A MODIFICATION OF THE KUHN-TUCKER THEOREM

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Summary:

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The problem of simultaneously maximizing a set of concave functions whose domain is defined by another set of concave functions is considered. It will be shown that any version of the Kuhn-Tucker theorem can be modified to deal with this problem.

author

Introduction:

Let $f(x) = \langle f_1(x), \cdots, f_m(x) \rangle$ and $g(x) = \langle g_1(x), \cdots, g_n(x) \rangle$ be respectively m and n-dimensional vector valued functions defined on a Euclidean space. We shall write $u \geqq 0$ if all components of the vector u are non-negative and $u > 0$ if all components of u are positive. Consider the following:

'Maximum problem'. For $z = (z_1, \cdots, z_m) \geqq o$ find a vector which maximizes.

$$z \cdot f(x) = \Sigma_i z_i f_i(x)$$

subject to the restrictions

$$x \geqq 0 \text{ and } g(x) \geqq 0.$$

If the corresponding Lagrangian form $L(x,y,z)$ defined by

$$L(x, y, z) = z \cdot f(x) + y \cdot g(x)$$

has a saddle-point (x', y') in $x \geqq 0$ and $y \geqq 0$, then the vector x' is a solution to the maximum problem. A pair (x', y') is called a saddle-point of a Lagrangian form L in $x \geqq 0$ and $y \geqq 0$ if $x' \geqq 0$, $y' \geqq 0$, and

(1) $L(x, y', z) \leqq L(x', y', z) \leqq L(x', y, z)$

for all $x \geqq 0$ and $y \geqq 0$. A solution to the maximum problem, however, does not in general correspond to a saddle-point of the Lagrangian form. The Kuhn-Tucker theorem is concerned with restrictions on f and g under which the correspondence does hold.

We shall use the following version of the Kuhn-Tucker theorem (due to Slater) in our discussion below:

Theorem 1. Assume that, for $x \geqq 0$, $h(x)$ is a concave real valued function. Assume also that $g(x)$ is a concave vector valued function such that there exists a vector $x^o \geqq 0$ for which $g(x^o) > 0$. Then the vector x'

maximizes $h(x)$ subject to the restrictions $x \geqq 0$, $g(x) \geqq 0$, if and only if, there is a vector $y' \geqq 0$ such that (x', y') is a saddle-point of the Lagrangian form $L(x, y) = h(x) + \sum_i y_i g_i(x)$.

The modification of the maximum problem which we wish to investigate here is the 'uniform maximum problem'. Find a vector x' that uniformly maximizes all components of $f(x)$ subject to the restrictions $x \geqq 0$ and $g(x) \geqq 0$.

Theorem 2. Assume that $f(x)$ and $g(x)$ are concave vector valued functions on $x \geqq 0$, and $g(x)$ satisfies the following condition:

There exists a vector $x^o \geqq 0$ such that $g(x^o) > 0$. Then a vector x' is a solution to the uniform maximum problem if and only if, the following conditions hold: for all $z \geqq o$, there exists a vector $y' = y'(z) \geqq 0$ such that the pair (x', y') is a saddle-point of the Lagrangian form $L(x, y, z) = z \cdot f(x) + y \cdot g(x)$.

Proof: 'If part'. Suppose that (x', y') is a saddle-point of L. From the last inequality in (1), it follows that

$$y' \cdot g(x') \leqq y \cdot g(x') \qquad \text{for all } y \geqq 0.$$

In particular $y' \cdot g(x') \leqq 0$. From the first inequality in (1) it follows that $z \cdot f(x) \leqq z \cdot f(x')$ whenever $x \geqq 0$ and $g(x) \geqq 0$. Since z is arbitrary we have $f_i(x) \leqq f_i(x')$ for all i. Thus, x' is a solution to the uniform maximum problem.

'Only if part'. Under the assumptions given, $z \cdot f(x)$ is a real valued concave function maximized by x'. By Theorem 1 it follows that there is a vector $y' \geqq 0$ such that (x', y') is a saddle-point of $L(x, y, z)$.

Note: Clearly, for any condition on f and g under which the solution to the maximum problem corresponds to a saddle-point of the corresponding Lagrangian form, our modification holds under the same condition. The extension of this result into abstruct spaces is immediate.

Reference:

M. Slater, "Lagrange Multipliers Revisited: A Contribution to Non-Linear Programming," Cowles Commission Discussion Paper, Math. 403, November 1950.