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PROGRESS REPORT

For Period 14 November to 14 February

RESEARCH AND STUDY IN SYSTEM OPTIMIZATION TECHNIQUES

Office of Astrodynamics and Guidance Theory

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RESEARCH AND STUDY IN SYSTEM OPTIMIZATION TECHNIQUES

SECTION 1

GENERAL

This is the first quarterly progress report submitted in accordance with the provisions of Modification 4 to Contract NAS 8-5222, "Research and Study in System Optimization Techniques." It covers the period from 14 November 1964 to 14 February 1965.

SECTION II

SUMMARY OF PRIOR PROGRESS

The results of the previous work done under the contract are reported in Final Report 1546-FTR 1.

The February 3 and 4 meeting of contractors on Guidance and Space Flight Theory was attended by Dr. E. B. Lee and D. K. Scharmack of Honeywell. Dr.Lee presented a paper entitled "Approximations to Linear Bounded Phase Coordinate Control Problems" and Mr. Scharmack presented a paper entitled "Nonlinear Optimal Feedback Control for Reentry"at the general session.

SECTION III

PROGRESS DURING REPORTING PERIOD

Stability of Motion Study

The new results that have been obtained in the area of stability of motion are contained in the attached appendix entitled "Lyapunov Functions and Their Related Systems with Applications to Finding Their Best Estimators." It includes a discussion of how Lyapunov functions can be applied to the fundamental problem of the determination of sets of initial states from which the desired terminal state of a dynamical system (with control) may be attained.

The paper also contains a mathematical analysis of the connection between differential equations and their Lyapunov functions. In the section on applications a method for obtaining estimates of Lyapunov functions via computer is developed. (see the summary)

The literature survey on methods and techniques for determining regions of stability is being carried on. As yet no new methods other than those previously examined have been found. However, some interesting related papers have been examined. The survey shall be continued as more of the prospective papers are collected.

Guidance Study

All the memos written at Honeywell on P-matrix predictive guidance have been read. It was decided to do computer simulations to evaluate P-matrix predictive guidance. The three stage Thor vehicle was chosen as the model and thrust and aero data was gathered on it. It was decided to use a cross product steering law. The predicted terminal error will be calculated as components in a guidance coordinate system. Thrust guidance will be used to drive two or three of the terminal error components to zero. Coding was started on the simulation program.

SECTION IV

PLANS FOR NEXT QUARTER

In the stability area the literature survey will be continued. Also some of the questions raised in the appended paper will be examined. One of these is "What should be the criteria for the choice of positive definite forms used in calculating Lyapunov functions?"

The computer program to evaluate P-matrix guidance will be coded and checked out. Simulations will be run to evaluate P-matrix guidance and modifications will be made to increase its ability to guide for off-nominal trajectories.

SECTION V

EXPENDITURES

Total funds expended on Modification 4 contract from its effective date of November 14, 1964 to February 7, 1965 have been $3694. This is 16 percent of the funds from Modification 4.

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APPENDIX

LYAPUNOV FUNCTIONS AND THEIR RELATED SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO FINDING THEIR BEST ESTIMATORS

by

D. L. Lukes

SECTION I

INTRODUCTORY THEORY

Introduction

Lyapunov functions have been used by mathematicians in the study of stability and asymptotic stability of the solutions of ordinary differential equations.* They can also be applied to determine regions of asymptotic stability and to find estimates of the disturbed motion about a reference trajectory. Consequently they have a high potential for being useful in many problems in applied mathematics.

Unfortunately, this use has been limited by the difficulty usually encountered in determining them for specific differential equations.** But, fortunately for the analyst, there is a one-to-many correspondence between a differential equation and its Lyapunov functions. Consequently, the technique of choosing a positive definite form and attempting to adjust its parameters to obtain a Lyapunov function for the system could conceivably be developed into a highly effective applied mathematical tool-particularly if the parameter adjustment could be done on high speed computers.

In this section the correspondence between systems and

*See L. Cesari [2, 107] for a discussion of Lyapunov's second method and bibliographical notes.

**V. I. Zubov [5] has developed a method for constructing Lyapunov functions as solutions to partial differential equations.

their Lyapunov functions is studied. On the basis of some of these results, functions $d = d(\omega)$ defined on the parameter space Ω are obtained in section II. The parameters in the form can be adjusted by minimizing $d(\omega)$ over Ω and so the process can be carried out on a computer.

Summary

We begin by establishing the correspondence between a Lyapunov function and the differential systems to which it applies. This is done first for the autonomous case (Theorem 1.3) and later to the more complicated case where time enters explicitly in the right hand sides of the system equations (Theorem 1.8).

For the autonomous case this correspondence is examined in more detail to arrive at cannonical forms for systems with quadratic Lyapunov functions (Corollaries 1.5 and 1.6).

Theorem 1.9 shows that if the Lyapunov derivative of a positive definite form V is negative in a deleted neighborhood of the origin, then a sufficient condition for global asymptotic stability is that its derivative in the direction ∇V be nonpositive.

We now prove some lemmas.

Lemma 1.1

Let $y \epsilon R^n$ and $y \neq 0$. Then

$y^1 = \{Ky\}_{K \epsilon \mathcal{K}}$ where

= all skew symmetric nxn matrices.

Proof:

$\{Ky\}_{K\in\mathcal{K}} \subseteq y^{\perp}$ since for arbitrayr $K\in\mathcal{K}$,

$(y,Ky) = (-Ky,y) = -y,Ky)$ so

$(y,Ky) = 0$.*

Now we show $\{Ky\}_{K\in\mathcal{K}} \supseteq y^{\perp}$:

Let $x\in y^{\perp}$ and $e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$. Then since $y \neq 0$, there exists an orthogonal matrix U for which $y = \|y\| Ue$. Thus, since $(x,y) = 0$, $(x,\|y\|Ue) = \|y\|(U^*x,e) = 0$. Thus $(U^*x,e) = 0$.

