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ON THE STABILITY AND STRUCTURE OF SHOCK WAVES IN INTERPLANETARY SPACE

F. L. Hinton

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The RAND Corporation
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PREFACE

This Memorandum is part of a continuing study of particles and fields conducted by The RAND Corporation under Contract NASr-21(05) for the National Aeronautics and Space Administration. Its subject is one presently ill-defined by experiment and one of interest to many geophysicists -- the nature of the structure of a shock wave formed by the interaction of the geomagnetic field and the solar wind.

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ABSTRACT

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This Memorandum postulates the structure of shock waves in a low-density plasma and uses it to explain several features of interplanetary shock waves observed by satellites and space probes. It shows in particular that unsteady flow behind the shock wave is caused (when the pressure there is anisotropic and satisfies the condition $\frac{P_{||} - P_{\perp}}{B^2/4\pi} > 1$) by the refraction of small-amplitude disturbances through the shock. The shock wave is stable, however, with respect to small perturbations in the fluid variables, because the problem of small-amplitude disturbances refracting through a fast hydromagnetic shock wave has a unique solution. The length for the unsteady flow's onset is characteristically a few ion cyclotron radii, which agrees with the experimental data. A possible explanation is offered for the increase in the flux of low-energy electrons seen by Explorer XII as the satellite passed inward behind the earth's bow shock on September 13, 1961.

Author

CONTENTS

I. INTRODUCTION

Recent satellite and space-probe experiments have provided experimental evidence for the existence of shock waves in interplanetary space, with thicknesses much smaller than the mean free path (which, in the interplanetary medium, is about 1 A.U.). It seems to be characteristic of these shock waves that the region behind them is a region of unsteady flow, resembling turbulence, with disordered magnetic fields and an isotropic flux of plasma ions. We will show here how such an unsteady region of flow is to be expected behind a shock wave in a low-density plasma. We will first investigate the stability and structure of such shock waves, and then show how our results apply to the shock waves observed in interplanetary space.

The Imp 1 magnetometer and plasma data provide evidence for a bow shock wave in front of the earth's magnetosphere in interplanetary space. Behind this shock wave, the magnetic field fluctuates rapidly in magnitude and direction (Ness, Scearce and Seek, 1964), and a hot isotropic flux of positive plasma ions is observed (Bridge, et al., 1964). The magnetometer experiment aboard Pioneer I (Sonett, Smith, and Sims, 1960; Sonett and Abrams, 1963) determine the low-frequency power spectrum of these magnetic-field fluctuations. Evidence for a second hydromagnetic shock wave in interplanetary space about 10^7 km from earth and traveling towards it was found in the Mariner II magnetometer and plasma data (Sonett, et al., 1964). The plasma flux increased, and the magnetic field increased in magnitude; disordered fields were present for many hours behind the shock wave.

The satellites and space probes mentioned above moved through the observed shock waves so rapidly that no measurements of the structure were possible.

Experimentally, we can distinguish between the following four regions in the flow (see Fig. 1): Region 1, the region ahead of the shock; Region s, the shock wave itself, whose structure has not yet been determined experimentally or theoretically; Region 2, the region behind the shock, where the magnetic fields and plasma flow are unsteady and disordered, whose thickness is very large compared to the thickness of Region s; Region 3, the region of steady flow and ordered fields, which follows the unsteady region after the fluctuations have been damped.

In this Memorandum, we investigate the consequences of dividing the shock wave, Region s, into two sub-regions, A and B, as shown in Fig. 2. We postulate the following structure for shock wave:

Region A: The shock wave proper, in which the electron gas undergoes a non-adiabatic transition, and whose thickness must therefore be no more than a few electron cyclotron radii. We suppose that there is an electric field in the direction opposite to the flow sufficiently strong (over a distance of a few Debye lengths) that the ion gas is slowed down, thus restoring charge neutrality behind this region. The postulated structure for Region A is similar to the structure discussed by Colgate (1959) for a strong shock wave in a collisionless plasma. Part of the energy of the ions' directed motion goes into heating the electrons, and part goes into compressing the ion gas. Because the transition region is so thin compared to an ion cyclotron radius, the compression of the ion gas is essentially one-dimensional.

Region B: The region just behind the shock wave proper, where unstable hydromagnetic waves grow in amplitude. Because we suppose the ion gas to undergo a compression that is essentially one-dimensional, the parallel ion pressure is expected to become much larger than the

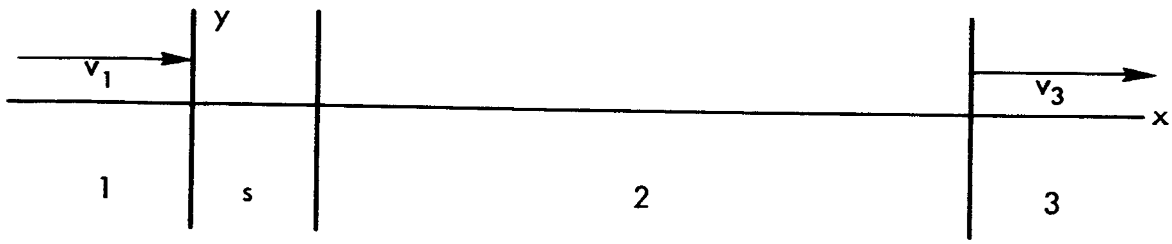

Fig. 1—The four regions in the flow which can be distinguished experimentally

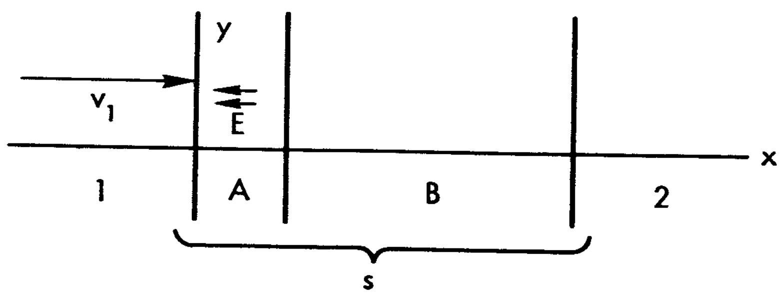

Fig. 2—The postulated shock structure

ion pressure perpendicular to the flow direction. When the magnetic field is also roughly in the direction of the flow, the hose instability criterion

$$\frac{P_{||} - P_{\perp}}{B^2/4\pi} > 1$$

may be satisfied, where $P_{||}$ and $P_{\perp}$ are the ion pressures parallel and perpendicular, respectively, to the magnetic field. Hose instability results in the growth of hydromagnetic waves (Parker, 1958). The amplitudes of these waves remain small enough that linear analysis may be used in a region whose thickness turns out to be a few ion cyclotron radii (see Sect. III below), and this is defined to be Region B.

In Region 2, the growing hydromagnetic waves become nonlinear, producing the disordered magnetic fields, isotropic ion flux, and unsteady flow observed experimentally; eventually the waves are damped by dissipative processes in this region, whose thickness may therefore be expected to be of the order of the mean free path. In the case of the earth's bow shock, the length of this region appears to be larger than the dimensions of the magnetosphere. The disordered fields behind the shock wave described by Sonett, et al. (1964), lasted many hours, indicating a length comparable to 1 A.U.

