6-2-64
NsG-444
1/p
[redacted]
NASA CR 56595
UNPUBLISHED PRELIMINARY DATA

A NOTE ON DIELECTRIC LENSES

P. L. E. Uslenghi
University of Michigan, Radiation Laboratory
Ann Arbor, Michigan

In the following we consider dielectric lenses with spherical or cylindrical symmetry, in which the index of refraction N is a function of the distance r from the center of the sphere or from the axis of the cylinder. Only the geometrical properties of optical ray paths are investigated; amplitude, phase and polarization of the electromagnetic wave are neglected. In the spherical case, the problem always reduces to the consideration of ray paths in a plane passing through the center of the sphere; in the cylindrical case, we shall limit ourselves to the case of rays lying in a plane perpendicular to the cylinder axis.

The following assumptions are made:

1) the radius of the sphere, or of the cylinder, is equal to unity;
2) the index of refraction $N(r)$ of the dielectric material relative to the surrounding medium is finite, except possibly at $r=0$, and $N(1) = 1$;
3) $N(r)$ is continuous with its first derivative in $0 < r < 1$.

The geometry of the problem is the same for both spherical and cylindrical lenses, and is illustrated in Fig. 1. An optical ray enters the lens at P_1 with an angle of incidence α, it is smoothly deviated along the path $P_1 P_2$, and leaves the lens at P_2. Let us indicate by R the optical length of the path $P_1 P_2$, by θ its angular extension as seen from the center $r = 0$, and by δ the deviation of the ray at P_2 from the direction of incidence; it is:

$$\delta = \theta + 2\alpha - \pi \quad . \qquad (1)$$

Following the usual procedure (see e.g. Toraldo di Francia, 1957 and Huynen, 1958), we set

$$r^2 N^2 = 1-\xi^2 \quad , \tag{2}$$

and introduce a new function $f(\xi)$ satisfying the relation:

$$2N^2 r\, dr = -f(\xi) d\xi \; . \tag{3}$$

Then it is:

$$R = \int_0^{\cos\alpha} \frac{f(\xi) d\xi}{\sqrt{\cos^2\alpha - \xi^2}} \quad , \tag{4}$$

$$\theta = \sin\alpha \int_0^{\cos\alpha} \frac{f(\xi) d\xi}{(1-\xi^2)\sqrt{\cos^2\alpha - \xi^2}} \quad , \tag{5}$$

$$\ln r = -\frac{1}{2} \int_0^{\xi} \frac{f(\eta) d\eta}{1-\eta^2} \quad . \tag{6}$$

Relations (2) and (6) give N as a function of r, in parametric form.

Formulas (4), (5) and (6) have been employed by various authors. Huynen (1958) has discussed the case in which

$$f(\xi) = a_0 + a_1 \xi \quad , \tag{7}$$

where a_0 and a_1 are constants.

In this note, the expressions for R, θ and $\ln r$ are derived for the case in which $f(\xi)$ is an analytic function of ξ in a neighborhood of $\xi = 0$. It is then pointed out that restrictions have to be placed on the coefficients of the power series representing $f(\xi)$, if one wants to have solutions with physical significance. In particular, the limitations on a_0 and a_1 for the simple case (7) are investigated in detail.

[redacted]

Let us split $f(\xi)$ into its even and odd components:

$$f(\xi) = f_e(\xi) + f_o(\xi) \quad , \tag{8}$$

where:

$$f_e(\xi) = \frac{f(\xi)+f(-\xi)}{2} \quad , \qquad f_o(\xi) = \frac{f(\xi)-f(-\xi)}{2} \quad . \tag{9}$$

Assume that for the range of values of ξ of interest to us, f_e and f_o may be represented by the uniformly convergent power series:

$$f_e(\xi) = \sum_{m=o}^{\infty} a_{2m}\xi^{2m} \quad , \qquad f_o(\xi) = \sum_{m=o}^{\infty} a_{2m+1}\xi^{2m+1} \quad , \tag{10}$$

where a_{2m} and a_{2m+1} are constants.