But it is clear that this requires

$$U^*x = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ 0 \end{pmatrix}$$ for some z_i, $i=1,2,\ldots n$.

*(x,y) and $\|x\| = \sqrt{x,x})$ denote the euclidean inner product and norm, respectively.

But $\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & z_1 \\ 0 & \cdots & \cdots & 0 & z_2 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & z_{n-1} \\ -z_1 & -z_2 & \cdots & -z_{n-1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Thus, $x = U\begin{pmatrix} z_1 \\ z_2 \\ z_{n-1} \\ 0 \end{pmatrix} = \left[U \begin{pmatrix} 0 & \cdots & \cdots & 0 & z_1 \\ 0 & \cdots & \cdots & 0 & z_2 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & z_{n-1} \\ -z_1 & -z_2 & \cdots & -z_{n-1} & 0 \end{pmatrix} U^* \right] U \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$$= \left[\frac{U \begin{pmatrix} 0 & \cdots & \cdots & 0 & z_1 \\ 0 & \cdots & \cdots & 0 & z_2 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & z_{n-1} \\ -z_1 & -z_2 & \cdots & -z_{n-1} & 0 \end{pmatrix}}{\|y\|} U^* \right] y \in \{Ky\}_{K \in \mathcal{K}} .$$

Q.E.D.

Lemma 1.2

Let $y \in R^n$ and $y \neq 0$. Then the solutions of the inequality $(x,y) < 0$ are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -\lambda & & & \alpha \\ & -\lambda & & \\ & & \ddots & \\ -\alpha & & & -\lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

where $\lambda > 0$ and the elements α are arbitrary.

Proof:

Let x satisfy $(x,y) < 0$ where $0 \neq y \in R^n$. Then $x = \mu y + v$ where μ is a scalar and $v \in y^{\perp}$. By the previous lemma, $v = Ky$ for some skew-symmetric matrix K. Also, $(x,y) = (\mu y + v, y) = \mu \|y\|^2 < 0$ so $\mu < 0$.

Conversely, every such x provides a solution.

Q.E.D.

We can now determine the systems with respect to which a given function is a Lyapunov function.

Theorem 1.3

Let $f \in C^1(R^n)$ and V satisfy:

(i) $0 \leq V \in C^2(R^n)$

(ii) $V(x) = 0 \Leftrightarrow x = 0$

(iii) $V(x) \to \infty$ as $\|x\| \to \infty$

(iv) $\nabla V(x) = 0 \Rightarrow x = 0$.

Then V is a Lyapunov function for $f \Leftrightarrow f(x) = [K(x) - \lambda(x) I] \nabla V(x)$ where $\lambda \in C^1$ and $0 < \lambda(x) \leq \frac{\|f(x)\|}{\|\nabla V\|}$ on $R^n - 0$ and K(x) is skew symmetric.

Proof:

($\Leftarrow$) If $f(x) = \left[K(x) - \lambda(x)I\right]\nabla V$ as specified in the theorem, then

$$(f,\nabla V) = (\nabla V, K\nabla V) - \lambda(\nabla V,\nabla V)$$

$$= -\lambda\|\nabla V\|^2 < 0 \text{ on } R^n - 0.$$

($\Rightarrow$) If $(f,\nabla V) < 0$ on $R^n - 0$, then by the previous lemma

$$f(x) = -\lambda(x)\,\nabla V(x) + K(x)\nabla V(x).$$

But $(f,\nabla V) = -\lambda\|\nabla V\|^2$ so

$$\lambda = \frac{-(f,\nabla V)}{\|\nabla V\|^2} \in C^1(R^n - 0) \text{ and } \lambda(x) > 0 \text{ for } x \neq 0.$$

Q.E.D.

Remarks:

(1) The theorem says that V corresponds to f as a Lyapunov function if and only if they can be related by a $\lambda(x)$ and $K(x)$ according to the equation $f(x) = \left[K(x) - \lambda(x)I\right]\nabla V(x)$.

We will now show that this equation can be solved for $V(x)$.

First we need a lemma.

$\Leftrightarrow$ denotes "if and only if"

$\Rightarrow$ and $\Leftarrow$ denote "implies"

Lemma 1.4

If K is any skew symmetric matrix then its eigenvalues lie on the imaginary axis so that if $\lambda \neq 0$ then $K - \lambda I$ is a non-singular matrix.

Proof:

From matrix theory we know that the unitary matrices are the exponentials of skew symmetric matrices. But the eigenvalues of unitary matrices have modulus 1. Then by the spectral mapping theorem the eigenvalues of skew symmetric matrices are on the imaginary axis.*

Q.E.D.

Thus, by the lemma, for $\lambda \neq 0$, $\nabla V = [K - \lambda I]^{-1} f$. But this partial differential equation can be solved [1 , 296].

$$V(x) = \int_0^{x_1} v_1(\sigma, x_2, x_3, \ldots, x_n) d\sigma + \int_0^{x_2} v_2(0, \sigma, x_3 \ldots, x_n) d\sigma$$

$$+ \; . \; . \; . \; . \; . + \int_0^{x_n} v_n(0, 0, \ldots, 0, \sigma) d\sigma$$

where $v(x) \equiv [K(x) - \lambda(x) I]^{-1} f(x)$.

(2) Whereas Theorem 1.3 provides the correspondence from a positive definite form to the systems respect to which it is a Lyapunov function, remark (1) provides the correspondence in the other direction.

*see F.R. Gantmacher [3]

That is, suppose $f(x)$ is given and we want to construct its Lyapunov functions, we would have to find the $\frac{n^2-n+2}{2}$ - tuples of functions to constitute the elements of $K(x)$ and $\lambda(x)$ which in turn would provide $v(x)$ which in turn would be integrated to get $V(x)$. Note that the elements of $K(x)$ and $\lambda(x)$ must be related through f so that the $\frac{n^2-2}{2}$ partial differential equations $\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i}$ hold.

