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A NONLINEAR ANALYSIS FOR SLOSHING FORCES AND MOMENTS ON A CYLINDRICAL TANK

by H. J. Weiss and T. R. Rogge

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GENERAL DYNAMICS/ASTRONAUTICS
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ABSTRACT

This analysis of the sloshing forces and moments on a cylindrical tank was prepared for use in conjunction with NASA CR-220 entitled "An Approximate Nonlinear Analysis of the Stability of Sloshing Modes Under Translational and Rotational Excitation," by T. R. Rogge and H. J. Weiss. It is not intended to be used as a separate report.

SECTION 1

DISCUSSION

1.1 PRESSURE. For the liquid tank system, the pressure is given by (see Reference 1, equations 2.38 and 3.14)

$$\frac{p - p_o}{\rho} = \alpha z + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + \frac{\partial\phi}{\partial r}(u_1 \cos\theta + u_2 \sin\theta)$$

$$+ \frac{1}{r}\frac{\partial\phi}{\partial\theta}(u_2 \cos\theta - u_1 \sin\theta) + \frac{\partial\phi}{\partial z}u_3 + \frac{1}{2}\left[u_1^2 + u_2^2 + u_3^2\right]$$

$$+ \omega_2 \frac{\partial\phi}{\partial r} z \cos\theta - \frac{\omega_2}{r} z \frac{\partial\phi}{\partial\theta}\sin\theta + \omega_3 \frac{\partial\phi}{\partial\theta} + r\frac{\partial\phi}{\partial z}(\omega_1 \sin\theta - \omega_2 \cos\theta)$$

$$- \omega_1 z\left(\frac{\partial\phi}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}\right) - \frac{\partial\phi}{\partial t} \tag{1}$$

where p_o is the ambient pressure.

The forces on the tank due to the liquid are

$$F_i = - \int_S (p - p_o)\, n_i \, dS \tag{2}$$

where n_i is the unit exterior normal to S, and S is the surface of the liquid in contact with the tank. On the free surface, $p = p_o$; thus, there is no contribution to the force at the free surface.

The moments are given by

$$M_i = \int_S \epsilon_{ijk}\, x_j\, n_k\, (p - p_o)\, dS \tag{3}$$

where ϵ_{ijk} is the third order alternating tensor.

In Reference 1, the potential is written as

$$\phi(r, \theta, z, t) = \psi(r, \theta, z, t) - u_1 r \cos\theta - u_2 r \sin\theta + z r \omega_1 \sin\theta$$
$$- z \omega_2 r \cos\theta \tag{4}$$

where ψ is approximated by

$$\psi = \epsilon^{\frac{1}{3}}\left[\psi_1 (\bar{r}, t) \cos \omega t + X_1 (\bar{r}, t) \sin \omega t\right]$$

$$+ \epsilon^{\frac{2}{3}} \left[\psi_0 (\bar{r}) + \psi_2 (\bar{r}) \cos 2\omega t + X_2 (\bar{r}) \sin 2\omega t\right]$$

$$+ \epsilon \left[\psi_3(\bar{r}) \cos 3\omega t + X_3 (\bar{r}) \sin 3\omega t\right] \tag{5}$$

$\bar{r}$ indicates (r, θ, z), and ϵ is a small parameter.

The perturbation velocities are given by

$$\left.\begin{aligned} u_i &= \epsilon \cos \omega t \\ \omega_i &= \frac{\epsilon}{h} \cos \omega t \end{aligned}\right\} \tag{6}$$

where h is the height of the undisturbed liquid in the tank.

The harmonic functions ψ_0, ψ_1, ψ_2, X_1, X_2 and X_3 are defined as (Reference 1, pp. 3-8 to 3-14).

$$\psi_0 = \text{constant} \tag{7.a}$$

$$\psi_1 = \left[f_1 (\tau) \cos \theta + f_3 (\tau) \sin \theta\right] J_1 \left(\lambda_{11} r\right) \frac{\cosh \left[\lambda_{11} (z + h)\right]}{\cosh \lambda_{11} h} \tag{7.b}$$

$$\psi_2 = \sum_{n=1}^{\infty} \hat{A}_{0n} J_0 \left(\lambda_{0n} r\right) \frac{\cosh \left[\lambda_{0n} (z + h)\right]}{\cosh \lambda_{0n} h}$$

$$+ \sum_{n=1}^{\infty} \left(\hat{A}_{2n} \cos 2\theta + \hat{B}_{2n} \sin 2\theta\right) J_2 \left(\lambda_{2n} r\right) \frac{\cosh \left[\lambda_{2n} (z + h)\right]}{\cosh \lambda_{2n} h} \tag{7.c}$$

