VON KARMAN CENTER

ADVANCED PROGRAMS DEPARTMENT

THERMAL STRAIN ANALYSIS OF ADVANCED MANNED SPACECRAFT HEAT SHIELDS

First Quarterly Status Report
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AEROJET
GENERAL TIRE
GENERAL

CONTRACT FULFILLMENT STATEMENT

This is the first of three quarterly progress reports submitted in partial fulfillment of the National Aeronautics and Space Administration Contract No. NAS 9-1986. This report covers the period from 1 September 1963 to 30 November 1963.

Approved by:

A. Zukerman
A. Zukerman, Project Manager

I. SUMMARY OF PROGRESS TO DATE

A. DERIVATION OF BASIC EQUILIBRIUM AND STRESS EQUATIONS AND THEIR FINITE DIFFERENCE ANALOGS

The derivation of the basic equations in the appropriate coordinate systems (spherical and toroidal) for the general non-axisymmetric case has been completed. The finite difference analogs to the partial difference equations have been derived for the general case (given in Appendix A for completeness). The above cited equations are based on the "thick-shell" theory which is appropriate for the overall thickness of the composite shell structure. The existence of very thin layers - the bond and sandwich face plates - within the structure are expected to cause numerical computation difficulties in the mixed derivatives for the thin layers if the thick wall formulation is utilized for these layers. Furthermore, such a treatment of the thin layers will require an excessive number of nodes. The possibility of adapting "thin-wall" theory for these layers was suggested by Mr. F. H. Brady. An analysis of this problem was carried out by Dr. D. H. Platus. This approach requires only a two-dimensional solution of the displacement - equilibrium equations at the median surface of the shell. The stress and strain distributions throughout the shell thickness are then obtained using the Kirchhoff bending hypothesis for thin shells. This effort method is summarized in Appendix B for a flat plate using Cartesian coordinates.

B. THE SINGULAR POINT

The equations for the general non-axisymmetric case possess a singularity on the geometric axis-of-symmetry. Inasmuch as this singularity is not an "essential-singularity", it should in principle be possible to formulate locally valid non-singular equations for this point. Since this point is common to all meridian planes, using it as a common node would reduce the total number of nodes considerably. An attempt was therefore made to derive such a formulation. A summary of this effort is presented in Appendix C. The additional programming required to utilize this formulation and the complications introduced would probably not justify the possible benefits (reduced total number of nodes). It was decided to establish the "singularity region" by the use of the simplified axisymmetric test case.

C. PROGRAMMING

Programming of the input data modeling and the equilibrium coefficient evaluation/storage subroutines is about 50% complete for the two media axisymmetric test case. Techniques for reducing round-off error arising in the use of the finite difference models of the partial derivatives are being studied in conjunction with an examination of the latest state-of-the-art in relaxation methods.

II. PLANNED ACTIVITIES FOR NEXT REPORTING PERIOD

A. DERIVATION OF EQUATIONS

The completion of the derivation of the "thin-wall" equations in the spherical-toroidal coordinate system will be accomplished during the next reporting period. The derivation of the finite difference analog of these equations will be initiated.

B. FORMULATION OF BOUNDARY CONDITIONS

The effort will be expanded during this period in formulating the boundary conditions for all cases under consideration.

C. PROGRAMMING

Effort will continue in programming the axisymmetric test case with the latest input incorporated. The objective of this test case is to establish optimum grid spacing, gain experience in the convergence problem and to establish the optimum grid layout near the singular point. It should be noted that the major part of the programming already completed and that planned is directly applicable to the general case.

III. PROBLEM AREAS

A. BASIC EQUATIONS

Numerical computation difficulties (accuracy degradation) are anticipated in the use of the "thin-shell" approximations for the bond and face plates. These problems are due to fourth derivatives required in these formulations. The extent of the difficulties and methods for their alleviation will be investigated upon completion of the derivation of the equations in the proper coordinate system.

B. THE SINGULAR POINT

As pointed out previously, the "simple" axisymmetric test case will be utilized to overcome this problem. The expected solution will be in the form of an "optimum" grid around the singular point.

C. OVERRELAXATION METHOD AND CONVERGENCE CRITERIA

An extensive effort is planned in this area with the test case providing the tool for testing approaches. Convergence criteria will be developed specifically suitable to the present formulation.

IV. PROGRAM CHANGES

The progress to date and the problems encountered during the last reporting period make it necessary to revise the original program schedule. These modifications are designed to assure timely achievement of the program objectives. The revised program schedule is shown in Figure 1. Reference to this revised program schedule will be made in the subsequent monthly reports.

THERMAL STRAIN ANALYSIS OF ADVANCED MANNED SPACECRAFT VEHICLES

PROGRAM SCHEDULE (REVISION 1)

