U(X)$ is at least semi-definite and nonvanishing on any solutions of (V-2).

(2) $W(s)$ is of opposite signs for s positive and negative. Thus $W(0) = 0$ and sign $W(-s)$ is different from sign $W(s)$.

(3) The function $\theta(X) = 0$ represents a closed surface about the origin.

If $U(X)$ is positive and $W(-s)$ is negative, then this construction would indicate that the equilibrium position of (V-2) is asymptotically stable with an estimate of the region of stability given by the set of X such that $V(X) = C$ is inscribed interior to the surface $\theta(X) = 0$. If $W(-s)$ is positive, then the conclusion is that the origin is unstable, while all solutions are bounded. If we indicate by V_i the surface $V_i = C_i$ inscribed by the surface $\theta(X) = 0$, and by V_o the surface $V_o = C_o$ which circumscribes the surface $\theta(X) = 0$, thus all solutions must approach a limit set Ω where Ω is in the region between V_o and V_i.

To construct a "V" function which has this property consider system (V-2) in the restricted form

(V-5) $$\dot{X} = A(X)X$$

where it is assumed that each element $a_{ij}(X)$ is at most a polynomial in the components of X. As a Liapunov function choose V to be of the form

(V-6) $$V = X^T Q(X)X$$

with

$$Q = (q_{ij}(X) = q_{ij}(0, \ldots, 0, x_i, 0, \ldots, x_j, 0, \ldots, 0)).$$

Thus each element q_{ij} of Q only depends upon the components x_i and x_j of X. The derivative of V is

(V-7)
$$\dot{V} = X^T[A^TQ + QA + \frac{dQ}{dt}]X$$
$$= X^T[A^TR + R^TA]X$$

where R is not symmetric in general. The elements r_{ij} of R can be defined in terms of the elements q_{ij}. This relation is given as

(V-8)
$$r_{ij} = q_{ij} + \epsilon_{ij}x_i \frac{\partial q_{ij}}{\partial x_i}$$

with ϵ_{ij} given by

(V-9)
$$\epsilon_{ij} = 1 \qquad i \neq j$$
$$= \frac{1}{2} \qquad i = j.$$

It is desired to constrain $\dot{V}$ to be of the form (V-4)

$$\dot{V} = U(X)W(\theta(X))$$

where U is required to be at least positive semi-definite. For U we may assume the quadratic form $U(X) = X^TSX$. The function $\theta(X) = 0$ must represent a closed curve, thus a reasonable choice for $\theta(X)$ is

$$\theta(X) = X^TP(X)X - k$$

with $P(X)$ positive definite. For $W(S)$ it is sufficient to choose an odd function such as $W(S) = S$ or $W(S) = S^3$, etc. For the first of these choices we have

(V-10)
$$\dot{V} = X^T[A^TR + R^TA]X = X^TSX[X^TP(X)X - k]$$
$$= X^T[SXX^TP(X) - Sk]X.$$

$$R(X) = \begin{pmatrix} q_{11} + \frac{x_1}{2} \frac{\partial q_{11}}{\partial x_1} & q_{12} + x_1 \frac{\partial q_{12}}{\partial x_1} \\ q_{12} + x_2 \frac{\partial q_{12}}{\partial x_2} & q_{22} + \frac{x_2}{2} \frac{\partial q_{22}}{\partial x_2} \end{pmatrix}.$$

The matrix $A^T R + R^T A$ becomes

$$A^T R + R^T A = \begin{pmatrix} -2q_{12} - 2x_2 \frac{\partial q_{12}}{\partial x_2} & \left(q_{11} + \frac{x_1}{2} \frac{\partial q_{11}}{\partial x_1} - q_{22} - \frac{x_2}{2} \frac{\partial q_{22}}{\partial x_2} + \epsilon(1-x_1^2)(q_{12} + x_2 \frac{\partial q_{22}}{\partial x_2}\right) \\ \left(q_{11} + \frac{x_1}{2} \frac{\partial q_{11}}{\partial x_1} - q_{22} - \frac{x_2}{2} \frac{\partial q_{22}}{\partial x_2} + \epsilon(1-x_1^2)(q_{12} + x_2 \frac{\partial q_{22}}{\partial x_2}\right) & \left[2q_{12} + 2x_1 \frac{\partial q_{12}}{\partial x_1} + \epsilon(1-x_1^2)(2q_{22} + x_2 \frac{\partial q_{22}}{\partial x_2})\right] \end{pmatrix}$$

We now wish to specify the right hand side of equation (V-11), namely the terms $SXX^T P(X) - Sk$. Since we wish S to be semi-definite, let it be the matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

while as yet we will keep P arbitrary.

$$P = \begin{pmatrix} P_{11}(x_1, x_2) & P_{12}(x_1 x_2) \\ P_{12}(x_1, x_2) & P_{22}(x_1 x_2) \end{pmatrix}.$$

Thus the matrix $SXX^T P(X) - Sk$ becomes

To satisfy these conditions, we wish to chose the elements of q_{ij} such that

$$A^T R + R^T A = SXX^T P(X) - Sk. \qquad \text{(V-11)}$$

In the solution of the above equation a number of difficulties should be observed. The matrix form on the left hand side is symmetric while the left hand side is not symmetric in general. Thus equation (V-11) is an equation only in the symmetric part of the left hand side. The other observation to be made is that when one obtains a variable matrix representation for a scalar function, this representation is not unique. Obviously the above construction is difficult for high order systems. To illustrate its application consider the following application to Van der Pol's equations.

Example 3.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = - x_1 + \epsilon x_2 - \epsilon x_1^2 x_2.$$

The matrix $A(X)$ thus has the form

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon(1 - x_1^2 \end{pmatrix}.$$

Let $Q(X)$ be the matrix

$$Q(X) = \begin{pmatrix} q_{11}(x_1) & q_{12}(x_1, x_2) \\ q_{12}(x_1, x_2) & q_{22}(x_2) \end{pmatrix}.$$

The matrix R takes the form

$$SXX^TP(X) - Sk = \begin{pmatrix} x_1^2p_{11} + x_1x_2p_{12} - k & x_1^2p_{12} + x_1x_2p_{22} \\ 0 & 0 \end{pmatrix}.$$

Equating the matrix $A^TR + RA^T$ to the symmetric part of $SXX^T(P(S) - Sk$ we obtain the three equations

$$- 2q_{12} - 2x_2 \frac{\partial q_{12}}{\partial x_2} = x_1^2p_{11} + x_1x_2p_{12} - k$$

$$q_{11} + \frac{x_1}{2} \frac{\partial q_{11}}{\partial x_1} - q_{22} - \frac{x_2}{2} \frac{\partial q_{22}}{\partial x_2} + \epsilon(1 - x_1^2)(q_{12} + x_2 \frac{\partial q_{12}}{\partial x_2}) =$$

$$\frac{x_1^2}{2} p_{12} + \frac{x_1x_2}{2} p_{22}$$

$$2q_{12} + 2x_1 \frac{\partial q_{12}}{\partial x_1} + \epsilon (1 - x_1^2)(2q_{22} + x_2 \frac{\partial q_{22}}{\partial x_2}) = 0.$$

The solutions of the above equations become extremely difficult unless some simplifying assumptions are made. One such assumption is that q_{22} is independent of x_1 and x_2. With this assumption the last of the above equations can be solved for q_{12} to give

$$\frac{\partial}{\partial x_1} (x_1q_{12}) + \epsilon\, q_{22}(1 - x_1^2) = 0$$

$$x_1q_{12} = \epsilon\, q_{22}(\frac{x_1^3}{3} - x_1) + C$$

where the constant of integration C may depend upon x_2. If C is chosen to be zero we have

$$q_{12} = \epsilon\ q_{22}(\frac{x_1^2}{3} - 1).$$

If this is used in the first of the above equations we have

$$-2q_{12} = x_1^2 p_{11} + x_1 x_2 p_{12} = k$$

or

$$k - 2\ \epsilon\ q_{22}(\frac{x_1^2}{3} - 1) = x_1^2 p_{11}(x_1,\ x_2) + x_1 x_2 p_{12}(x_1,\ x_2).$$

This last equation requires a dependence of p_{11} and p_{12} upon x_2 such that the right hand side is independent of x_2. If we assume that $p_{12} = 0$ and p_{11} depends only upon x_1, this first equation is consistent with the last. Applying these restrictions to the second equation we must have $p_{22} = 0$, and solving for q_{11} we obtain

$$q_{11} = q_{22} - \epsilon^2 q_{22}(\ - \frac{x_1^4}{9} + \frac{2}{3} x_1^2 - 1)$$

with

$$p_{11} = -\frac{2}{3}\epsilon\ q_{22},\quad p_{12} = 0,\quad p_{22} = 0$$

$$k = -2\ \epsilon\ q_{22}.$$

The assumptions we have made upon the parameters have been too restrictive. We obtained a function $\theta(X)$ which does not represent a closed curve for $\theta(X) = 0$. The V and $\dot{V}$ thus obtained are

$$\dot{V} = \frac{2}{3}\epsilon\ x_1^2[3 - x_1^2]$$

$$V = \frac{\epsilon^2}{9} x_1^6 - \frac{2}{3}\epsilon^2 x_1^4 + (1 + \epsilon^2)x_1^2 + \frac{2}{3}\epsilon\ x_1^3 x_2 - 2\ \epsilon\ x_1 x_2 + x_2^2.$$

The matrix A(X) is positive definite so that we have obtained boundedness. Rather than reworking the original equations under less restrictive assumptions we can choose V of the same form as obtained but keep free parameters as coefficients. This approach gives

$$V = a_2 x_1^6 - a_1 x_1^4 + a_o x_1^2 + 2b_o x_1^3 x_2 - 2b_1 x_1 x_2 + x_2^2.$$

Differentiating we obtain for $\dot{V}$ the expression

$$\dot{V} = (2a_o - 2b_1\epsilon - 2)x_1x_2 + (2\epsilon - 2b_1)x_2^2 + (6a_2 - 2b_o\epsilon)x_1^5x_2$$

$$+ (2b_o\epsilon - 4a_1 + 2b_1\epsilon)x_1^3x_2 + (6b_o - 2)x_1^2x_2^2 - 2b_o x_1^4 + 2b_1 x_1^2.$$

We now choose the parameters so that the first three terms vanish. Thus

$$a_2 = \frac{b_o\epsilon}{3} \qquad b_1 = \epsilon \qquad a_o = 1 + \epsilon^2$$

and $\dot{V}$ becomes

$$\dot{V} = x_1^2[2\epsilon + (6b_o - 2)x_2^2 - 2b_o x_1^2 + (2b_o\epsilon - 4a_1 + 2\epsilon^2)x_1x_2].$$

In order for the quantity inside the brackets to represent a closed curve we must have

$$6b_o - 2 < 0 \quad \text{and} \quad -2(6b_o - 2)b_o - (b_o\epsilon - 2a_1 + \epsilon^2)^2 > 0.$$

Thus $0 < b_o < \frac{1}{3}$. The second inequality can be satisfied for $a_1 = \frac{1}{2}[b_o\epsilon + \epsilon^2]$, $b_o = \frac{k}{6}$ $\quad 0 \leq k \leq 2$. With these choices we have

$$\dot{V} = x_1^2[2\epsilon - (2 - k)x_2^2 - \frac{k}{3} x_1^2].$$

Thus $\dot{V}$ vanishes on the ellipse given by

$$\frac{k}{3} x_1^2 + (2 - k)x_2^2 = 2\epsilon$$

and $\dot{V}$ is positive semi-definite inside and negative semi-definite outside this ellipse. V is given by

$$V = (\epsilon^2 + 1)x_1^2 - (\frac{k}{6} + \epsilon) \frac{\epsilon}{2} x_1^4 + \frac{k}{18} \epsilon x_1^6 + \frac{2k}{6} x_1^3 x_2 - 2\epsilon x_1 x_2 + x_2^2.$$

For $k = \frac{3}{2}$ and $\epsilon = 1$

$$V = \frac{x_1^6}{12} - \frac{5}{8} x_1^4 + 2x_1^2 + x_1 x_2(\frac{x_1^2}{2} - 2) + x_2^2$$

and is positive definite for all X. Thus the limit set or cycle must be in the region between the curves given by $V_o = c_o$ which circumscribes the circle of radius 2 and $V_i = c_i$ which is inscribed inside the circle of radius 2.

A concept closely related to the concept of boundedness is the concept of total stability or stability under constantly acting disturbances. This concept may be defined as follows.

Consider the system

(V-1) $$\dot{X} = F(X, t)$$

and the perturbed system

(V-12) $$\dot{X} = F(X, t) + G(X, t)$$

where it is assumed $F(0, t) = 0$.