$$X_1 = \left[f_2 (\tau) \cos \theta + f_4 (\tau) \sin \theta\right] J_1 \left(\lambda_{11} r\right) \frac{\cosh \left[\lambda_{11} (z + h)\right]}{\cosh \lambda_{11} h} \tag{8.a}$$

$$X_2 = \sum_{n=1}^{\infty} \hat{C}_{0n} J_0 \left(\lambda_{0n} r\right) \frac{\cosh\left[\lambda_{0n} (z + h)\right]}{\cosh \lambda_{0n} h}$$

$$+ \sum_{n=1}^{\infty} \left(\hat{C}_{2n} \cos 2\theta + \hat{D}_{2n} \sin 2\theta\right) J_2 \left(\lambda_{2n} r\right) \frac{\cosh\left[\lambda_{2n} (z + h)\right]}{\cosh \lambda_{2n} h} \qquad (8.b)$$

$\tau = \frac{1}{2}\epsilon^{2/3}\omega t$, J_m are Bessel functions of the first kind of order m, and λ_{mn} are the solutions of $J'_m (\lambda_{mn} a) = 0$. The coefficients $\hat{A}_{0n}, \hat{A}_{2n}, \cdots, \hat{D}_{2n}$ are defined as in Reference 1, p. 3-11.

Within the order of the approximations used, neither ψ_3 nor X_3 contribute to the pressure, or therefore to the forces and moments.

Substitute (4) into (1)

$$\frac{p - p_0}{\rho} = \alpha z + \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] - \frac{z^2}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{u_3^2}{2}$$

$$+ \frac{3}{2} r^2 \left(\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta\right) + 2\frac{\partial\psi}{\partial z}\left(r\omega_1 \sin\theta - r\omega_2 \cos\theta + \frac{u_3}{2}\right)$$

$$- 3 r^2 \omega_1 \omega_2 \sin\theta \cos\theta + u_3 \omega_1 r \sin\theta - u_3 \omega_2 r \cos\theta - z\omega_2 u_1$$

$$+ z\,\omega_1 u_2 - u_2 \omega_3 r \cos\theta + z\,\omega_3 \omega_2 r \sin\theta$$

$$+ u_1 \omega_3 r \sin\theta + \omega_3 \frac{\partial\psi}{\partial\theta} + z\,\omega_3 \omega_1 r \cos\theta$$

$$+ \dot{u}_1 r \cos\theta + \dot{u}_2 r \sin\theta - zr\dot{\omega}_1 \sin\theta + z\dot{\omega}_2 r \cos\theta - \frac{\partial\psi}{\partial t} \qquad (9)$$

There are two types of steady-state motion to consider: planar and nonplanar. For planar motion the generalized coordinates, $f_i(\tau)$ are

$$\left.\begin{aligned} f_1 &= \Upsilon \\ f_3 &= \Upsilon Q \\ f_2 &= f_4 = 0 \end{aligned}\right\} \qquad (10)$$

where Υ is a parameter that is independent of time and Q is defined as in Reference 1, (3.61).

Thus, $\left(\text{see Reference 1, (3.41)}\right)$

$$\left.\begin{aligned}
\hat{A}_{0n} &= \hat{A}_{2n} = \hat{B}_{2n} = 0 \\
\hat{D}_{2n} &= -\gamma^2 Q\, \Omega_{2n} \\
\hat{C}_{2n} &= \frac{\gamma^2\, \Omega_{2n}}{2} (Q^2 - 1) \\
\hat{C}_{0n} &= -\frac{\gamma^2\, \Omega_{0n}}{2} (Q^2 + 1) \\
\psi_2 &= X_1 = 0
\end{aligned}\right\} \qquad (11)$$

where Ω_{2n} and Ω_{0n} are defined in Reference 1, p. A-12. Thus, from (7), (8), and (11), we obtain the specific forms of the harmonic functions.

$$\psi_1 = \gamma \left[\cos\theta + Q \sin\theta\right] J_1 \left(\lambda_{11} r\right) \frac{\cosh\left[\lambda_{11} (z + h)\right]}{\cosh\lambda_{11} h} \qquad (12)$$

$$X_2 = -\frac{\gamma^2 (Q^2 + 1)}{2} \sum_{n=1}^{\infty} \Omega_{0n} J_0 \left(\lambda_{0n} r\right) \frac{\cosh\left[\lambda_{0n} (z + h)\right]}{\cosh\lambda_{0n} h}$$

$$+ \gamma^2 \sum_{n=1}^{\infty} \Omega_{2n} \left[\frac{(Q^2 - 1)}{2} \cos 2\theta - Q \sin 2\theta\right] J_2 \left(\lambda_{2n} r\right)$$

$$\frac{\cosh\left[\lambda_{2n} (z + h)\right]}{\cosh\lambda_{2n} h} \qquad (13)$$

$$\psi (r, \theta, z, t) = \epsilon^{1/3} \psi_1 \cos\omega t + \epsilon^{2/3} \left[\psi_0 + X_2 \sin 2\omega t\right] + O(\epsilon) \qquad (14)$$

where $O(\epsilon)$ includes the higher harmonic terms. Neglect terms of the order $O(\epsilon)$ and compute