(3) Suppose $f \epsilon C^1(R^n)$, $f(x) = 0 \Leftrightarrow x = 0$ and $V(x)$ satisfies (i) - (iv). Then $K(x)$ and $\lambda(x)$ can be calculated so that $f(x) = [K(x) - \lambda(x)I]\nabla V(x)$. Furthermore suppose $\lambda(x) > 0$ on some deleted neighborhood of the origin. Then f is globally asymptotically stable if det.$[K(x) - \lambda(x)I] \neq 0$ for $x \neq 0$.

<u>Corollary 1.5</u>

The C^1 autonomous differential systems on R^n which have quadratic Lyapunov functions can be represented (module a non-singular linear transformation) as:

$$\dot{x} = [K(x) - \lambda(x)P]x \quad \text{where}$$

$K = K(x)$ is skew symmetric, $\lambda = \lambda(x) > 0$ on $R^n - 0$ and $\lambda \epsilon C^1(R^n - 0)$, and P is a positive definite symmetric nxn matrix.

Every such system has the quadratic Lyapunov function $V(x) = \|x\|^2$. Also the form of the system is invariant with respect to orthogonal transformations.

Proof:

Assume the hypotheses; namely that $\dot{x} = f(x) \in C^1(R^n)$ and $(f,\nabla V) < 0$ on $R^n - 0$ where $V(x) = \frac{1}{2}(x,Px)$, P positive definite and symmetric. Then $\nabla V(x) = Px$ so by the previous corollary,

$f(x) = [K(x) - \lambda(x)I]Px$ where $\lambda > 0$ on $R^n - 0$ and $\lambda \epsilon C^1(R^n - 0)$ and $K(x)$ is skew symmetric.

Now make the change of variable $y = \sqrt{P}\, x$ where $\sqrt{P}$ is the unique symmetric positive definite square root of P. Then

$$\dot{y} = \sqrt{P}\,\dot{x} = \sqrt{P}\,[K - \lambda I]\Big|_{x = (\sqrt{P})^{-1}y} \sqrt{P}\, y \text{ so}$$

$$\dot{y} = [(\sqrt{P}\, K\, \sqrt{P}) - \lambda P]\Big|_{x = (\sqrt{P})^{-1}y} y. \text{ But}$$

$\sqrt{P}\, K(\sqrt{P})$ is skew symmetric and $0 < \lambda(\sqrt{P}^{-1}y)$ on $R^n - 0$ and also $C^1(R^n - 0)$.

Conversely, every system of the form $\dot{x} = [K(x) - \lambda(x)P]x$ has $V(x) = \|x\|^2$ as a Lyapunov function since $\nabla V(x) = 2x$ and $([K - \lambda P]x,\ 2x) = 2(x,Kx) - 2\,\lambda(x,Px)$

$$= 2\lambda(x,Px) \text{ for } x \neq 0.$$

To see that the given form of the differential equation is invariant with respect to orthogonal linear transformations, let $y = Ux$ where U is an orthogonal matrix.

Then $\dot{y} = U\dot{x} = U[K - \lambda P] U^*y$

$$= [UKU^* - \lambda UPU^*]y.$$

But UKU* is skew symmetric and UPU* is positive definite symmetric.

Q.E.D.

Now by a judicious choice of U and a possible reparameterization of the solutions we can obtain an even further reduction to a cannonical form:

Corollary 1.6

Every C^1 autonomous differential system on R^n which has a quadratic Lyapunov function (in a coordinate system (x)) is geometrically equivalent to a system

$$\dot{x} = [K(x) - \Lambda]x \quad \text{where}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}, \; \lambda_i > 0 \text{ and constant } i = 1,2,\ldots n \text{ and}$$

$$K(x) = \begin{pmatrix} 0 & & & \\ & 0 & \ddots & \alpha(x) \\ & & \ddots & \\ & -\alpha(x) & & \ddots \; 0 \end{pmatrix}.$$

Remarks:

(1) Every such system has the Lyapunov function $V(x) = \|x\|^2$.

(2) Every such system in which $\alpha(x) \epsilon C^1(R^n)$ has a linear part.

Proof:

In the previous corollary, choose U so as to diagonalize P. Also reparameterize by $\frac{ds}{dt} = \lambda(x)$.

We shall now solve the nonhomogeneous inequality which occurs in the analysis of non-autonomous systems. It is a generalization of lemma 1.2.

Lemma 1.7

Let $y \epsilon R^n$ and $y \neq 0$. Let c be a fixed scalar.

Then the solutions of the inequality $(x,y) < c$ are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -\lambda & & & \alpha \\ & -\lambda & & \\ -\alpha & & \ddots & \\ & & & -\lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + c \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

where $\lambda > 0$, the elements of α are arbitrary and β satisfies the equation $(\beta,y) + \lambda c = 1$.

Proof: Let $x = (x_1, x_2, \ldots, x_n)^*$ be an arbitrary element in R^n. Then $(x_1, x_2, \ldots, x_n, 1)^*$ in R^{n+1} can be represented as $u - \lambda v$ where λ is some scalar, $v = (y_1, y_2, \ldots, y_n, -c)^*$ and $u \epsilon v^{\perp}$. But if x is a solution of the inequality $(x,y) < c$, then $(x,y) - c = -\lambda \|v\|^2 < 0$. But $v \neq 0$ so $\lambda > 0$.