In Section II below, we prove the stability of Region A by assuming that it may be idealized as a plane, fast hydromagnetic shock.*

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*The terminology is that used by Bazer and Ericson (1959). A hydromagnetic shock wave is fast if the normal Alfvén velocity behind it is less than the normal flow velocity behind and relative to the shock wave. The normal phase velocities of small-amplitude waves behind and relative to a fast shock wave are positive for six of the modes and negative for the seventh, which is the only mode in the rear region that can travel upstream to the shock wave.

Assuming the thickness of Region A is much smaller than wavelengths of interest, we show, following Gardner and Kruskal (1964), that such a shock is stable with respect to small perturbations of the fluid variables. (This still leaves the possibility, however, that there is instability with respect to the degrees of freedom characteristic of the plasma's discrete nature.)

In Section III, we discuss the structure of Region B and calculate its thickness, assuming the dominant frequency is 0.1 times the ion cyclotron frequency.

In Section IV, we compare our theoretical results with some experimental results concerning the interplanetary medium.

II. THE STABILITY OF THE FAST HYDROMAGNETIC SHOCK WAVE

In an inviscid compressible fluid that is a perfect conductor, a shock wave is a surface at which the fluid variables have jump discontinuities. The shock conditions governing the jumps in these quantities are derived in Appendix B. While investigating the stability of a shock, we consider an arbitrary small perturbation in the fluid variables, $\delta\rho, \delta\vec{v}, \delta\vec{B}$, and δs, where ρ is the density, $\vec{v}$ is the veolcity, $\vec{B}$ is the magnetic field, and s is the specific entropy, and we consider the perturbation in the position of the shock, $\delta\varphi$. If there is one initial small perturbation that tends to grow exponentially with time, the shock is called unstable.

An arbitrary small disturbance at t = 0 can be expanded in terms of normal modes; in the case of plane symmetry, we use a Fourier expansion. If the unperturbed shock is the surface x = 0, let the perturbed shock be the surface $\varphi(\vec{x},t) = x + \delta\varphi(y,z,t) = 0$. Consider the following mode of the shock perturbation:

$$\delta\varphi(y,z,t) = \widetilde{\delta\varphi}\, e^{i(\vec{\ell} \cdot \vec{x} - \omega t)} \quad ,$$

where $\vec{\ell}$ lies in the yz plane. The linearized conservation equations provide a dispersion relation for small disturbances which have the space and time dependence $e^{i(\vec{k} \cdot \vec{x} - \omega t)}$. (See App. C.) This is a relation between ω and the components of $\vec{k}$, and there are seven possible values for k, the x-component of the propagation vector, that satisfy the dispersion relation. Hence, the perturbations in the fluid variables are given by

$$U(\vec{x},t) = \left(\sum_{j=1}^{7} \widetilde{U}_j \, e^{ik_j x} \right) e^{i(\vec{\ell} \cdot \vec{x} - \omega t)} \quad ,$$

where

$$U = \begin{pmatrix} \delta\rho \\ \delta\vec{v} \\ \delta\vec{B}_t \\ \delta s \end{pmatrix}$$

is a seven-component column vector, and the $\tilde{U}_j$ are the amplitudes of the seven normal modes of the fluid. If Im $\omega > 0$, then all disturbances grow exponentially in time, and the shock is unstable.

Since the linearized shock conditions (see below) are to be satisfied at $x = 0$ by the small disturbances U, the frequency ω must be the same for all disturbances, and the yz-component of the propagation vector must be equal to $\vec{\ell}$ for all disturbances: $\vec{k} = (k, \ell_y, \ell_z)$. Furthermore, the boundary conditions $U \rightarrow 0$ as $x \rightarrow \pm \infty$ imply that Im $k < 0$ for $x < 0$ and > 0 for $x > 0$. Because the eigenvalues of ω/k ahead of the shock ($x < 0$) are all positive, Im $k < 0$ implies Im $\omega < 0$. Therefore, since we are looking for unstable modes, with Im $\omega > 0$, the disturbances in front of the shock must have zero amplitudes. Also, the amplitude of the one mode for which $\frac{\omega}{k_7} < 0$ behind the shock must have zero amplitude: $\tilde{U}_7 = 0$.

The coefficients $\tilde{U}_j$ are determined by the initial condition,

$$U(\vec{x},0) = \left(\sum_{j=1}^{7} \tilde{U}_j e^{ik_j x} \right) e^{i\vec{\ell} \cdot \vec{x}} \qquad ,$$

and by the linearized shock conditions. If it is possible to find non-zero amplitudes $\tilde{U}_j$ for the other six modes behind the shock and for $\delta\tilde{\varphi}$, for some given values of $\vec{\ell}$ and ω, with Im $\omega > 0$, then there is an initial condition $U(\vec{x},0)$ which grows exponentially, and the shock is unstable. (This is the stability problem as posed by Gardner and

Kruskal, 1964; although we have adopted a different method of proof below, much of the formalism is adapted from their work, and generalized to include the effects of non-isotropic pressure behind the shock.)

The shock conditions may be linearized by setting $\varphi = x + \delta\varphi$, and $\rho = \rho_0 + \delta\rho$, etc. The unperturbed shock then satisfies the following conditions:

$$[\rho v_x] = 0 \quad , \qquad [B_x] = 0 \quad ,$$

$$\hat{x}\,[P_\perp + \frac{B^2}{8\pi}] + (\rho v_x)[\vec{v}] + (\frac{B_x}{4\pi})\,[(\nu - 1)\vec{B}] = 0 \quad ,$$

$$(\rho v_x)\left[e + \frac{v^2}{2} + \frac{P_{||}}{\rho}\right] = 0 \quad .$$

We have taken $\vec{v}$ parallel to $\vec{B}$, and in the xy plane; $\vec{E} = 0$ on both sides of the shock in this coordinate system. It is always possible to choose the coordinate system in this way if the magnetic field does not lie in the plane of the shock, as shown by de Hoffmann and Teller (1950). (See appendices for derivation and explanation of symbols.)

The perturbed quantities satisfy the following boundary conditions, neglecting terms of second order in the small quantities;

$$[\delta(\rho v_x)] = i\omega\delta\varphi[\rho] - ik_y\delta\varphi[\rho v_y] \quad ,$$

$$\left[\delta(\rho v_x)\left(e + \frac{v^2}{2} + \frac{P_\perp}{\rho}\right)\right] + (\rho v_x)\left[Tds + \frac{1}{3}(P_{||} - P_\perp)\frac{\delta\rho}{\rho^2} + \vec{v}\cdot\delta\vec{v} + a^2\frac{\delta\rho}{\rho} + b^2\frac{\delta s}{\rho}\right]$$

$$+[\nu \frac{\delta B_x}{4\pi} \quad vB] + \left(\frac{B_x}{4\pi}\right)[\nu(\vec{v} \cdot \delta\vec{B} + \vec{B} \cdot \delta\vec{v})]$$

$$= i\omega\delta\varphi[\rho(e + \frac{v^2}{2}) + \frac{B^2}{8\pi}] - ik_y\delta\varphi[(\rho v_y)(e + \frac{v^2}{2} + \frac{P_\perp}{\rho}) + \frac{\nu B_y Bv}{4\pi}] \quad ,$$

$$[a^2\delta\rho + b^2\delta s + \frac{\vec{B} \cdot \delta\vec{B}}{4\pi}] + 2(\rho v_x)[\delta v_x] + [v_x{}^2\delta\rho] + (\frac{B_x}{2\pi})[(\nu-1)\delta B_x]$$

$$= -ik_y\delta\varphi\{(\rho v_x)[v_y] + (\frac{B_x}{4\pi})[(\nu-1)B_y]\} \quad ,$$