ψ_t, ψ_r, ψ_z, and ψ_θ.

$$\psi_t = -\epsilon^{1/3}\,\omega\,\psi_1 \sin\omega t + 2\,\epsilon^{2/3}\,\omega X_2 \cos 2\omega t \tag{15.a}$$

$$\psi_k = \epsilon^{1/3}\,\frac{\partial\psi_1}{\partial k}\cos\omega t + \epsilon^{2/3}\,\frac{\partial X_2}{\partial k}\sin 2\omega t,\quad k = r,\ \theta,\ z \tag{15.b}$$

$$\frac{\partial\psi_1}{\partial r} = \gamma\lambda_{11}\,(\cos\theta + Q\sin\theta)\,J_1'\left(\lambda_{11}\,r\right)\,\frac{\cosh\left[\lambda_{11}\,(z+h)\right]}{\cosh\lambda_{11}\,h} \tag{16.a}$$

$$\frac{\partial\psi_1}{\partial\theta} = \gamma\,(-\sin\theta + Q\cos\theta)\,J_1\left(\lambda_{11}\,r\right)\,\frac{\cosh\left[\lambda_{11}\,(z+h)\right]}{\cosh\lambda_{11}\,h} \tag{16.b}$$

$$\frac{\partial\psi_1}{\partial z} = \gamma\lambda_{11}\,(\cos\theta + Q\sin\theta)\,J_1\left(\lambda_{11}\,r\right)\,\frac{\sinh\left[\lambda_{11}\,(z+h)\right]}{\cosh\lambda_{11}\,h} \tag{16.c}$$

$$\frac{\partial X_2}{\partial r} = -\frac{\gamma^2\,(Q^2+1)}{2}\sum_{n=1}^{\infty}\Omega_{0n}\,\lambda_{0n}\,J_0'\left(\lambda_{0n}\,r\right)\,\frac{\cosh\left[\lambda_{0n}\,(z+h)\right]}{\cosh\lambda_{0n}\,h}$$

$$+\ \gamma^2\sum_{n=1}^{\infty}\Omega_{2n}\,\lambda_{2n}\left[\frac{(Q^2-1)}{2}\cos 2\theta - Q\sin 2\theta\right]J_2'\left(\lambda_{2n}r\right)$$

$$\frac{\cosh\left[\lambda_{2n}\,(z+h)\right]}{\cosh\lambda_{2n}\,h} \tag{17.a}$$

$$\frac{\partial X_2}{\partial\theta} = -2\gamma^2\sum_{n=1}^{\infty}\Omega_{2n}\left[\frac{(Q^2-1)}{2}\sin 2\theta + Q\cos 2\theta\right]J_2\left(\lambda_{2n}\,r\right)$$

$$\frac{\cosh\left[\lambda_{2n}\,(z+h)\right]}{\cosh\lambda_{2n}\,h} \tag{17.b}$$

$$\frac{\partial X_2}{\partial z} = -\frac{\gamma^2 (Q^2+1)}{2} \sum_{n=1}^{\infty} \Omega_{0n} \lambda_{0n} J_0 \left(\lambda_{0n} r\right) \frac{\sinh \left[\lambda_{0n} (z+h)\right]}{\cosh \lambda_{0n} h}$$

$$+ \gamma^2 \sum_{n=1}^{\infty} \Omega_{2n} \lambda_{2n} \left[\frac{(Q^2-1)}{2} \cos 2\theta - Q \sin 2\theta\right] J_2 \left(\lambda_{2n} r\right)$$

$$\frac{\sinh \left[\lambda_{2n} (z+h)\right]}{\cosh \lambda_{2n} h} \tag{17.c}$$

Disregarding terms of O (ϵ), then for $k = r, \theta, z$

$$\psi_k^{\,2} = \epsilon^{\frac{2}{3}} \left(\frac{\partial \psi_1}{\partial k}\right)^2 \cos^2 \omega t \tag{18}$$

(Note in obtaining (18), the term O (ϵ) in $\psi_k^{\,2}$ -- namely, $\epsilon \left(\frac{\partial \psi_1}{\partial k}\right) \left(\frac{\partial X_2}{\partial k}\right)$ $\cos \omega t \sin 2\omega t$ -- introduces a harmonic term of higher order and is therefore neglected.)