Definition 3. The solution $X = 0$ of (V-1) is said to be stable under constantly acting perturbations if for every $\epsilon > 0$ there exists two constants $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$ such that if $\|X(t_o)\| < \delta_1$ and $\|G(X, t)\| < \delta_2$ then $\|X(t)\| < \epsilon$ for all $t \geq t_o$.

The main results on the theory of total stability are contained in a theorem due to Malkin[26].

Theorem 5: If the solution $X = 0$ of (V-1) is uniformly asymptotically stable, then it is stable under constantly acting forces.

Since for autonomous systems asymptotic stability implies uniform asymptotic stability then asymptotic stability implies total stability. As yet no results have appeared to give estimates of how large the region of total stability is and its relation to the region of asymptotic stability.

B. Non-Autonomous Systems. The problem of constructing Liapunov functions for non-autonomous systems remains one of the main undeveloped areas. Some procedures have been advocated for linear systems by Szegö[27], Roitenberg[28] and others, but even here the useable results are few unless one is restricted to second order systems. The main reason for this is probably the tremendous complexity of such systems. Even for linear systems, familar procedures break down and intuition can lead one astray.

In the design of control systems for boost vehicles the design engineer is confronted with the time varying system representing the perturbations about a nominal trajectory. The design of such systems in general is not based on the time varying nature, but rather the problem is assumed to be stopped at some point on the trajectory. The perturbation equations become constant coefficient equations and familar linear stationary procedures are used. In some cases a single design is valid throughout the whole control regime, while in other certain parameters representing gains and time constants are programmed as a function of time. These time varying values are obtained by smoothing throughout the set of discrete values obtained from the constant coefficient analysis.

In all of such procedures the design engineer is reasoning as follows: Consider the time varying closed loop system

$$\dot{X} = A(t)X. \tag{V-13}$$

If all the eigenvalues $\lambda_k(t)$ of the matrix $A(t)$ always have negative real parts, then the solutions of (V-13) are asymptotically stable. Unfortunately the above reasoning is fallacious since a simple counter-example due to Zubov[2] will show that solutions of (V-13) may be unstable even though the eigenvalues of A are negative and constant.

Example 4. Consider the system

$$\dot{x}_1 = -(1 + 9\cos^2 6t - 12\sin 6t\cos 6t)x_1 + (12\cos^2 6t + 9\sin 6t\cos 6t)x_2$$

$$\dot{x}_2 = -(12\sin^2 6t - 9\cos 6t\sin 6t)x_1) - (1 + 9\sin^2 6t + 12\sin 6t\cos 6t)x_2$$

The characteristic equation of the above is

$$|A(t) - \lambda I| = \lambda^2 + 11\lambda + 10 = 0$$

with the two eigenvalues $\lambda_1 = -10$, $\lambda_2 = -1$. Thus asymptotic stability would appear to be insured. The fundamental matrix of solutions of this system has for its elements the expressions

$$\varphi_{11} = \frac{1}{5}\cos 6t(e^{2t} + 4e^{-13t}) + \frac{1}{5}\sin 6t(2e^{2t} - 2e^{-13t})$$

$$\varphi_{12} = \frac{1}{5}\sin 6t(4e^{2t} + e^{-13t}) + \frac{\cos 6t}{5}(2e^{2t} - 2e^{-13t})$$

$$\varphi_{21} = -\frac{1}{5}\sin 6t(e^{2t} + 2e^{-13t}) + \frac{\cos 6t}{5}(2e^{2t} - 2e^{13t})$$

$$\varphi_{22} = \frac{1}{5}\cos 6t(4e^{2t} + e^{-13t}) + \frac{1}{5}\sin 6t(e^{-13t} - 2e^{2t}).$$

Thus all solutions diverge with time. In fact not only does negative eigenvalues fail to insure stability but positive eigenvalues do not insure instability. The logical question that immediate arises is: if the approach used in the past is invalid , why has it worked so well? Unfortunately it is difficult to build a strong case against success. It would appear intuitively that if the time variation of the system is sufficiently slow, then the stability can be determined by considering the eigenvalues of the system. Here intuition is correct. The formal statement of such a result due to Rosenbrock[29] is as follows.

<u>Theorem 4</u>: Consider the system (V-13)

$$\dot{X} = A(t)X$$

where for all $t \geq t_o$ every element $a_{ij}(t)$ of $A(t)$ is differentiable and satisfies $|a_{ij}(t)| \leq a$. Let all eigenvalues of $A(T)$ be such that $Real(\lambda(A) \leq - \epsilon < 0$. Then there exists some $\delta > 0$ such that if $|\frac{\partial a_{ij}}{\partial t}| \leq \delta$, then the equilibrium position $X = 0$ is asymptotically stable.

In this theorem, which is an existence theorem, no method of determining the suitable bounds upon δ are given. When the system (V-18) is in companion form with non-repeated eigenvalues, then some bounds can be determined upon the elements of $A(T)$. Since for this form of $A(X)$, the elements are identical to the coefficients of the characteristic equation, these bounds may be transfered to suitable bounds on the eigenvalues of A. For an expression for such bounds see Rosenbrock[29].

Since for a linear autonomous system a suitable Liapunov function is given by a quadratic form, it is logical to start from this point. Therefore consider the quadratic form

$$V = X^T QX.$$

Its derivative becomes

$$\dot{V} = X^T[A^T(t)Q + QA(t)]X.$$

We desire $\dot{V}$ to be negative definite to insure stability. For the autonomous case given any positive definite matrix P we could always obtain solutions Q to the equation

$$A^TQ + QA = -P \qquad \text{(V-14)}$$

providing A was stable. This procedure can also be used for a time varying case. Thus for any P, Q is given by

$$Q = \int_0^\infty e^{A^T(t)u} P(t) e^{A(t)u} du$$

providing the integral converges. Unfortunately Q in such a determination will not be constant. An alternative approach is given by applying the Hurwicz criteria to the matrix $A^TQ + QA$ to obtain sufficient conditions to insure stability.

Example 5. Consider the linear system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 3x_2 + f(t)x_2.$$

Let V be the quadratic form X^TQX where

$$Q = \begin{pmatrix} 5/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

then $\dot{V}$ is given by

$$\dot{V} = - X^T P X$$

where

$$P = \begin{pmatrix} 1 & - f(t)/4 \\ - f(t)/4 & 1 - f(t)/2 \end{pmatrix}.$$

For $\dot{V}$ to be negative definite, we require

$$1 - f(t)/2 > 0$$

$$1 - f(t)/2 > - \frac{f^2(t)}{16} .$$

The first inequality requires

$$f(t) < 2 \quad \text{for all} \quad t$$

while the second gives

$$- 4(1 + \sqrt{2}) < f(t) < 4(\sqrt{2} - 1).$$

Results which are slightly better than those obtained by the method above were given by Zubov[2]. His results are as follows: Let V be the quadratic form

$$V = X^T Q(t) X.$$

Then $\dot{V}$ is given by

$$\dot{V} = X^T [A^T Q + QA + \dot{Q}] X = X^T P(t) X.$$

Let $\lambda_1(t)$ and $\lambda_n(t)$ be the smallest and largest eigenvalue of $P(t)$. Let $\mu_1(t)$ and $\mu_n(t)$ be the smallest and largest eigenvalue of $Q(t)$. The solutions of (V-13) are such that

(V-15)
$$X^T(t_o)X(t_o)\ \frac{\mu_1(t_o)}{\mu_n(t)}\ e^{\int_{t_o}^{t} \frac{\lambda_1(s)}{\alpha_1(s)}ds} \leq X^T(t)X(t) \leq X(t_o)\ X^T(t_o)X(t_o)\frac{\mu_n(t_o)}{\mu_1(t)}\ e^{\int_{t_o}^{t} \frac{\lambda_n(s)}{\alpha_2(s)}ds}$$

where α_1 and α_2 are defined as

(V-16)
$$\alpha_1 = \frac{1}{2}\ [(1 - \operatorname{sgn}(\lambda_1))\mu_1 + (1 + \operatorname{sgn} \lambda_1)\mu_n]$$

(V-17)
$$\alpha_2 = \frac{1}{2}\ [(1 + \operatorname{sgn} \lambda_n)\mu_1 + (1 - \operatorname{sgn} \lambda_n)\mu_n].$$

If the right hand side of (V-15) is bounded, then $X = 0$ is stable. If the right hand side approaches zero as $t \to \infty$, then we have asymptotic stability.

In particular if $Q(t)$ is chosen as the identity matrix, and then $P(t) = A^T + A$ or twice the symmetric part of A. Then (V-15) takes the form

(V-18)
$$X^T(t_o)X(t_o)e^{\int_{t_o}^{t}\lambda_1(s)ds} \leq X^T(t)X(t) \leq X^T(t_o)X(t_o)e^{\int_{t_o}^{t}\lambda_n(s)ds}.$$

These results are stronger than those given by applying the Hurewicz criteria since the Hurewicz determinants may oscillate in sign with time. Thus no conclusion can be reached, while in many such cases (V-15) or (V-18) will still give useful results.

Example 6. Consider the system of example 5

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 3x_2 + f(t)x_2.$$

If we use the same function V, then the matrix

$$P = \begin{pmatrix} -1 & f(t)/4 \\ f(t)/4 & 1 - f(t)/2 \end{pmatrix}.$$

The eigenvalues of P are

$$\lambda_{1,\ 2} = -1 + \frac{f(t)}{4}(1 \pm \sqrt{2}).$$

Thus

$$\lambda_n = -1 + (1 + \sqrt{2})\frac{f(t)}{4}\left(\frac{1 + \operatorname{sgn} f(t)}{2}\right) + \left(\frac{1 - \sqrt{2}}{4}\right)f(t)\left(\frac{1 - \operatorname{sgn} f(t)}{2}\right)$$

If $f(t)$ satisfies the inequality

$$f(t) < \frac{4}{1 + \sqrt{2}} \quad \text{for all } t.$$

Then the largest eigenvalue of P is negative and we have asymptotic stability. $f(t)$ can exceed this value without destroying the asymptotic stability providing we have

$$\int_0^t f(s)ds < \frac{4}{1 + \sqrt{2}}\, t.$$

If the matrix $A(t)$ is diagonal, then the stability is determined almost by inspection. This has led many to consider the problem of performing a suitable transformation. Thus if we have the system

$$\dot{X} = A(t)X$$

and we make the transformation $X = Q(t)Y$ then we obtain

$$Q(T)\dot{Y} + \dot{Q}(T)Y = A(T)Q(T)Y \tag{V-19}$$

or by premultiplying by $Q^{-1}(t)$ we obtain

$$\dot{Y} = [Q^{-1}AQ - Q^{-1}\dot{Q}]Y. \tag{V-20}$$

This just seems to transform the main difficulty to the problem of finding suitable transformation.

For the nonlinear time varying system, the state of the art for the construction of Liapunov functions is for all intents non-existent. For specific problems "V" functions have been obtained but other than attempts to bound the time varying coefficients by constants and analyzing the resulting autonomous system, no general procedures are available.

CHAPTER VI

DISCRETE SYSTEMS

The transformation of the concepts of stability for differential equations may be carried over directly to systems or difference equations with little modification for autonomous systems. Thus we consider as a model the set of difference equations

(VI-1) $$X(t_{k+1}) = F(X(t_k)).$$

Often we will use the notation

(VI-2) $$X_{k+1} = F(X_k)$$

for (VI-1). We will assume that $X = 0$ is an equilibrium point. The numbers t_k represent discrete values of time. The difference $t_{k+1} - t_k$ is assumed to be a constant for all k unless otherwise specified.

Definition VI-1. The solution $X = 0$ is stable if given any $\epsilon > 0$ and a t_o there exists a $\delta(\epsilon, t_o)$ such that for $\|X(t_o)\| < \delta$ implies that $\|X(t_k)\| < \epsilon$ for all $t_k > t_o$.

Definition VI-2. The solution $X = o$ is said to be asymptotically stable if $X = 0$ is stable and $\lim_{k \to \infty} \|X(t_k)\| \to 0$.

From the statement of the theorems of Liapunov for the stability of continuous systems, the appropriate theorems for discrete systems follows immediately. These will be stated with the analog of LaSalle's extension.