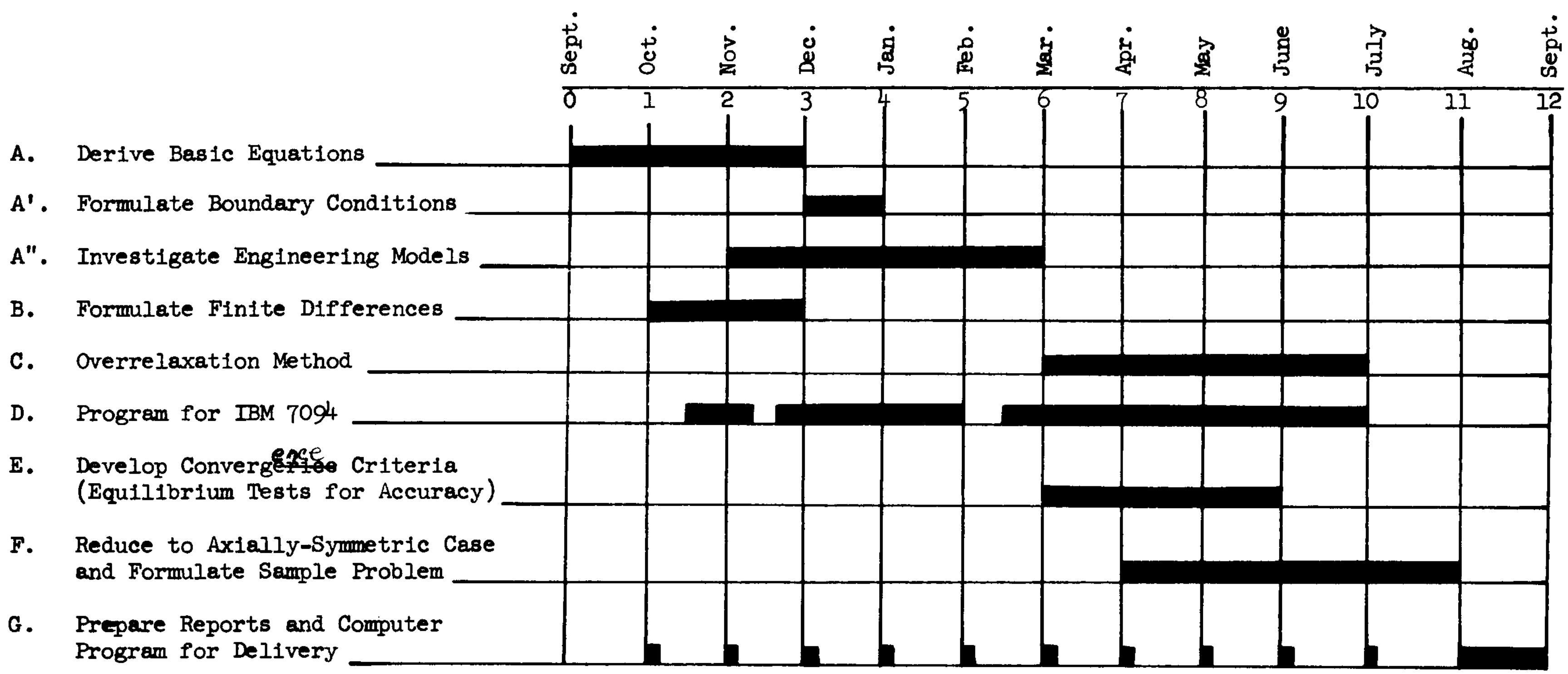


Figure 1

APPENDIX A

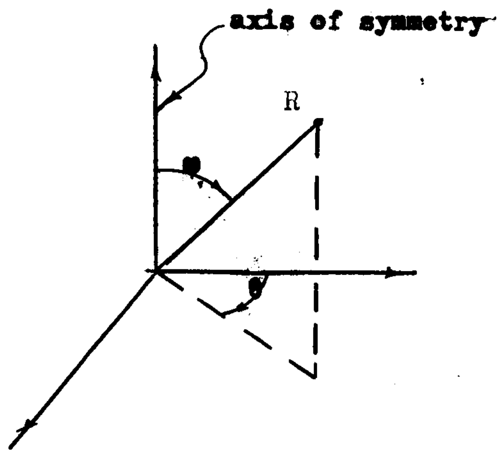
Derivation of Equilibrium Equations in Terms of Displacements in Spherical and Toroidal Coordinates

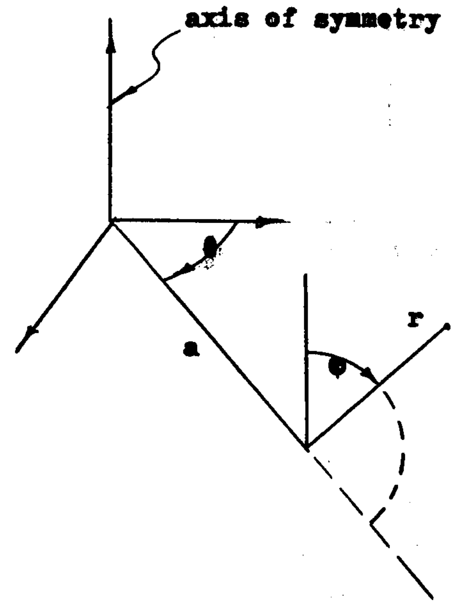
Orthogonal curvilinear coordinates $(\alpha_1, \alpha_2, \alpha_3)$

Element of arc ds defined by

$$ds^2 = \sum_{i=1}^{3} g_{ii}\, d\alpha_i^2 \qquad (1)$$

Where g_{ii} are the metric coefficients





	Spherical Coordinates	Toroidal Coordinates
α_1	R	r
α_2	φ	φ
α_3	θ	θ
g_{11}	1	1
g_{22}	R^2	r^2
g_{33}	$R^2\sin^2\varphi$	$(a + r\sin\varphi)^2$
$g \equiv \sqrt{g_{11}g_{22}g_{33}}$	$R^2\sin\varphi$	$r(a + r\sin\varphi)$

Note: Toroidal coordinates reduce to spherical coordinates in the limit as $a \longrightarrow 0$.

Derivation of Equilibrium Equations in Terms of Displacements in Spherical and Toroidal Coordinates (cont.)

Equations of equilibrium with zero body force:

$$\sum_{j=1}^{3}\left[\frac{\partial}{\partial \alpha_j}\left(\frac{g g_{ii}\tau_{ij}}{\sqrt{g_{ii}g_{jj}}}\right) - \frac{1}{2}\cdot\frac{g\tau_{jj}}{g_{jj}}\frac{\partial g_{jj}}{\partial \alpha_i}\right] = 0 \qquad (2)$$

Where $g = \sqrt{g_{11}g_{22}g_{33}}$ and τ_{ii} and τ_{ij} are normal and shear components of stress, respectively.

Substituting the respective components of α_i and g_{ii} in Eq. (2) and performing the indicated differentiations and summations, there are obtained the following equilibrium equations in terms of stresses for each coordinate system:

Spherical Coordinates

$$\frac{\partial \tau_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{R\varphi}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{2\tau_{RR} - \tau_{\varphi\varphi} - \tau_{\theta\theta} + \tau_{R\varphi}\cot\varphi}{R} = 0 \qquad (3)$$

$$\frac{\partial \tau_{R\varphi}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{\varphi\theta}}{\partial \theta} + \frac{3\tau_{R\varphi} + \left(\tau_{\varphi\varphi} - \tau_{\theta\theta}\right)\cot\varphi}{R} = 0 \qquad (4)$$

$$\frac{\partial \tau_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{\varphi\theta}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{3\tau_{R\theta} + 2\tau_{\varphi\theta}\cot\varphi}{R} = 0 \qquad (5)$$

Toroidal Coordinates

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{1}{(a+r\sin\varphi)}\frac{\partial \tau_{r\theta}}{\partial \theta} \qquad (6)$$

$$+ \frac{(a+2r\sin\varphi)\tau_{rr} - (a+r\sin\varphi)\tau_{\varphi\varphi} - \tau_{\theta\theta}r\sin\varphi + \tau_{r\varphi}r\cos\varphi}{r(a+r\sin\varphi)} = 0$$

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} + \frac{1}{a+r\sin\varphi}\frac{\partial \tau_{\varphi\theta}}{\partial \theta}$$

$$+ \frac{(2a+3r\sin\varphi)\tau_{r\varphi} + \left(\tau_{\varphi\varphi} - \tau_{\theta\theta}\right)r\cos\varphi}{r(a+r\sin\varphi)} = 0 \qquad (7)$$

Derivation of Equilibrium Equations in Terms of
Displacements in Spherical and Toroidal Coordinates (cont.)

