UNPUBLISHED PRELIMINARY DATA

A THEORETICAL STUDY OF NONLINEAR FUEL SLOSHING IN AN ELASTIC CIRCULAR TANK*

by

Wen Hwa Chu
Senior Research Engineer
Department of Mechanical Sciences
Southwest Research Insitute

Investigations on breathing vibrations of an elastic shell containing liquid were made in Refs. 1, 2, and 3. In Ref. 3, it was shown that linear theory and experiment for natural frequencies are in good agreement if enough terms are used to represent the mode shape.

In Ref. 4, nonlinear phenomena were reported. The amplitude frequency curve shows characteristics of softening. Aside from the jump phenomena, more complex response of the shell resembling beats was encountered. In the mean time, the liquid exhibits large amplitude oscillations with a mode shape different from linear fuel sloshing but at about the same low frequencies. For a better understanding of this phenomena, preliminary studies were made to determine the approximate steady-state response and the stability boundary, as follows.

A. Formulation of the Problem

A preliminary investigation has been made of the problem of fuel sloshing in a tank consisting of an elastic thin cylindrical shell, with the following basic assumptions:

(1) Linear shell theory; i.e., displacements of the middle surface are assumed small compared with the thickness of the shell, and strains are assumed directly proportional to stresses in the shell.

(2) Potential flow theory; i.e., the fluid is incompressible and nonviscous, and the flow is irrotational.

(3) Boundary conditions are expressed in terms up to the third order in liquid amplitudes, but include only terms of the first order in the amplitudes of shell displacement.

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Assumption (1) is based on experimental observation, of empty shell vibrations, which indicates very little of the nonlinear softening effect. Assumption (2) is made for the sake of mathematical tractability. Assumption (3) is made in order to allow large liquid amplitudes while the amplitudes of shell vibrations remain relatively small.

Since inclusion of the higher order terms would have greatly complicated the boundary condition equations, attention was limited to terms of only the first three orders in fluid amplitudes. Although it may be possible to consider an infinite series in fluid amplitudes, the effort required to derive corresponding boundary condition equations would be prohibitive, especially since these equations would still be restricted to the region of convergence of the infinite series.

B. Methods of Solution

Solutions to the class of problems defined above may be constructed in the following manner. By assumption (2) the fluid motion is derivable from a velocity potential which satisfies Laplace's equation throughout the volume occupied by the fluid. Exact solutions to the Laplace equation for a circular cylinder can be found by the method of separation of variables. Solutions which approximately satisfy the shell equations and the boundary condition equations can then be constructed, using the exact solutions to Laplace's equations obtained above, by the well-known method of Galerkin. Alternate methods of solution include perturbation techniques, in which each unknown is expanded in a power series of some small parameter, and other approximate methods (Ref. 5). Numerical methods would be limited to transient solutions, since steady solutions, which are the present objective, would be obscured by the accumulation of errors after many time steps. However, a seminumerical method could be used and seems to be promising.

C. A Specific Example

An upright cylindrical tank filled with a liquid to a depth equal to three-fourths its length has been considered. The actual vibrations of this tank-liquid system are extremely complicated, and therefore preclude exhaustive theoretical treatment. However, the need for qualitative theoretical results led to drastic simplifications as regards assumed vibration modes. Thus, even though the free-liquid-surface condition was expressed as an infinite series, the liquid modes were assumed to consist of one cos (4θ)-mode, one cos (8θ)-mode, and two axisymmetric modes, one of which is constant. Likewise, the shell was assumed to vibrate in a single breathing mode. A set of ordinary differential equations results from substituting these assumed modes into the general formulation. The steady state amplitudes can then be obtained approximately by the method of harmonic balance which is again limited to a few terms.

The stability equations are obtained by considering small perturbations from the steady state solution or by differentiation of the original equation. The small or differential disturbance was assumed to be in proportion to $e^{\lambda t}$ which leads to an eigenvalue problem for determining λ. The sign of the real parts indicates the stability: if any of the real parts are positive the oscillation is unstable.

The rank of the matrix can be reduced by the assumption that the oscillations are slowly varying, an assumption which can first be used to reduce the order of the differential equations. Thus, for the present problem, with the assumption of slowly varying amplitudes, a 6×6 matrix was obtained; without it a 16×16 matrix was obtained. The problem was further simplified to a 3×3 matrix for checking purposes. This 3×3 matrix was found to be unsatisfactory for predicting stability bounds, as its eigenvalues are stable for all three branches of its response curve.