Theorem VI-1: If there exists a function $V(X_k)$ which is positive definite, such that the difference

Theorem VI-1: If there exists a function $V(X_k)$ which is positive definite, such that the difference

$$\Delta V(X_k) = V(X_{k+1}) - V(x_k) = V(F(X_k)) - V(X_k)$$

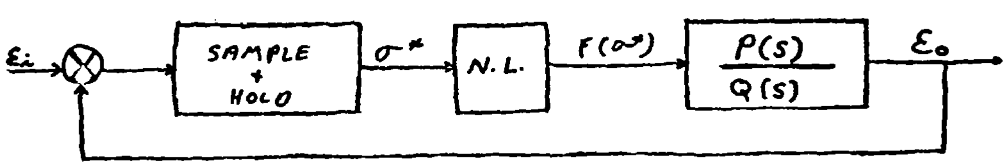
is negative semi-definite, then the solution $X = 0$ is stable.

Theorem VI-2: If there exists a function $V(X_k)$ which is positive definite such that the difference

$$\Delta V(X_k) = V(X_{k+1}) - V(x_k) = V(F(X_k)) - V(X_k)$$

is negative definite or negative semi-definite with ∇V not vanishing identically on any solution sequence X_k, then the solution $X = 0$ is asymptotically stable.

The discrete system expressed by equation (VI-1) is a reasonable model for many sampled data or digital control systems. Unfortunately such problems when encountered are in terms of block-diagrams or mixed systems of continuous differential equations and discrete algebraic equations. Thus one needs transformations to place such systems into the discrete notation. A typical sampled-data system may have the block diagram representation



From previous developments we have the describing equations for the above system

$$\dot{X} = AX + BF(\sigma^*)$$

$$\epsilon_o = C^T X - rF(\sigma^*)$$

$$\sigma^* = \epsilon_i(t_k) - \epsilon_o(t_k) \qquad t_k \leq t \leq t_{k+1}.$$

The first step in the transformation to discrete form is to obtain a discrete representation for the first of the above two equations. To facilitate this consider the linear continuous system

(VI-3) $$\dot{X} = AX + BU.$$

Its solution is given by

(VI-4) $$X(t) = \Phi(t, t_o)X(t_o) + \int_{t_o}^{t} \Phi(t, s)BU(s)ds.$$

In the above solution let $t = t_{k+1}$ and $t_o = t_k$.

$$X(t_{k+1}) = \Phi(t_{k+1}, t_k)X(t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)BU(s)ds.$$

In the interval $t_k \leq t < t_{k+1}$, we will assume that U is constant. Then U may be taken outside of the above integral to give

(VI-5) $$X_{k+1} = \Phi(t_{k+1}, t_k)X_k + [\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)ds]BU_k.$$

The matrix Φ used in the above representation is called the transition matrix and it satisfies the following relations.

(VI-6) $$\Phi(t, S) = \Phi^{-1}(s, t)$$

(VI-7) $$\Phi(t, s)\Phi(s, r) = \Phi(t, r)$$

(VI-8) $$\Phi(t, t) = I.$$

If the matrix A in (VI-3) is constant that $\Phi(t, s)$ is defined as

$$\Phi(t, s) = e^{A(t-s)} = I + A(t-s) + \frac{A^2}{2!}(t-s)^2 + \dots + \frac{A^n}{n!}(t-s)^n + \dots .$$

Thus for constant A (VI-5) takes the form

(VI-9) $$X_{k+1} = e^{A(t_{k+1}-t_k)}X_k + [\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-s)} ds]BU_k.$$

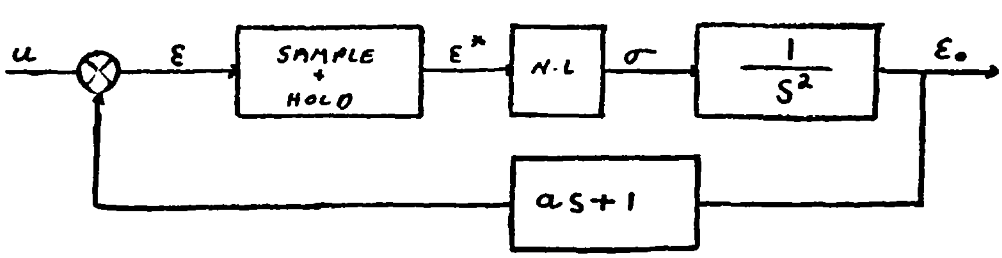
If the sampling period $t_{k+1} - t_k = T$ is constant and if A is nonsingular (VI-9) takes the form

(VI-10) $$X_{k+1} = e^{AT}X_k + \int_0^T e^{A(T - s)}ds\ BU_k$$

(VI-11) $$= e^{AT}X_k + A^{-1}(e^{AT} - I)BU_k.$$

From the above it becomes apparent to transform a sampled data system into discrete form it is necessary to obtain a solution to the associated differential equations. For linear systems these solutions are relatively easy to obtain. Consider the sampled system

Example VI-1.



The sample hold unit replaces the continuous function ϵ by a piece-wise continuous function ϵ^* defined as

$$\epsilon^*(t) = \epsilon(t_k) \qquad t_k \le t \le t_{k+1}.$$

Thus the system has the representation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma$$

$$\epsilon = (-1 \ -a) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u(t).$$

The matrix e^{At} is given by

$$e^{At} = I + A + \frac{A^2}{t!} + \dots$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \dots + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \doteq \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

The discrete state vector is given by (VI-10)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{k+1} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_x + \int_0^T \begin{pmatrix} 1 & T-s \\ 0 & 1 \end{pmatrix} ds \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma(t_k)$$

or finally we have

$$x_1(t_{k+1}) = x_1(t_k) + Tx_2(t_k) + \frac{T^2}{2}\sigma(t_k)$$

$$x_2(t_{k+1}) = x_2(t_k) + T\sigma(t_k)$$

$$\sigma(t_k) = F[u(t_k) - x_1(t_k) - ax_2(t_k)].$$

If the differential equations describing the continuous part of the above system, were nonlinear then the transformation to discrete form is much more difficult. In general only an approximation to the true discrete equation can be obtained since one can not in general solve the nonlinear equations. Most often such transformations are obtained by approximating the equations by a set of difference equations. Thus

(VI-12) $$\dot{x} = \frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k}$$

(VI-13) $$\ddot{x} = \frac{x(t_{k+2}) - 2x(t_{k+1}) + x(t_k)}{(t_{k+1} - t_k)^2}.$$

For fixed $T = (t_{k+1} - t_k)$ the nonlinear system

$$\ddot{x} + 2x^2\dot{x} + \sin x = U(t_k)$$

takes the approximate form

$$x(t_{k+2}) - 2x(t_{k+1}) + x(t_k) + 2Tx^2(t_k)[x(t_{k+1}) - x(t_k)] + T^2 \sin x(t_k)$$
$$= T^2U(t_k).$$

If the state variable $x_1 = x(t_k)$, $x_2(t_k) = x(t_{k+1})$ are used the system takes the form

$$x_1(t_{k+1}) = x_2(t_k)$$
$$x_2(t_{k+1}) = -x_1(t_k) - T^2 \sin x_1(t_k) + 2x_2(t_k)$$
$$- 2Tx_1^2(t_k)x_2(t_k) + 2Tx_1^3(t_k) + t^2U(t_k).$$

If the original system was first transformed into the state vector representation

$$y_1 = x \qquad \dot{y}_1 = y_2$$
$$y_2 = \dot{x} \qquad \dot{y}_2 = - 2y_1^2y_2 - \sin y_1 + U(t_k)$$

then these equations take the discrete form

$$y_1(t_{k+1}) = y_1(t_k) + Ty_2(t_k)$$
$$y_2(t_{k+1}) = y_2(t_k) - 2Ty_1^2(t_k)y_2(t_k) - T \sin y_1(t_k) + T\ (t_k).$$

The construction of Liapunov functions for discrete systems follows analogously from the corresponding constructions for continuous systems. Some constructions which are valid for discrete systems are not valid for continuous systems. The simplest of such functions is the norm of the vector. Consider the system (VI-2) with V chosen as the norm

(VI-14) $$V = \|X_k\|.$$

The difference

(VI-15)
$$\Delta V = \|X_{k+1}\| - \|X_k\|$$
$$= \|F(X_k)\| - \|X_k\|.$$

If for some norm, $\|F(X)\| < \|X\|$ then F is said to be a contraction. If F is a contraction it is obvious that ΔV as given by (VI-15) is negative, thus the system (VI-2) is asymptotically stable. The difficulty lies in determining a suitable norm for which it can be shown that F is a contraction. Some of the more commonly used norms are the following

$$\|X\|_T = \sum_{i=1}^{n} c_i |x_i| \qquad c_i \text{ positive}$$

$$\|X\|_S = \max_i c_i |x_i| \qquad c_i \text{ positive}$$

$$\|X\|_Q = (X^T Q X)^{1/2} \qquad Q \text{ positive definite.}$$

To illustrate how these may be applied consider the system (VI-2) in the special form

$$X_{k+1} = F(X_k) = A(X_k)X_k \,. \tag{VI-16}$$

As a Liapunov function consider

$$V = \|X\|_S = \max_i c_i |x_i(t_k)|. \tag{VI-17}$$

The difference

$$\Delta V(X_k) = \max_i c_i |x_i(t_{k+1})| - \max_i c_i |x_i(t_k)|$$

$$= \max_i c_i \left| \sum_{j=1}^{n} a_{ij} x_j \right| - \max_i c_i |x_i(t_k)|.$$

Let us examine the first term in this last equation

$$\max_i c_i |\Sigma\, a_{ij} x_j| \le \max_i c_i \sum_{j=1}^{n} |a_{ij}|\,|x_j| = \max_i \sum_{j=1}^{n} \frac{c_i}{c_j} |a_{ij}| c_j |x_j|$$

$$\le \max_i \sum_j \frac{c_i}{c_j} |a_{ij}| \cdot \max_j c_j |x_j|.$$

Thus $A(X_k)X(k)$ is a contraction if

$$\max_i \sum_j \frac{c_i}{c_j} |a_{ij}| \; \|X\|_S < \|X\|_S .$$

For this to be a contraction it is sufficient for

$$\text{(VI-18)} \qquad \max_i \sum_{j=1}^{n} \frac{c_i}{c_j} |a_{ij}(X_k)| < 1 \quad \text{for all } X$$

and if (VI-18) is satisfied (VI-16) is asymptotically stable. The only requirement on the numbers c_i is that they be positive.

This construction has given conditions on the rows of the matrix A sufficient to ensure asymptotic stability. Similar conditions on the column vectors of A can be obtained by considering

$$\text{(VI-19)} \qquad V = \|X\|_T = \Sigma\, c_i |x_i(t_k)| .$$

In terms of (VI-16) the difference becomes

$$\Delta V = \sum_{i=1}^{n} c_i |x_i(t_{k+1})| - \sum_{i=1}^{n} c_i |x_i(t_k)|$$

$$= \sum_{i=1}^{n} c_i \left| \sum_{j=1}^{n} a_{ij} x_j \right| - \sum_{i=1}^{n} c_i |x_i(t_k)| .$$

Examining the first of the above terms

$$\sum_{i=1}^{n} c_i \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} c_i |a_{ij}| |x_j| = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{c_i}{c_j} |a_{ij}| c_j |x_j|$$

$$\leq \left(\max_j \sum_{i=1}^{n} \frac{c_i}{c_j} |a_{ij}|\right) \sum_{j=1}^{n} c_j |x_j| .$$

For F to be a contraction we require

(VI-20) $$\max_i \sum_{i=1}^{n} \frac{c_i}{c_j} (a_{ij}) < 1.$$

Example VI-2. As an application of the above construction consider the system

$$x_1(t_{k+1}) = \frac{1}{3} x_1(t_k) - \frac{2}{3} x_2(t_k) + \frac{3}{5} f(x_2(t_k))$$

$$x_2(t_{k+1}) = -\frac{1}{3} x_1(t_k) + \frac{2}{3} x_2(t_k) - \frac{3}{10} f(x_2(t_k)).$$

For this system the matrix $A(X)$ takes the form

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} + \frac{3}{5} \frac{f}{x_2} \\ -\frac{1}{3} & \frac{2}{3} - \frac{3}{10} \frac{f(x)}{x_2} \end{pmatrix}.$$

For F to be a contraction it is sufficient to consider the column vectors for arbitrary positive c_i. This gives the two inequalities

$$\left|\frac{1}{3}\right| + \frac{c_2}{c_1} \left| -\frac{1}{3}\right| < 1$$

$$\frac{c_1}{c_2} \left| -\frac{2}{3} + \frac{3}{5} \frac{f}{x_2} \right| + \left|\frac{2}{3} - \frac{3}{10} \frac{f}{x_2} \right| < 1.$$

The first inequality is satisfied for

$$c_2 < 2c_1.$$

With $\frac{c_1}{c_2} < \frac{1}{2}$, the second inequality is satisfied for

$$0 < \frac{f(x_2)}{x_2} < \frac{10}{3} .$$

For all $f(x_2)$ satisfying this condition, the above system is asymptotically stable.

