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ROCKET TRAJECTORY OPTIMIZATION

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ROCKET TRAJECTORY OPTIMIZATION

H. Hinz
Systems Research Section

April 1965

Final Technical Summary Report

Contract NAS 8-1549
National Aeronautics and Space Administration
Marshall Space Flight Center
Huntsville, Alabama

Approved by: Charles E. Mack, Jr.

Charles E. Mack, Jr.
Director of Research

FOREWORD

This final report summarizes the results of the Rocket Trajectory Optimization Project conducted under Contract NAS 8-1549. The contract was administered by the Aero-Astrodynamics Laboratory (formerly Aeroballistics Division) of NASA's Marshall Space Flight Center, Huntsville, Alabama. The work was carried from March 1961 to December 1964 by the Research Department of Grumman Aircraft Engineering Corporation. Mr. William E. Miner was the cognizant technical director for the Astrodynamics and Guidance Theory Division, Marshall Space Flight Center.

During this period twelve individual reports (Refs. 1-12) have been submitted as contributions to "Progress Reports On Studies in the Fields of Space Flight and Guidance Theory," published by Marshall Space Flight Center. Ten of these reports have appeared in Progress Report Nos. 1-5, and two have been submitted for publication in Progress Report No. 7.

At the request of the Future Projects Branch, two computer programs (IBM cards and program listings) and a typical solution for each have been forwarded to Marshall Center. Also prepared for the computer programming personnel at Marshall Center, is a report (Ref. 13) that describes in detail the equations and logic of the computer program for three dimensional variable low thrust interplanetary trajectory optimization.

The primary objectives throughout the investigation have been 1) to develop techniques and computer programs for calculating optimum trajectories of low thrust vehicles, and 2) to extend the variational theory of optimum rocket flight wherever needed.

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SUMMARY

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This report summarizes work carried out under Contract NAS 8-1549 on Rocket Trajectory Optimization. The report sections reflect the major areas of investigation pursued and note the references that contain further treatment of the subject under discussion. The reference section contains a complete list of all publications generated in the course of the contract investigations. Basically, three types of trajectory problems were investigated: interplanetary low thrust, geocentric low thrust, and geocentric high thrust.

A successive approximation technique closely related to the gradient method has been used to compute three dimensional, optimal, low thrust, Earth-to-Mars trajectories. Two computer programs have been developed, one for constant thrust applications, the other for variable thrust-limited problems. The programs generate planetary ephemeris information, and optimize the trajectories for rendezvous and intercept missions.

Optimum, geocentric, low thrust, orbital transfers have been computed using gradient-type methods and a recently developed generalized Newton-Raphson method. Circle-to-circle transfers involving up to 26 revolutions about the earth have been calculated with the Newton-Raphson technique.

A third type of problem - optimum, geocentric, high thrust, orbital transfer - has been treated also using both of the successive approximation techniques. For this purpose, the generalized Newton-Raphson method had to be extended in application to handle inequality constraints on the thrust magnitude control variable.

Author

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INTRODUCTION

The problem of determining economical trajectories for low thrust vehicles has been attacked by the classical variational method with only limited success. The main difficulties are associated with the two-point boundary value problem, arising in numerical solution of the Euler-Lagrange equations of optimal flight, and with the inequality constraints imposed on the control variables. It is believed that these difficulties represent an even greater obstacle to solving more complicated and realistic three dimensional problems using classical techniques.

Successive approximation methods have been explored at Grumman in an attempt to overcome the two-point boundary value difficulty. Of the three methods described in Ref. 1, the "Min H*" scheme, classed among the direct methods of the calculus of variations, has been utilized for most of the computer programs. More recent efforts have resulted in development of the generalized Newton-Raphson operator technique, an indirect method that so far has proven to be five to ten times faster and considerably easier to program.

Two general types of low thrust trajectory optimization problems have been studied concurrently utilizing these successive approximation techniques. The first concerns one-way, interplanetary missions for which the vehicle may achieve rendezvous with a target planet in less than one revolution around the sun. The second deals with geocentric orbital transfer missions requiring several hundred revolutions about the earth to achieve higher energy orbits or escape conditions.

From a computational viewpoint the geocentric case is considerably more difficult to treat because of excessive accumulation of truncation and round-off errors resulting from the large number of integration intervals. It has been suggested that the terms "weak" and "strong" central force field be used to distinguish between the two problems, inasmuch as the gravitational acceleration of the earth at an altitude of 200 miles is about 1500 times greater than the gravitational attraction of the sun at a distance of one astronomical unit.