Furthermore, by lemma 1.1, u can be written as $u = \hat{K}v$ where $\hat{K}$ is a skew symmetric $(n+1)x(n+1)$ matrix,

$$\hat{K} = \begin{pmatrix} 0 & & & \hat{\alpha} \\ & 0 & & \\ & & \ddots & \\ -\hat{\alpha} & & & 0 \end{pmatrix}.$$

Thus we see that x solves the inequality if and only if $(x_1, x_2, \ldots, x_n, 1)^* = u - \lambda v$ for $\lambda > 0$ and $u \epsilon v^{\perp}$. That is, the solutions of the inequality $(x,y) < c$ are the solutions of the system

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{pmatrix} = \begin{pmatrix} -\lambda & & & & \\ & -\lambda & & \hat{\alpha} & \\ & & \ddots & & \\ -\hat{\alpha} & & & \ddots & \\ & & & & -\lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ -c \end{pmatrix}.$$

But we can write

$$\begin{pmatrix} -\lambda & & & & \\ & -\lambda & & \hat{\alpha} & \\ & & \cdot & & \\ & -\hat{\alpha} & & \cdot & \\ & & & & -\lambda \end{pmatrix} = \left(\begin{array}{ccccc|c} -\lambda & & & & & -\beta_1 \\ & -\lambda & & & \alpha & -\beta_2 \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ -\alpha & & & & \cdot & \cdot \\ & & & & -\lambda & -\beta_n \\ \hline \beta_1 & \beta_2 & \cdots & & \beta_n & -\lambda \end{array}\right)$$

so the solutions of the inequality can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} -\lambda & & & & \\ & -\lambda & & & \alpha \\ & & \cdot & & \\ & & & \cdot & \\ & -\alpha & & & \cdot \\ & & & & -\lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} + c \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{pmatrix}$$

where the equation $(\beta, y) + \lambda c = 1$ must also be satisfied.

Q.E.D.

We shall now make the appropriate assumptions and definitions so that we can prove the theorem corresponding to theorem 1.3 for the nonautonomous case. For a discussion of how nonautonomous systems arise in applications we refer the reader to Section II.

Consider a system of differential equations $\dot{x}_i = f_i(x_1, x_2, \ldots x_n, t)$ $i = 1, 2, \ldots n$ where the right hand sides together with their partial derivatives $\frac{\partial f_i}{\partial x_j}(x_1,x_2,\ldots,x_n,t)$ are continuous in a domain Γ of the space of variables $x_1,x_2,\ldots,x_n,t$.* The differential equation will also be written as a vector equation $\dot{x} = f(x,t)$. We make the same assumptions on a function $v = v(x,t)$ and its partial derivatives $\frac{\partial v(x,t)}{\partial x_i}$, $i = 1,2,\ldots n$ in Γ. We further assume that Γ has the form $\Gamma = N_{\epsilon_o}(v) \equiv [(x,t) \| 0 \leq v(x,t) < \epsilon_o,$ $t_o \leq t \leq T]$ where ϵ_o, t_o and T are fixed constants and that $v(x,t) = 0$ and $f(x,t) = 0$ if and only if $x = 0$, for each $t \epsilon [t_o,T]$. We also require that for each fixed $t_1 \epsilon [t_o,T]$ the level surfaces $v(x,t_1) = \text{constant} \leq \epsilon_o$ are simple closed surfaces (topological n-spheres) about $(0,t_1)$. For example, if for each fixed $t_1 \epsilon [t_o,T]$, $v(x,t_1)$ is defined for all $x \epsilon R^n$, then the previous condition is satisfied if $v(x,t) \to \infty$ as $\|x\| \to \infty$ and $\nabla_x v(x,t) = 0$ if and only if $x = 0$, for each fixed $t \epsilon [t_o,T]$.

A condition sufficient for trajectories passing through a point (x^1,t_1) in $N_{\epsilon_o}(v)$ at time $t_1 < T$ to remain in the set $N_{\epsilon_1}(v) \equiv [(x,t) \| 0 \leq v(x,t) \leq v(x^1,t_1)]$ for future time up to T is that $\frac{dv(x,t)}{dt} < 0$ along the trajectories in $N_{\epsilon_o}(v)$ for which $x \neq 0$.

Definition

When f and v satisfy all of the above conditions we say

*These conditions are sufficient for the existence and uniqueness of the solutions of the differential equations, see Pontryagin [4,159].

that f(x,t) is v-stable on $N_{\epsilon_o}(v)$.

We can now apply lemma 1.7 to determine the v-stable systems.

Theorem 1.8

The systems which are v-stable on $N_{\epsilon_o}(v)$ are those which can be written in the form:

$$f(x,t) = [K(x,t) - \lambda(x,t)I]\nabla_x v(x,t) - \frac{\partial v(x,t)}{\partial t}\beta(x,t)$$

on $N_{\epsilon_o}(v)$ where $\lambda(x,t) > 0$ for $x \neq 0$, K is skew symmetric and β satisfies the equation $(\beta, \nabla_x v) - \lambda \frac{\partial v}{\partial t} = 1$.

Proof: The proof is a simple application of lemma 1.7 and the fact that $\frac{dv(x,t)}{dt} = (f, \nabla_x v) + \frac{\partial v}{\partial t}$ along a trajectory.

Remarks:

(1) Theorem 1.8 is a generalization of Theorem 1.3.

(2) The way that this theorem could be applied would be to choose a v and $N_{\epsilon_o}(v)$ for the problem and then try to choose the parameters K, λ and β so that the resulting system f(x,t) matches up with the system being analyzed. Notice that the equation that β must satisfy is a linear algebraic equation.

We now return to the autonomous case.

Theorem 1.9

Suppose $f(x) \in C^1(R^n)$ and that $f(x) = 0 \Leftrightarrow x = 0$.

Furthermore suppose V satisfies:

(i) $0 \leq V \in C^2(R^n)$

(ii) $V(x) = 0 \Leftrightarrow x = 0$

(iii) $V(x) \to \infty \Rightarrow x = 0$, and

$(f, \nabla V) < 0$ in a deleted neighborhood of the origin.

Then if $(\nabla V, \nabla(f, \nabla V)) \leq 0$ in R^n the system $\dot{x} = f(x)$ is globally asymptotically stable.

Proof:

Suppose the hypotheses of the theorem hold.

If $(f, \nabla V) < 0$ on $R^n - 0$ then V is a Lyapunov function and the system is globally asymptotically stable.