$$[\delta v_x \rho v_y] + (\rho v_x)[\delta v_y] + [v_x v_y \delta\rho] + (\frac{B_x}{4\pi})[(\nu-1)\delta B_y] + [\frac{\nu-1}{4\pi} B_y \delta B_x]$$

$$= i\omega\delta\varphi[\rho v_y] - ik_y\delta\varphi[P_\perp + \rho v_y{}^2 + (\nu-1/2)\frac{B_y{}^2}{4\pi}] \quad ,$$

$$(\rho v_x)[\delta v_z] + (\frac{B_x}{4\pi})[(\nu-1)\delta B_z] = -ik_z\delta\varphi[P_\perp + \frac{B^2}{8\pi}] \quad ,$$

$$[v_x\delta B_y] - [v_y\delta B_x] + [B_y\delta v_x] - (B_x)[\delta v_y] = i\omega\delta\varphi[B_y] \quad ,$$

$$[v_x\delta B_z] - (B_x)[\delta v_z] = 0 \quad ,$$

$$[\delta B_x] = -ik_y\delta\varphi[B_y] \quad .$$

To convert from the variable δe to δs, we have used the following relation (see App. A):

$$\delta e - \frac{P_\perp}{\rho^2}\delta\rho = Tds + \frac{1}{3}(P_{||} - P_\perp)\frac{\delta\rho}{\rho^2} \quad .$$

These are the boundary conditions on the small quantities at the shock. They are useful in this form if the propagation vector $\vec{k}$ lies along one of the coordinate axes, say, the x-axis. In this case, because we have (from App. C) $\hat{k} \cdot \delta\vec{B} = \delta B_x = 0$ and $k_y = 0$, the last condition is identically satisfied. We have really only seven boundary conditions relating the quantities $\delta\rho$, $\delta\vec{v}$, $\delta\vec{B}_t$, δs before and behind the shock and $\delta\varphi$. These conditions determine the solution of the problem of refraction of a small-amplitude wave through a fast hydromagnetic shock. The seven quantities $\delta\rho$, $\rho\vec{v}$, $\delta\vec{B}_t$, δs in the incident wave are considered as given, and we can solve to find $\delta\varphi$ and the seven quantities $\delta\rho$, $\delta\vec{v}$, $\delta\vec{B}_t$, δs in the refracted wave. For the refracted wave, there are only six modes which carry energy away from the shock, so only six of the quantities $\delta\rho$, $\delta\vec{v}$, $\delta\vec{B}_t$, δs are independent. Hence, the six amplitudes of the diverging modes and $\delta\varphi$ are the seven quantities which are uniquely determined by the solution of the above equations.

It is interesting to consider the role of the entropy waves in the refraction problem. There are two cases:

(1) Propagation vector $\vec{k}$ not perpendicular to magnetic field $\vec{B}$. The quantities to be determined are the amplitudes of 1 fast wave, 2 Alfvén waves, 2 slow waves, and 1 entropy wave (δs).

(2) Propagation vector $\vec{k}$ perpendicular to magnetic field $\vec{B}$. The quantities to be determined are the amplitude of 1 fast wave, and the 5 quantities δs, $\delta\vec{v}_t$, $\delta\vec{B}_t$ in the entropy waves.

To determine these quantities, we have a system of seven non-homogeneous equations. Since the problem of the refraction of an incident small disturbance through a shock is a physical problem, we can argue that it must have a unique solution. It follows that the system determinant is not zero. When the amplitude of the incident wave is zero, the system of equations is homogeneous, and has only the zero solution, since the system determinant is not zero. Symbolically, if

$$X = \begin{pmatrix} \delta\varphi \\ A_1 \\ \vdots \\ A_6 \end{pmatrix},$$

where A_i is the amplitude of the ith mode in refracted wave, and we are given

$$V = \begin{pmatrix} \delta\rho_1 \\ \delta\vec{v}_1 \\ \delta\vec{B}_{t_1} \\ \delta s_1 \end{pmatrix},$$

then the system may be written $MX = V$. On physical grounds, we have argued that det $M \neq 0$, so that M^{-1} exists, and the unique solution is $X = M^{-1} V$. In particular, if $V = 0$, then $X = 0$.

Now the problem of instability, as formulated above, is equivalent to the problem of finding a refracted wave with non-zero amplitude when the incident wave has zero amplitude. Because, as we have shown, this is impossible for a fast shock, it is stable. This completes the proof that a fast hydromagnetic shock is stable, with no restrictions on the geometry of the shock.* The generality of the proof is a consequence of the unique solution for the refraction problem for fast shocks. (This is the "evolutionary condition" used by Jeffrey and Taniuti (1964, p. 125) to select the physically relevant solutions of the generalized Rankine--Hugoniot relations for a system of conservation equations.)

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*Gardner and Kruskal (1964) have proved it only for fast parallel and perpendicular shocks -- that is, when the magnetic field is either parallel, or perpendicular, to the shock normal. Although, since the special coordinate system with $\vec{E} = 0$ does not exist, the linearized shock relations do not hold in the above form for perpendicular shocks, similar equations can be written for perpendicular shocks, and the same reasoning used.

III. THE STRUCTURE OF REGION B

In this section, we will assume that the ion pressure in the direction of the magnetic field is considerably greater than the ion pressure perpendicular to the magnetic field in Region B, i.e., $P_{||_b} > P_{\perp_b}$. This pressure anisotropy in Region B is assumed to lead to hose instability (Parker, 1958). This will reduce $P_{||_b}$ and cause the field in Region B to have a disordered structure. This in turn will cause the ion flux to be isotropic, rather than predominantly in the flow direction. The way in which hose instability comes about is as follows: Waves of infinitesimally small amplitude ahead of the shock and incident upon it are refracted through the shock, generally resulting in waves of all six outgoing modes. If $\nu > 1$, then one of the Alfvén modes will grow spatially, for a given frequency, where

$$\nu \equiv \frac{P_{||_b} - P_{\perp_b}}{B^2/4\pi} \quad .$$

As an example, consider an incident Alfvén wave with $\vec{k}$ in the plane of $\vec{v}$ and $\vec{B}$, e.g., the xy plane; suppose the only non-zero quantities in the incident wave are δv_{z_1} and δB_{z_1}. In the refracted waves, it is consistent to set $\delta\rho_b = \delta v_{x_b} = \delta v_{y_b} = \delta B_{x_b} = \delta B_{y_b} = \delta s_b = 0$, and $\delta\varphi = 0$. Then δv_{z_b} and δB_{z_b} are determined by the following relations:

$$(\rho v_x)[\delta v_z] + \left(\frac{B_x}{4\pi}\right) [(\nu-1)\delta B_z] = 0 \quad ,$$

$$[v_x \delta B_z] - (B_x)[\delta v_z] = 0 \quad .$$

The incident wave is

$$\begin{pmatrix} \delta v_{z_1} \\ \delta B_{z_1} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{4\pi\rho_1} \end{pmatrix} c_i \, e^{ik_y y} \, e^{i(k_i \cos\theta_i x - \omega t)} \quad .$$

The refracted wave is

$$\begin{pmatrix} \delta v_{z_b} \\ \delta B_{z_b} \end{pmatrix} = \left\{ \begin{pmatrix} -i\sqrt{\nu-1} \\ 4\pi\rho_b \end{pmatrix} c_+ e^{i(k_+ \cos\theta_+ x)} + \begin{pmatrix} i\sqrt{\nu-1} \\ 4\pi\rho_b \end{pmatrix} c_- e^{ik_- \cos\theta_- x} \right\} e^{i(k_y y - \omega t)} \quad .$$