D. Attempts to Determine the Eigenvalues

The following four computer programs were available for use in computing the eigenvalues:

(1) A Jacobi-like method for real variables obtained from the University of Rochester;

(2) The direct and inverse power method obtained from the University of Texas;

(3) Determination of roots after expansion into a polynomial, based on SwRI programs;

(4) A complex code of the Jacobi-like method, also from the University of Rochester.

Considerable difficulties were encountered with the first two programs, as the magnitudes and signs of the real parts were often uncertain, varying significantly with the number of iterations and with the convergence criteria. The methods were first tested on the simplified 3×3 matrix. The Jacobi-like method (1) failed using several trial numbers of iterations and refinements, although this program was very successful in previous applications. After several trials of both programs (1) and (2), using different numbers of iterations and different convergence criterions, the power method (2) finally yielded the correct answer, which was also obtained by the polynomial method (3).

Subsequent efforts to apply the power method to the 6×6 matrix were again unsuccessful, since the (small) real parts of the first eigenvalue continued to fluctuate in sign and magnitude with convergence criteria, and further this method failed to converge for other eigenvalues with these criteria.

The polynomial method (3) was next tried on the 6×6 matrix, but because instability was indicated on the stable branch of the response curve the results are somewhat doubtful. This method was also applied to the 16×16 matrix which somehow failed to converge and might require adjustment in the initial guesses. Also, there is some doubt, when using the polynomial method, as to the accuracy of the coefficients of the polynomial which are obtained from powers of the matrix.

For all the methods, the difficulty could well rest with the matrix being ill-conditioned, or due to a precision problem in the computations.

The eigenvalue problem can be written as $\lambda Ax - Bx = 0$. In attacking this problem, an attempt was made to improve the conditioning of matrices A and B by reducing the largest element of each column of A and B to order unity. This was done by absorbing different factors into the components of the eigenvectors. After modification, the Jacobi-like method (4) seemed to yield reasonable answers. However, for the real code, there is no checking program when the eigenvalues are complex. Unfortunately, eigenvalues of the 6×6 matrix indicated a very mild instability everywhere. Surprisingly, this instability remained mild even for quite large amplitudes of shell vibration at resonance. It is doubtful that discarding the assumption of slowly varying amplitude, as discussed in Section C, would lead to satisfactory results.

The complex code of the Jacobi-like method was used for one case, but the result was inaccurate due to large off-diagonal terms in the eigenvalue matrix. Numerical integration of six amplitude governing equations was then tried. The results are encouraging as divergence was computed for a nondimensional force equal to 0.002 at the approximate breathing natural frequency. Motion practically maintained its shape (over 40 cycles) for a nondimensional force equal to 0.0005 at a frequency of 40 cps above or below the natural frequency. The philosophy of this approach is that the truncation error would grow into significant values if the motion is unstable. Additional investigation of this approach will require additional funds not available at present. If additional funds are made available, numerical integration of the simplified stability equation could also be performed.

Finally, Routh's criterion (Ref. 6) and Hurwitz's criterion (Ref. 7) came to the attention of the writer. In these references, there is also a necessary condition for stability in that the coefficients of the polynomial governing the eigenvalues must be of the same sign. On the lower stable branch below resonance, the 3×3 and the 6×6 matrix all satisfy this condition, while the 16×16 matrix, without the assumption of slow varying amplitude, yields a result of doubtful precision. It seems that comparisons of the Hurwitz criterion with the method of numerical integration would be rewarding.

CONCLUSIONS

The nonlinear liquid motion in an elastic circular tank under going breathing vibrations is a laborious task. This task is amplified due to certain "ill-conditions" of the matrix, the real part of the eigenvalues of which determines the stability. So far, the computer programs available have failed to yield reliable eigenvalues. The method of numerical integration has been employed, but, unfortunately, additional funds will be required before the stability boundary can be determined.

In the mathematical analysis, the breathing problem is different in that dynamic terms are larger due to the high breathing frequency and that balance of the small constant terms led to appreciable time independent cos (0θ) and cos (8θ) components of the free surface elevation; that is, they are qualitatively in agreement with experimental observations.

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