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\varphi\theta}}{\partial\varphi} + \frac{1}{(a+r\sin\varphi)}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{(a+3r\sin\varphi)\tau_{r\theta} + 2\tau_{\varphi\theta}r\cos\varphi}{r(a+r\sin\varphi)} = 0 \qquad (8)$$

Hooke's Law Including Temperature Terms

$$\tau_{ii} = \lambda\,\theta + 2\mu e_{ii} - (3\lambda+2\mu)\int_{T_o}^{T}\alpha(T)dT \qquad (9)$$

$$\tau_{ij} = 2\mu e_{ij} \qquad (10)$$

where

$$\theta = e_{11} + e_{22} + e_{33}$$

and λ and μ are the Lame' constants defined in terms of Poisson's ratio ν and Young's modulus E according to

$$\left.\begin{aligned} \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \mu &= \frac{E}{2(1+\nu)} \end{aligned}\right\} \qquad (11)$$

Strain - Displacement Relations

$$e_{ii} = \frac{\partial}{\partial\alpha_i}\frac{u_i}{\sqrt{g_{ii}}} + \frac{1}{2g_{ii}}\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial\alpha_k}\frac{u_k}{\sqrt{g_{kk}}} \qquad (12)$$

Derivation of Equilibrium Equations in Terms of Displacements in Spherical and Toroidal Coordinates (cont.)

$$e_{ij} = \frac{1}{2\sqrt{g_{ii}g_{jj}}} \left[g_{ii} \frac{\partial}{\partial \alpha_j} \left(\frac{u_i}{\sqrt{g_{ii}}} \right) + g_{jj} \frac{\partial}{\partial \alpha_i} \left(\frac{u_j}{\sqrt{g_{jj}}} \right) \right] , \quad i \neq j \qquad (13)$$

Let u, v, w be components of displacement in the three principal directions r or R, φ and θ. Then substitution of these displacements in Eqs. (12) and (13), with the metric coefficients of page 1, yields the strain-displacement relations for the two coordinate systems:

Spherical Coordinates

$$\left.\begin{aligned}
e_{RR} &= \frac{\partial u}{\partial R} \\
e_{\varphi\varphi} &= \frac{1}{R}\frac{\partial v}{\partial \varphi} + \frac{u}{R} \\
e_{\theta\theta} &= \frac{1}{R\sin\varphi}\frac{\partial w}{\partial \theta} + \frac{u}{R} + \frac{v\cot\varphi}{R} \\
e_{R\varphi} &= \frac{1}{2}\left(\frac{1}{R}\frac{\partial u}{\partial \varphi} - \frac{v}{R} + \frac{\partial v}{\partial R} \right) \\
e_{\varphi\theta} &= \frac{1}{2}\left(\frac{1}{R}\frac{\partial w}{\partial \varphi} - \frac{w\cot\varphi}{R} + \frac{1}{R\sin\varphi}\frac{\partial v}{\partial \theta} \right) \\
e_{R\theta} &= \frac{1}{2}\left(\frac{1}{R\sin\varphi}\frac{\partial u}{\partial \theta} - \frac{w}{R} + \frac{\partial w}{\partial R} \right)
\end{aligned}\right\} \qquad (14)$$

Toroidal Coordinates

$$e_{rr} = \frac{\partial u}{\partial r}$$

$$e_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r}$$

$$e_{\theta\theta} = \frac{1}{a+r\sin\varphi}\frac{\partial w}{\partial\theta} + \frac{u\sin\varphi}{a+r\sin\varphi} + \frac{v\cos\varphi}{a+r\sin\varphi}$$

$$e_{r\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial\varphi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)$$

$$e_{\varphi\theta} = \frac{1}{2}\left(\frac{1}{a+r\sin\varphi}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial w}{\partial\varphi} - \frac{w\cos\varphi}{a+r\sin\varphi}\right) \qquad (15)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{a+r\sin\varphi}\frac{\partial u}{\partial\theta} + \frac{\partial w}{\partial r} - \frac{w\sin\varphi}{a+r\sin\varphi}\right)$$

Equilibrium Equations in Terms of Displacements

Expressing the stresses in terms of displacements using Hooke's law, Eqs. (9) and (10), with the strain-displacement relations, Eqs. (14) and (15), the equilibrium equations, Eqs. (3) - (8), may be written in terms of displacements in the form

$$\begin{aligned}
& A_k \frac{\partial^2 u}{\partial\alpha_1^2} + B_k \frac{\partial^2 u}{\partial\alpha_2^2} + C_k \frac{\partial^2 u}{\partial\alpha_3^2} + D_k \frac{\partial^2 u}{\partial\alpha_1\partial\alpha_2} + E_k \frac{\partial^2 u}{\partial\alpha_2\partial\alpha_3} \\
& + F_k \frac{\partial^2 u}{\partial\alpha_1\partial\alpha_3} + G_k \frac{\partial u}{\partial\alpha_1} + H_k \frac{\partial u}{\partial\alpha_2} + I_k \frac{\partial u}{\partial\alpha_3} + J_k u \\
& + \bar{A}_k \frac{\partial^2 v}{\partial\alpha_1^2} + \bar{B}_k \frac{\partial^2 v}{\partial\alpha_2^2} + \bar{C}_k \frac{\partial^2 v}{\partial\alpha_3^2} + \bar{D}_k \frac{\partial^2 v}{\partial\alpha_1\partial\alpha_2} + \bar{E}_k \frac{\partial^2 v}{\partial\alpha_2\partial\alpha_3} \\
& + \bar{F}_k \frac{\partial^2 v}{\partial\alpha_1\partial\alpha_3} + \bar{G}_k \frac{\partial v}{\partial\alpha_1} + \bar{H}_k \frac{\partial v}{\partial\alpha_2} + \bar{I}_k \frac{\partial v}{\partial\alpha_3} + \bar{J}_k v \qquad (16) \\
& + \bar{\bar{A}}_k \frac{\partial^2 w}{\partial\alpha_1^2} + \bar{\bar{B}}_k \frac{\partial^2 w}{\partial\alpha_2^2} + \bar{\bar{C}}_k \frac{\partial^2 w}{\partial\alpha_3^2} + \bar{\bar{D}}_k \frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} + \bar{\bar{E}}_k \frac{\partial^2 w}{\partial\alpha_2\partial\alpha_3} \\
& + \bar{\bar{F}}_k \frac{\partial^2 w}{\partial\alpha_1\partial\alpha_3} + \bar{\bar{G}}_k \frac{\partial w}{\partial\alpha_1} + \bar{\bar{H}}_k \frac{\partial w}{\partial\alpha_2} + \bar{\bar{I}}_k \frac{\partial w}{\partial\alpha_3} + \bar{\bar{J}}_k w = \\
& \frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{kk}}}\frac{\partial T}{\partial\alpha_k}, \quad k = 1,2,3
\end{aligned}$$

TABLE I

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

SPHERICAL COORDINATES

($\varphi \neq 0$)

	k = 1	k = 2	k = 3
A_k	$\lambda+2\mu$	0	0
B_k	μ/R^2	0	0
C_k	$\mu/(R^2\sin^2\varphi)$	0	0
D_k	0	$(\lambda+\mu)/R$	0
E_k	0	0	0
F_k	0	0	$(\lambda+\mu)/(R\sin\varphi)$
G_k	$2(\lambda+2\mu)/R$	0	0
H_k	$\mu\cot\varphi/R^2$	$2(\lambda+2\mu)/R^2$	0
I_k	0	0	$2(\lambda+2\mu)/(R^2\sin\varphi)$
J_k	$-2(\lambda+2\mu)/R^2$	0	0
$\bar{A}_k$	0	μ	0
$\bar{B}_k$	0	$(\lambda+2\mu)/R^2$	0
$\bar{C}_k$	0	$\mu/(R^2\sin^2\varphi)$	0
$\bar{D}_k$	$(\lambda+\mu)/R$	0	0
$\bar{E}_k$	0	0	$(\lambda+\mu)/(R^2\sin\varphi)$
$\bar{F}_k$	0	0	0
$\bar{G}_k$	$(\lambda+\mu)\cot\varphi/R$	$2\mu/R$	0
$\bar{H}_k$	$-(\lambda+3\mu)/R^2$	$(\lambda+2\mu)\cot\varphi/R^2$	0
$\bar{I}_k$	0	0	$(\lambda+3\mu)\cot\varphi/(R^2\sin\varphi)$
$\bar{J}_k$	$-(\lambda+3\mu)\cot\varphi/R^2$	$-(\lambda+2\mu)/(R^2\sin^2\varphi)$	0

TABLE I

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

SPHERICAL COORDINATES (cont.)