The existence of boundary conditions at $x = 0$ requires that k_y and ω be the same for all three waves, which leads to Snell's law for the angles of incidence and refraction: $k_y = k_i \sin\theta_i = R_e k_+ \sin\theta_+ = R_e k_- \sin\theta_-$ or

$$\frac{\sin\theta_i}{\sin\theta_+} = \frac{v_b(v_1 + A_1)}{v_b^2 + (\nu-1)A_b^2} \quad ,$$

and $\theta_- = \theta_+$, since $R_e k_- = R_e k_+$. The boundary conditions give a non-homogeneous system of two equations, which enables us to solve for the amplitudes of the refracted Alfvén modes, c_+ and c_-, in terms of the amplitude of the incident Alfvén wave, c_i. In this case, we can explicitly demonstrate the fact that the system determinant is not zero:

$$\Delta = -2i\sqrt{\nu-1} \; \{1 + \frac{\nu-1}{\mathcal{A}_2^{\,2}}\} \neq 0 \quad ,$$

because we are assuming that $\nu > 1$. The solution of the equations is

$$c_+ = \{1 + \mathcal{A}_1 + \frac{\nu-1}{\mathcal{A}_2} \left(\sqrt{\frac{\rho_b}{\rho_1}} + \frac{1}{\mathcal{A}_2} \right) + i\sqrt{\nu-1} \left(\sqrt{\frac{\rho_b}{\rho_1}} - \frac{\mathcal{A}_1}{\mathcal{A}_2} \right) \} \frac{c_i}{\Delta} \quad ,$$

$$c_- = \{-1 - \frac{1}{\mathcal{A}_1} - \frac{(\nu-1)}{\mathcal{A}_2} \left(\sqrt{\frac{\rho_b}{\rho_1}} + \frac{1}{\mathcal{A}_2} \right) + i\sqrt{\nu-1} \left(\sqrt{\frac{\rho_b}{\rho_1}} - \frac{1}{\mathcal{A}_1 \mathcal{A}_2} \right) \} \frac{c_i}{\Delta} ,$$

where

$$\mathcal{A}_1 = \frac{v_{x_1}}{B_{x_1} / \sqrt{4\pi\rho_1}}$$

and

$$\mathcal{A}_2 = \frac{v_{x_b}}{B_{x_b} / \sqrt{4\pi\rho_b}} .$$

Assuming that $\nu > 1$ behind the shock, the refracted + mode grows to about 2.7 times its amplitude in a distance

$$\lambda = \frac{\left[v_b^{\,2} + (\nu-1)\, A_b^{\,2} \right] \cos\theta_b}{\sqrt{\nu-1}\, A_b \omega \cos\alpha_b}$$

from the shock, while the refracted - mode is attenuated to about 1/2.7 of its amplitude in that distance. Hence, the thickness of Region B is of the order of a few λ's. The higher frequency waves grow faster (i.e., in a shorter distance), but the mechanism of the instability works only for frequencies small compared to the ion cyclotron frequency (see Longmire, 1963, p. 124). The fastest growing waves might have a frequency of about 0.1 times the ion cyclotron frequency, ω_i. This should be the dominant frequency in the disordered field region, and with this frequency in λ, we have the thickness of Region B. Since ν is not too large, and v_b is of the same order of magnitude as A_b, we have approximately

$$\lambda \simeq \frac{v_b}{\omega_i} ,$$

which is the cyclotron radius for an ion with the speed v_b. We conclude that the thickness of Region B is a few ion cyclotron radii.

Noerdlinger (1964) has also described instabilities that could explain the fluctuating magnetic field in Region B. His analysis is based on the assumption that the perpendicular pressure $P_{\perp}$ is greater than the parallel pressure $P_{||}$. "The anisotropy of the plasma behind the shock front leads to instability with respect to transverse waves." His calculations also agree fairly well with the noise spectrum observed by Sonett, Smith and Sims (1960). His analysis probably applies on the eastern sunlit side of the earth's bow wave, whereas the analysis given in this paper is meant to apply to the western sunlit side (see next section).

IV. COMPARISON WITH EXPERIMENTAL RESULTS ON THE INTERPLANETARY MEDIUM

In the solar wind, the magnetic field direction seems to lie roughly either in the direction of, or opposite to, the flow, but is at such an angle that it is roughly normal to the earth's bow shock on its western sunlit side. (See Fig. 3.) At each point on the shock, we can investigate the shock structure in a coordinate system moving in a direction which lies in the plane of $\vec{v}$ and $\vec{B}$ and is tangent to the shock, with velocity such that the resultant flow velocity is parallel to $\vec{B}$. We assume that the shock has the same structure, over distances small compared to its radii of curvature, as an infinite plane shock. In the preferred coordinate system at each point on the shock, $\vec{v}$ and $\vec{B}$ are parallel; both are roughly parallel to the shock normal in the western sunlit side of the shock. Since this is the part of the shock described by the early Imp 1 data, it seems reasonable to compare these experimental data with the theoretical results for an idealized plane shock model in which the magnetic field and the flow velocity are both parallel to the shock normal, i.e., a parallel shock. The results are not too critically dependent upon this idealization, as long as the magnetic field and flow velocity are both roughly normal to the shock. In the case of a parallel shock, we have $\alpha = \theta$, so the characteristic length for the spatial growth of the unstable hydromagnetic waves is

$$\lambda = \frac{v_b^2 + (\nu - 1) A_b^2}{\sqrt{\nu - 1}\, A_b \omega} .$$

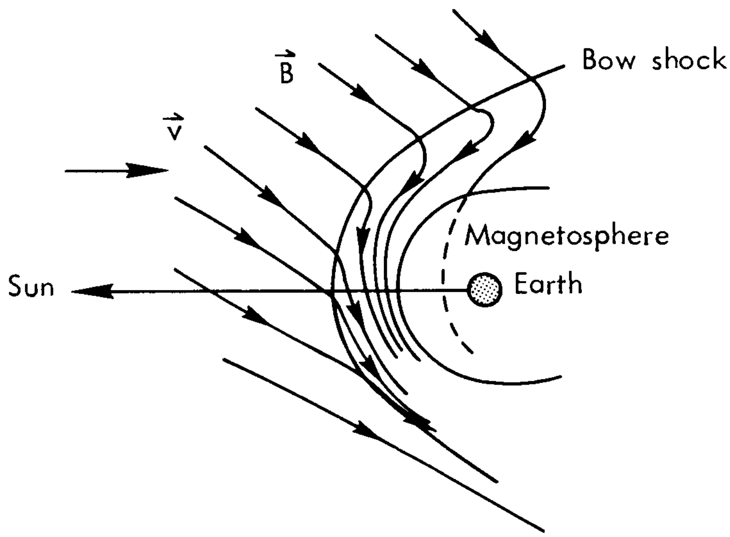

Fig.3 — The direction of the magnetic field near the earth's bow shock

If $\nu \simeq 2$, $v_b \simeq 10^7$ cm/sec, $A \simeq 1.5 \times 10^7$ cm/sec, and $\omega = 0.1\omega_i \simeq 0.1$ sec^{-1}, then $\lambda \simeq 2000$ km $\simeq 1/3$ R_e. Since λ should be measured along the field lines, and the field is roughly as shown in Fig. 3, the distance measured along the normal to the shock is probably about 0.1 R_e.