We now consider the contrary case. Thus, there exists $x \neq 0$ in R^n such that $(f(x), \nabla V(x)) = 0$ (by the intermediate value theorem which applies since $(f, \nabla V)$ is continuous in $R^n - 0$ which is connected and since $(f, \nabla V) < 0$ in a deleted neighborhood of the origin.)

Define $T = [x: (f(x), \nabla V(x)) = 0,\ x \neq 0]$ which we note to be a nonempty closed set. Then by (iii) and the continuity of V, $\min_{x \in T} V(x)$ occurs for a point in T which we denote by x^*.

Thus $V(x^*) = \min_{x \in T} V(x)$.

Next we define

$$D^* \equiv [x: 0 < V(x) < V(x^*)].$$

This is an open, arc-wise connected set about the origin.

To see the arcwise connectedness of a set $[x: 0 < V(x) < \alpha]$ where $\alpha > 0$, suppose there were a component other than the one about the origin - call it θ. But θ is bounded since $0 < V(\theta) < \alpha$ and $V(x) \to \infty$ as $\|x\| \to \infty$. Thus $\min_{x \in \bar{\theta}} V(x)$ occurs on $\bar{\theta}$.

If $\min_{x \in \bar{\theta}} V(x)$ occurs at a point in θ, then it is necessary that $\nabla V = 0$ at the point, which is impossible, since θ does not contain the origin.

The remaining possibility is that $\min_{x \in \bar{\theta}} V(x)$ occurs on $\partial\theta$, the boundary of θ. But $\partial\theta = [x: V(x) = \alpha]$ so we would have $V(\theta) = \alpha$ and so again $\nabla V = 0$, this time on all of θ. Thus there are no other components.

(a) Therefore we have $(f, \nabla V) \neq 0$ on D^* which is connected, but $(f, \nabla V) < 0$ on part of D^* so $(f, \nabla V) < 0$ on all of D^*.

(b) We now show that $x^* - \epsilon \nabla V(x^*)$ is in D^* for all sufficiently small $\epsilon > 0$:

$V(x^* - \epsilon \nabla V(x^*)) = V(x^*) - \epsilon \|\nabla V(x^*)\|^2 + O(\epsilon^2)$. But $\nabla V(x^*) \neq 0$ so for all sufficiently small $\epsilon > 0$, $V(x^* - \epsilon \nabla V(x^*)) < V(x^*)$. Also $x^* - \epsilon \nabla V(x^*) \neq 0$ for all sufficiently small $\epsilon > 0$ since $x^* \neq 0$.

Therefore $0 < V(x^* - \epsilon \nabla V(x^*)) < V(x^*)$ for all sufficiently small $\epsilon > 0$.

(c) Thus from (a) and (b) we have $(f[x^* - \epsilon\nabla V(x^*)],$ $\nabla V[x^* - \epsilon \nabla V(x^*)]) < 0$ for all sufficiently small $\epsilon > 0$.

Now we shall show that this leads to a contradiction of our hypotheses.

(d) Expanding,

$$(f[x^* - \epsilon\nabla V(x^*)], \nabla V[x^* - \epsilon\nabla V(x^*)]) =$$

$$= (f(x^*), \nabla V(x^*)) - \epsilon(f(x^*), (\frac{\partial\nabla V}{\partial x^*}) \nabla V(x^*))$$

$- \epsilon(\nabla V(x^*), (\frac{\partial f}{\partial x^*}) \nabla V(x^*)) + h(0(\epsilon^2))$ where $\|h(0(\epsilon^2))\| = 0(\epsilon^2)$ as $\epsilon \to 0$. Thus, letting $\epsilon \to 0$ and dividing by -1,

$$(f(x^*), (\frac{\partial\nabla V}{\partial x^*}) \nabla V(x^*)) + (\nabla V(x^*), (\frac{\partial f}{\partial x^*}) \nabla V(x^*)) > 0.$$

But $(\frac{\partial\nabla V}{\partial x})$ is a symmetric matrix so

$$(\nabla V(x^*), (\frac{\partial\nabla V}{\partial x^*}) f(x^*) + (\frac{\partial f}{\partial x^*}) \nabla V(x^*)) > 0$$

But this says $(\nabla V, \nabla(f,\nabla V)) > 0$ at x^* which is contrary to our hypothesis.

Q.E.D.

SECTION II

APPLICATIONS, CALCULATION OF BEST ESTIMATORS

Some Remarks About Section I

One of the objectives in Section I was to study the connection between a differential equation and its Lyapunov functions. We saw that part of the form of a differential equation can be changed without requiring a change in the Lyapunov function. That is, if we have a system $f(x)$ with Lyapunov function $V(x)$ then we can perturb the system by adding any term of the form $K(x)\,\nabla V(x)$ (where $K(x)$ is skew symmetric) to get another system with the same Lyapunov function. This has the important implication that once a Lyapunov function has been found for a system there are a variety of design changes that can be made in the physical system which will not affect the stability of the system. Similar remarks could be made about nonautonomous systems.

Another conclusion that can be drawn from theorem 1.3 is that if the parameters in a positive definite form $\tilde{V}$ can be adjusted so that it will be a Lyapunov function for $f(x)$, then the parameter adjustment could be done by equating coefficients in an equation $(f,\nabla\tilde{V}) = -\lambda(x)\,\|\nabla\tilde{V}\|^2$ for some $\lambda(x) > 0$. A corresponding remark could be made about the nonautonomous case.

It should be noted that the conditions in the theorems could be changed to give local results. Of course that is all that is required in many applied problems since the physical system operates in only a small part of the phase space.

We now discuss how some applied problems fall into the category that has been considered.

How Lyapunov Functions Might Apply to the Study of Equations of Disturbed Motion

We shall now indicate how Lyapunov functions might be used to obtain estimates of the disturbed motion about a reference trajectory.