($\varphi \neq 0$)

	k = 1	k = 2	k = 3
$\bar{\bar{A}}_k$	0	0	μ
$\bar{\bar{B}}_k$	0	0	μ/R^2
$\bar{\bar{C}}_k$	0	0	$(\lambda+2\mu)/(R^2\sin^2\varphi)$
$\bar{\bar{D}}_k$	0	0	0
$\bar{\bar{E}}_k$	0	$(\lambda+\mu)/(R^2\sin\varphi)$	0
$\bar{\bar{F}}_k$	$(\lambda+\mu)/(R\sin\varphi)$	0	0
$\bar{\bar{G}}_k$	0	0	$2\mu/R$
$\bar{\bar{H}}_k$	0	0	$\mu\cot\varphi/R^2$
$\bar{\bar{I}}_k$	$-(\lambda+3\mu)/(R^2\sin\varphi)$	$-(\lambda+3\mu)\cot\varphi/(R^2\sin\varphi)$	0
$\bar{\bar{J}}_k$	0	0	$-\mu/(R^2\sin^2\varphi)$

TABLE II

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

TOROIDAL COORDINATES

	k = 1	k = 2	k = 3
A_k	$\lambda+2\mu$	0	0
B_k	μ/r^2	0	0
C_k	$\mu/(a+r\sin\varphi)^2$	0	0
D_k	0	$(\lambda+\mu)/r$	0
E_k	0	0	0
F_k	0	0	$(\lambda+\mu)/(a+r\sin\varphi)$
G_k	$(\lambda+2\mu)(a+2r\sin\varphi)/[r(a+r\sin\varphi)]$	0	0
H_k	$\mu\cos\varphi/[r(a+r\sin\varphi)]$	$\dfrac{2(\lambda+2\mu)r\sin\varphi+(\lambda+3\mu)a}{r^2(a+r\sin\varphi)}$	0

TABLE II

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

TOROIDAL COORDINATES (cont.)

	k = 1	k = 2	k = 3
I_k	0	0	$\frac{2(\lambda+2\mu)r\sin\varphi+(\lambda+\mu)a}{r(a+r\sin\varphi)^2}$
J_k	$-(\lambda+2\mu)\left[1/r^2+\sin^2\varphi/(a+r\sin\varphi)^2\right]$	$(\lambda+2\mu)a\cos\varphi/\left[r(a+r\sin\varphi)^2\right]$	0
$\bar{A}_k$	0	μ	0
$\bar{B}_k$	0	$(\lambda+2\mu)/r^2$	0
$\bar{C}_k$	0	$\mu/(a+r\sin\varphi)^2$	0
$\bar{D}_k$	$(\lambda+\mu)/r$	0	0
$\bar{E}_k$	0	0	$(\lambda+\mu)/\left[r(a+r\sin\varphi)\right]$
$\bar{F}_k$	0	0	0
$\bar{G}_k$	$(\lambda+\mu)\cos\varphi/(a+r\sin\varphi)$	$\mu(a+2r\sin\varphi)/\left[r(a+r\sin\varphi)\right]$	0
$\bar{H}_k$	$-(\lambda+3\mu)/r^2$	$(\lambda+2\mu)\cos\varphi/\left[r(a+r\sin\varphi)\right]$	0
$\bar{I}_k$	0	0	$(\lambda+3\mu)\cos\varphi/(a+r\sin\varphi)^2$
$\bar{J}_k$	$-\frac{(\lambda+3\mu)r\sin\varphi\cos\varphi+\mu a\cos\varphi}{r(a+r\sin\varphi)^2}$	$-\frac{(\lambda+2\mu)r^2+\mu a^2+(\lambda+3\mu)ar\sin\varphi}{r^2(a+r\sin\varphi)^2}$	0
$\bar{\bar{A}}_k$	0	0	μ
$\bar{\bar{B}}_k$	0	0	μ/r^2
$\bar{\bar{C}}_k$	0	0	$(\lambda+2\mu)/(a+r\sin\varphi)^2$

TABLE II

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

TOROIDAL COORDINATES (cont.)

	k = 1	k = 2	k = 3
$\bar{\bar{D}}_k$	0	0	0
$\bar{\bar{E}}_k$	0	$(\lambda+\mu)/[r(a+r\sin\varphi)]$	0
$\bar{\bar{F}}_k$	$(\lambda+\mu)/(a+r\sin\varphi)$	0	0
$\bar{\bar{G}}_k$	0	0	$\mu(a+2r\sin\varphi)/[r(a+r\sin\varphi)]$
$\bar{\bar{H}}_k$	0	0	$\mu\cos\varphi/[r(a+r\sin\varphi)]$
$\bar{\bar{I}}_k$	$-(\lambda+3\mu)\sin\varphi/(a+r\sin\varphi)^2$	$-(\lambda+3\mu)\cos\varphi/(a+r\sin\varphi)^2$	0
$\bar{\bar{J}}_k$	0	0	$-\mu/(a+r\sin\varphi)^2$

TABLE III

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

POLAR COORDINATES - FOR APPROXIMATE TWO DIMENSIONAL LOCAL

SOLUTION FOR ANY (θ = constant) CROSS-SECTIONAL PLANE

	k = 1	k = 2
A_k	$\lambda+2\mu$	0
B_k	μ/R^2	0
D_k	0	$(\lambda+\mu)/R$
G_k	$(\lambda+2\mu)/R+\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial R}$	$\frac{1}{R}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial\varphi}$

TABLE III

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

POLAR COORDINATES - FOR APPROXIMATE TWO DIMENSIONAL LOCAL

SOLUTION FOR ANY (θ = constant) CROSS-SECTIONAL PLANE

(cont.)

	k = 1	k = 2
H_k	$\frac{1}{R^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$	$\frac{1}{R}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial R} + (\lambda+3\mu)/R^2$
J_k	$\frac{1}{R}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial R} - (\lambda+2\mu)/R^2$	$\frac{1}{R^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial\varphi}$
$\bar{A}_k$	0	μ
$\bar{B}_k$	0	$(\lambda+2\mu)/R^2$
$\bar{D}_k$	$(\lambda+\mu)/R$	0
$\bar{G}_k$	$\frac{1}{R}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$	$\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial R} + \mu/R$
$\bar{H}_k$	$\frac{1}{R}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial R} - \frac{(\lambda+3\mu)}{R^2}$	$\frac{1}{R^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial\varphi}$
$\bar{J}_k$	$-\frac{1}{R^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$	$-\frac{1}{R}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial R} - \mu/R^2$

NOTE: Temperature dependent material property derivative terms are also included. Only applicable coefficients are listed. By replacing R by r, the above coefficients are applicable in the torus cross-section region.