In addition to the two low thrust studies, an attempt has been made to calculate optimum trajectories of high thrust vehicles for which the full-throttle durations are expected to be short relative to coasting periods.

The propulsion system for all of the numerical studies is thrust-limited with a constant specific impulse. For the interplanetary and high thrust studies, the engines are capable of shutdown and restart. Attention is focused upon nominal trajectory optimization, and no consideration is given to guidance, perturbation effects, or error analysis.

During the investigation, several analyses dealing with the variational theory of optimum rocket flight have been carried out. Two of these, singular extremals and discontinuous variational problems, are also reported on herein.

INTERPLANETARY LOW THRUST MISSIONS

Two IBM 7090 computer programs have been developed for determining optimum, three dimensional, low thrust, interplanetary trajectories (Refs. 2 and 13). The programs are identical with the exceptions that one treats constant thrust rockets, and the other deals with variable thrust-limited engines. A successive approximation scheme that employs the Min H* operation described in Ref. 1 is used to optimize the trajectories, and a penalty function technique is used to handle terminal constraints on position and velocity components and fuel consumption.

For the system model and equations of motion, it is assumed that the entire flight is under the effects of solar gravity only. The orbits of the planets of departure and destination are taken as elliptic and noncoplanar. For each trajectory to be optimized, the planetary orbital elements are computed for the date of departure, using ephemeris information, and are taken as constant throughout the flight. Thus, sufficient realism is obtained without resorting to tape-stored data or other lengthy computational procedures.

Although the computer programs are capable of dealing with missions initiated from any position in the solar system, it usually is assumed that the vehicle has been launched from some planet of departure and boosted to a velocity sufficient for escape from that planet. Thus, the initial components of position and velocity of the vehicle with respect to the heliocentric-inertial system are taken to be identical with those of the planet of departure. By appropriate selection of the penalty constants that govern convergence to the desired terminal conditions, it is possible to study missions such as orbital transfer, rendezvous, and intercept.

Optimization is achieved by determining the time varying control functions that minimize the final value of time. These functions consist of the two thrust steering angles and the thrust magnitude within limits from zero to some maximum value. For the constant thrust application, the fuel expended is not limited, and minimizing time is equivalent to minimizing fuel. For the variable thrust case, the total propellant allocated is less than that required for the corresponding constant thrust example, and as a consequence one will obtain coasting arcs or thrust magnitudes less than maximum.

Optimum, Earth-to-Mars, rendezvous trajectories have been computed for departure dates from January 1965 to September 1973, covering a range of four synodic periods of Mars (synodic period of Mars is 2.135 years). For this phase of the numerical studies (Ref. 6), the thrust magnitude is constant and continuous. The results indicate that there are no distinct or highly significant differences between the three dimensional, optimum trajectories and those obtained from the previous circle-to-circle coplanar studies. This is due to the small inclination of Mars' orbit (1.85°). The differences that do exist appear to be associated mainly with the eccentricity of the planets' orbits.

Examination of the trajectories indicates that there are three types of optimum transfer. For one type, the vehicle's trajectory is entirely within the orbits of the Earth and Mars. Because the planets are in a more favorable position, this class of transfers includes the "minimum minimorum" of the minimum time rendezvous trajectories. At progressively later launch dates, Mars falls behind the Earth, resulting in a second type of transfer for which the vehicle flies out past the Martian orbit and "waits" for the planet to overtake it. At still later launch dates, Mars is so far behind the Earth that the vehicle "decides" that rather than wait for Mars, it is more "profitable" in terms of time and fuel to increase its angular velocity, passing closer to the Sun than the planet Mercury, and eventually catching up with Mars. Although this maneuver requires an additional revolution about the Sun, the transfer time is less.

To better understand the performance capability of low thrust propulsion systems, a limited vehicle parameter variation was carried out. For this phase of the study, the launch date and the vehicle's thrust/initial weight ratio, T/W_0, was varied, and the specific impulse of the rocket engine was kept fixed at 5685 seconds. The following table summarizes the results. Minimum trip times are given for four series of launch dates, each series occurring once every synodic period during which the planets are in the most favorable position. The results indicate clearly that 1971 is the vintage year for low thrust trajectories. Examination of the trajectories for these superior 1971 launch dates reveals that rendezvous occurs when Mars is near perihelion of its orbit, or preferably just past perihelion.