In the treatment of the stability of differential equations, it was found to be useful to have a construction for the linear problem since this became the basis for other constructions. This is equally true for the treatment of discrete systems. Consider the linear system

(VI-21) $$X_{k+1} = AX_k.$$

As a Liapunov function consider for V

$$V = \|X\|_Q^2 = X_k^T Q X_k.$$

The difference becomes

(VI-22)
$$\begin{aligned} \Delta V &= X_{k+1}^T Q X_{k+1} - X_k^T Q X_k \\ &= X_k^T A^T Q A X_k - X_k^T Q X_k \\ &= X_k^T [A^T Q A - Q] X_k = - X_k^T [P] X_k. \end{aligned}$$

As in the continuous case the question arises, is it possible for any given positive definite matrix P to construct a definite matrix Q such that

(VI-23) $$A^T Q A - Q = - P?$$

The answer is in the affirmative by considering the following: Define $Q(n)$ as

$$Q(n) = \sum_{k=0}^{n} (A^T)^n PA^n.$$

Then

$$A^TQ(n)A - Q(n) = \sum_{k=0}^{n} (A^T)^{n+1} PA^{n+1} - \sum_{k=0}^{n} (A^T)^n \; PA^n$$

$$= (A^T)^{n+1} PA^{n+1} - P.$$

If all eigenvalues of A are less than unity in absolute value then

$$\lim_{n \to \infty} Q(n) = Q$$

with

$$A^TQA - Q = - P.$$

In continuous systems one often approaches the problem of stability by examining either the linear part to determine the local behavior and then extending the analysis by means of the Zubov construction to obtain more complete results as to the region of asymptotic stability. Similar procedures can be applied to discrete systems since the Zubov construction is applicable. Thus consider the system (VI-2)

(VI-2) $$X_{k+1} = F(X_k).$$

The analogue of the partial differential equation to be solved is the difference equation

(VI-24) $$\Delta V(X_k) = V(X_{k+1}) - V(X_k) = - W(X_k)(1 - V(X_k)).$$

The principal results relating to the solution of (VI-25) and its relationship to the region of asymptotic stability is given by O'Shea[30] and may be summarized in the theorems that follow.

Theorem VI-3: If the linear approximation of (VI-2) is asymptotically stable, then for any positive definite quadratic form $W(X_k)$, equation (VI-24) has a solution $V(X_k)$ defined for all X in the domain of asymptotic stability.

The solution V is given by the converging infinite product

$$V(X_k) = 1 - \frac{1}{\prod_{n=k}^{\infty} (1 + W(X_n))} . \qquad \text{(VI-25)}$$

This may be obtained by dividing both sides of (VI-24) by $-(1 - V(X_k))$ and adding 1 to both sides

$$1 + \frac{V(X_{k+1}) - V(X_k)}{-(1 - V(X_k))} = 1 + W(X_k)$$

or

$$\frac{1 - V(X_{k+1})}{1 - V(X_k)} = 1 + W(X_k). \qquad \text{(VI-26)}$$

If we take the logarithm of both sides and sum $k = m$ to $k = m - n$ we obtain

$$\sum_{k=m}^{m+n} \text{Ln}(1 - V(X_{k+1}) - \text{Ln}(1 - V(X_k)) = \sum_{k=m}^{m+m} \text{Ln}(1 + W(X_k)$$

or

$$\text{Ln}(1 - V(X_{k+m+n+1})) - \text{Ln}(1 - V(X_m) = \sum_{k=m}^{m+n} \text{Ln}(1 + W(X_k)).$$

Since by hypothesis $V(X_n) \to 0$ as n increases to ∞, we obtain

(VI-27) $$- \ln(1 - V(X_m)) = \sum_{k=m}^{\infty} \ln(1 + W(X_k)$$

or finally

(VI-25) $$V(X_m) = 1 - \frac{1}{\Pi(1 + W(X_k))}.$$

The bound of the region of asymptotic stability is given by $V(X_k) < 1$. If for a given value of X_k we have $V(X_k) < 1$, this does not imply that X_k is in the region of asymptotic stability. This point is clarified by the following

Theorem VI-4: If $0 \leq V(X_k) < 1$ is in a simply connected region containing the origin, then any X in this region belongs to the domain of asymptotic stability and $V = 1$ is the boundary of this domain.

If $F(X)$ in equation (VI-2) is an analytic function, then a solution of (VI-24) may be obtained in terms of a Taylor series which converges in some domain about the equilibrium. The main disadvantage of this approach is that it is difficult to obtain the general term of the series so that its region of convergence may be determined. Both of these procedures suffer from the disadvantage that rarely does a closed form expression for V result from these constructions.

Example VI-3. (O'Shea[30]).

$$x_1(t_{k+1}) = x_1^2(t_k) - x_2^2(t_k)$$

$$x_2(t_{k+1}) = 2x_1(t_k)x_2(t_k).$$

Let

$$V(x_k) = x_1^2 + x_2^2.$$

Then the difference

$$\Delta V(X_k) = V(X_{k+1}) - V(X_k)$$
$$= x_1^2(t_{k+1}) + x_2^2(t_{k+1}) - x_1^2(t_k) - x_2^2(t_k)$$
$$= [x_1^2(t_k) - x_2^2)]^2 + (2x_1(t_k)x_2(t_k))^2 - x_1^2(t_k) - x_2^2(t_k)$$
$$= (x_1^2(t_k) + x_2^2(t_k))^2 - (x_1^2(t_k) - x_2^2(t_k))$$
$$= (x_1^2 + x_2^2)[1 - (x_1^2 + x_2^2)].$$

Thus the origin is asymptotically stable for $\|X\| < 1$. Therefore the above results are applicable. Let $W(X)$ in (VI-24) be the function $x_1^2(t_k) + x_2^2(t_k)$. We now wish to examine the expression $\prod_{n=k}^{\infty} (1 + W(X^n))$

$$\prod_{n=k}^{\infty} (1 + W(X_n)) = [1 + x_1^2(t_k) + x_2^2(t_k)][1 + x_1^2(t_{k+1}) + x_2^2(t_{k+1})]$$
$$\ldots \quad [1 + x_1^2(t_{k+n}) + x_2^2(t_{k+n})$$
$$= [1 + x_1^2 + x_2^2][1 + (x_1^2 - x_2^2)^2 + (2x_1x_2)^2] \ldots [1 + x_1^2(t_{k+n}) + x_2^2(t_{k+n})] \ldots$$
$$= [1 + x_1^2 + x_2^2][1 + (x_1^2 + x_2^2)^2][1 + (x_1^2 + x_2^2)^4] \ldots [1 + (x_1^2 + x_2^2)^{2n} \ldots .$$

If this product is multiplied out we obtain

$$\prod_{n=k}^{\infty} (1 + W(X_n)) = 1 + (x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2 + \ldots + (x_1^2 + x_2^2)^m + \ldots$$
$$= \frac{1}{1 + (x_1^2 + x_2^2)} \quad \text{for} \quad (x_1^2 + x_2^2) < 1.$$

Thus $V(X_k)$ becomes

$$V(X_k) = 1 - \frac{1}{\pi(1 + W(X_k)}$$
$$= 1 - \frac{1}{\dfrac{1}{1 - (x_1^2 + x_2^2)}} = x_1^2 + x_2^2.$$

Therefore the complete region of asymptotic stability is given by

$$V = x_1^2 + x_2^2 < 1.$$

An alternate procedure for determining the region of asymptotic stability is based on the following: Consider the system (VI-2)

$$X_{k+1} = F(X_k). \qquad \text{(VI-2)}$$

Assume that there exist a "V" function which is positive definite throughout the whole space and such that ΔV is negative definite for $\|X\| < s$. Let C be the minimum value of $V(X)$ for $\|X\| = s$. Then we have that the origin is asymptotically stable for all X such that $V(X) = C$. Thus the curve $V(X) = C$ becomes the boundary of asymptotic stability given by this choice of V. The actual region of asymptotic stability may be much larger. We now consider a sequence of Liapunov functions as follows: Let

$$V_1(X_k) = V(F(X_k))$$
$$V_2(X_k) = V_1(F(X_k))$$
$$\vdots$$
$$V_n(X_k) = V_{n-1}(F(X_k)).$$

The functions V_n are Liapunov functions which are positive definite and are such that the region of asymptotic stability is given by $V_n(X_k) = C$. Thus we can iterate and remap the boundary of the region of asymptotic stability. If the function $F(X(t_k))$ is a contraction in the norm $V(X_k) = X^TQX$, then the procedure will expand the boundary at each step, and we have the set S_n contained in the set S_{n-1} for all n where S_n is the set of all X such that $V_n(X) \leq C$. If F is not a contraction in our original norm we have no assurance than S_n contains S_{n-1}, but we may still improve on the region of asymptotic stability by taking the union of these regions. To illustrate this procedure which is based on the inverse mapping technique due to O'Shea[31] consider the following exampl

Example VI-4 (O'Shea[31]).

$$x_1(t_{k+1}) = x_2^2(t_k)$$
$$x_2(t_{k+1}) = x_1^2(t_k).$$

As an initial choice for V consider $V = x_1^2 + x_2^2$. Then ΔV is given by

$$\begin{aligned}\Delta V &= x_1^2(t_{k+1}) + x_2^2(t_{k+1}) - x_1^2(t_k) - x_2^2(t_k)\\ &= x_2^4(t_k) + x_1^4(t_k) - x_1^2(t_k) - x_2^2(t_k)\\ &= - x_1^2(1 - x_1^2) - x_2^2(1 - x_2^2).\end{aligned}$$

Thus ΔV is negative definite for $\|X\| < 1$. The region of asymptotic stability is given by

$$V = x_1^2(t_k) + x_2^2(t_{k+1}) < 1.$$

Its boundary becomes

$$V(X_k) = 1.$$

If we now apply the mapping procedure we have that this boundary is after one iteration

$$\begin{aligned}V_1(X_k) &= V(F(X_k) = x_2^4 + x_1^4 = 1\\ V_2(X_k) &= V_1(F(X_k)) = x_1^8 + x_2^8 = 1\\ &\vdots\\ V_n(X_k) &= V_{n-1}(F(X_k)) = x_1^{2^{n+1}} + x_2^{2^{n+1}} = 1.\end{aligned}$$

In the limit as n increases without bound the region given by $V_n(X_k) = 1$ approaches the square which circumscribes the unit circle. Thus we have asymptotic stability for all X such that $\|X\| = \max |X_i| = 1$. This mapping may be applied to regions of instability as well as to regions of stability.

Example VI-5. (O'Shea[31])

$$x_1(t_{k+1}) = x_1^2(t_k) + x_2^2(t_k)$$
$$x_2(t_{k+1}) = x_1(t_k).$$

Let

$$V = x_1^2(t_k) + x_2^2(t_k).$$

Then ΔV is given by

$$\begin{aligned}\Delta V &= x_1^2(t_{k+1}) + x_2^2(t_{k+1}) - x_1^2(t_k) - x_2^2(t_k) \\ &= (x_1^2(t_k) + x_2^2(t_k))^2 + x_1^2(t_k) - x_1^2(t_k) - x_2^2(t_k) \\ &= (x_1^2 + x_2^2)^2 - x_2^2.\end{aligned}$$

For $x_1^2 + x_2^2 > 1$ ∇V is positive definite and we have instability. The boundary of this region is

$$V = 1.$$

If the preceding mapping is applied we have

$$\begin{aligned}V_1(X_k) &= V_o(F(X_k)) = (x_1^2 + x_2^2)^2 + x_1^2 = 1 \\ V_2(X_k) &= V_1(F(X_k)) = ((x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2)^2 \\ &= (x_1^2 + x_2^2)^4 + (x_1^2 + x_2^2)(2x_1^2 + 1) + x_1^4 = 1.\end{aligned}$$

Consider as a second choice of a Liapunov function the following

$$W(X_k) = 2x_1^2 + x_2^2.$$

Then

$$\begin{aligned}\Delta W(X_k) &= 2x_1^2(t_{k+1}) + x_2^2(t_{k+1}) - 2x_1^2(t_k) - x_2^2(t_k) \\ &= 2(x_1^2 + x_2^2)^2 + x_1^2 - 2x_1^2 - x^2 \\ &= x_1^2[2(x_1^2 + x_2^2) - 1] + x_2^2[2(x_1^2 + x_2^2) - 1]\end{aligned}$$

for $x_1^2 + x_2^2 < \frac{1}{2}$. ΔW is negative definite, thus we have asymptotic stability for all X_k such that

$$W = 2x_1^2 - x_2^2 < \frac{1}{2}.$$

The boundary of the region of asymptotic stability is given by $W = \frac{1}{2}$. It is obvious that the complete region of asymptotic stability has its boundary between $W = \frac{1}{2}$ and $V = 1$. If we apply the above mappings we have that this boundary lies in the space between $W_n = \frac{1}{2}$ and $V_n = 1$.