Derivation of Equilibrium Equations in Terms of
Displacements in Spherical and Toroidal Coordinates (cont.)

Equations for Stresses in Terms of Displacements

From Hooke's law, Eqs. (9) and (10),

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij}\left[\lambda\theta - (3\lambda + 2\mu)\int_{T_o}^{T} \alpha\,(T)dT\right] \qquad (17)$$

Where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = 1, \quad i = j$$
$$= 0, \quad i \neq j$$

and

$$\theta \equiv e_{11} + e_{22} + e_{33}$$

Writing the strains in terms of displacements from either Eqs. (14) or (15) and shortening the nomenclature by defining the stresses

$$\tau_1 \equiv \tau_{rr} \text{ or } \tau_{RR}$$
$$\tau_2 \equiv \tau_{\varphi\varphi}$$
$$\tau_3 \equiv \tau_{\theta\theta}$$
$$\tau_4 \equiv \tau_{r\varphi} \text{ or } \tau_{R\varphi}$$
$$\tau_5 \equiv \tau_{\varphi\theta}$$
$$\tau_6 \equiv \tau_{r\theta} \text{ or } \tau_{R\theta}$$

Eq. (17) may be written in terms of displacements according to

$$\tau_\ell + \Delta_\ell\,(3\lambda+2\mu)\int_{T_o}^{T}\alpha(T)dT = \alpha_\ell u_r + \beta_\ell u_\varphi + \gamma_\ell u_\theta + \delta_\ell u$$
$$+ \bar{\alpha}_\ell v_r + \bar{\beta}_\ell v_\varphi + \bar{\gamma}_\ell v_\theta + \bar{\delta}_\ell v \qquad (18)$$
$$+ \bar{\bar{\alpha}}_\ell w_r + \bar{\bar{\beta}}_\ell w_\varphi + \bar{\bar{\gamma}}_\ell w_\theta + \bar{\bar{\delta}}_\ell w$$

Where

$$\Delta_\ell = 1 \quad \text{if} \quad \ell = 1,2,3$$
$$= 0 \quad \text{if} \quad \ell = 4,5,6$$

TABLE IV

COEFFICIENTS OF STRESS EQUATIONS

SPHERICAL COORDINATES

($\varphi \neq 0$)

ℓ	1	2	3	4	5	6
α_ℓ	$\lambda+2\mu$	λ	λ	0	0	0
β_ℓ	0	0	0	μ/R	0	0
γ_ℓ	0	0	0	0	0	$\mu/(R\sin\varphi)$
δ_ℓ	$2\lambda/R$	$2(\lambda+\mu)/R$	$2(\lambda+\mu)/R$	0	0	0
$\bar{\alpha}_\ell$	0	0	0	μ	0	0
$\bar{\beta}_\ell$	λ/R	$(\lambda+2\mu)/R$	λ/R	0	0	0
$\bar{\gamma}_\ell$	0	0	0	0	$\mu/(R\sin\varphi)$	0
$\bar{\delta}_\ell$	$\frac{\lambda\cot\varphi}{R}$	$\lambda\cot\varphi/R$	$(\lambda+2\mu)\cot\varphi/R$	$-\mu/R$	0	0
$\bar{\bar{\alpha}}_\ell$	0	0	0	0	0	μ
$\bar{\bar{\beta}}_\ell$	0	0	0	0	μ/R	0
$\bar{\bar{\gamma}}_\ell$	$\lambda/(R\sin\varphi)$	$\lambda/(R\sin\varphi)$	$(\lambda+2\mu)/(R\sin\varphi)$	0	0	0
$\bar{\bar{\delta}}_\ell$	0	0	0	0	$-\mu\cot\varphi/R$	$-\mu/R$

TABLE V

COEFFICIENTS OF STRESS EQUATIONS

TOROIDAL COORDINATES

ℓ	1	2	3	4	5	6
α_ℓ	$\lambda+2\mu$	λ	λ	0	0	0
β_ℓ	0	0	0	μ/R	0	0
γ_ℓ	0	0	0	0	0	$\mu/(a+r\sin\varphi)$
δ_ℓ	$\frac{\lambda(a+2r\sin\varphi)}{r(a+r\sin\varphi)}$	$\frac{\lambda+2\mu}{r}+\frac{\lambda\sin\varphi}{a+r\sin\varphi}$	$\frac{\lambda}{r}+\frac{(\lambda+2\mu)\sin\varphi}{a+r\sin\varphi}$	0	0	0
$\bar{\alpha}_\ell$	0	0	0	μ	0	0
$\bar{\beta}_\ell$	λ/r	$(\lambda+2\mu)/r$	λ/r	0	0	0
$\bar{\gamma}_\ell$	0	0	0	0	$\mu/(a+r\sin\varphi)$	0
$\bar{\delta}_\ell$	$\frac{\lambda\cos\varphi}{a+r\sin\varphi}$	$\frac{\lambda\cos\varphi}{a+r\sin\varphi}$	$\frac{(\lambda+2\mu)\cos\varphi}{a+r\sin\varphi}$	$-\mu/r$	0	0
$\bar{\bar{\alpha}}_\ell$	0	0	0	0	0	μ
$\bar{\bar{\beta}}_\ell$	0	0	0	0	μ/r	0
$\bar{\bar{\gamma}}_\ell$	$\lambda/(a+r\sin\varphi)$	$\frac{\lambda}{a+r\sin\varphi}$	$\frac{\lambda+2\mu}{a+r\sin\varphi}$	0	0	0
$\bar{\bar{\delta}}_\ell$	0	0	0	0	$-\frac{\mu\cos\varphi}{a+r\sin\varphi}$	$-\frac{\mu\sin\varphi}{a+r\sin\varphi}$

TABLE VI

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

POLAR COORDINATES - FOR APPROXIMATE TWO DIMENSIONAL LOCAL SOLUTION FOR ANY (θ = constant) CROSS-SECTIONAL PLANE

	1	2	4
α_ℓ	$\lambda+2\mu$	λ	0
β_ℓ	0	0	μ/R
δ_ℓ	λ/R	$(\lambda+2\mu)/R$	0
$\bar{\alpha}_\ell$	0	0	μ
$\bar{\beta}_\ell$	λ/R	$(\lambda+2\mu)/R$	0
$\bar{\delta}_\ell$	0	0	$-\mu/R$

NOTE: Only applicable values of ℓ and coefficients are listed. By replacing R by r, the above coefficients are applicable in the torus cross-section region.