The component of the thrust steering angle in the plane of the ecliptic repeatedly displays the following characteristic motion. Early in flight this vector component points generally

T/W_o	Jan.-Feb. 1967	Mar.-Apr. 1969	May-July 1971	Aug.-Sept. 1973
8.467×10^{-5}	204	186	166	180
2×10^{-4}	132	120	108	116
5×10^{-4}	80	72	66½	72
1×10^{-3}	-	-	45¼	47

in the direction of motion and away from the sun. About half-way in flight it rotates almost abruptly from a position pointing away from the sun to a position pointing toward the sun. Thereafter, the vector is generally in the direction of motion and toward the sun. This steering program is quite similar to the one derived from the linearized, near-circular, orbital transfer studies (Ref. 4). As expected, the component of the thrust normal to the plane of the ecliptic is generally quite small ($\pm 5^o$). It is this control component that must change the small inclination of the vehicle's orbital plane by 1.85^o without unduly penalizing the in-plane, energy-producing component of the thrust vector.

In addition to the variation in T/W_o, the effect of changing the specific impulse, I_s, was also briefly examined. For values of I_s 100 per cent larger, the transfer time increased only slightly by 2-3 per cent. The real significance of the more efficient engines, is, of course, in the greater payloads delivered.

Optimum, Earth-to-Mars, intercept trajectories were also computed for vehicles with constant thrust engines. Because of the removal of constraints upon the terminal values of the velocity components, there is an appreciable savings in transfer time -- as much as 40 per cent. Whether there is a proportional savings in fuel consumed for the corresponding fly-by-and-return mission could be determined only by detailed analysis of the round trip case.

Optimum, Earth-to-Mars, variable thrust, rendezvous trajectories were determined for several launch dates in the vicinity of January 1967 and May 1971. These dates correspond to times when the planets are in very favorable positions. As in the previous circle-to-circle coplanar studies, the thrust magnitude programs have a bang-bang throttle characteristic, the transfers consisting of an initial full throttle period, a coasting period, and a final full throttle period. The significant feature of these variable thrust-limited vehicles is that a significant reduction in fuel requirements may be achieved if the transfer time is permitted to be slightly longer. For example, the results of the January 1967 launch indicate that a 5 per cent sacrifice in time yields a 30 per cent savings in fuel. The ratio, however, does not remain constant, i.e., if the time is permitted to increase 15 per cent the savings in fuel is only 50 per cent.

OPTIMAL LOW THRUST NEAR-CIRCULAR ORBITAL TRANSFER

An attempt has been made to obtain analytically, for a simplified system model, an optimum circle-to-circle orbital transfer solution that includes an abrupt 180° thrust reversal similar to that noted in some of the coplanar Earth-Mars trajectories obtained numerically. Such a solution is of interest in the accuracy/time study of numerical integration. It is further conjectured that such an analysis may provide reliable insight to the characteristics of the general low thrust problem.

The equations of motion employed for the analysis (Ref. 4) are expressed in coordinates of a rotating axis system, with the origin moving in a circular orbit about the earth. By assuming that the vehicle's thrust/mass ratio is constant, and that the motion is always near this origin, the equations become linear with constant coefficients. Because the equations contain the undetermined thrust steering angle, solutions can be expressed in terms of superposition integrals containing this angle. By the indirect method of the variational calculus, an optimum thrust direction program is formulated in terms of two unknown constants. These constants are associated with the constraints on terminal values.

Analytical evaluation of the unknown constants appears to be unpromising because of the complex integrations involved. Instead, a digital program has been developed that numerically integrates the equations of motion and, by a systematic search procedure, automatically evaluates the constants and generates the thrust direction time histories. Optimum solutions have been calculated for circle-to-circle orbital transfers up to 10 revolutions about the earth. Because the problem is linear, and because one of the equation parameters may be eliminated by nondimensionalizing time, it turns out that the results computed with only one set of data are sufficient to describe in detail the optimum circle-to-circle orbital transfer for any specified values of the vehicle's thrust/mass ratio, the vehicle's initial orbital period, and the duration of powered flight. Due to the linearizing approximations, the numerical results become more inaccurate for the longer durations of flight.