Consider an ordinary differential equation

$\dot{y} = g(y,t)$ to be called the equation of undisturbed motion and let $\bar{y} = \bar{y}(t,t_o,y_o)$ be a solution satisfying $\bar{y}(t_o,t_o,y_o) = y_o$. We are interested in analyzing the behavior of the system in a neighborhood of the reference trajectory $\bar{y}$. Define a new variable $x = y - \bar{y}$ and consider the new differential equation

$\dot{x} = f(x,t) \equiv g(x + \bar{y},t) - g(\bar{y},t)$. The latter equation is called the equation of disturbed motion. Any solution $x = x(t,t_o,x_o)$ with $x(t_o,t_o,x_o) = x_o$ of this equation gives the vector difference between corresponding points (in time) on the reference trajectory and the trajectory generated by the equation of undisturbed motion passing through $y_o + x_o$ at time t_o. Notice that $f(0,t) \equiv 0$ for all t.

It should be noted that no linearization has been made and that an important feature of the Lyapunov technique is that with it an attempt is made at obtaining bounds on the variation in x without linearizing the equation of disturbed motion.

Thus we are led to a nonautonomous differential equation

$\dot{x} = f(x,t)$ where $f(0,t) \equiv 0$. We shall now consider the motion of this system in the space-time configuration space indicated by fig. 1.

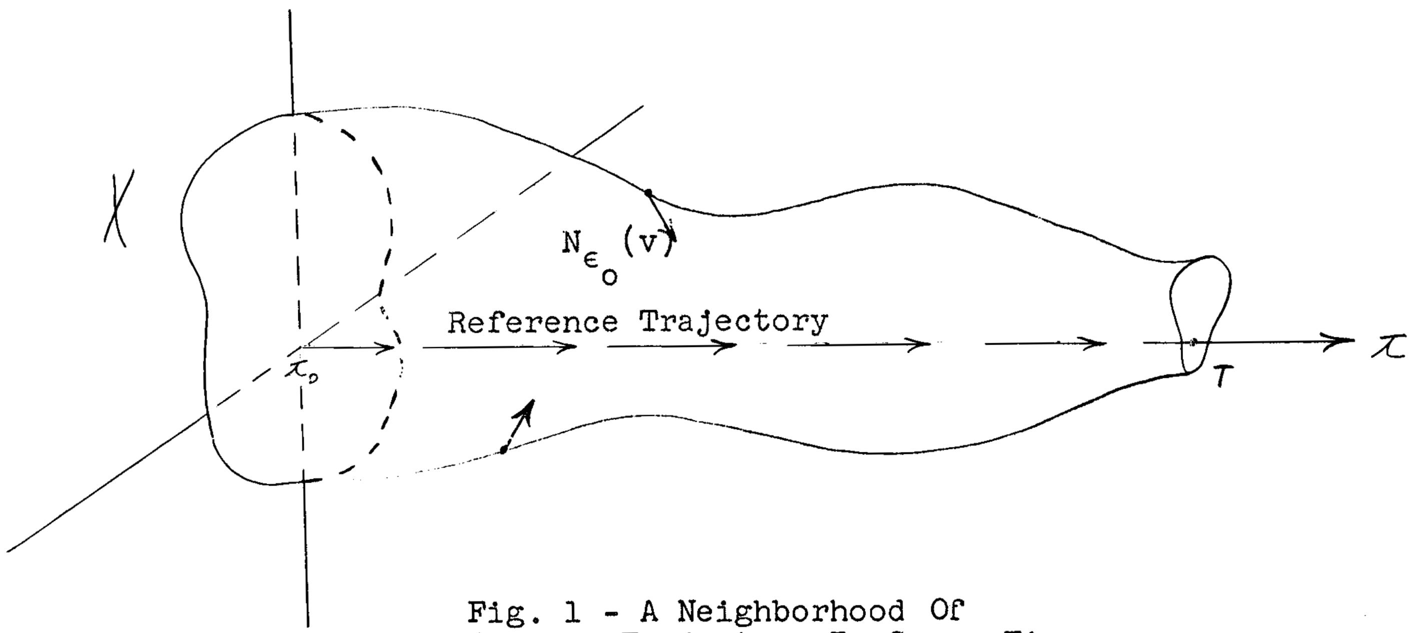

Fig. 1 - A Neighborhood Of
The Reference Trajectory In Space-Time

In certain applied problems it is important that for various kinds of neighborhoods of the reference trajectory the trajectories intersecting the neighborhood remain in the neighborhood for future time. In principle Lyapunov functions can be applied to verify the existence of these neighborhoods and also to determine their size and shape.

In some applications to the guidance and control of aero-space vehicles the system equation for the controlled system $\dot{x} = f(x,t)$ and the reference trajectory $\bar{y}$ are obtained only after much work in the face of many design constraints. Obviously it is important that these constraints not be violated when the system is disturbed away from the reference trajectory. Also it is an important problem to predict the size of the disturbances that can be tolerated. For example, it would be valuable to be able to quickly predict the variance in initial conditions that could be tolerated in a rocket launching problem without critically violating the constraints on the burnout velocity and position. This problem would be solved if a Lyapunov function such as in theorem 1.8 could be found for the system.

We shall now consider the problem of computing Lyapunov functions. First we discuss the autonomous case and later the nonautonomous case.

Calculation of Best Estimators $\hat{V}$ for Autonomous Systems

Let f(x) be a C^1 differential system which is asymptotically stable at the origin. Suppose that we have selected a positive definite form $\tilde{V}(x)$ with free parameters in a set Ω and for each choice of the parameters conditions (i) - (iv) of Theorem 1.3 are satisfied. The problem to be considered is how to adjust the parameters of $\tilde{V}(x)$ so as to make it into a Lyapunov function for f(x).