Substituting expansions (10) into relations (4), (5) and (6) and integrating term by term, it is found that

$$R = \frac{\pi}{2}\left\{a_o + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{m!\, 2^m} a_{2m}(\cos\alpha)^{2m}\right\} +$$

$$+ \sum_{m=o}^{\infty} a_{2m+1}(\cos\alpha)^{2m+1} \sum_{h=o}^{m} \frac{(-1)^h}{2h+1} \binom{m}{h} \quad , \tag{11}$$

$$\theta = \frac{\pi}{2} a_o + \left(\frac{\pi}{2} - \alpha\right) a_1 + \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{a_{2m}(\cos\alpha)^{2m}}{\sum\limits_{k=o}^{m} \binom{m}{k}(-\sin^2\alpha)^k} \left\{1 + \frac{1}{(m-1)!\, 2^{m-1} \sin\alpha} \times\right.$$

$$\left. \times \sum_{h=o}^{m-1} (2h-1)!!\,(2m-2h-3)!! \sum_{\ell=1}^{m-h} \binom{m}{h+\ell}(-\sin^2\alpha)^{\ell}\right\} +$$

$$+\sum_{m=1}^{\infty} a_{2m+1}(\cos\alpha)^{2m}\left\{(\frac{\pi}{2}-\alpha)\sum_{h=o}^{m}\binom{m}{h}(\tan\alpha)^{2m-2h}-\right.$$

$$\left.-\sum_{h=o}^{m-1}\sum_{\ell=1}^{m-h}\binom{m}{h}\frac{(-1)^{m-h-\ell}}{2(m-h-\ell)+1}(\tan\alpha)^{2\ell-1}\right\}, \quad (12)$$

$$\ln r^4 = a_o \ln\left|\frac{1-\xi}{1+\xi}\right| + a_1 \ln(1-\xi^2) + \sum_{m=1}^{\infty} a_{2m}\left\{\ln\left|\frac{1-\xi}{1+\xi}\right| + 2\sum_{h=1}^{m}\frac{\xi^{2h-1}}{2h-1}\right\} +$$

$$+\sum_{m=1}^{\infty} a_{2m+1}\left\{\ln(1-\xi^2)+2\sum_{h=1}^{m}\frac{\xi^{2h}}{2h}\right\}, \quad (13)$$

where

$$(2h-1)!! = 1x3x5x\ldots x(2h-1) \quad ,$$

is the semi-factorial of (2h-1) (in particular $(-1)!!=1$).

Huynen (1958) derived an expression equivalent to (13), for the case in which $f(\xi)$ is a polynomial of degree n in ξ ; he also gave the first few terms of expansions (11) and (12).

The coefficients a_{2m} and a_{2m+1} cannot be arbitrarily chosen; in fact, the angle θ must obviously be positive for the entire range $0 \leqslant \alpha < \frac{\pi}{2}$. Suppose that $f(\xi)$ is a polynomial of degree n in ξ , then the fundamental limitation on the choice of the coefficients is:

$$\theta(a_o, a_1, \ldots, a_n; \alpha) > 0, \text{ for all } 0 \leqslant \alpha < \frac{\pi}{2} \ , \quad (14)$$

i.e. the representative point $(a_o, a_1, \ldots, a_n)$ must belong to a certain region of the (n+1)-dimensional space $(a_o, a_1, \ldots, a_n)$.

From formulas (2) and (3) it is seen that the relationship between N and r remains unchanged when both ξ and $f(\xi)$ change sign, i.e. when ξ and all the coefficients a_{2m} change sign (see also formula (13)). The ambiguity that thus arises in the choice of the representative point $(a_o, a_1, \ldots)$ for a given lens $N=N(r)$ is eliminated by requiring that inequality (14) be satisfied.

In order to elucidate this concept, let us examine in detail the case of formula (7), for which relation (12) reduces to

$$\theta = \frac{\pi}{2} a_o + \left(\frac{\pi}{2} - \alpha\right) a_1 ; \tag{15}$$

then relation (14) gives:

$$a_o + a_1 > 0, \quad a_o \geqslant 0 , \tag{16}$$

and therefore the representative point (a_o, a_1) must belong to the shaded region of Fig. 2.