Equations at the Axis of Symmetry

Certain of the coefficients in the displacement equilibrium and stress equations become singular at the axis of symmetry ($\varphi = 0$). For the non-axially-symmetric case the axis of symmetry has no special physical significance and this point can be avoided. For the axially-symmetric case, however, the axis of symmetry is generally quite important and the singular coefficients may be evaluated by the use of L'Hôspital's rule. For example, the coefficient H_1 in the displacement equilibrium equations in spherical coordinates is $\mu \cot\varphi / R^2$ which becomes infinite as φ approaches zero. From Eq. (16), this term multiplies the displacement component $\frac{\partial u}{\partial \varphi}$. The conditions for axial symmetry are

$$w(R,\varphi,\theta) = \frac{\partial f}{\partial \theta} = 0 \text{, where f is any function of } R,\varphi,\theta, \tag{19}$$

from which it can be shown that

$$v = \frac{\partial u}{\partial \varphi} = \frac{\partial^2 v}{\partial \varphi^2} = 0 \text{ at } \varphi = 0. \tag{20}$$

Hence, since $\frac{\partial u}{\partial \varphi}$ approaches zero while H_1 approaches infinity, L'Hôspital's rule is applicable to the product

$$\frac{\mu \cot\varphi}{R^2} \frac{\partial u}{\partial \varphi}$$

as $\varphi \to 0$. Taking the limit, there is obtained

$$\begin{aligned} \lim_{\varphi \to 0} \frac{\mu \cot\varphi}{R^2} \cdot \frac{\partial u}{\partial \varphi} &= \frac{\mu}{R^2} \lim_{\varphi \to 0} \frac{\frac{\partial u}{\partial \varphi} \cos\varphi}{\sin\varphi} \\ &= \frac{\mu}{R^2} \lim_{\varphi \to 0} \frac{\frac{\partial^2 u}{\partial \varphi^2} \cos\varphi - \frac{\partial u}{\partial \varphi} \sin\varphi}{\cos\varphi} \\ &= \frac{\mu}{R^2} \cdot \frac{\partial^2 u}{\partial \varphi^2}. \end{aligned}$$

Derivation of Equilibrium Equations in Terms of Displacements in Spherical and Toroidal Coordinates (cont.)

Hence, for this case, the coefficient H_1 becomes zero and the coefficient B_1 which multiplies $\frac{\partial^2 u}{\partial \varphi^2}$ is increased by μ/R^2. Applying this limiting process to all the singular terms the following sets of coefficients are obtained:

TABLE VII

COEFFICIENTS OF EQUILIBRIUM EQUATIONS ON AXIS OF SYMMETRY ($\varphi = 0$) FOR AXIALLY - SYMMETRIC CASE

SPHERICAL COORDINATES

k	1	2	3	k	1	2	3	k	1	2	3
A_k	$\lambda+2\mu$	0	0	$\bar{A}_k$	0	0	0	$\bar{\bar{A}}_k$	0	0	0
B_k	$2\mu/R^2$	0	0	$\bar{B}_k$	0	0	0	$\bar{\bar{B}}_k$	0	0	0
C_k	0	0	0	$\bar{C}_k$	0	0	0	$\bar{\bar{C}}_k$	0	0	0
D_k	0	0	0	$\bar{D}_k$	$2(\lambda+\mu)/R$	0	0	$\bar{\bar{D}}_k$	0	0	0
E_k	0	0	0	$\bar{E}_k$	0	0	0	$\bar{\bar{E}}_k$	0	0	0
F_k	0	0	0	$\bar{F}_k$	0	0	0	$\bar{\bar{F}}_k$	0	0	0
G_k	$2(\lambda+2\mu)/R$	0	0	$\bar{G}_k$	0	0	0	$\bar{\bar{G}}_k$	0	0	0
H_k	0	0	0	$\bar{H}_k$	$-2(\lambda+3\mu)/R^2$	0	0	$\bar{\bar{H}}_k$	0	0	0
I_k	0	0	0	$\bar{I}_k$	0	0	0	$\bar{\bar{I}}_k$	0	0	0
J_k	$-2(\lambda+2\mu)/R^2$	0	0	$\bar{J}_k$	0	0	0	$\bar{\bar{J}}_k$	0	0	0

TABLE VIII

COEFFICIENTS OF STRESS EQUATIONS ON AXIS OF SYMMETRY ($\varphi = 0$) FOR AXIALLY - SYMMETRIC CASE SPHERICAL COORDINATES

ℓ	1	2	3	4	5	6
α_ℓ	$\lambda+2\mu$	λ	λ	0	0	0
β_ℓ	0	0	0	μ/R	0	0
γ_ℓ	0	0	0	0	0	0
δ_ℓ	$2\lambda/R$	$2(\lambda+\mu)/R$	$2(\lambda+\mu)/R$	0	0	0
$\bar{\alpha}_\ell$	0	0	0	μ	0	0
$\bar{\beta}_\ell$	$2\lambda/R$	$2(\lambda+\mu)/R$	$2(\lambda+\mu)/R$	0	0	0
$\bar{\gamma}_\ell$	0	0	0	0	0	0
$\bar{\delta}_\ell$	0	0	0	$-\mu/R$	0	0
$\bar{\bar{\alpha}}_\ell$	0	0	0	0	0	0
$\bar{\bar{\beta}}_\ell$	0	0	0	0	0	0
$\bar{\bar{\gamma}}_\ell$	0	0	0	0	0	0
$\bar{\bar{\delta}}_\ell$	0	0	0	0	0	0

Temperature Dependence of Elastic Constants

If, in addition to the coefficient of thermal expansion, the elastic constants are strongly dependent on temperature, then additional terms must be included in the displacement equilibrium equations to account for the special derivatives of these constants. Differentiating the stress component τ_{ii} with respect to coordinate α_i, for example, from Equation (9), there is obtained

$$\frac{\partial \tau_{ii}}{\partial \alpha_i} = \lambda \frac{\partial \theta}{\partial \alpha_i} + \theta \frac{\partial \lambda}{\partial \alpha_i} + 2\mu \frac{\partial e_{ii}}{\partial \alpha_i} + 2\, e_{ii} \frac{\partial \mu}{\partial \alpha_i}$$

$$- (3\lambda + 2\mu)\, \alpha(T) \frac{\partial T}{\partial \alpha_i} - \frac{\partial}{\partial \alpha_i} - (3\lambda + 2\mu) \int_{T_o}^{T} \alpha\,(T)\, dT$$

$$= \theta \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \alpha_i} + 2\, e_{ii} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \alpha_i} - \frac{\partial}{\partial T} (3\lambda + 2\mu) \frac{\partial T}{\partial \alpha_i} \int_{T_o}^{T} \alpha\,(T)\, dT \tag{21}$$

$$+ \lambda \frac{\partial \theta}{\partial \alpha_i} + 2\mu \frac{\partial e_{ii}}{\partial \alpha_i} - (3\lambda + 2\mu)\, \alpha\,(T) \frac{\partial T}{\partial \alpha_i} \;,$$

where the first three terms to the right of the equal sign have not been accounted for in the coefficients of Equation (16). Representing the additional terms by primed quantities, Equation (16) becomes

$$(A_k + A'_k) \frac{\partial^2 u}{\partial \alpha_1^2} + (B_k + B'_k) \frac{\partial^2 u}{\partial \alpha_2^2} + \ldots = \frac{(3\lambda + 2\mu)\, \alpha(T)}{\sqrt{g_{kk}}} \frac{\partial T}{\partial \alpha_k} \tag{22}$$

$$+ \frac{1}{\sqrt{g_{kk}}} \frac{\partial}{\partial T} (3\lambda + 2\mu) \frac{\partial T}{\partial \alpha_k} \int_{T_o}^{T} \alpha(T)\, dT, \quad k = 1,\, 2,\, 3$$

The coefficients A'_k, B'_k, ... are tabulated below for spherical and toroidal coordinates, and for the special point in spherical coordinates on the axis of symmetry for the case of axial symmetry.

