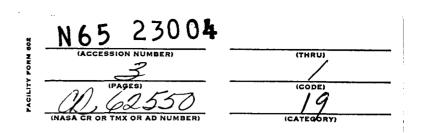
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A MODIFICATION OF THE KUHN-TUCKER THEOREM

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A MODIFICATION OF THE KUHN-TUCKER THEOREM

Summary:

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The problem of simultaneously maximizing a set of concave functions whose domain is defined by another set of concave functions is considered. It will be shown that any version of the Kuhn-Tucker theorem can be modified to deal with this problem.

Introduction:

Let $f(x) = \langle f_1(x), \dots, f_m(x) \rangle$ and $g(x) = \langle g_1(x), \dots, g_n(x) \rangle$ be respectively m and n-dimensional vector valued functions defined on a Euclidean space. We shall write $u \ge 0$ if all components of the vector u are non-negative and u > 0 if all components of u are positive. Consider the following:

'Maximum problem'. For z = $(z_1, \dots, z_m) \ge 0$ find a vector which maximizes.

$$z \cdot f(x) = \sum_{i} z_{i} f_{i}(x)$$

subject to the restrictions

$$x > 0$$
 and $g(x) \ge 0$.

If the corresponding Lagrangian form L(x,y,z) defined by

$$L(x, y, z) = z \cdot f(x) + y \cdot g(x)$$

has a saddle-point (x', y') in $x \ge 0$ and $y \ge 0$, then the vector x' is a solution to the maximum problem. A pair (x', y') is called a saddle-point of a Lagrangian form L in $x \ge 0$ and $y \ge 0$ if $x' \ge 0$, $y' \ge 0$, and

(1) $L(x, y', z) \le L(x', y', z) \le L(x', y, z)$

for all $x \ge 0$ and $y \ge 0$. A solution to the maximum problem, however, does not in general correspond to a saddle-point of the Lagrangian form. The Kuhn-Tucker theorem is concerned with restrictions on f and g under which the correspondence does hold.

We shall use the following version of the Kuhn-Tucker theorem (due to Slater) in our discussion below:

Theorem 1. Assume that, for $x \ge 0$, h(x) is a concave real valued function. Assume also that g(x) is a concave vector valued function such that there exists a vector $x^{O} \ge 0$ for which $g(x^{O}) > 0$. Then the vector x'

maximizes h(x) subject to the restrictions $x \ge 0$, $g(x) \ge 0$, if and only if, there is a vector $y' \ge 0$ such that (x', y') is a saddle-point of the Lagrangian form $L(x, y) = h(x) + \leq y_i g_i(x)$.

The modification of the maximum problem which we wish to investigate here is the 'uniform maximum problem'. Find a vector x' that uniformly maximizes all components of f(x) subject to the restrictions $x \ge 0$ and $g(x) \ge 0$.

Theorem 2. Assume that f(x) and g(x) are concave vector valued functions on $x \ge 0$, and g(x) satisfies the following condition:

There exists a vector $x^{O} \ge 0$ such that $g(x^{O}) > 0$. Then a vector x' is a solution to the uniform maximum problem if and only if, the following conditions hold: for all $z \ge 0$, there exists a vector $y' = y'(z) \ge 0$ such that the pair (x', y') is a saddle-point of the Lagrangian form $L(x, y, z) = z \cdot f(x) + y \cdot g(x)$.

Proof: 'If part'. Suppose that (x', y') is a saddle-point of L. From the last inequality in (1), it follows that

$$y' \cdot g(x') \leq y \cdot g(x')$$
 for all $y \geq 0$.

In particular $y' \cdot g(x') \leq 0$. From the first inequality in (1) it follows that $z \cdot f(x) \leq z \cdot f(x')$ whenever $x \geq 0$ and $g(x) \geq 0$. Since z is arbitrary we have $f_i(x) \leq f_i(x')$ for all i. Thus, x' is a solution to the uniform maximum problem.

'Only if part'. Under the assumptions given, $z \cdot f(x)$ is a real valued concave function maximized by x'. By Theorem 1 it follows that there is a vector $y' \ge 0$ such that (x', y') is a saddle-point of L(x, y, z).

Note: Clearly, for any condition on f and g under which the solution to the maximum problem corresponds to a saddle-point of the corresponding Lagrangian form, our modification holds under the same condition. The extension of this result into abstruct spaces is immediate.

Reference:

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M. Slater, "Lagrange Multipliers Revisited: A Contribution to Non-Linear Programming," Cowles Commission Discussion Paper, Math. 403, November 1950.

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