If we also require that N(0) be finite and non-zero and that $(dN/dr)_{r=0} = 0$, then the representative point must belong either to the straight line DGF for which

$$a_o + a_1 = 2, \quad N(0) = 4^{a_o/4} , \tag{17}$$

or to the straight line DS for which

$$a_1 - a_o = 2 , \quad N(0) = 4^{-a_o/4} . \tag{18}$$

In Fig. 2, the point D(0, 2) corresponds to the free space lens ($N=1$, $\delta=0$), the point G(1, 1) to the Luneberg lens ($N = \sqrt{2-r^2}$; Luneberg, 1944) and the point F(2, 0) to Maxwell's fish eye ($N=2/(1+r^2)$), while the point H(2, 2) represents the isotropic lens ($N = \sqrt{(2-r)/r}$, $\delta = \pi$; see e.g. Eaton, 1953) .

Let us now impose the further restriction that the optical rays make less than one complete turn around the origin, i.e. that $\theta < 2\pi$; then the representative points (a_0, a_1) cannot be chosen outside the rhomboid $0AB_0C$ of Fig. 3. The permissable points of Fig. 2, which are external to the rhomboid $0AB_0C$ of Fig. 3 correspond to the so-called higher-order lenses, which were first considered by Stettler (1955).

If we require that all incident rays make at least ℓ, but less than (ℓ+1), complete turns around the origin before leaving the lens (ℓ is any positive integer), then $2\pi\ell \leqslant \theta < 2\pi(\ell+1)$, and therefore the representative point (a_0, a_1) must belong to the rhomboid $B_{\ell-1}A_\ell B_\ell C_\ell$, which is obtained by shifting $0AB_0C$, an amount 4ℓ along the positive a_0 axis (see Fig. 3).

That portion of the permissible region of Fig. 2 which is not covered by the rhomboids of Fig. 3 represents lenses for which the number of complete turns of the optical ray around the origin may be changed by varying the angle α of incidence.

The investigation that we developed for the simple case of formula (7) could be repeated for the more complicated cases in which $f(\xi)$ is a polynomial of degree n = 2, 3, etc. In the case n = 2, the angle θ is given by the relation:

$$\theta = \frac{\pi}{2} a_0 + (\frac{\pi}{2} - \alpha)a_1 + \frac{\pi}{2}(1-\sin\alpha)a_2 \quad , \qquad (19)$$

which follows from (12).

Acknowledgement

The research reported in this paper was sponsored by the National Aeronautics and Space Administration, Langley Research Center, under Grant NsG-444.

References

Eaton, J. E. (1953), U. S. Naval Research Laboratory Report 4110.

Huynen, J. R. (1958), IRE Wescon Convention Record, Part 1, pp. 219-230.

Luneberg, R. K. (1944), The Mathematical Theory of Optics, Brown University Press, Providence, R. I.

Stettler, R. (1955), Optik, 12, pp. 529-543.

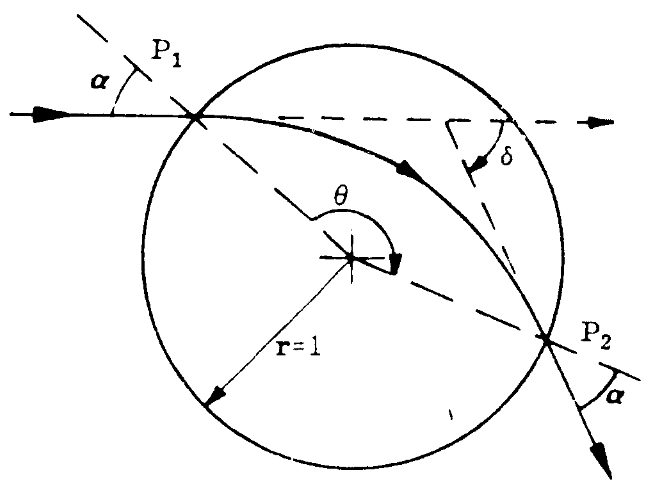
Toraldo di Francia, G. (1957), Annali Mat. Pura Appl., 44, pp. 35-44.