The concept of boundedness or Lagrange stability has been extended to sampled data systems by Pearson[32]. Once again the appropriate definition only requires a small change in language from those for the continuous case.

<u>Definition</u> VI-3: A discrete system (VI-2) is said to be bounded if for every $\alpha > 0$ there exists a $\beta(\alpha) > 0$ such that if $\|X_k\| < \alpha$ then $\|X_{k+n}\| < \beta$ for all n.

Definition VI-4: A discrete system (VI-2) is said to be ultimately bounded if (VI-2) is bounded region Ω containing the origin such that all solution sequences approach Ω asymptotically as $k \to \infty$.

The corresponding theorems for the relation between Liapunov functions and boundedness are

Theorem VI-5: Let Ω be the set given by $\|X_k\| > r$ and assume that there exists a function $V(X_k)$ which is positive definite in Ω and such that $V(X_k) \to \infty$ as $\|X_k\| \to \infty$. If the difference $\Delta V = V(X_{k+1}) - V(X_k)$ is negative semi-definite for all X in Ω then all solutions of (VI-2) are bounded.

Theorem VI-6: Let Ω be the same as above, if there exist a $V(X_k)$ which is definite in Ω while ΔV is definite of opposite sign then the solutions of (VI-2) are ultimately bounded.

The stability of discrete systems of the Lur'e type play an important role in analysis of many guidance systems. Such problems usually arise from a combination of continuous and discrete sugsystems. Thus many practical systems are described by the equations

$$\text{(VI-28)} \qquad \begin{aligned} \dot{X} &= A_1 X + Bf_1(\sigma_k) \\ \sigma &= C^T X_k \end{aligned}$$

where in general the equation for σ_k comes from the digitized guidance loop. This system takes the discrete form

$$\begin{aligned} X_{k+1} &= e^{A_1 T} X_k + [\int_0^T e^{A(T-s)} ds] B_1 f(\sigma_k) \\ \sigma_k &= C^T X_k . \end{aligned}$$

If we define the matrix $A = e^{A_1 T}$ and $B = \int_0^T e^{A(T-s)} ds \; B_1$ then we have the discrete system

$$\text{(VI-29)} \qquad \begin{aligned} X_{k+1} &= AX_k + Bf(\sigma_k) \\ \sigma_k &= C^T X_k \\ \sigma_{k+1} &= C^T AX_k + C^T Bf(\sigma_k), \end{aligned}$$

which becomes the Lur'e problem of direct control in discrete form. One major difference between the discrete and continuous problem must be emphasized. In the continuous case the problem was to obtain conditions for asymptotic stability for which the only restrictions on $f(\sigma)$ were

$$\sigma f(\sigma) > 0 \quad \int_0^\infty f(\sigma)d\sigma \to \infty.$$

For the discrete system the same conditions can not be used for $f(\sigma_k)$ since every function $f(\sigma_k) = r\sigma_k$ is in the above region for $r > 0$. In the closed loop discrete case we will never have stability for all gains, therefore we must restrict $f(\sigma)$. Thus in (VI-29) we assume that $f(\sigma)$ is such that

$$f(0) = 0 \quad 0 \leq \sigma_k f(\sigma) \leq k_m \sigma_k^2.$$

In addition it is assumed that the curve $f(\sigma)$ is differentiable and such that $\left|\frac{df(\sigma)}{d\sigma}\right| \leq \mu$. As a Liapunov function consider the form

(VI-30) $$V = X_k^T Q X_k + d \int_0^{\sigma_k} f(s)ds.$$

The difference ΔV along solutions of (VI-29) are

(VI-31) $$\Delta V = X_k^T [A^T QA - Q]X_k + f(\sigma_k)B^T QAX_k + X_k^T A^T QBf(\sigma_k) + f^2(\sigma_k)B^T QB + \int_{\sigma_k}^{\sigma_{k+1}} f(s)ds.$$

If we apply the mean value theorem to the last term in equation (VI-31) we obtain

$$\text{(VI-32)} \qquad \int_{\sigma_k}^{\sigma_{k+1}} f(s)ds \leq f(\sigma_k)[\sigma_{k+1} - \sigma_k] + \frac{\mu}{2}(\sigma_{k+1} - \sigma_k)^2$$

thus we have

$$\text{(VI-33)} \qquad \begin{aligned} \Delta V \leq & - X^T P X + f(\sigma_k) B^T Q A X_k + X_k^T A^T Q B f(\sigma_k) + f^2(\sigma_k) B^T Q B \\ & + d f(\sigma_k)[\sigma_{k+1} - \sigma_k] + \frac{\mu}{2}(\sigma_{k+1} - \sigma_k)^2. \end{aligned}$$

If the identity $-\alpha[\sigma_k f(\sigma_k)) - C^T X_f f(\sigma_k)]$ is added to the above expression, we obtain after much algebraic manipulation

$$\text{(VI-34)} \qquad \Delta V \leq - X_k^T R X_k + f(\sigma_k)[D^T X + X^T D] - S f^2(\sigma_k) - \alpha f(\sigma_k)[\sigma_k - \frac{1}{k_m} f(\sigma_k)]$$

where the matrix R, the vector D and the scalar S are defined as

$$\text{(VI-35)} \qquad R = P - \frac{d\mu}{2}(A^T - I) C C^T (A - I)$$

$$\text{(VI-36)} \qquad D^T = B^T Q A - \frac{d\mu}{2} B^T C C^T - \frac{\alpha C^T}{2} + \frac{d\mu}{2} B C C^T A + \frac{\alpha C^T A}{2} - \frac{\alpha C^T}{2}$$

$$\text{(VI-37)} \qquad S = \frac{\alpha}{k_m} - B^T Q B - \alpha C^T B - \frac{d\mu}{2} B^T C C^T B^T$$

where $A^T Q A - Q = - P$. If the constants d and α can be chosen such that R is positive definite, S is greater than zero and $S - D^T R^{-1} D \geq$ zero, then the right hand side is at least negative semi-definite, and thus the origin is asymptotically stable.

Observe that one still has the basic problem of how to choose the matrix P in order to maximize the region of stability in terms of the parameter space. Also observe that the maximum slope of $f(\sigma)$ had to be bounded. If this assumption is relaxed, then one has difficulty in incorporating the term involving the integral of the nonlinearity in the Liapunov function. This restriction on $f(\sigma)$ can be overcome if the problem is treated as one of direct control. Consider once again the system (VI-29)

$$X_{k+1} = AX_k + Bf(\sigma_k) \qquad \text{(VI-29)}$$

$$\sigma_k = C^T X_k$$

where we assume $0 < \sigma f(\sigma) < k\sigma^2$. Consider for V

$$V = X^T QX. \qquad \text{(VI-30)}$$

Then the difference of V along the solutions of (VI-29) becomes

$$\Delta V = X^T [A^T Q A - Q] + f(\sigma_k)[B^T QAX_k + X_k^T A^T QB] + B^T QBf^2(\sigma_k). \qquad \text{(VI-30)}$$

If the two quantities $-(\sigma_k f(\sigma) - C^T Xf(\sigma_k)$ and $\pm \frac{f^2(\sigma)}{k}$ are added to (VI-30) we obtain

$$\Delta V = X^T [A^T QA - Q] + f(\sigma)[(B^T QA + \frac{C^T}{2})X_k + X_k^T (A^T QB + \frac{C}{2}] \qquad \text{(VI-31)}$$

$$- [\frac{1}{k} - B^T QB]f^2(\sigma) - (\sigma - \frac{f(\sigma)}{k})f(\sigma_k)$$

Since A is assumed to have eigenvalues which are less than unity in absolute value, then (VI-31) takes the form

(VI-32) $$V = - Y_k^T[R]Y_k - f(\sigma_k)(\sigma_k - f\frac{(\sigma_k)}{k})$$

where $Y_k^T = [X_k^T, f(\sigma_k)]$

$$R = \begin{bmatrix} P & -(A^TQB + \frac{C}{2}) \\ -(A^TQB + \frac{C}{2})^T & (\frac{1}{k} - B^TQB) \end{bmatrix}$$

The origin is asymptotically stable if R is neg tive definite. This requires

$$\frac{1}{k} > B^TQB \quad \text{and} \quad \frac{1}{k} - B^TQB - (A^TQB + \frac{C}{2})^T P^{-1}(A^TQB + \frac{C}{2}) \geq 0.$$

This last result could have been obtained directly from the equations (VI-35), (VI-36) and (VI-37) by choosing d to be zero.

CHAPTER VII

MODERN CONTROL APPLICATIONS OF THE DIRECT METHOD

The role of the direct method in the synthesis of linear and nonlinear control systems has been greatly overshadowed by its success as an analysis tool. In actuality its use in synthesis is in many respects the easier of the two problems. In the analysis of a nonlinear system, the system is well defined and it is necessary to seek out a Liapunov function which demonstrates its stability properties. Such functions can be very illusive. In the area of synthesis, one may choose almost any function at random as long as it is definite. The requirement that its derivative be definite of opposite sign automatically places restrictions upon the parameters of the system. Unfortunately these restrictions may pose considerable problems of mechanization. An alternative choice of the "V" function may lead to very simple mechanization. The inability to relate such requirements à prori in the choice of the "V" function has posed the large problems in synthesis. Some of the results from optimal control theory should mitigate these difficulties.

The vast majority of control systems designed to date are based upon a ruthless linearization of all encountered nonlinearities. Linear systems are understood by most engineers and they can relate non-mathematical performance criteria such as peak overshoot, natural frequency, etc., to their linear analyses. When they first encounter the direct method much of their intuitive feel is lost. This limitation is more a limitation of nonlinear systems rather than a particular limitation of the direct method.

Even with these admitted difficulties, the direct method when treated as a philosophical approach or point of view leads to many useful designs. Its relation to the fundamental concepts of optimal and adaptive control theory are too intimate to be ignored.

A. <u>Control based on negative</u> $\dot{V}$. The problem of synthesis takes the form

$$\dot{X} = F(X, U) \tag{VII-1}$$

where X is assumed to be an n-vector and U an m-vector. U represents the control variables. The object of the synthesis problem is to choose $U = U(X, t)$ as a function of the state variables and or time, such that the system, in addition to being stable, performs in some desirable manner. In many cases the control variables U will be restricted in some manner, for example, the norm may be required to be less than a given amount.

For many such problems the synthesis procedure may be as simple as constructing a Liapunov function and choosing the control U to make $\dot{V}$ as negative as possible. For example consider the linear system

<u>Example</u> VII-1.

$$\dot{X} = AX + BU \tag{VII-2}$$

where it is assumed that $|U_i| < 1$ and that the system is completely controllable. This last restriction is required to insure that the system can be stabilized. We do not assume that it is necessarily stable. Consider as a Liapunov function the positive definite quadratic form

$$V = X^T QX.$$

Then for $\dot{V}$ we have

$$\dot{V} = X^T[A^TQ + QA]X + U^TB^TQX + X^TQBU.$$

Therefore we wish to choose U in such a manner that $\dot{V}$ is as negative as possible. Such a choice gives

(VII-3) $$U = -\,\mathrm{Sgn}(B^TQX)$$

which is a bang-bang controller. If A is not stable, then this gives a stable operation near the equilibrium position. One still has a degree of freedom in the choosing of the matrix Q. If the original uncontrolled system is either stable or asymptotically stable, then U can be chosen to not only stabilize but to meet auxiliary performance criteria. Such procedures are applicable to nonlinear systems. Consider the problem of a tumbling space vehicle.