An examination of the optimum time history solutions reveals many interesting characteristics and has encouraged further search for an analytically integrated closed-form solution. For example, it is noted that the time variation of the thrust steering angle during the first half of an optimum transfer is always antisymmetrical with respect to the midpoint. Similarly, the time histories of the two components of both the velocity and acceleration also have this "exactly" symmetrical or antisymmetrical property. "Exactly" refers to an accuracy of six significant figures for the numerically integrated results and suggests that the symmetry property would be exact if the integration were error-free.

Another interesting characteristic is observed whenever the duration of powered flight is equal to some integer number of the orbital period. For these cases, the thrust direction remains circumferential and the vehicle passes through a higher energy circular orbit condition at the end of each revolution. The possibility that the optimal thrust direction program may be circumferential for a circle-to-circle orbital transfer involving many revolutions about a central force field should offer an exploitation avenue in connection with the guidance problem.

GEOCENTRIC VERY LOW THRUST ORBITAL TRANSFER

Min H* Method

Two of the problems associated with the optimization of low thrust geocentric trajectories by successive approximation techniques arise as a consequence of the large number of revolutions about the earth required by a vehicle to achieve significant energy changes. One of these problems is the sizable accumulation of round-off truncation error resulting from the many integration intervals required. The second difficulty, brought about by optimization requirements, is the need to store the control functions as functions of time. If the functions are rapidly changing ones, the amount of storage required may become computationally prohibitive.

To provide the precision necessary for geocentric trajectory computation, a new variation-of-parameters method has been developed (Refs. 1 and 5). In addition to achieving the required accuracy, the method avoids solution of Kepler's equation and does not exhibit singularities or ambiguities such as encountered in other variation-of-parameter schemes. The logarithmic spiral was employed as a closed-form check solution, and trajectories were computed in single precision for 60 revolutions about the earth and compared with another better-known perturbation technique. The new method required less computing time and proved accurate to five significant figures as compared to only three significant figures for the other method.

To prevent the computer storage requirements of the Min H* optimization procedure from becoming excessive, a perturbation technique has been developed (Ref. 1) that uses the concept of Green's functions (or influence functions). These functions, which are easily derived analytically for near-elliptical motion, represent the first-order perturbations of the position and velocity components due to a small increment in each occurring at some previous time. By appropriate integration and application of Euler's equation, an optimum thrust direction program is derived in terms of the Green's functions and certain parameters. These parameters, which are constant for unperturbed two-body motion, may be assumed to vary slowly with time so as to account for second and higher order effects. And, since the perturbations produced by low thrust engines are extremely small, the parameters should indeed vary slowly, thereby considerably reducing the number of points requiring stored information.

However, programming the Green's functions and the variation-of-parameters procedure, together with the associated transformations and logic, was noted to be excessively complex. For this reason, the Green function approach has been discarded and replaced by a formula more consistent with the basic variation-of-parameters equations.

The new formula (Ref. 5) is based on the application of the multiplier rule to the differential equations for these slowly varying state variables. The result is a thrust direction formula that is a function of the state variables, a set of multipliers, and the time. An examination of the time derivates of the multipliers shows them to be of the same order of magnitude as the time derivatives of the state variables, i.e., they both contain the thrust/mass ratio, a small number, as a common factor. Therefore, it may be concluded that the thrust direction is adequately represented by storing these multipliers at widely spaced intervals, and the Min H* scheme can then be applied directly to optimizing the trajectory with respect to these quantities. In this manner the storage required for the thrust program can be reduced. The formula and the multiplier equations have been incorporated into the computer program, and preliminary results bear out the expectations.

To gain experience the problem chosen is that of optimum transfer of a vehicle with a relatively high thrust level of 10^{-3}g's from a circular, low altitude, earth orbit to a 24-hour orbit. Early results demonstrate that the parameters and multipliers vary slowly so that a large integration interval of approximately 1000 seconds can be used. With an initial guess of constant circumferential thrust, several descents of the Min H* scheme result in a thrust program for which the angle varies periodically with the period of the orbit, and this is achieved with only minor changes in the magnitudes of the multipliers. Though complete convergence has not been achieved, the new thrust program resembles closely the analytical results for circle-to-circle orbital transfer (Ref. 4).