Substitute equations (15) through (18) into (9)

$$\frac{p - p_o}{\rho} = \alpha z + \frac{1}{2} \epsilon^{\frac{2}{3}} \cos^2 \omega t \left\{ \left[\gamma \lambda_{11} (\cos\theta + Q \sin\theta) J_1' \left(\lambda_{11} r\right) \right.\right.$$

$$\left.\frac{\cosh \left[\lambda_{11} (z+h)\right]}{\cosh \lambda_{11} h}\right]^2 + \left[\frac{\gamma}{r}(-\sin\theta + Q\cos\theta) \quad J_1 \left(\lambda_{11} r\right)\right.$$

$$\left.\frac{\cosh \left[\lambda_{11} (z+h)\right]}{\cosh \lambda_{11} h}\right]^2$$

$$\left.+ \left[\gamma \lambda_{11} (\cos\theta + Q\sin\theta) \; J_1 \left(\lambda_{11} r\right) \frac{\sinh \left[\lambda_{11} (z+h)\right]}{\cosh \lambda_{11} h}\right]^2 \right\}$$

$$+ \epsilon^{\frac{1}{3}} \gamma\omega \sin \omega t \, (\cos\theta + Q \sin\theta) \, J_1\left(\lambda_{11} r\right) \frac{\cosh \lambda_{11} (z+h)}{\cosh \lambda_{11} h}$$

$$- 2\, \epsilon^{\frac{2}{3}} \omega \cos 2\omega t \left[- \frac{\gamma^2 (Q^2 + 1)}{2} \sum_{n=1}^{\infty} \Omega_{0n} J_0\left(\lambda_{0n} r\right) \frac{\cosh \lambda_{0n} (z+h)}{\cosh \lambda_{0n} h} \right.$$

$$+ \gamma^2 \sum_{n=1}^{\infty} \Omega_{2n} \left\{ \frac{(Q^2 - 1)}{2} \cos 2\theta - Q \sin 2\theta \right\} J_2 \left(\lambda_{2n} r\right)$$

$$\left. \frac{\cosh \lambda_{2n} (z+h)}{\cosh \lambda_{2n} h} \right]$$

$$- r\omega\epsilon \sin \omega t \cos\theta - r\omega\epsilon \sin \omega t \sin \theta + \frac{z r \epsilon \omega}{h} \sin \omega t \sin\theta$$

$$- \frac{z r \epsilon \omega}{h} \sin \omega t \cos \theta \tag{19}$$

disregarding terms of order greater than O (ϵ) and higher harmonic terms.

1.2 FORCES. On the sides of the tanks, $n_1 = \cos\theta$, $n_2 = \sin\theta$, $n_3 = 0$. On the bottom of the tanks, $n_1 = n_2 = 0$, $n_3 = -1$.

$$F_1 = -\rho a \int_0^{2\pi} \int_{-h}^{0} \frac{\left(p - p_o\right)}{\rho} \cos\theta \, d\theta \, dz, \qquad r = a \tag{20.a}$$

$$F_2 = -\rho a \int_0^{2\pi} \int_{-h}^{0} \frac{\left(p - p_o\right)}{\rho} \sin\theta \, d\theta \, dz, \qquad r = a \tag{20.b}$$

$$F_3 = \rho \int_0^{2\pi} \int_0^{a} \frac{\left(p - p_o\right)}{\rho} r \, dr \, d\theta, \qquad z = -h \tag{20.c}$$

Note that because of the trigonometric functions appearing in $\frac{p - p_o}{\rho}$ and because of the form of n_i there are considerable simplifications in evaluation of F_i. Thus

$$\int_0^{2\pi} \sin^2 \theta \, d\theta = \int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$$

$$\int_0^{2\pi} \Big| \cos\theta, \ \sin\theta, \ \cos\theta\sin\theta, \ \cos^2\theta\sin\theta, \ \sin^2\theta\cos\theta, \ \cos^3\theta, \ \sin^3\theta,$$
$$\cos 2\theta\cos\theta, \ \sin 2\theta\cos\theta, \ \sin 2\theta\sin\theta, \ \cos 2\theta\sin\theta \Big| \ d\theta = 0 \qquad (21)$$