Example VII-2. The equations are given by

(VII-4)
$$I_x\dot{p} + (I_y - I_z)qr = T_1 + m_{12}T_2 + m_{13}T_3$$
$$I_y\dot{q} + (I_z - I_x)pr = m_{12}T_1 + T_2 + m_{23}T_3$$
$$I_z\dot{r} + (I_x - I_y)pq = m_{13}T_1 + m_{23}T_2 + T_3.$$

The numbers m_{ij} represent the misalignment in the application of the thrust vector. We assume that the components of thrust are bounded where without loss in generality we assume $|T_i| \le 1$. Since the above system without control is a conservative system with a stable equilibrium position, we can construct a V function from the integrals of motion. Let V be such a function

(VII-5) $$V = I_xp^2 + I_yq^2 + I_zr^2.$$

If we represent the original system in vector notation we have

$$\dot{X} = F(X) + Q^{-1}MU$$

where Q^{-1} in the inverse of the diagonal inertia matrix, M is the misalignment matrix. Observe that V is

$$V = X^T Q X.$$

Thus for $\dot{V}$ we obtain

$$\dot{V} = [F^T QX + X^T QF + X^T MU + U^T MX$$

but Q was chosen such that $F^T QX + X^T QF \equiv 0$. Therefore we have

(VII-6) $$\dot{V} = 2\, U^T MX.$$

If U is chosen such that

$$\bar{U} = -\operatorname{sgn}\left(\frac{MX}{2}\right),$$

$\dot{V}$ is negative definite and we have asymptotic stability. This control gives for the components of U

(VII-7)
$$T_1 = -\operatorname{sgn}(p + c_1{}^2 q + c_{13} r)$$
$$T_2 = -\operatorname{sgn}(c_{12} p + q + c_{23} r)$$
$$T_3 = -\operatorname{sgn}(c_{13} p + c_{23} q + r).$$

Once again we obtain an asymptotically stable system with a bang-bang control. If the problem discussed above in example 2 is such that there are no constraints upon the control vector, then not only can the thrust be chosen such that the origin is asymptotically stable, but one can generate a linear control with exponential stability. Thus consider

Example 2 - continued.

$$\dot{X} = F(X) + Q^{-1} Mu$$

where Q is the moment of inertia matrix and M represents the thrust misalignment matrix. Once again we consider V of the form

$$V = X^T QX$$

and $\dot{V}$ takes the form

$$\dot{V} = [F^T Q + QF] + X^T MU + U^T MX$$
$$= X^T MU + U^T MX = 2X^T MU.$$

Now if U is chosen as

$$U = - kM^{-1}QX$$

we obtain

$$\dot{V} = - 2kX^T QX = - 2kV.$$

Thus we have a linear feedback control which is asymptotically stable. If we examine the above equation we obtain

$$\dot{V} = - 2kV$$
$$V = V_o e^{- 2kt}.$$

Thus we have

$$\|X(t)\|_Q^2 = \|X(o)\|_Q^2 \, e^{- 2kt}$$

or

$$\|X(t)\|_Q = \|X(o)\|_Q e^{- kt}$$

and all solutions of the nonlinear system have exponential asymptotic stability. Other choices of U could be made to give almost any desired type of response.

Not only may the use of the direct method lead to greater stability, but it may also improve the performance of a system. This procedure as shown by LaSalle[33] is as follows. Consider the control system

$$\dot{X} = AX + BU \tag{VII-2}$$

where A is assumed stable. As a cost function or performance index consider the following

$$\mathcal{P}(X, U) = \int_0^\infty (X^T PX + U^T CU)ds. \tag{VII-8}$$

As a Liapunov function consider the quadratic form $V = X^T QX$ where Q is such that $A^T Q + QA = - P$. Without the application of control we have

$$\dot{V} = - X^T PX. \tag{VII-9}$$

Integrating we obtain

$$\begin{aligned} \int_0^\infty \frac{dV}{ds} ds &= - \int_0^\infty X^T PXds \\ V\Big|_0^\infty &= - \int_0^\infty X^T PX\, ds. \end{aligned} \tag{VII-10}$$

Since the uncontrolled system is asymptotically stable (VII-10) reduces to

$$V(0) = + \int_0^\infty (X^T PX)ds = \mathcal{P}(X, 0). \tag{VII-11}$$

For the controlled system using the same V function we have for $\dot{V}$

(VII-12) $$\dot{V} = - X^T PX + U^T B^T QX + X^T QBU.$$

Let R be any positive definite symmetric matrix and define the relation between U and X as

$$- \frac{RU}{2} = B^T QX \quad \text{or} \quad U = - 2R^{-1} B^T QX.$$

$\dot{V}$ in equation (VII-11) now takes the form

(VII-13) $$\dot{V} = - X^T PX - U^T RU.$$

Observe that $\dot{V}$ is more negative. Thus stability has been improved. Integrating (VII-13) we obtain

(VII-14)
$$\int_0^\infty \frac{dV}{ds} ds = V(\infty) - V(0) = - \int_0^\infty X^T PXds - \int_0^\infty U^T RUds$$

$$\text{Or} \quad \int_0^\infty X^T PXds = V(0) - \int_0^\infty U^T RUds.$$

The value of the performance order with control is

(VII-15)
$$\mathcal{P}(X, U) = \int_0^\infty X^T PX \, ds + \int_0^\infty U^T CU$$

$$= V(0) - \int_0^\infty U^T RU + \int_0^\infty U^T CU ds.$$

The difference in the value of the performance function becomes

(VII-16) $$\mathcal{P}(X, 0) - \mathcal{P}(X, U) + \int_0^\infty U^T (R - C)U \, ds.$$

Thus the value of the performance with control is less than that of the uncontrolled system providing R is chosen such that $R-C$ is positive definite. An obvious choice of R is $R = \lambda C$ with λ greater than unity.

B. Adaptive Control. In the area of adaptive control the second method of Liapunov may give insight as to the methods of approach. For model reference adaptive systems, the concept of adaption may alternatively be thought of as a stability problem. Since a control is sought to force the plant to follow the model, then the error between the plant and the model is required to be asymptotically stable. To illustrate this approach consider the following:

Example VII-3 (Rang[34]). Assume the object to be controlled is described by the equation

(VII-17) $$\ddot{x} + a_o\dot{x} + a_1 x = a_2 u(t)$$

where the quantities a_o, a_1 and a_2 are either constant or slowly varying, but in either case unknown. Assume a model of the form

(VII-18) $$\ddot{y} + b_o\dot{y} + b_1 y = b_2 f(t)$$

where it is assumed that the coefficients b_i are known and the system (VII-18) is asymptotically stable. It is desired to determine how to choose the control u such that (VII-17) is forced to respond like (VII-18).

Let $z = x - y$ be the error between the plant and the model. Thus z satisfies the equation

(VII-19) $$\ddot{z} + b_o\dot{z} + b_1 z = a_2 u - b_2 f + (b_o - a_o)\dot{x} + (b_1 - a_1)x.$$

If the control u was chosen as

$$\text{(VII-20)} \qquad u = \frac{1}{a_2}[b_2 f - (b_o - a_o)\dot{x} - (b_1 - a_1)x]$$

then the plant characteristics would be replaced by those of the model. Unfortunately we do not know the plant parameters, therefore we choose a controller of a similar form namely

$$\text{(VII-21)} \qquad u = g_2[b_2 f - (b_o - g_o)\dot{x} - (b_1 - g_1)x]$$

where the unknown functions g_o, g_1, and g_2 will be assumed to be functions of the error z and its derivative $\dot{z}$. We assume that they will be determined by integration as follows

$$\text{(VII-22)} \qquad \begin{aligned} \dot{g}_o &= h_o(z, \dot{z}) \\ \dot{g}_1 &= h_1(z, \dot{z}) \\ \dot{g}_2 &= h_2(z, \dot{z}) \end{aligned}$$

where we assume the functions $h_i(0, 0) = 0$. Ultimately we desire the functions g_i to approach the unknown plant characteristics a_i. Thus for the system given by the equations (VII-22) and (VII-19) we desire a Liapunov function given in terms of the desired equilibrium point $(x, \dot{x}, g_o, g_1, g_2) = (0, 0, a_o, a_1, \frac{1}{a_2})$. Thus consider a Liapunov function which is a quadratic form in the deviations of the variables from the equilibrium.

$$\text{(VII-23)} \qquad \begin{aligned} V = \frac{\dot{z}^2}{2} + \frac{b_o}{2} z\dot{z} + \frac{b_1}{2} z^2 + \frac{k_o}{2}(g_o - a_o)^2 \\ + \frac{k_1}{2}(g_1 - a_1)^2 + \frac{k_2 a_2}{2}(g_2 - \frac{1}{a_2}). \end{aligned}$$

Differentiating we obtain

$$\dot{V} = (\dot{z} + \frac{b_o}{2} z)\ddot{z} + \frac{b_o}{2} z^2 + b_1 z\dot{z} + \frac{k_o}{2}(g_o - a_o)\dot{g} + k_1(g_1 - a_1)g_1$$

(VII-24)
$$+ k_2 a_2 (g_2 - \frac{1}{a_2})\dot{g}_2 = - \frac{b_o}{2} (\dot{z}^2 + b_o z\dot{z} + b_1 z^2) + [\dot{z} + \frac{b_o}{2} z]$$

$$[a_2 u - b_2 f(b_o - a_o)\dot{x} + (b_1 - a_1)x] + k_o(g_o - a_o)h_o + k_1(g_1 - a_1)h_1$$

$$+ k_2(a_2 g_2 - 1)h_2.$$

If the identity

(VII-25)
$$\frac{u}{g_2} - [b_2 f - (b_o - g_o)\dot{x} - (b_1 - g_1)x] = 0$$

is added to (VII-24) we obtain

$$\dot{V} = - \frac{b_o}{2} (\dot{z}^2 + b_o z\dot{z} + b_1 z^2) + (g_o - a_o)[k_o h_o + \dot{x}(\dot{z} + \frac{b_o}{2} z)]$$

(VII-26)
$$+ (g_1 - a_1)[k_1 h_1 + x(\dot{z} + \frac{b_o}{2} z)] + (a_2 g_2 - 1)[k_2 h_2 + (\dot{z} + \frac{b_o}{2} z)\frac{u}{g_2}]$$

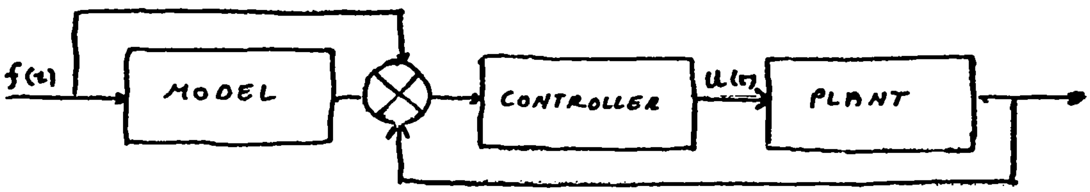
Thus $\dot{V}$ will be negative semi-definite if we set the coefficients of $(g_o - a_o)$, $(g_1 - a_1)$, and $(a_2 g_2 - 1)$ equal to zero. This gives the following

(VII-27)
$$h_o = - \frac{\dot{x}}{k_o} (\dot{z} + \frac{b_o}{2} z)$$

(VII-28)
$$h_1 = - \frac{x}{k_1} (\dot{z} + \frac{b_o}{2} z)$$

(VII-29)
$$h_2 = - \frac{u}{g_a k_2} (\dot{z} + \frac{b_o}{2} z).$$

If the term f(t) is kept nonzero, then the system will eventually reach the desired equilibrium position. The mechanization of the system is given by the block diagram



This scheme does not necessarily lead to a good adaptive system since it is rather slow and does not follow rapidly changing plant characteristics. Another disadvantage is that even though it can be extended to systems of higher order, it cannot be used for systems with zeros in its transfer function.

An alternate procedure for the solution of the same problem under different hypotheses was presented by Grayson[35]. The equation for the plant and modes remain the same. The difference equation (VII-19) is

(VII-19) $$\ddot{z} + b_0\dot{z} + b_1 z = a_2 u - b_2 f + (b_0 - a_0)\dot{x} + (b_1 - a_1)x.$$

The control u is assumed to be of the form

(VII-21) $$u = (1 - g_2)f - g_0\dot{x} - g_1 x.$$

If the V function

(VII-30) $$V = \frac{1}{2} z^2 + \frac{1}{2} b_0 z\dot{z} + \frac{1}{2} b_1 z^2$$

is differentiated we obtain

$$\dot{V} = (\dot{z} + \frac{b_o}{2} z)\ddot{z} + \frac{b_o}{2} \dot{z}^2 + b_1 z\dot{z}$$

$$= (\dot{z} + \frac{b_o}{2} z)[- b_o\dot{z} - b_1 z + a_2 u - b_2 f + (b_o - a_o)\dot{x} + (b_1 - a_1)x] + \frac{b_o}{2} \dot{z}^2 + b$$

$$= - \frac{b_o}{2} V + (\dot{z} + \frac{b_o}{2} z)[f[a_2(1 - g_2) - b_2)] + (b_o - a_o - a_2 g_o)\dot{x} + (b_1 - a_1 - a_2 g_1$$

Now if g_2, g_1 and g_o are chosen to ensure $\dot{V}$ is negative then we have asymptotic stability. If the plant parameters are limited and cannot vary too widely from the model, and a_2 is nonzero, then the following inequalities hold

$$|a_o - b_o| < c_o$$

$$|a_1 - b_1| < c_1$$

$$|a_2 - b_2| < c_2$$

$$0 < \frac{1}{a_2} < c_3$$

and the quantities g_i can be chosen as follows

(VII-31) $$g_1 = c_3 c_1 \operatorname{sgn} x(\dot{z} + \frac{b_o}{2} z)$$

(VII-32) $$g_o = c_3 c_o \operatorname{sgn} \dot{x}(\dot{z} + \frac{b_o}{2} z)$$

(VII-33) $$g_2 = c_3 c_2 \operatorname{sgn} f(\dot{z} + \frac{b_o}{2} z).$$

An advantage of this last procedure is that it permits a system to track a plant which is changing rather rapidly; in addition it is not restricted to systems with no zeros in the transfer function but can be generalized to such cases. Hiza and Li[36] have extended this approach to the case where the model is of one order lower than the plant. The main disadvantage to this approach is that one requires á priori bounds upon the variation of the parameters, but for most practical systems these are reasonable restrictions.