For the frequency, we have used the highest (approximate) frequency for which the hose instability would be expected to occur, 0.1 times the ion cyclotron frequency, which is about $\omega_i \simeq 1$ sec^{-1} for $B \simeq 10\gamma$. Since these waves grow faster than waves of lower frequency, we might expect that the dominant frequency in the disordered field behind the shock would be about $\omega = 0.1$ sec^{-1}. The magnetometer data from Pioneer I seems to have a periodicity of about ten seconds (Sonett, Smith and Sims, 1960), and the power spectrum generally shows a steep decrease near $\omega = 0.1$ rad/sec (Sonett and Abrams, 1963), in agreement with this prediction.

The Imp 1 magnetometer data (Ness, Scearce and Seek, 1964) indicate that the characteristic length for development of the turbulence behind the earth's bow shock is about 0.1 R_e, in agreement with our calculation of λ. This fact, plus the fact that the Pioneer I data give a spectrum in agreement with the growing hydromagnetic-wave hypothesis, provides the strongest experimental support for the above theory. Evidence exists also, however, for the presence of growing hydromagnetic waves of lower frequency and, hence, of longer characteristic growth length, as described in the following paragraph.

Measurements of the magnetic fields between 5.2 and 15.4 R_e by instruments aboard the interplanetary probe Pioneer V (Coleman, 1964) show that the magnetic field beyond the geomagnetic cavity is disordered and that both the mean field and the amplitude of the field variations decrease with increasing geocentric distance. Freeman, Van Allen and Cahill (1963) have observed that the flux of electrons of a few KEV energy as measured by Explorer XII on September 13, 1961, increases from about 12 R_e inward to about 9 R_e. Hence, both the amplitudes of hydromagnetic waves and the flux of low-energy electrons increase inwards in the disordered region behind the earth's bow shock. It is possible that these electrons are being accelerated to the observed energies by the process of Fermi acceleration, caused by their interaction with low-frequency hydromagnetic waves. Since these grow in distances of about 1 or 2 R_e, according to the above theory, the flux of accelerated electrons would increase appreciably over the same distance, e.g., from the shock wave at about 12 R_e to about 9 R_e. Both of the above observations are therefore consistent with the hypothesis of growing hydromagnetic waves of lower frequency than $0.1\ \omega_i$.

V. SUMMARY

We have shown how several features of shock waves in interplanetary space can be understood. Although a fast shock is stable with respect to small perturbations of the fluid variables, the region behind the shock will contain disordered magnetic fields and unsteady flow if the hose instability criterion is satisfied. This will always happen when small-amplitude disturbances are refracted through the shock. The characteristic length for the growth of the unstable hydromagnetic waves has been calculated for a special case and agrees well with experimental data from Imp 1. We have also shown how the acceleration of low-energy electrons by Fermi processes behind the shock can explain the increase in electron flux seen by Explorer XII as the satellite passed inward behind the earth's bow shock on September 13, 1961.

Appendix A

THERMODYNAMICS OF NON-ISOTROPIC GASES

Chew, Goldberger and Low (1956) have shown that in the development of fluid equations for a plasma the general form of the gas pressure tensor is $P_{ij} = P_{\perp} \delta_{ij} + (P_{||} - P_{\perp}) \frac{B_i B_j}{B^2}$, when the heat flow along the magnetic field is negligible and the magnetic field is large. By definition, $P_{ij} = n \langle m v_i v_j \rangle$, $P_{||} = n \langle m v_{||}^2 \rangle$, and $P_{\perp} = \frac{1}{2} n \langle m v_{\perp}^2 \rangle$, where the brackets indicate an average over many particles in some small volume. The velocity averages are taken with the use of the velocity distribution functions for the species of particles which are present, and the velocity is measured with respect to the average motion of the particles.

Assuming we can neglect interparticle forces, the internal energy e of the gas is the kinetic energy per gram. Since $\frac{1}{2} P_{kk}$ is the kinetic energy per cubic centimeter (where a sum over k from 1 to 3 is assumed), the internal energy is $e = \frac{3}{2} \frac{P}{\rho}$, where the average pressure $P = \frac{1}{3} P_{kk} = \frac{1}{3} (P_{||} + 2P_{\perp})$, or 1/3 the trace of the pressure tensor, and ρ is the density in grams/cm^3. Defining the temperature by $kT = \frac{1}{3} \langle mv^2 \rangle$, we have the equation of state $P = nkT$, or $P = \rho RT$, where $R = \frac{nk}{\rho}$. The second law of thermodynamics for reversible processes may be written

$$de = Tds + \frac{P}{\rho^2} d\rho = Tds + \frac{1}{3} (P_{||} + 2P_{\perp}) \frac{d\rho}{\rho^2} \quad ,$$

where s is the entropy per gram.

In the application that interests us, the kinetic energy due to the motion of the ions parallel to the magnetic field will not be small compared to that of the electrons, so we can assume that ion acoustic waves are not present (Stix, 1962, p. 41). Since we are interested in frequencies that are small compared to the plasma frequency we can ignore electron plasma waves as well, and we can, therefore, set $\delta P_{||} = 0$. It follows that $\delta P = \frac{2}{3}\,\delta P_{\perp}$, and the perpendicular pressure can be used as a thermodynamic variable, rather than the average pressure. Furthermore, because we have $\frac{P_{\perp}}{\rho^2} = \text{const}$ under conditions when the guiding center approximation is valid, (i.e., when the thermodynamic conditions are adiabatic) we have therefore

$$\left(\frac{\partial P_{\perp}}{\partial \rho}\right)_s = 2\,\frac{P_{\perp}}{\rho} \quad .$$

Hence, $P_{\perp}$ is a function only of ρ and s. Thus we can write $\delta P_{\perp} = a^2\,\delta\rho + b^2\,\delta s$, where

$$a^2 = \left(\frac{\partial P_{\perp}}{\partial \rho}\right)_s$$

and

$$b^2 = \left(\frac{\partial P_{\perp}}{\partial s}\right)_{\rho} \quad .$$

Because for an ideal gas

$$\frac{a^2}{b^2} = \frac{c_p}{\rho}$$

where c_p is the specific heat at constant pressure, we have

$$a^2 = 2\,\frac{P_{\perp}}{\rho}$$

and

$$b^2 = 2\frac{P_\perp}{c_p}$$

(since $a^2 = 2\frac{P_\perp}{\rho}$).