TABLE IX

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS
FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS
SPHERICAL COORDINATES

	k = 1	k = 2	k = 3
A'_k	0	0	0
B'_k	0	0	0
C'_k	0	0	0
D'_k	0	0	0
E'_k	0	0	0
F'_k	0	0	0
G'_k	$\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial R}$	$\frac{1}{R}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial\varphi}$	$\frac{1}{R\sin\varphi}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial\theta}$
H'_k	$\frac{1}{R^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$	$\frac{1}{R}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial R}$	0
I'_k	$\frac{1}{R^2\sin^2\varphi}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\theta}$	0	$\frac{1}{R\sin\varphi}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial R}$
J'_k	$\frac{2}{R}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial R}$	$\frac{2}{R^2}\frac{\partial}{\partial T}(\lambda+\mu)\frac{\partial T}{\partial\varphi}$	$\frac{2}{R^2\sin\varphi}\frac{\partial}{\partial T}(\lambda+\mu)\frac{\partial T}{\partial\theta}$
$\bar{A}'_k$	0	0	0
$\bar{B}'_k$	0	0	0
$\bar{C}'_k$	0	0	0
$\bar{D}'_k$	0	0	0
$\bar{E}'_k$	0	0	0
$\bar{F}'_k$	0	0	0
$\bar{G}'_k$	$\frac{1}{R}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$	$\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial R}$	0
$\bar{H}'_k$	$\frac{1}{R}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial R}$	$\frac{1}{R^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial\varphi}$	$\frac{1}{R^2\sin\varphi}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial\theta}$

TABLE IX

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS

FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS

SPHERICAL COORDINATES (cont.)

	k = 1	k = 2	k = 3
$\bar{I}'_k$	0	$\frac{1}{R^2 \sin^2\varphi} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\theta}$	$\frac{1}{R^2 \sin\varphi} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\varphi}$
$\bar{J}'_k$	$\frac{\partial\lambda}{\partial T} \frac{\partial T}{\partial R} \frac{\cot\varphi}{R} - \frac{1}{R^2} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\varphi}$	$\frac{\cot\varphi}{R^2} \frac{\partial\lambda}{\partial T} \frac{\partial T}{\partial\varphi} - \frac{1}{R} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial R}$	$\frac{\cot\varphi}{R^2 \sin\varphi} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial\theta}$
$\bar{\bar{A}}'_k$	0	0	0
$\bar{\bar{B}}'_k$	0	0	0
$\bar{\bar{C}}'_k$	0	0	0
$\bar{\bar{D}}'_k$	0	0	0
$\bar{\bar{E}}'_k$	0	0	0
$\bar{\bar{F}}'_k$	0	0	0
$\bar{\bar{G}}'_k$	$\frac{1}{R \sin\varphi} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\theta}$	0	$\frac{\partial\mu}{\partial T} \frac{\partial T}{\partial R}$
$\bar{\bar{H}}'_k$	0	$\frac{1}{R^2 \sin\varphi} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\theta}$	$\frac{1}{R^2} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\varphi}$
$\bar{\bar{I}}'_k$	$\frac{1}{R \sin\varphi} \frac{\partial\lambda}{\partial T} \frac{\partial T}{\partial R}$	$\frac{1}{R^2 \sin\varphi} \frac{\partial\lambda}{\partial T} \frac{\partial T}{\partial\varphi}$	$\frac{1}{R^2 \sin^2\varphi} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial\theta}$
$\bar{\bar{J}}'_k$	$-\frac{1}{R^2 \sin\varphi} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\theta}$	$-\frac{\cot\varphi}{R^2 \sin\varphi} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\theta}$	$-\frac{1}{R} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial R} - \frac{\cot\varphi}{R^2} \frac{\partial\mu}{\partial T} \frac{\partial T}{\partial\varphi}$

TABLE X

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS

FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS

TOROIDAL COORDINATES

	k = 1	k = 2	k = 3
A'_k	0	0	0
B'_k	0	0	0
C'_k	0	0	0
D'_k	0	0	0
E'_k	0	0	0
F'_k	0	0	0
G'_k	$\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial r}$	$\frac{1}{r}\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial \varphi}$	$\frac{1}{(a+r\sin\varphi)}\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial \theta}$
H'_k	$\frac{1}{r^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial \varphi}$	$\frac{1}{r}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial r}$	0
I'_k	$\frac{1}{(a+r\sin\varphi)^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial \theta}$	0	$\frac{1}{(a+r\sin\varphi)}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial r}$
J'_k	$\left[\frac{1}{r}+\frac{\sin\varphi}{(a+r\sin\varphi)}\right]\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial r}$	$\frac{1}{r^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial \varphi}+\frac{\sin\varphi}{r(a+r\sin\varphi)}\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial \varphi}$	$\frac{\sin\varphi}{(a+r\sin\varphi)^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial \theta}+\frac{\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial \theta}}{r(a+r\sin\varphi)}$
$\overline{A}'_k$	0	0	0
$\overline{B}'_k$	0	0	0
$\overline{C}'_k$	0	0	0
$\overline{D}'_k$	0	0	0
$\overline{E}'_k$	0	0	0
$\overline{F}'_k$	0	0	0
$\overline{G}'_k$	$\frac{1}{r}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial \varphi}$	$\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial r}$	0
$\overline{H}'_k$	$\frac{1}{r}\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial r}$	$\frac{1}{r^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial \varphi}$	$\frac{1}{r(a+\sin\varphi)}\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial \theta}$

TABLE X

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS

TOROIDAL COORDINATES (cont.)

	k = 1	k = 2	k = 3
$\bar{I}'_k$	0	$\frac{1}{(a+r\sin\varphi)^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\theta}$	$\frac{1}{r(a+r\sin\varphi)}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$
$\bar{J}'_k$	$\frac{\cos\varphi}{(a+r\sin\varphi)}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial r}-\frac{1}{r^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$	$\frac{\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial\varphi}-\frac{1}{r}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial r}$	$\frac{\cos\varphi}{(a+r\sin\varphi)^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial\theta}$
$\bar{\bar{A}}'_k$	0	0	0
$\bar{\bar{B}}'_k$	0	0	0
$\bar{\bar{C}}'_k$	0	0	0
$\bar{\bar{D}}'_k$	0	0	0
$\bar{\bar{E}}'_k$	0	0	0
$\bar{\bar{F}}'_k$	0	0	0
$\bar{\bar{G}}'_k$	$\frac{1}{(a+r\sin\varphi)}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\theta}$	0	$\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial r}$
$\bar{\bar{\bar{H}}}'_k$	0	$\frac{1}{r(a+r\sin\varphi)}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\theta}$	$\frac{1}{r^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\varphi}$
$\bar{\bar{\bar{I}}}'_k$	$\frac{1}{(a+r\sin\varphi)}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial r}$	$\frac{1}{r(a+r\sin\varphi)}\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial\varphi}$	$\frac{1}{(a+r\sin\varphi)^2}\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial\theta}$
$\bar{\bar{\bar{J}}}'_k$	$-\frac{\sin\varphi}{(a+r\sin\varphi)^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\theta}$	$-\frac{\cos\varphi}{(a+r\sin\varphi)^2}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial\theta}$	$-\frac{\frac{\partial\mu}{\partial T}}{(a+r\sin\varphi)}\left[\frac{\partial T}{\partial r}\sin\varphi+\frac{\partial T}{\partial\varphi}\frac{\cos\varphi}{r}\right]$

Derivation of Equilibrium Equations in Terms of
Displacements in Spherical and Toroidal Coordinates (cont.)