and

$$F_1 = - \frac{\rho a \pi \gamma \epsilon^{\frac{1}{3}} \omega}{\lambda_{11}} \sin\omega t \ J_1\left(\lambda_{11} a\right) \tanh \lambda_{11} h \qquad (22.a)$$

$$F_2 = - \frac{\rho a \pi \gamma \epsilon^{\frac{1}{3}} Q \omega}{\lambda_{11}} \sin\omega t \ J_1\left(\lambda_{11} a\right) \tanh \lambda_{11} h \qquad (22.b)$$

$$F_3 = \rho \Bigg\{ - \alpha \pi a^2 h + \frac{1}{2} \epsilon^{\frac{2}{3}} \cos^2 \omega t \Bigg[\frac{\gamma^2 \pi \lambda_{11}{}^2 \left(Q^2 + 1\right)}{\cosh^2 \lambda_{11} h} \int_0^a r \left[J_1'\left(\lambda_{11} r\right)\right]^2 dr$$

$$+ \frac{\gamma^2 \pi \left(Q^2 + 1\right)}{\cosh^2 \lambda_{11} h} \int_0^a \frac{J_1{}^2\left(\lambda_{11} r\right)}{r} dr$$

$$+ \epsilon^{\frac{2}{3}} \omega \cos 2\omega t \sum_{n=1}^{\infty} \frac{\pi \gamma^2 \left(Q^2 + 1\right)}{\cosh \lambda_{11} h} \Omega_{0n} \int_0^a r J_0\left(\lambda_{0n} r\right) dr \Bigg\} \qquad (22.c)$$

1.3 MOMENTS. The first moment is calculated from (3) as

$$M_1 = \int_S x_2 n_3 \left(p - p_o\right) dS - \int_S x_3 n_2 \left(p - p_o\right) dS \qquad (23)$$

From the geometry of the boundary, then,

$$M_1 = -\rho \int_0^{2\pi} \int_0^{a} \frac{p - p_o}{\rho} \sin\theta \, r^2 \, dr \, d\theta \Bigg|_{z=-h}$$

$$- \rho a \int_0^{2\pi} \int_{-h}^{0} \frac{p - p_o}{\rho} \sin\theta \, z \, dz \, d\theta \Bigg|_{r=a} \tag{24.a}$$

Similarly

$$M_2 = \rho \int_0^{2\pi} \int_0^{a} \frac{p - p_o}{\rho} \cos\theta \, r^2 \, dr \, d\theta \Bigg|_{z=-h}$$

$$+ \rho a \int_0^{2\pi} \int_{-h}^{0} \frac{p - p_o}{\rho} \cos\theta \, z \, dz \, d\theta \Bigg|_{r=a} \tag{24.b}$$

$$M_3 = \int_S \epsilon_{3jk} \, x_j \, n_k \left(p - p_o\right) dS$$

$$= a \int_0^{2\pi} \int_{-h}^{0} \left(p - p_o\right) \left[x_1 n_2 - x_2 n_1\right] d\theta \, dz$$

$$= a \int_0^{2\pi} \int_{-h}^{0} \left(p - p_o\right) (\cos\theta \sin\theta - \sin\theta \cos\theta) \, r \, d\theta \, dz = 0 \tag{24.c}$$

Substitute (19) into (24a) and (24b) to obtain

$$M_1 = -\frac{\rho Q \pi \epsilon^{\frac{1}{3}} \gamma \omega \sin \omega t}{\cosh \lambda_{11} h} \int_0^a r^2 J_1 \left(\lambda_{11} r\right) dr$$

$$- \frac{\rho a \pi \epsilon^{\frac{1}{3}} Q \gamma \omega \sin \omega t \ J_1 \left(\lambda_{11} a\right) \left[1 - \cosh \lambda_{11} h\right]}{\lambda_{11}^2 \cosh \lambda_{11} h} \tag{25.a}$$

$$M_2 = \frac{\rho a \pi \epsilon^{\frac{1}{3}} \gamma \omega \sin \omega t \ J_1 \left(\lambda_{11} a\right) \left[1 - \cosh \lambda_{11} h\right]}{\lambda_{11}^2 \cosh \lambda_{11} h}$$

$$+ \frac{\rho \pi \epsilon^{\frac{1}{3}} \gamma \omega \sin \omega t}{\cosh \lambda_{11} h} \int_0^a r^2 J_1 \left(\lambda_{11} r\right) dr \tag{25.b}$$

REFERENCE

1. T. R. Rogge and H. J. Weiss, An Approximate Nonlinear Analysis of the Stability of Sloshing Modes Under Translational and Rotational Excitation, General Dynamics/Astronautics Report GD|A-DDE64-059, November 1964. (NASA CR-220, 1965)

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