While developing the closed-form check solution of the logarithmic spiral, a new closed-form reference solution (Ref. 3) was developed that appears particularly applicable for computation of low thrust trajectories. The thrust is tangential in direction and varies in magnitude directly with the total velocity of the vehicle and inversely with the square of the radius from the control body. This program has been so chosen that the solution possesses as many arbitrary constants as the order of system of differential equations governing the motion. Thus, an Encke or variation-of-parameters analysis of neighboring trajectories is practicable. Exploitation of this new reference solution has not been carried out.

Generalized Newton-Raphson Method

In addition to the Min H* optimization technique, which has been utilized for most of the rocket trajectory studies, a new method has been recently developed that is referred to as the generalized Newton-Raphson method. This method was originally derived to solve nonlinear two-point boundary value problems (Ref. 25). More recently it has been successfully applied to optimization of low thrust, interplanetary trajectories (Ref. 10). The method departs from the usual indirect procedure of successively integrating the nonlinear equations and adjusting arbitrary initial conditions of the Lagrange multipliers until the remaining boundary conditions are satisfied. Instead, an operator is introduced that produces a sequence of sets of functions that satisfy the nonlinear system formed by the state equations and the Euler-Lagrange equations. Under appropriate conditions, this sequence converges uniformly and rapidly (quadratically) to the solution of the nonlinear boundary value problem. Another outstanding feature is that the initial approximation does not need to satisfy the differential equation. Thus, simple first guesses that have physical meaning (not like the gross uncertainty of the initial values of the multipliers) have provided satisfactory solutions for the problems studies.

Utilizing the interplanetary computer program based on the generalized Newton-Raphson method, and changing the data from heliocentric to geocentric, optimum, coplanar, circle-to-circle transfers have been calculated for low thrust vehicles that are initially in a parking orbit about the earth. For this program, the final values of the radius and velocity components are prescribed and the final value of time is minimized. With minor modifications, a second program has been developed for which the final value of time is prescribed and the final radius of the circular orbit is maximized.

For a thrust/initial weight ratio, (T/W_0), of 10^{-3} g's, optimal solutions have been generated for progressively increasing values of final time up to durations involving 21.4 revolutions about the earth (26 revolutions for 10^{-4} g's). However, this appears to be the limit for this method employing geocentric polar coordinates and a simple second order predictor-corrector integration procedure. For longer durations, convergence to an accuracy of four significant figures has not been achieved, presumably because of truncation/round-off errors.

For values of final time up to a few orbital periods, the results are quite similar to those obtained from previous linear analyses (Ref. 4). For longer durations, differences are noted in the performance achieved and in the character of the optimum thrust steering angle. The altitude gained for the nonlinear study is superior to the linear results, as may be expected, due to the more realistic mathematical model that takes into account the decrease in gravitational attraction and reduction in vehicle mass as the duration of the maneuver increases. Although the thrust steering angle oscillates about zero (circumferential) at the orbital frequency, as in the linear analyses, the amplitude of motion behaves differently. Instead of being constant with time, the amplitude is generally largest during the first orbit and then decreases gradually.

The solution for the transfer involving 21.4 revolutions required five iteration cycles, a total of 24 trajectories, 49 seconds of IBM 7094 computer time, and about 425 integration intervals per trajectory. Generally, we have noted that a minimum of 20 integration intervals per revolution is required.

Although the linear solutions derived in Ref. 4 are inaccurate for the long duration transfers, it is possible to utilize the linear formulas, which are very accurate for one revolution, and continuously rectify for change in gravity and mass. The integrals to be evaluated and the new formulas are quite simple. For the 19-revolution example, the error is reduced from 77 per cent (linear formula) to 1/4 per cent (rectified formula). This averaging process can be improved further, if necessary, for longer duration transfers.

A limited vehicle parameter variation was carried out and is summarized in the following table for a fixed value of final time equal to 40.29 hours. In the following table I_s is the specific impulse of the engine; ΔR is the change in radius of the circular orbit; and N is the number of revolutions about the earth to complete the maneuver.

T/W_o (g's)	I_s (sec)	ΔR (miles)	N (rev)
.0025	5000	10,658.	13.064
.001	5000	2,134.	20.046
.0005	5000	893.6	23.123
.00025	5000	413.9	24.792
.0001	5000	158.2	25.852
.001	1000	2,323.	19.824

GEOCENTRIC HIGH THRUST ORBITAL TRANSFER

Min H* Method

The applications of successive approximation techniques are not limited to low thrust trajectories or to smooth and continuous control functions. For the interplanetary studies, a comparatively high thrust of 10^{-3} earth g's (3.83 sun g's at the heliocentric radius of Mars) has been dealt with successfully. Also, for variable thrust trajectories with specified limits on thrust magnitude and fuel consumption, the solution converges to clearly defined regions of full thrust and coast.