In general, the momentum flux tensor for the fluid has the form

$$\Pi_{ij} = P_{ij} + \rho v_i v_j + \frac{B^2}{8\pi}\delta_{ij} - \frac{1}{4\pi} B_i B_j \quad ,$$

where P_{ij} is the gas pressure tensor, and $\rho v_i v_j$ is the momentum flux due to mass motion ($\vec{v}$ is the velocity of mass motion). The terms involving the magnetic field $\vec{B}$ are the terms in the Maxwell stress tensor, when the energy in the electric field $\vec{E}$ is negligible compared to the energy in the magnetic field. With the above form for the gas pressure tensor, we have:

$$\overleftrightarrow{\Pi} = (P_\perp + \frac{B^2}{8\pi})\overleftrightarrow{I} + \rho\,\overleftrightarrow{vv} + (\nu-1)\frac{\overleftrightarrow{BB}}{4\pi} \quad ,$$

where

$$\nu \equiv \frac{P_{||}-P_\perp}{B^2/4\pi} \quad .$$

The general form of the energy flux vector is

$$\vec{w} = \rho\vec{v}(e + \frac{v^2}{2}) + \overleftrightarrow{P}\cdot\vec{v} + \frac{c}{4\pi}\vec{E}\times\vec{B} \quad .$$

The first term is the flux of internal energy and kinetic energy of mass motion; the second term is the work done by a unit volume of the gas; and the last term is the Poynting flux of electromagnetic energy. In a coordinate system such as we will usually choose for simplicity, where $\vec{v}$ is parallel to $\vec{B}$, the last term is zero. In this case, we have:

$$\vec{w} = \rho\vec{v}\left(e + \frac{v^2}{2} + \frac{P_\perp}{\rho} + \nu\frac{B^2}{4\pi\rho}\right) = \rho\vec{v}\left(e + \frac{v^2}{2} + \frac{P_{||}}{\rho}\right) \quad .$$

Appendix B

CONSERVATION LAWS AND SHOCK CONDITIONS

By taking moments of the Boltzmann equation, one can get equations which express the conservation of mass, momentum, and energy in the fluid and the electromagnetic field. To get a closed set of equations, with the same number of unknown quantities as equations, we need to make some assumptions. If the effective frequency and wave vector k for the time and space variation of the fluid variables are such that ω/k is much larger than the mean thermal velocity of the particles, then the effect of the heat flow tensor can be neglected (Bernstein and Trehan, 1960). We assume that this is true, noting that the speed of shock waves in interplanetary space is usually an order of magnitude higher than the mean thermal speed.

Furthermore, the gradients induced by small-amplitude waves will be assumed to be so small that heat flow is negligible. We will use the conservation equations in the following form.

Conservation of mass: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Conservation of momentum: $$\frac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot \overleftrightarrow{P} = 0$$

Conservation of energy: $$\frac{\partial}{\partial t} \left[\rho \left(e + \frac{v^2}{2}\right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \vec{w} = 0$$

(The last equation can be written $\frac{ds}{dt} = 0$ when the derivative exists.)

We will use the electromagnetic equations for a medium with infinite conductivity:

$$\nabla \cdot B = 0, \quad \frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \mathrm{curl}\, \vec{E} = 0, \quad \text{where } \vec{E} = \frac{1}{c}\vec{B} \times \vec{v} \quad .$$

The shock conditions are obtained by integrating the conservation laws and electromagnetic equations across the surface $\varphi(\vec{x},t) = 0$, which is the shock. For any function U we have

$$\frac{d}{dt}\int U d^3\vec{x} = \int \frac{\partial U}{\partial t} d^3\vec{x} + \oint_{S=0} U\vec{q} \cdot d\vec{S} \quad ,$$

where q is the velocity of the surface S = 0. Since dS = 0 in the surface, we have

$$\frac{\partial S}{\partial t} + \nabla S \cdot \vec{q} = 0 \quad ,$$

where $\vec{q} = \frac{d\vec{r}}{dt}$. Now if U is the density of a quantity which is conserved, we have $\frac{\partial U}{\partial t} + \nabla \cdot \vec{F} = 0$, where $\vec{F}$ is the flux of the conserved quantity. Using the divergence theorem, we have therefore:

$$\frac{d}{dt}\int U d^3\vec{x} = \oint_{S=0} d\vec{S} \cdot (U\vec{q} - \vec{F}) \quad .$$

We apply this formula to a small pillbox-shaped surface S = 0 enclosing a small part of the surface $\varphi = 0$; in the limit as the height of the pillbox goes to zero, we have $0 = [U\vec{q} - \vec{F}] \cdot \hat{n}$, where $[Q] \equiv Q_2 - Q_1$. But $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$, and $\vec{q} \cdot \nabla \varphi = -\frac{\partial\varphi}{\partial t}$, in this limit, so we have $\frac{\partial\varphi}{\partial t}[U] + \nabla\varphi \cdot [\vec{F}] = 0$. This is the general form of the shock conditions. (The equation $\frac{ds}{dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$ cannot be integrated across the shock because the derivative $\frac{d}{dt}$ does not exist on the shock surface.) We obtain the following shock conditions:

$$\frac{\partial\varphi}{\partial t}[\rho] + \nabla\varphi \cdot [\rho\vec{v}] = 0 \quad , \qquad \frac{\partial\varphi}{\partial t}[\rho\vec{v}] + \nabla\varphi \cdot [\overleftrightarrow{P}] = 0,$$

$$\frac{\partial\varphi}{\partial t}[\rho(e + \frac{v^2}{2}) + \frac{B^2}{8\pi}] + \nabla\varphi \cdot [\vec{w}] = 0 \quad ,$$

$$\frac{\partial\varphi}{\partial t}[\vec{B}] + \nabla\varphi \times [\vec{E}] = 0 \quad , \qquad \nabla\varphi \cdot [\vec{B}] = 0 \quad .$$

When the position of the shock does not change in time, these reduce to the time-steady shock conditions, of the form $\hat{n} \cdot [\vec{F}] = 0$. In general, the shock conditions do not hold in this form when the flow behind the shock is unsteady. But if the time average of the equation

$$\frac{d}{dt}\int U \, d^3\vec{x} = \oint d\vec{S} \cdot (U\vec{q} - \vec{F})$$

is taken over a long time, and if we assume the volume integral is bounded as a function of time, then we have

$$\overline{\oint d\vec{S} \cdot (U\,\vec{q} - \vec{F})} = 0 \quad .$$

The surface integral can be taken over a fixed cylinder, so that $\vec{q} = 0$. If the cylinder can be chosen so that the contribution from its sides averages to zero, then we have $\hat{n} \cdot \overline{[\vec{F}]} = 0$, where the flux is evaluated at points on either side of the shock, and the average is taken over a long time.

Appendix C

SMALL-AMPLITUDE WAVES IN A NON-ISOTROPIC HYDROMAGNETIC MEDIUM

With the forms for $\overleftrightarrow{P}$ and $\vec{w}$ given in Appendix A, the conservation laws and electromagnetic equations give the following MHD equations, assuming that the derivatives all exist:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad ,$$

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} + \nabla \left(P_{\perp} + \frac{B^2}{8\pi} \right)$$

$$- \vec{B} \cdot \nabla \left[\left(1 - \frac{P_{||} - P_{\perp}}{B^2/4\pi} \right) \frac{\vec{B}}{4\pi} \right] = 0 \quad ,$$

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) s = 0 \quad , \qquad \nabla \cdot \vec{B} = 0 \quad ,$$

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{B} + \vec{B} \left(\nabla \cdot \vec{v} \right) - \left(\vec{B} \cdot \nabla \right) \vec{v} = 0 \quad .$$

To find the equations for small-amplitude waves, we linearize these equations by setting $Q = Q_0 + \delta Q$, where $\frac{\delta Q}{Q_0} \ll 1$, for each quantity in the equations. We assume that

$$\delta \nu \equiv \delta \left(\frac{P_{||} - P_{\perp}}{B^2/4\pi} \right) = - \frac{2P_{||}}{B^4/4\pi} \vec{B} \cdot \delta \vec{B} \quad .$$