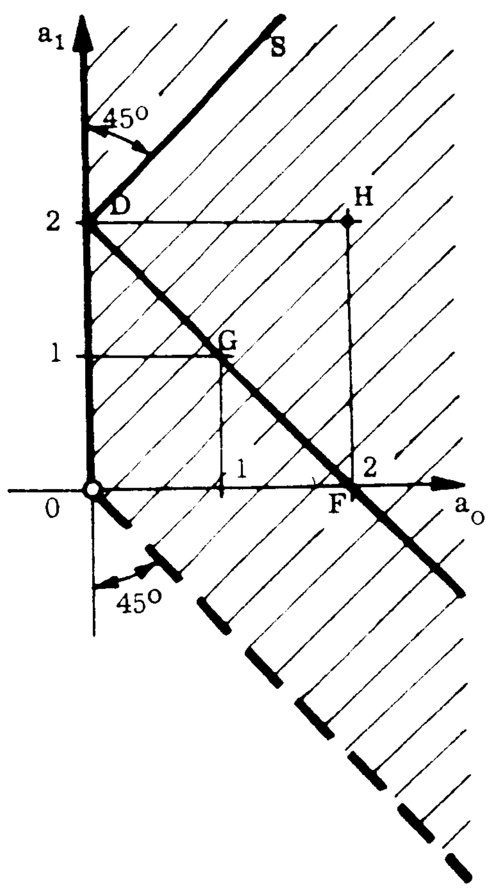
Captions for Figures

1 Geometry for the Optical Ray Path

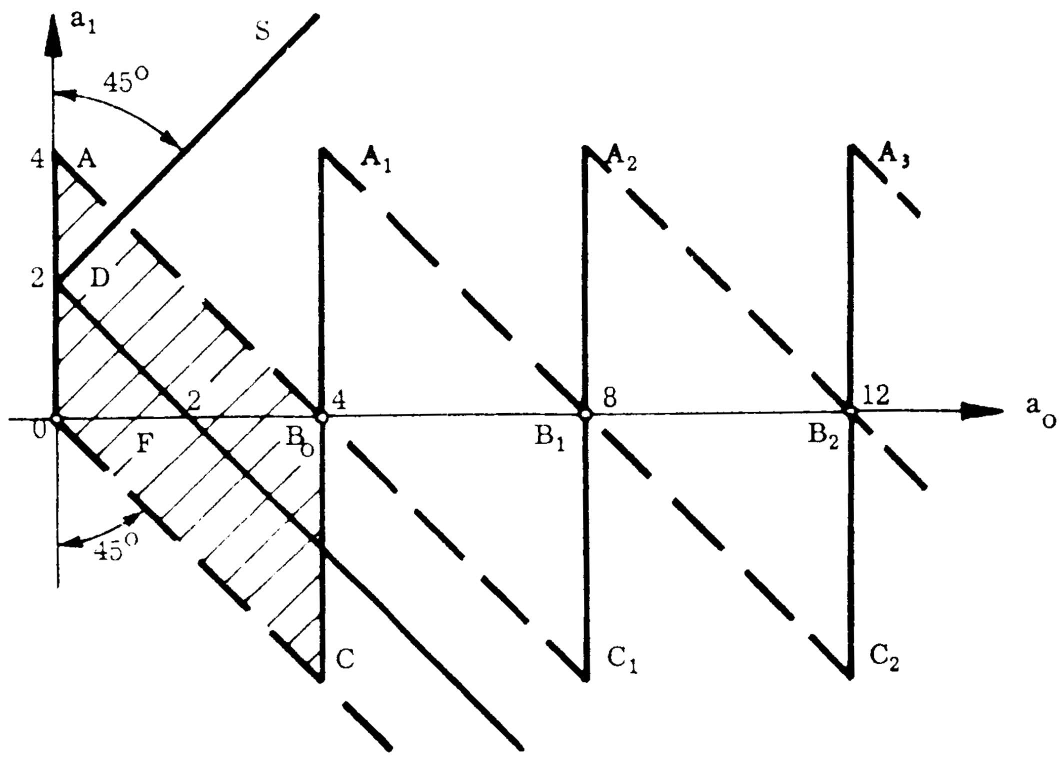
2 The Permissible Domain in the (a_o, a_1) Plane

3 Fundamental Lenses (shaded area) and Higher-Order Lenses


P1
α
δ
θ
r=1
P2
α



a_1
S
45°
D
H
2
1
G
1
2
0
F
a_o
45°



a_1
S
45°
4
A
A_1
A_2
A_3
2
D
2
4
8
12
a_o
0
F
B_o
B_1
B_2
45°
C
C_1
C_2