Based on this success, a computer program using the Min H* method has been developed to treat three dimensional circle-to-circle, orbital transfer and intercept problems. If the fuel consumption for thrust-limited rockets is limited, the optimum minimum-time solution should consist of two or three full thrust periods separated by coasting arcs. Exploratory studies were made for a vehicle with a thrust/initial weight ratio of 8×10^{-2} and a specific impulse of 400 seconds (chemical rocket). Boundary conditions correspond to geocentric orbital transfer for initial and final orbits at 200 and 4000 miles altitude, and inclined to each other by as much as 30°. As in all of the other thrust-limited studies, minimum transfer time is used as the optimization criterion, and the final value of mass is specified.

Initial attempts to solve the orbital transfer problem were unsuccessful. However, by eliminating the terminal constraints on the velocity components, the problem was changed to optimum orbital intercept. For this example, the successive approximations converged without difficulty to a bang-bang solution with two arcs: an initial full-throttle period followed by a coasting arc to the point of intercept.

Generally, computer programs for trajectory optimization employ a constant numerical integration interval. However, it has been noted that for some of the more "sensitive" optimization problems where convergence is not attainable, it is necessary to repeat integration of the final time interval of each trajectory using significantly smaller steps. This makes it possible to calculate more accurately the minimum of the function composed of the terminal value of time to be minimized and the penalty terms to be reduced to zero. Under appropriate conditions, the minimization of this function will result in successive approximations that tend toward the solution of the original problem of minimum time with boundary conditions satisfied.

To locate and evaluate the minimum of this function more accurately, the high thrust computer program was revised to include a variable final integration interval, rather than just smaller intervals. This revision provided convergence, but at a very unsatisfactory rate.

Generalized Newton-Raphson Method

Instead of continuing the high thrust studies using the Min H* method, an alternative approach has been explored. Because of its rapid convergence, the generalized Newton-Raphson method has been extended in application to variational problems with bounded control variables (Ref. 12). Utilizing the method of Valentine (Ref. 26), a new variable is introduced such that the inequality constraint may be replaced by an equivalent equality constraint. The resulting nonlinear system of state and Euler-Lagrange equations now consists of differential and algebraic equations, whereas for the constant thrust example, which is not a bounded control problem, the resulting equations can be reduced to a system of only differential equations. Thus, the generalized Newton-Raphson method had to be modified in order to deal with the additional algebraic equations resulting from the inequality constraint on thrust magnitude.

To gain experience, the method was applied to the more simple problem of coplanar, circle-to-circle, Earth-Mars transfer, for which check solutions are available (Ref. 1). Employing a constant integration interval, the iterated solutions converged very nearly to the true optimum solution, but did not stabilize. Instead, the solutions oscillated at a steady small amplitude about the optimum solution, such that the final value of the radius for each iteration cycle alternated between two values that differed by about 1 per cent, each value being close to the specified terminal radius.

The difficulty apparently is due to the step-by-step integration procedure that is carried out at a prescribed constant time interval, resulting in thrust discontinuities that may occur only at the ends of the integration intervals. The difficulty is easily eliminated and convergence is achieved by appreciably reducing the integration step size only in those intervals where a discontinuity is detected.

Application of the new method to geocentric high thrust trajectory optimization has not been attempted.

SINGULAR EXTREMALS

Variational problems in which the control variable appears linearly in the system differential equations yield to conventional treatment if the optimal control has a bang-bang characteristic. Typical examples of such solutions are the optimal control characteristics of coplanar Earth-Mars orbital transfer (Ref. 1), and three dimensional, Earth-Mars rendezvous trajectories (Refs. 2 and 6). In each of these examples, the thrust magnitude program converges without difficulty to clearly defined regions of full thrust and coast.

However, difficulties arise with the possibility of intervals wherein the optimal control may be intermediate between the specified limits, such segments of the solution being singular subarcs in classical variational terminology. Until very recently, general theory has not been available for determining singular arcs, deciding as to whether they are minimizing even locally, i.e., over a short time interval, or for determining their role as subarcs of a composite solution.