In view of Theorem 1.3 we can now consider the systems of differential equations $\tilde{f}$ which correspond to $\tilde{V}$, that is,

$$\tilde{f}(x) = [K(x) - \lambda(x)I]\nabla\tilde{V}(x)$$

where K(x) is any skew symmetric matrix function of x and $\lambda(x)$ is any positive definite form. We now propose approximating f by $\tilde{f}$ by suitably determining K, λ and the free parameters in $\tilde{V}$.

Since K is allowed to depend upon x we first choose it so as to solve the problem, $\min_K \|f - \tilde{f}\|$. But

$$\|f - \tilde{f}\|^2 = \|f - K\nabla\tilde{V} + \lambda\nabla\tilde{V}\|^2$$

$$= \|f - K\nabla\tilde{V}\|^2 + 2\lambda(f - K\nabla\tilde{V},\ \nabla\tilde{V}) + \lambda^2\|\nabla\tilde{V}\|^2$$

$$= \|f - K\nabla\tilde{V}\|^2 + 2\lambda(f,\nabla\tilde{V}) + \lambda^2\ \|\nabla\tilde{V}\|^2.$$

Thus K occurs in only the first term and it is easy to see that $\min_K \|f - K\nabla\tilde{V}\|^2 = \dfrac{(f,\nabla\tilde{V})^2}{\|\nabla\tilde{V}\|^2}$. Thus

$$\min_K \|f - \tilde{f}\|^2 = \frac{(f,\nabla\tilde{V})^2}{\|\nabla\tilde{V}\|^2} + 2\lambda(f,\nabla\tilde{V}) + \lambda^2\|\nabla\tilde{V}\|^2.$$

Next we minimize over λ(x).

$$\min_{\lambda\geq 0}\ \min_K \|f - \tilde{f}\|^2 = \begin{cases} \dfrac{(f,\nabla\tilde{V})^2}{\|\nabla\tilde{V}\|^2} & \text{when } (f,\nabla\tilde{V}) > 0 \\ 0 & \text{when } (f,\nabla\tilde{V}) \leq 0. \end{cases}$$

Now it would seem appropriate to conclude the approximation by a least squares type determination of the parameters by choosing a collection of base points $\hat{x}$ filling out a neighborhood of the origin under investigation. In short we are led to adjusting the free parameters in the form $\tilde{V}$ by minimizing the non-negative real valued function $d(\omega)$ defined on the parameters space Ω by the formula:

$$d(\omega) = \min_{\lambda \geq 0} \min_{K} \sum_{\hat{x}} \frac{\|f - \tilde{f}\|^2}{\|f\|^2} = \sum_{(f,\nabla\tilde{V})>0} \left(\frac{f}{\|f\|}, \frac{\nabla\tilde{V}}{\|\nabla\tilde{V}\|}\right)^2 .$$

Whenever $d(\omega)$ has a minimum on Ω we call the corresponding $\tilde{V}$ a best estimator and denote it by $\hat{V}$.

Thus once the form $\tilde{V}$ has been selected and the base points $\hat{x}$ specified, then the computation for adjusting the parameters to obtain $\hat{V}$ reduces to a standard numerical problem of minimizing a real valued function $d(\omega)$ of several variables over Ω. Of course $\hat{V}(x)$ depends upon $\tilde{V}$, $\hat{x}$ and f. Once the calculation is terminated, the region where $(f(x), \nabla\hat{V}(x)) < 0$ will already be calculated.

Calculation of Best Estimators $\hat{v}$ for Nonautonomous Systems

We now carry out the corresponding calculations for the nonautonomous case. Again we assume that a positive definite form $\tilde{v}(x,t)$ with parameter space Ω has been selected. As in the autonomous case we attempt to obtain an algorithm for adjusting the parameters of $\tilde{v}$ by approximating f by systems $\tilde{f}$ which are $\tilde{v}$ stable with respect to $\tilde{v}$. Thus we define

$$d(\omega) = \min_{\lambda \geq 0} \min_{K} \sum_{(\hat{x},t)} \frac{\|f - \tilde{f}\|^2}{\|f\|^2} \quad \text{on } \Omega$$

where $\tilde{f}$ is to have the form

$$\tilde{f}(x,t) = [K(x,t) - \lambda(x,t)I]\ \nabla_x \tilde{v}(x,t) - \frac{\partial \tilde{v}(x,t)}{\partial t}\,\beta(x,t) \quad ;$$

where K is skew symmetric and $\beta(x,t)$ is any solution of $(\beta,\nabla_x\tilde{v})-\lambda\,\frac{\partial\tilde{v}}{\partial t} = 1.$

Here $(\hat{x,t})$ denotes a collection of base points in the neighborhood of the reference trajectory in the space-time configuration space. As before, the algorithm for adjusting the parameters in $\tilde{v}$ will be to minimize $d(\omega)$ over Ω. When this minimum is realized at some point in Ω the corresponding $\tilde{v}$ will be called a best estimator and be denoted by $\hat{v}$.

We now calculate the two minimums indicated in the formula for $d(\omega)$.

$$\|f-\tilde{f}\|^2 = \|f\|^2 - 2(f,\tilde{f}) + \|\tilde{f}\|^2.$$

$$(f,\tilde{f}) = (f,K\nabla_x\tilde{v}) - \lambda(f,\nabla_x\tilde{v}) - \frac{\partial\tilde{v}}{\partial t}(\beta,f).$$

$$\|\tilde{f}\|^2 = \|K\nabla_x\tilde{v}\|^2 - \frac{\partial\tilde{v}}{\partial t}(\beta,K\nabla_x\tilde{v})$$

$$+ \lambda^2\|\nabla_x\tilde{v}\|^2 + \lambda\frac{\partial\tilde{v}}{\partial t}(\beta,\nabla\tilde{v})$$

$$-\frac{\partial\tilde{v}}{\partial t}(\beta,K\nabla_x\tilde{v}) + \lambda\frac{\partial\tilde{v}}{\partial t}(\beta,\nabla_x\tilde{v}) + \left(\frac{\partial\tilde{v}}{\partial t}\right)^2\|\beta\|^2$$

$$= \|K\nabla_x\tilde{v}\|^2 - 2\,\frac{\partial\tilde{v}}{\partial t}(\beta,K\nabla_x\tilde{v})$$

$$+ \lambda^2\|\nabla_x v\|^2 + 2\lambda\frac{\partial\tilde{v}}{\partial t}(\beta,\nabla\tilde{v}) + \left(\frac{\partial\tilde{v}}{\partial t}\right)^2\|\beta\|^2.$$