(We have already assumed that $\delta P_{||} = 0$, and for frequencies small compared to the ion cyclotron frequency the adiabatic relation $\delta \left(\frac{P_{\perp}}{B^2} \right) = 0$ holds.) Furthermore, we eliminate the variable $\delta P_{\perp}$ by setting $\delta P_{\perp} = a^2\delta\rho + b^2\delta s$. If all small quantities have the space and time variation $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$, then the linearized equations are

$$(-\omega + \vec{k}\cdot\vec{v}_0)\delta\rho + \rho_0\,\vec{k}\cdot\delta\vec{v} = 0 \quad ,$$

$$\rho_0\,(-\omega + \vec{k}\cdot\vec{v}_0)\delta\vec{v} + \vec{k}(a^2\,\delta\rho + b^2\delta s + \frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi})$$

$$+ \frac{(\nu-1)}{4\pi}\,(\vec{k}\cdot\vec{B}_0)\,\delta\vec{B} - \frac{2P_{||}}{B_0^{\;4}}\,(\vec{k}\cdot\vec{B}_0)\,\vec{B}_0\,(\vec{B}_0\cdot\delta\vec{B}) = 0 \quad ,$$

$$(-\omega + \vec{k}\cdot\vec{v}_0)\delta s = 0 \quad , \qquad \vec{k}\cdot\delta\vec{B} = 0 \quad ,$$

$$(-\omega + \vec{k}\cdot\vec{v}_0)\delta\vec{B} + \vec{B}_0\,(\vec{k}\cdot\delta\vec{v}) - (\vec{B}_0\cdot\vec{k})\,\delta\vec{v} = 0 \quad .$$

If we introduce components normal to the wave front (n) and tangential to it (t), then we have seven equations in the seven unknowns $\delta\rho$, $\delta\vec{v}$, $\delta\vec{B}_t$, and δs, because $\delta B_n = 0$. The equations may be written as follows:

$$-c^*\delta\rho + \rho_0\delta v_n = 0$$

$$\left(\frac{B_0^{\;2}}{4\pi} - \frac{2P_{||}}{B_0^{\;2}}\,B_n^{\;2}\right)\frac{\vec{B}_t\cdot\delta\vec{B}_t}{B_0^{\;2}} - \rho_0 c^*\delta v_n + a^2\delta\rho + b^2\delta s = 0$$

$$-\frac{2P_{||}}{B_0^{\;2}}\,\frac{B_n\vec{B}_t}{B_0^{\;2}}\,(\vec{B}_t\cdot\delta\vec{B}_t) - \rho_0 c^*\delta\vec{v}_t + \frac{\nu-1}{4\pi}\,B_n\delta\vec{B}_t = 0$$

$$-c^*\delta s = 0$$

$$-c^*\delta\vec{B}_t + \vec{B}_t\delta v_n - B_n\delta\vec{v}_t = 0 \quad ,$$

where $c^* = \frac{\omega}{|\vec{k}|} - \frac{\vec{k}}{|\vec{k}|} \cdot \vec{v}_0$ is the speed of the wave relative to the fluid.

1. Entropy waves:

$$\delta s \neq 0, \qquad c^* = 0, \qquad \omega = \vec{k} \cdot \vec{v}_0, \qquad \frac{\partial \omega}{\partial \vec{k}} = \vec{v}_0, \qquad \delta v_n = 0 \quad .$$

(a) If $B_n \neq 0$ then $\delta\vec{v}_t = 0$, $\delta\vec{B}_t = 0$, and $\delta\rho = -\frac{b^2}{a^2}\delta s$. The variation in density and temperature (and hence entropy) is such that there is no variation in pressure; it is convected with the fluid.

(b) If $B_n = 0$, then in general, $\delta\vec{v}_t \neq 0$, $\delta\vec{B}_t \neq 0$, and

$$a^2\delta\rho + b^2\delta s + \frac{\vec{B}_t \cdot \delta\vec{B}_t}{4\pi} = 0 \quad .$$

For propagation perpendicular to the field direction, the tangential components of velocity and magnetic field may vary.

If $c^* \neq 0$, then $\delta s = 0$; take components parallel and perpendicular to the magnetic field:

$$- c^*\delta\rho + \rho_0\delta v_n = 0$$

$$- \rho_0 c^* \delta v_n + a^2\delta\rho + \frac{B_t \delta B_{t_{||}}}{4\pi}\left(1 - \frac{8\pi P_{||} B_n^2}{B_0^4}\right) = 0 \quad .$$

$$-\rho_0 c^* \delta v_{t_{||}} + \left[\frac{(\nu-1)}{4\pi} - \frac{2P_{||} B_t^2}{B_0^4}\right] B_n \delta B_{t_{||}} = 0 \quad .$$

$$- c^* \delta B_{t_{||}} + B_t \delta v_n - B_n \delta v_{t_{||}} = 0 \quad .$$

$$-\rho_0 c^* \delta v_{t_\perp} + \frac{(\nu-1)}{4\pi} B_n \delta B_{t_\perp} = 0 \quad .$$

$$- c^* \delta B_{t_\perp} - B_n \delta v_{t_\perp} = 0 \quad .$$

2. Alfvén waves:

$$\delta\rho = \delta v_n = \delta B_{t_{||}} = \delta v_{t_{||}} = 0 \quad .$$

The only equations involving the components $\delta v_{t_\perp}$ and $\delta B_{t_\perp}$ are the last two. If

$$\delta v_{t_\perp} \neq 0, \ \delta B_{t_\perp} \neq 0, \text{ then } c^{*2} + (\nu-1) \frac{B_n^2}{4\pi\rho_0} = 0 \quad .$$

(a) Stable waves $(0 \leq \nu < 1)$

$$c^* = \pm \sqrt{1-\nu} \frac{B_n}{\sqrt{4\pi\rho_0}} = \pm \sqrt{1-\nu} \quad A \cos\theta$$

where

$$A = \frac{B}{\sqrt{4\pi\rho_0}}$$

and

$$\cos\theta = \frac{B_n}{B} = \frac{\vec{k} \cdot \vec{B}}{|\vec{k}| \cdot B} \quad .$$

Therefore,

$$\omega = \left(\pm\sqrt{1-\nu} \frac{\vec{B}}{\sqrt{4\pi\rho_0}} + \vec{v}_0 \right) \cdot \vec{k} \quad .$$

The phase velocity of the waves is $\omega/k = (v_0 \pm \sqrt{1-\nu}\ A) \cos\theta$.

The group velocity is

$$\frac{\partial\omega}{\partial\vec{k}} = \vec{v}_0 \pm \sqrt{1-\nu}\ \frac{\vec{B}}{\sqrt{4\pi\rho_0}} ,$$

which always has a positive component in the direction of $\vec{v}_0$, behind a fast shock. We have the following two modes:

$$+\text{ mode: } k_+ = \frac{\omega}{(v_0 + \sqrt{1-\nu}\ A)\cos\theta} , \quad \delta v_{t_\perp} = -\frac{\sqrt{1-\nu}}{\sqrt{4\pi\rho_0}}\ \delta B_{t_\perp} .$$

$$-\text{ mode: } k_- = \frac{\omega}{(v_0 - \sqrt{1-\nu}\ A)\cos\theta} , \quad \delta v_{t_\perp} = \frac{\sqrt{1-\nu}}{\sqrt{4\pi\rho_0}}\ \delta B_{t_\perp} .$$