An investigation of singular extremals has been carried out (Refs. 8 and 9) in an attempt to develop methods of analysis for such problems that occur frequently in optimal rocket flight. Although the studies have not yielded useful general procedures, these efforts have been the forerunners of some recent significant findings of Kelley (Ref. 27), and Kopp and Moyer (Ref. 28).

THROTTLEABLE MULTISTAGE ROCKET FLIGHT

The optimal flight of a multistage rocket involves jump discontinuities in mass whenever a stage has consumed all of its fuel and is separated. Such discontinuities do not present variational difficulty if the mass is specified as a function of time. However, for multistage rockets where the engine is throttleable or capable of shutdown and restart, and where the throttle position is treated as the control variable to be optimized, mass is a discontinuous state variable for which the associated Lagrange multiplier will require special treatment. This particular application has motivated a study of the general class of discontinuous variational problems that require the solution extremals to jump when a given hypersurface has been reached. Two techniques for calculating the change in the Lagrange multipliers are presented in Ref. 7.

The first approach is geometrical and therefore loses most of its usefulness as the number of state variables is increased beyond two. However, it does provide diagnostic insight. The boundary of the reachable set is regarded as a wavefront that is determined by wavelets. This principle is used to construct the wavefront just after the discontinuity and to determine the normal to its tangent plane. Of course, the Lagrange multiplier vector is parallel to this normal.

The second approach is due to Cicala (Ref. 29) and has recently been utilized by Jazwinski (Ref. 30). It is analytic and therefore much more flexible and powerful. The jump conditions are adjoined to the original integral resulting in an augmented integral that is varied in the usual manner. The stationary requirement provides a set of nonlinear equations, the solution of which determines the discontinuous changes in the Lagrange multipliers.

Application of these methods to the throttleable multistage rocket problem establishes that when a stage is dropped, all of the multipliers are continuous except the one associated with the mass, and that the discontinuous multiplier is changed such that the Hamiltonian remains continuous.

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12. Kenneth, P., and Taylor, G., "Computation of Optimum Interplanetary Low Thrust Trajectories with Bounded Thrust Magnitude by Means of the Generalized Newton-Raphson Method," Progress Report No. 7 on Studies in the Fields of Space Flight and Guidance Theory, NASA-MSFC Report to be published.

Also submitted, but not published for general distribution, is the following report that describes in detail the equations and logic of the computer program (IBM cards) forwarded to the Future Projects Branch (Astrodynamics and Guidance Theory Division):

13. Hinz, H.K., Taylor G.E., and Cardone, W.R., "Three Dimensional Variable Low Thrust Interplanetary Trajectory Optimization Program (Details of Computer Program)," prepared by the Grumman Research Department for NASA-MSFC, December 1962.

Publications

As a result of the three-year contract effort, the following papers have been published, or will appear, in the open literature. All publications acknowledge that the work was fully or partially supported by NASA Contract NAS 8-1549 with the Aeroballistics Division/Aero-Astrodynamics Laboratory of Marshall Space Flight Center:

14. Kelley, H.J., Kopp, R.E., and Moyer, H.G., "Successive Approximation Techniques for Trajectory Optimization," Proceedings of the IAS Symposium on Vehicle Systems Optimization, Garden City, N.Y., November 28-29, 1961.

15. Lindorfer, W., and Moyer, H.G., "Application of a Low-Thrust Trajectory Optimization Scheme to Planar Earth-Mars Transfer," ARS Journal, p. 260, February 1962.

16. Kelley, H.J., "Method of Gradients," Chapter 6 in Optimization Techniques, edited by G. Leitmann, Academic Press, 1962.

17. Pinkham, G., "Reference Solution for Low-Thrust Trajectories," ARS Journal, p. 775, May 1962.

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20. Kopp, R.E., and McGill, R., "Several Trajectory Optimization Techniques — Part I: Discussion," Computing Methods in Optimization Problems, edited by A.V. Balakrishnan and L.W. Neustadt, Academic Press, 1964.

21. Moyer, H.G., and Pinkham, G., "Several Trajectory Optimization Techniques — Part II: Application," Computing Methods in Optimization Problems, edited by A.V. Balakrishnan and L.W. Neustadt, Academic Press, 1964.

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28. Kopp, R.E., and Moyer, H.G., "Necessary Conditions for Singular Extremals," Paper No. 65-63, AIAA Second Aerospace Sciences Meeting, New York, January 25-27, 1965.

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