Thus,

$$\|f-\tilde{f}\|^2 = \|K\nabla_x\tilde{v}\|^2 - 2\frac{\partial\tilde{v}}{\partial t}(\beta,K\nabla_x\tilde{v}) - 2(f,K\nabla_x\tilde{v})$$

$$+ \lambda^2\|\nabla_x\tilde{v}\|^2 + 2\lambda\frac{\partial\tilde{v}}{\partial t}(\beta,\nabla_x\tilde{v}) + 2\lambda(f,\nabla_x\tilde{v})$$

$$+ 2\frac{\partial\tilde{v}}{\partial t}(\beta,f) + \left(\frac{\partial\tilde{v}}{\partial t}\right)^2\|\beta\|^2 + \|f\|^2$$

$$= \|K\nabla_x\tilde{v} - \left(\frac{\partial\tilde{v}}{\partial t}\beta + f\right)\|^2 - \|\frac{\partial\tilde{v}}{\partial t}\beta + f\|^2$$

$$+ \ . \ . \ .$$

$$+ \ . \ . \ .$$

$$= \|K\nabla_x\tilde{v} - \left(\frac{\partial\tilde{v}}{\partial t}\beta + f\right)\|^2$$

$$+ \lambda^2\|\nabla_x\tilde{v}\|^2 + 2\lambda\frac{\partial\tilde{v}}{\partial t}(\beta,\nabla_x\tilde{v}) + 2\lambda(f,\nabla_x\tilde{v}).$$

But $\min\limits_K \|K\nabla_x\tilde{v} - \left(\frac{\partial\tilde{v}}{\partial t}\beta + f\right)\|^2 = \frac{\left(\frac{\partial\tilde{v}}{\partial t}\beta + f,\ \nabla_x\tilde{v}\right)^2}{\|\nabla_x\tilde{v}\|^2}$

$$= \left[\frac{\frac{\partial\tilde{v}}{\partial t}(\beta,\nabla_x\tilde{v}) + (f,\nabla_x\tilde{v})}{\|\nabla_x\tilde{v}\|^2}\right]^2 .$$

Now we use the fact that β is restricted to be a solution of the equation $(\beta,\nabla_x\tilde{v}) = 1 + \lambda\frac{\partial\tilde{v}}{\partial t}$ in the two equations above to get

$$\min_K \|f-\tilde{f}\|^2 = \frac{\left[(f,\nabla_x\tilde{v}) + \frac{\partial\tilde{v}}{\partial t}(1 + \lambda\,\frac{\partial\tilde{v}}{\partial t})\right]^2}{\|\nabla_x\tilde{v}\|^2}$$

$$+ 2\lambda\left[(f,\nabla_x\tilde{v}) + \frac{\partial\tilde{v}}{\partial t}\,(1 + \lambda\,\frac{\partial\tilde{v}}{\partial t})\right] + \lambda^2\|\nabla_x\tilde{v}\|^2.$$

Thus we see

$$\min_{\lambda\geq 0}\ \min_K\ \|f-\tilde{f}\|^2 = \begin{cases} \dfrac{\left[(f,\nabla_x\tilde{v}) + \frac{\partial\tilde{v}}{\partial t}\right]^2}{\|\nabla_x\tilde{v}\|^2} & \text{when } (f,\nabla_x\tilde{v}) + \frac{\partial\tilde{v}}{\partial t} > 0 \\ 0 & \text{when } (f,\nabla_x\tilde{v}) + \frac{\partial\tilde{v}}{\partial t} \leq 0 \end{cases}.$$

Therefore we get

$$d(\omega) = \min_{\lambda\geq 0}\ \min_K \sum_{\hat{(x,t)}} \frac{\|f-\tilde{f}\|^2}{\|f\|^2}$$

$$= \sum_{(f,\nabla_x\tilde{v}) + \frac{\partial\tilde{v}}{\partial t} > 0} \left[\left(\frac{f}{\|f\|},\ \frac{\nabla_x\tilde{v}}{\|\nabla_x\tilde{v}\|}\right) + \frac{\frac{\partial\tilde{v}}{\partial t}}{\|f\|\,\|\nabla_x\tilde{v}\|}\right]^2 .$$

Thus we see that the formula for $d(\omega)$ reduces to the one previously derived for the autonomous case.

We have not discussed the problem of how $\tilde{V}$ or $\tilde{v}$ and Ω is to be selected. A study of this problem would be of value to the technique for computing best estimators discussed above.

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SYMBOLS COMMONLY USED

Ω	parameter space
$\tilde{V}(x)$	positive definite form
$\tilde{v}(x,t)$	positive definite form
$y^{\perp}$	orthogonal complement
K	skew symmetric matrix
$\mathcal{K}$	class of skew symmetric matrices
(x,y)	euclidean inner product
$\|x\|$	euclidean norm
U	orthogonal matrix
R^n	n-dimensional euclidean space
$C^1(R^n)$	class of functions with continuous first partial derivatives on R^n
$f(x)$	right hand side for a system of autonomous differential equations
$f(x,t)$	right hand side for a system of nonautonomous differential equations
R^n-0	R^n with the origin deleted
$\nabla V(x)$	gradient of $V(x)$
I	identity nxn matrix
P	positive definite symmetric matrix
$\bar{\Theta}$	closure of the set Θ
x^*, D^*	* denotes a superscript
$(x_1,x_2,\ldots,x_n)^*$	column vector (transposed row)
U^*, P^*	* denotes transpose of a matrix