C. Optimal Control. Returning to the problem of optimal control, one often may use the Zubov construction to give a Liapunov function in terms of the performance index. Under suitable conditions this partial differential equation treated as a function of u may be minimized thus permitting U to be solved in terms of the gradient of V. Thus given the system

(VII-33) $$\dot{X} = F(X, t, U)$$

with a cost function of the form

$$\mathcal{P} = \int_t^\infty L(X, U, s)ds$$

then a Liapunov function V is sought such that

(VII-34) $$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial t} + \nabla V^T \cdot F(X, t, U) = - L(X, U, t).$$

The quantity to be minimized with respect to U is the function

(VII-35) $$\min_U \{\frac{\partial V}{\partial t} + \nabla V^T F(X, t, U) + L(X, U, t)\}.$$

Equation (VII-36) for U = U optimal becomes the Hamilton-Jacobi partial differential equation of optimal control theory. If L is positive definite

$$F(0, 0, t) = 0, \quad L(0, 0, t) = 0,$$

then the existence of a solution to (VII-35) implies asymptotic stability.

Example VII-4. Consider the optimal control problem due to Kalman[37].

(VII-36) $$\dot{X} = AX + BU$$

(VII-37) $$\mathcal{P} = \int_t^{\infty} [X^T P X + U^T R U] ds.$$

Since (VII-37) and (VII-38) are autonomous assume a V function which is autonomous. Its derivative becomes

(VII-38) $$\dot{V} = \nabla V^T [AX + BU]$$

but it is desired to choose V such that

(VII-39) $$\dot{V} = \nabla V^T [AX + BU] = - X^T P X - U^T r U$$

Thus for an optimum solution we desire to find

$$\min_U \{ \nabla V^T [AX + BU] + X^T P X + U^T R U \}.$$

This minimization gives

(VII-40) $$U = \frac{- R^{-1} B^T}{2} \nabla V.$$

If V is assumed to be a quadratic form

$$V = X^T Q X$$

then

$$\nabla V = 2QX \quad \text{and} \quad U = - R^{-1} B^T Q X$$

and $\dot{V}$ becomes

$$\dot{V} = X^T[(QBR^{-1}B^T - A^T)Q + Q(A - BR^{-1}B^TQ)]X = - X^RPX - X^TQBR^{-1}B^TQX.$$

Thus we require

$$X^T[(- QBR^{-1}B^T + A^T)Q + Q(A - BR^{-1}B^TQ)] + P + QBR^{-1}B^TQ]X = 0.$$

This implies that Q must be a solution to

$$A^TQ + QA + P - QBR^{-1}B^TQ = 0.$$

When a optimization problem is such that one cannot solve (VII-34) and (VII-35), then various iteration schemes are avilable. Consider the system (VII-1)

(VII-1) $$\dot{X} = F(X, U)$$

and an associated performance index

(VII-41) $$= \int_t^\infty L\{X, U\}dt$$

where $L\{X, U\}$ is assumed to be positive definite. Assume that there exists a nominal control $U_n(X)$ which stabilizes (VII-1). Such a control can always be found if (VII-1) is linearized and for L we use the quadratic approximation. In terms of this solution an iterative procedure can be applied based upon the procedures described in Section A. Similar procedures were used by Aoki[38].

The major contributions of the direct method to the solution of practical control problems has been via an application of Lur'e construction and its extensions. This contribution would be much more extensive if it were not for the formidable number of computations one must resort to in lieu of any optimal way of choosing the matrix of the quadratic form. Let us reexamine the application of these construction techniques from the point of view of the computational requirements. The model for the problem of Lur'e in the discrete form points out what is involved. The equations have the representation

$$\dot{X} = AX + Bf(\sigma_k)$$
$$\sigma_k = C^T X_k.$$

The first step in such a solution is the transformation to discrete form. To accomplish this we require the two computations

$$(1) \qquad e^{AT} = A_1 \qquad\qquad (2) \quad \int_0^T e^{A(T-s)} ds B = B_1.$$

Once these computations are made the system then has the discrete representation

$$X_{k+1} = A_1 X_k + B_1 f(\sigma_k)$$
$$\sigma_k = C^T X_k.$$

Invariably at this stage, the matrix A_1 possesses one or more eigenvalues with absolute value equal to or greater than unity. Thus we are forced to stabilize this matrix before we proceed. This stabilization is accomplished by replacing the nonlinearity by the expression

$$f(\sigma_k) = k\sigma_k + g(\sigma_k)$$

where the value k must be chosen such that

$$A_2 = (A + kB_1C^T)$$

has all eigenvalues less than unity in absolute value. In addition one must ascertain the range of the parameter $k_m < k < k_M$ for which stability is insured. We now construct a V function of the form

$$V = X_k^TQX_k + d\int_0^{\sigma_k} g(s)ds$$

and obtain the difference ΔV in the form

$$\Delta V \leq -X^TRX_k + g(\sigma)[D^TX_k + x_kD] - Sg^2(\sigma_k) + \alpha g(\sigma_k)[\sigma_k - \frac{1}{k_M} g(\sigma_k)]$$

where

$$R = -(A^TQA_2 - Q) - \frac{d\mu}{2}[A_2^T - I]CC^T(A_2 - I)$$

$$D^T = B_1^T QA_2 - \frac{d\mu}{2} B_1^T CC^T - \alpha\frac{C^T}{2} + \frac{d\mu}{2} B_1CC^TA_2 + \frac{\alpha}{2} C^TA_2$$

$$S = \frac{\alpha}{k_M} - B_1^TQB_1 - dC^TB_1 - \frac{d\mu}{2} B_1^TCC^TB.$$

For asymptotic stability it is sufficient for the following to hold for some α and d.

(1) R to be positive definite

(2) S be positive

(3) $S - D^TR^{-1}D$ be positive.

Observe that to insure definiteness we must compute the matrix Q from the relation

$$A_2^T Q A_2 = -P.$$

Once Q_1, α and d are specified the resulting quantities R_1, D^T, and S can be computed along with a check of the above sufficient conditions for stability. Thus the major computations are

(1) $A_1 = e^{AT}$, $B_1 = \int_0^T e^{A(T-S)} ds \, B$

(2) find k_m and k_M with $k_m < k < k_M$ such that $(A_1 + kB_1C^T)$ is stable in the discrete sense

(3) for given P compute Q such that

$$A_2^T Q A_2 - Q = -P$$

(4) choose α and d such that R is positive definite, and $S - D^T R^{-1} D$ is equal to or greater than zero.

For continuous systems step one in the above computation can be eliminated. If the result of the above computation does not lead to a useful result then the values of either α, d or the matrix P must be modified and steps (3) and (4) repeated.

Fortunately the required computations for the above already exist in a computer program developed for NASA under Contract No. NAS2-1107 by Dr. R. E. Kalman and Mr. T. S. Englar. This program, entitled "An Automatic Synthesis Program for Optimal Filters and Control Systems", is primarily designed to implement the solution of the linear optimal control problem with a quadratic performance index whose solution was given by Kalman. This program accepts matrices as inputs up to a maximum dimension of fifteen by fifteen and is adequate to handle all of the anticipated problems of this nature.

A problem representing a seventh order discrete guidance parameter study was placed into the above described program. Unfortunately results were not available in time for inclusion in this report but will be made available in a separate communication at a later date.

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

From the material presented in the previous sections it is obvious that even though the second method is a powerful tool of analysis and bears many close ties with much of modern control theory, it has serious difficiencies when applied to realistic problems. These deficiencies result primarily from the inability to construct optimal "V" functions to determine regions in the parameter space which insure stability. A second major deficiency is the inability to relate to a given "V" function qualitative design goals.

Even with these limitations it must be realized that this is the only method to date to treat nonlinear systems. The above limitations will be overcome as familiarity with the second method is developed at the level of the working design engineer. Adequate computer programs will be developed as the need for such methods increase. From a practical point of view, it would appear that the direct method will have its largest role of application in the synthesis and analysis of discrete systems. For continuous systems, linear synthesis procedures coupled with computer simulation have been adequate for such a large percentage of systems that the role of the direct method has been only considered for the exceptional problem. For discrete systems the situation is somewhat different. Here there is not a long heritage of familiar procedures. As the discrete systems grow increasingly complex, the direct method will become the major analytic tool.

In the area of future development, it is felt that this should be primarily in the area of computer program development. Adequate procedures presently exist for generating "V" functions for any given problem, but for high order systems one must fall back upon computers to make any practical use of such procedures. A second problem is the determination of definiteness or indefiniteness of homogeneous functions of degree greater than the second. This is primarily a research task and is not amendable to computer solution.

Appendix:

During the first quarter of this program an extensive survey of the literature both domestic and translated was made. The results of this survey are presented here.

Bibliography on Liapunov's Second Method.

During the first three months of this program a survey of the literature on the theory and application of Liapunov's second method was made. The purpose of this survey was to obtain all applications to problems in the control field and optimistically form a table of appropriate "V" functions for different classes of problems. Unfortunately, this goal was not realized, since, other than the Lur'e problem, no general class of control problems has been documented with appropriate construction of "V" functions. Applications of the second method are plentiful, and methods of construction are also plentiful, but these methods still require considerable ingenuity of the user.

In this paper, we have tentatively classified the various papers into five different categories. This classification is somewhat arbitrary and many of the papers actually could be classified into many different categories. To review each paper is beyond the capabilities of the writer, but rather we will survey each category and indicate the more interesting of papers.

A. General Theory.

The basis of the Liapunov Second Method is contained in the memoir (6) which was first written in 1896. Although the results were known to mathematicians, interest in the second method was not generated in this country until about ten years ago. Texts have appeared within the past few years along with translations of the more significant books from Russia.

For a general treatment of the theory, without the requirement of a strong mathematics background, (4) is excellent, (3) gives the most detailed treatment of the subject and the natural extensions of the method with applications. (4) is also noteworthy for its complete bibliography. The texts (2), (7), and (8) translated from the Russians are more difficult to study due to the absence of matrix notation.

From an engineering point of view, most interest in the second method has been with respect to autonomous systems. For non-autonomous systems the concepts of stability become more complex because of the need for more types of stability and the construction of Liapunov functions become considerably more involved. (1) gives a detailed treatment of the various concepts of stability and their interrelations.

From an engineering point of view, one wishes to construct a "V" function which gives the strongest type of stability in the largest region. In applications, one often obtains one without the other. This problem is somewhat aleviated as a result of the extensions due to LaSalle (5). These extensions permit one to obtain in many cases conclusions on asymptotic stability from a Liapunov function which only insures stability.

In most applications, one, by use of the Liapunov second method, obtains sufficient conditions for stability. Converse theorems showing the necessity of the second method were obtained by Massera, Kurzweil and others (9), (10), (11), and (12).

B. General Construction of Liapunov Functions.

The construction of a Liapunov function for a specific nonlinear system presents a challenge to the designer. The problem may be separated into two parts (a) determining whether a given function is definite or indefinite, (b) assuming (a) has been solved, construct a function which is definite while its derivatives are definite of the opposite sign. The solution to problem (a) has not as yet been found. In general for quadratic forms, conditions for definiteness of the function can be stated in terms of the elements of the matrix of the quadratic form. In some cases scalar nonlinearities may be integrated to give an analogue of stored energy, this accounts for the wide spread use of V functions formed from the combination of these two. The most famous of such constructions is due to Lur'e which will be treated later. For many applications, particularly for conservative systems, Liapunov functions are formed from some of the integrals of motion. In physical cases these integrals correspond to the total momentum or total energy in the system.