(b) Unstable waves ($\nu > 1$)

$$c^* = \pm i\sqrt{\nu-1}\ A\cos\theta , \quad \omega = |\vec{k}| \cos\theta\,(v_0 \pm i\sqrt{\nu-1}\ A) .$$

$$+\text{ mode: } k_+ = \frac{(v_0 - i\sqrt{\nu-1}\ A)\,\omega}{[v_0^2 + (\nu-1)A^2]\cos\theta} , \quad \delta v_{t_\perp} = \frac{-i\sqrt{\nu-1}}{\sqrt{4\pi\rho_0}}\ \delta B_{t_\perp} .$$

$$-\text{ mode: } k_- = \frac{(v_0 + i\sqrt{\nu-1}\ A)\,\omega}{[v_0^2 + (\nu-1)A^2]\cos\theta} , \quad \delta v_{t_\perp} = \frac{i\sqrt{\nu-1}}{\sqrt{4\pi\rho_0}}\ \delta B_{t_\perp} .$$

Let $k = |\vec{k}| \cos\alpha$; since the waves vary as $e^{i(\vec{k}\cdot\vec{x} - \omega t)} = e^{i(kx + \ell_y y + \ell_z z - \omega t)}$, we have the following behavior for the two modes:

+ mode:

$$\left\{ e^{i(\ell_y y + \ell_z z)}\ e^{i\omega\left(\frac{x}{\cos\theta}\ \frac{v_0\cos\alpha}{v_0^2 + (\nu-1)A^2} - t\right)} \right\} \cdot e^{\frac{\sqrt{\nu-1}\ A\omega\cos\alpha}{\cos\theta(v_0^2 + (\nu-1)A^2)}\ x} ,$$

$$- \text{ mode:} \qquad \{\ldots\} \qquad e^{\dfrac{-\sqrt{\nu-1}\, A\omega \cos\alpha}{\cos\theta\, (v_0^2 + (\nu-1) A^2)}\, x} \quad .$$

The + mode grows spatially, and the - mode is damped spatially, with a characteristic length for growth (or attenuation) of

$$\lambda = \frac{[v_0^2 + (\nu-1) A^2] \cos\theta}{\sqrt{\nu - 1}\; A\omega \cos\alpha} \quad .$$

3. Fast and Slow Waves:

The remaining equations may be written as follows:

$$\begin{pmatrix} -c^* & \rho_0 & 0 & 0 \\ a^2 & -\rho_0 c^* & 0 & \left(1 - \dfrac{8\pi P_{||} B_n^2}{B_0^4}\right) B_t/4\pi \\ 0 & 0 & -\rho_0 c^* & \left[\dfrac{-2P_{||} B_t^2}{B_0^4} + \dfrac{(\nu-1)}{4\pi}\right] B_n \\ 0 & B_t & -B_n & -c^* \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta v_n \\ \delta v_{t||} \\ \delta B_{t||} \end{pmatrix} = 0 \quad .$$

The four eigenvalues of c^* are given by

$$c^{*2} = \frac{1}{2}\left[a^2 + \frac{B^2}{4\pi\rho_0} - \nu\frac{B_n^{\ 2}}{4\pi\rho_0}\right]$$

$$\pm\sqrt{\frac{1}{4}\left[a^2 + \frac{B^2}{4\pi\rho_0} - \nu\frac{B_n^{\ 2}}{4\pi\rho_0}\right]^2 + \left[\left(1 - 2\frac{B_t^{\ 2}}{B^2}\right)\nu/4\pi - \frac{1}{4\pi}\right]\frac{a^2B_n^{\ 2}}{\rho_0} - \frac{a^4B_n^{\ 2}B_t^{\ 2}}{B^4}}\ .$$

The minus sign corresponds to the slow wave modes. These are unstable, with $c^{*2} < 0$, for values of ν and θ such that

$$\nu > \frac{1 + \frac{a^2}{A^2}\frac{B_t^{\ 2}}{B^2}}{1 - 2\frac{B_t^{\ 2}}{B^2}} = \frac{1 + \frac{a^2}{A^2}\sin^2\theta}{1 - 2\sin^2\theta}\ .$$

For $\vec{k}$ parallel to $\vec{B}_0$ ($\theta = 0$) they are the same as the unstable Alfvén waves (but with a different polarization) and are unstable for $\nu > 1$, as we found in the preceding paragraph.

The above criterion for instability is not correct for $\theta \neq 0$, because $P_{||}$ and $P_{\perp}$ contain little information about the velocity distribution function. In general, the fluid equations alone cannot correctly treat instabilities arising from a non-Maxwellian distribution, though they happen to be sufficient for the hose instability for Alfvén waves. These last are the only specific results used in this Memorandum.

REFERENCES

Bazer, J., and W. B. Ericson, "Hydromagnetic Shocks," Astrophys. J., 129, 758-785, 1959.

Bernstein, I. B., and S. K. Trehan, "Plasma Oscillations," Nuclear Fusion, 1, 3, 1960.

Bridge, H., A. Egidi, A. Lazarus, E. Lyon, and L. Jacobson, "Preliminary Results of Plasma Measurements on IMP-A," paper presented at COSPAR Meeting, Florence, Italy, 1964.

Chew, G. F., M. L. Goldberger, and F. E. Low, "The Boltzmann Equation and the One-Fluid Hydromagnetic Equations in the Absence of Particle Collisions," Proc. Roy. Soc., A236, 112-118, 1956.

Coleman, P. J., Jr., "Characteristics of the Region of Interaction Between the Interplanetary Plasma and the Geomagnetic Field: Pioneer 5," J. Geophys. Res., 69, 3051-3076, 1964.

Colgate, S. A., "Collisionless Plasma Shock," Phys. Fluids, 2, 485, 1959.

Freeman, J. W., J. A. Van Allen, and L. J. Cahill, "Explorer 12 Observations of the Magnetospheric Boundary and the Associated Solar Plasma on September 13, 1961," J. Geophys. Res., 68, 2121-2130, 1963.

Gardner, C. S., and M. D. Kruskal, "Stability of Plane Magnetohydrodynamic Shocks," Phys. Fluids, 7, 700-706, 1964.

de Hoffmann, F., and E. Teller, "Magneto-Hydrodynamic Shocks," Phys. Rev., 80, 692-703, 1950.

Jeffrey, A., and T. Taniuti, Non-Linear Wave Propagation, Academic Press, New York, 1964.

Longmire, C. L., Elementary Plasma Physics, Interscience Publishers, New York, 1963.

Ness, N. F., C. S. Scearce, and J. B. Seek, "Initial Results of the Imp 1 Magnetic Field Experiment," J. Geophys. Res., 69, 3531-3569, 1964.

Noerdlinger, P. D., "Wave Generation Near the Outer Boundary of the Magnetosphere," J. Geophys. Res., 69, 369-375, 1964.

Parker, E. N., "Dynamical Instability in an Anisotropic Ionized Gas of Low Density," Phys. Rev., 109, 1874-1876, 1958.

Sonett, C. P., and I. J. Abrams, "The Distant Geomagnetic Field, 3, Disorder and Shocks in the Magnetopause," J. Geophys. Res., 68, 1233-1263, 1963.

Sonett, C. P., D. S. Colburn, L. Davis, Jr., E. J. Smith, and P. J. Coleman, Jr., "Evidence for a Collision-Free Magnetohydrodynamic Shock in Interplanetary Space," Phys. Rev. Letters, 13, 153-156, 1964.

Sonett, C. P., E. J. Smith, and A. R. Sims, "Surveys of the Distant Geomagnetic Field: Pioneer I and Explorer VI," in Space Research, H. K. Kallmann-Bijl (ed.), North-Holland Publishing Company, Amsterdam, pp. 921-937, 1960.

Stix, T. H., The Theory of Plasma Waves, McGraw-Hill Book Company, New York, 1962.