Specific methods of construction other than the above have been proposed. In general none of these methods give a solution to the determination of the definiteness of a given function, but assuming that this problem is solvable, they do lead to methods of construction of "V" functions. The most noteworthy of these methods is the variable gradient method proposed by Gibson and Schultz (20). The method consists in the assumption of a vector ∇V, From this vector $\dot{V}$ can be formed by taking the dot product of ∇V with $\dot{X}$. The elements of ∇V are kept free with the restriction that the matrix $\frac{\partial}{\partial X} \nabla V$ is symmetric. This ensures that ∇V may be integrated independent of the path. The elements of ∇V are then chosen to force $\dot{V}$ to be at least negative semi-definite. The principle advantage is that it enables one to handle the specific non-linearity in the system.

Another approach to the construction of Liapunov functions was proposed by Zubov (8). This approach has been treated in detail in (17). (21), (22), and (23) use an approach which is similar in that it is based on the same differential equation, but the construction proceeds along different lines. The Zubov approach starts with the partial differential equation,

$$\nabla V^T F(X) = \Phi(X)[1 + F^T F][1 - V]$$

where Φ is a positive definite quadratic form. For this construction it is assumed that the original system is asymptotically stable. The Zubov construction does give the region of stability and an approximation scheme for its determination.

Many extensions of the concept of a positive definite quadratic form have been proposed. These usually proceed with a quadratic form for the linear system and then generalize it by permitting the elements to become functions of the state variables. The methods vary in the original matrix of the quadratic form and the manner in which this form is extended. (16) and (15) are the two main references to this approach.

An interesting approach from a geometric point of view was proposed by Infante and Clark (14), but unfortunately appears to be restricted to second order systems. The approach is based on constructing a function which is the integral of the original system modified by the addition of extra terms which are kept free. The cross product of the tangent vector of the original system with the tangent vector of the modified system is required to be negative.

Geiss and Reiss (19) advocate an approach based on starting with a form which is semi-definite and integrating this by parts using the differential equations to aid in the integration.

It is difficult to determine which of these methods is preferable in any given case since each requires a certain degree of ingenuity of the user. On some sample problems the writer obtained comparable results by all of the above methods. The only factor which would recommend one approach over another would be the ease of construction and this appears to be subjective. The author has found the variable gradient procedure the easiest to apply.

Papers (24) to (34) contain various applications of constructions of "V" functions consisting of quadratic forms and integrals of the nonlinearizaties. Papers (35) to (44) are constructions of functions from one or more first integrals of motion.

C. The Lur'e and Aizerman Problem.

The problem of Lur'e is perhaps the most completely documented nonlinear control problem. The basic problem may be stated as follows: given the system

$$\dot{X} = AX + B\delta$$
$$\sigma = C^T X - r\delta$$

where the equation for the controller is given in various forms:

a) Direct control $\quad \delta = F(\sigma) \quad r = 0 \quad F(0) = 0 \quad \sigma F(\sigma) > 0$

b) Indirect control $\quad \dot{\delta} = F(\sigma) \quad F(0) = 0 \quad \sigma F(\sigma) > 0.$

Obtain conditions on the vectors B and C to ensure asymptotic stability of the null solution. Lur'e (45) considered a "V" function of the form of a combination of a quadratic form plus an integral of the non-linearity. Before constructing the "V" function Lur'e first transformed the system to one in which A was assumed diagonal and the vector B consisted of all ones. Differentiating V he obtained $\dot{V}$ in the form

$$\dot{V} = - X^T RX + F(\sigma)Q^T X - rF^2(\sigma).$$

Conditions for asymptotic stability were obtained by setting the coefficient of $F(\sigma)$ to zero giving a set of algebraic equations which requires solution. If the quadratic form of the "V" function is assumed to be

$$V = X^T SX.$$

Then R is given by

$$- R = A^T S + SA.$$

Lur'e chose for the matrix R a particularly simple form, namely $R = \alpha\alpha^T$. The condition for stability thus becomes conditions on the elements α_i in the resolving equations. A detailed treatment of this problem may be found in (46) and (47). Lur'e assumed falsely that by his approach global asymptotic stability was insured, but LaSalle (63) demonstrated conditions under which this was true. The form of the resolving equations become quite complicated for systems of higher dimension, these having been tabulated by Rekasius (77) and Rozenvasser (78). Kalman (61) proves the conditions under which the resolving equations are solvable.

Lefschetz (64) and Yakubovich (83) independently recast the Lur'e problem in matrix notation and obtained simpler conditions for asymptotic stability. In the Lefschetz construction a single condition is given, this may be obtained as follows:

$$\dot{X} = AX + BF(\sigma) \qquad \dot{\sigma} = C^T X - rF(\sigma).$$

Let

$$V = X^T QX + \int_0^\sigma F(t)dt.$$

$$\dot{V} = X^T[A^T Q + QA]X + X^T[QB + \frac{C}{2}]F(\sigma) + F(\sigma)[B^T Q + \frac{C^T}{2}]X - rF^2(\sigma).$$

then

$$Y = \begin{pmatrix} X \\ F(\sigma) \end{pmatrix}$$

where

$$\dot{V} = -Y^T \begin{bmatrix} S & -\alpha \\ -\alpha^T & r \end{bmatrix} Y \qquad \begin{aligned} -S &= A^T Q + QA \\ \alpha &= QB + \frac{C}{2}. \end{aligned}$$

Thus for $\dot{V}$ to be negative definite we must have $r - \alpha^T S^{-1} \alpha > 0$. Since Q depends upon the choice of S, it is obvious that this inequality becomes a complicated expression in terms of the elements of S. It would be desirable to choose S such that $\alpha^T S^{-1} \alpha$ is minimized. Results in this direction have been reported by Morozan (70). This construction due to Lefschetz lends itself to the consideration of multiple nonlinearities with little loss. Application to the lateral equations of an aircraft with two nonlinearities has been treated by Chang Jen-Wei (56) and (55). Others have approached the Lur'e problem using a different matrix for the quadratic form giving simpler conditions for stability (51), (52), and (54).

The problem of Lur'e has been extended to the case where the controller equation has taken a more complicated form. Chang Ssu-Ying (57) assumes a controller of the form

$$a\ddot{\delta} + b\dot{\delta} + c\delta = F(\sigma).$$

Meyer (69), Maigerin (68) and Letov (65) consider a controller in which the effect of the load is taken into account. This gives the controller equation in the form

$$\dot{\delta} = F(\sigma)\phi(w) \qquad \phi(w) = \begin{cases} 1 & w \geq 1 \\ \sqrt{w} & 0 < w < 1 \\ 0 & w < 0 \end{cases}$$

where w may take various forms such as:

$$w = 1 - d\delta \operatorname{sgn}(\sigma)$$

$$w = 1 - (d\delta + f\dot{\delta})\operatorname{sgn}(\sigma).$$

Meyer (69) gives complete results for the non-critical case and partial results in the critical case. Rozenvasser (80) considers the case where the controller depends upon t while Gorbatenko (59) considers the case where $F(\sigma)$ is of the form $F(\sigma) = \Sigma k_i \sigma^i$. Yakubovich (82) considers a controller with hysteresis.

The Aizerman problem may be considered as a converse problem to the Lur'e problem. In the Lur'e problem one finds conditions under which asymptotic stability exists for arbitrary non-linearity in a given region. The Aizerman problem or conjecture may be stated as follows:

Let

$$\dot{X} = AX + F(\sigma)$$
$$\sigma = C^T X.$$

Assume that the origin is asymptotically stable for $F(\sigma) = k\sigma$ with $k_1 \leq k \leq k_2$ then is the origin asymptotically stable for all $F(\sigma)$ such that

$$k_1\sigma^2 \leq \sigma F(\sigma) \leq k_2\sigma^2.$$

In general the answer to this problem is in the negative although for some classes of problems it is true. The conjecture is verified for a class of third order systems by Bergen and Williams (53) and Pliss (73). Mufti (72) solves the Aizerman problem for the systems,

$$\dot{X} = aX + F(Y) \qquad\qquad \dot{X} = F(X) + aY$$
$$\text{and}$$
$$\dot{Y} = bX + cY \qquad\qquad \dot{Y} = bX + cY.$$

D. Problems in Optimal and Adaptive Control.

The relationship between Liapunov's functions and performance criteria was first suggested by Kalman and Bertram (93) where they advocated the use of positive definite performance criteria thus insuring asymptotic stability. LaSalle (99) shows how one by use of a Liapunov function may aid in the choice of a control that improves the stability and performance of a system. Kalman (94) has given the optimal solution of the linear system with quadratic performance index and has shown that this index is a Liapunov function. Al'Brekht (85) constructs a Liapunov function for the nonlinear system in terms of a series expansion assuming the solution to the linear problem. Aoki (86) proposes an algorithm to successively optimize the control system by constructing a sequence of approximations to a Liapunov function.

Applications of the Liapunov second method have been made to the area of model reference adaptive control. The main idea behind this application is that the eventual agreement between the learning model coefficients and the plant coefficients may be interpreted as a problem in asymptotic stability. Rang and Johnson (90) and (105) design a nonlinear continuous controller by means of a Liapunov function for a system with a single input where it is assumed that the model is of the same order as the system. Grayson (88) designs a similar system by use of a discontinuous control. Kiza and Li (89) extend Grayson's approach to a time varying plant and a model of lower order than the plant. Donalson and Leondes (87) who have poineered in the area of model reference systems apply the concept of eventual stability due to LaSalle and Rath (99) to analyse a class of adaptive systems.

The class of control problems discussed so far have been primarily continuous systems. Extensions of most of the preceding work has been made to discrete or sampled systems. Once again the main reference theoretically is due to Kalman and Bertram (93). Pearson (104) treats a discrete form of the Lur'e problem and obtains necessary conditions for the existence of a quadratic form Liapunov function. Kodama (96) and (97) obtain sufficient conditions for global asymptotic stability of a discrete version of the Aizerman type equation. Counter examples of Aizerman's conjecture are given. O'Shea (102) extends Zubov's method to discrete systems. Approximate methods of solving for the Liapunov function are described. Jury and Lee (91) describe a class of nonlinear sampled data systems of the Lur'e type. Kadota and Bourne (92) apply the construction of Liapunov functions to a class of systems of the pulse-modulated type.

E. Boundedness and Non-stationary Systems.

In many practical cases an engineer is willing to forego stability providing his solutions are bounded near the equilibrium point. This practical stability or Lagrange Stability as it is called is an outgrowth of the Liapunov theory. Boundedness in a sense is the stability of the point at infinity. The most detailed treatment of boundedness is due to Yoshizawa (113) and (114). Pliss (110) treats the boundedness of non-linear equations of third order. Rekasius (111) constructs Liapunov functions for control systems with step inputs to obtain boundedness.

The problem of constructing Liapunov functions for time varying systems becomes much more complicated. Part of the difficulty is in the construction of positive definite time varying matrixes, this problem is facilitated if the matrix is diagonal. One may start with diagonalizing the linear portion of the system, but this transforms the difficulty to the diagonalization of time varying systems. In general one starts with a Liapunov form based upon a non-time varying system and then tries to find bounds on the time variation to ensure negative definiteness of $\dot{V}$. Szegö (126) constructs a Liapunov function of the form $V = X^T(C(t)X$ where the elements of $C(t)$ are solved from the equation:

$$C(t)A + (C(t)A)^T = C(t)B(t) - \lambda(t)I$$

where $\lambda(t)$ is a scalar and $B(t)$ is assumed semi-definite. Roitenberg (125), Razumikhin (124), Chetaev (117) consider constant coefficient quadratic forms in order to obtain stability bounds. Pozharitskii (122) considers combinations of the first integrals of motion. Persidskii (121) gives conditions on $V(t, X)$ to obtain instability. Bhatia (115) gives a detailed treatment of stability of nonlinear time varying second order systems.

An area of much current research is the field of functional differential equations. Functional differential equations in many cases represent a more realistic approach to practical problems, since most systems have distributed parameters and contain lags. Unfortunately, the handling of such problems poses unsurmountable difficulties. Problems involving delays have been treated in the literature. For nonlinear systems results by examining the linearized delay equations have been treated by Razumikhin (123). Stability theorems for functional equations have been developed by Hale (128) and (129). Applications to partial differential equations have been made by Fowler (130) and Wang (131).

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PMM Prikladnaia Matematika I Mekhanika
AT Automatika I Telemekhanika

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