Axis of Symmetry with Axial Symmetry

The only non-aero terms in the coefficients of Table IX on the axis of symmetry in the axially-symmetric case are the following:

$$G_1' = \frac{\partial}{\partial T} (\lambda+2\mu) \frac{\partial T}{\partial R}$$

$$J_1' = \frac{2}{R} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R}$$

$$\bar{H}_1' = \frac{2}{R} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R}$$

The integral term in Equation (22) is also non-zero for the equilibrium equation corresponding to k = 1.

Finite Difference Formulation

The difference analogs to the partial differential equations are constructed on a grid network as shown in Figure 2, for which α_1 = constant lines are ordered by the subscript i, α_2 = constant lines by the subscript j, α_3 = constant lines by the subscript k, and the intersection of grid lines (nodes) by the triple subscript i, j, k.

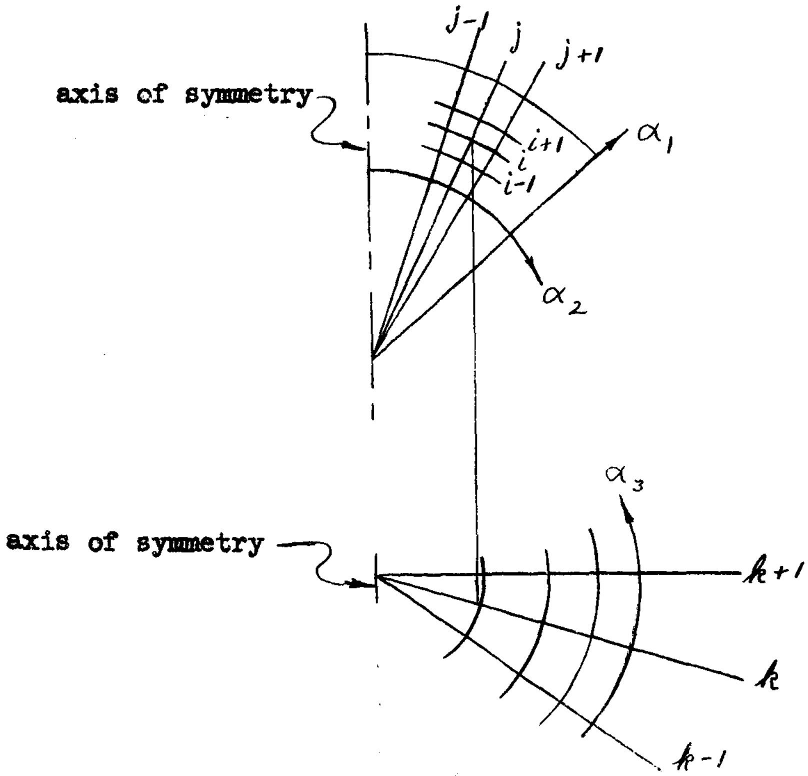


Figure 2. Grid Notation for Finite Difference Formulation

For the general case the grid spacing will be irregular and the increments in the vicinity of a node will be designated by the following:

$$
\begin{aligned}
h_{11} &= (\alpha_1)_{i+1} - (\alpha_1)_i & h_{21} &= (\alpha_2)_{i+1} - (\alpha_2)_i & h_{31} &= (\alpha_3)_{i+1} - (\alpha_3)_i \\
h_{12} &= (\alpha_1)_{i+2} - (\alpha_1)_i & h_{22} &= (\alpha_2)_{i+2} - (\alpha_2)_i & h_{32} &= (\alpha_3)_{i+2} - (\alpha_3)_i \\
h_{13} &= (\alpha_1)_i - (\alpha_1)_{i-1} & h_{23} &= (\alpha_2)_i - (\alpha_2)_{i-1} & h_{33} &= (\alpha_3)_i - (\alpha_3)_{i-1} \\
h_{14} &= (\alpha_1)_i - (\alpha_1)_{i-2} & h_{24} &= (\alpha_2)_i - (\alpha_2)_{i-2} & h_{34} &= (\alpha_3)_i - (\alpha_3)_{i-2}
\end{aligned}
$$

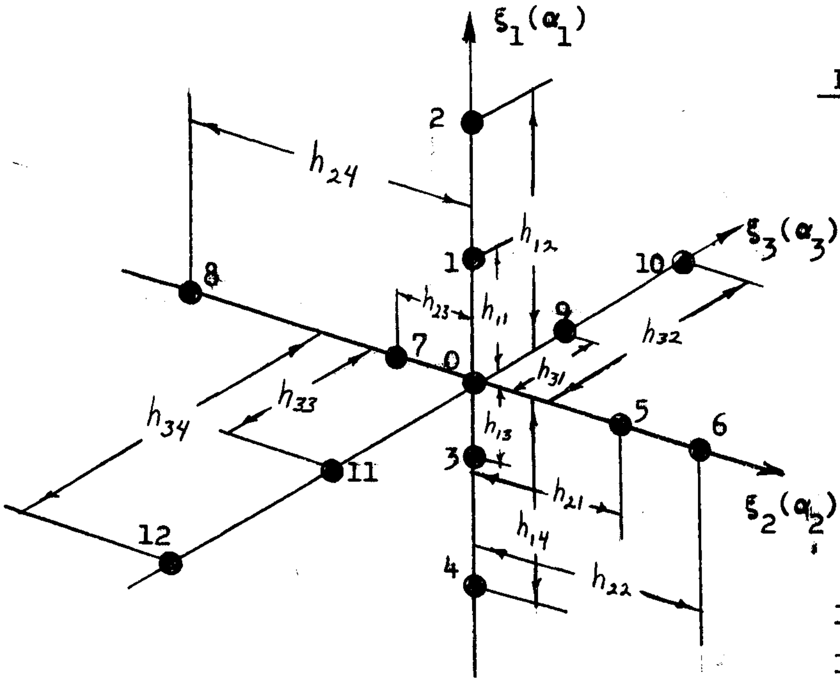
Let $f\ (\alpha_1, \alpha_2, \alpha_3)$ be any function of the coordinates such that it and its partial derivatives (up to any order required in the analysis) are continuous, and expand the function about the point i, j, k. Using a new coordinate system with origin at i, j, k and with ξ_1, ξ_2, ξ_3 directed along α_1, α_2, α_3, respectively, the function $f\ (\xi_1, \xi_2, \xi_3)$ is written

$$
\begin{aligned}
f\ (\xi_1, \xi_2, \xi_3) = f_{i,j,k} &+ B_1 \xi_1 + B_2 \xi_2 + B_3 \xi_3 + B_4 \xi_1 \xi_2 \\
&+ B_5 \xi_2 \xi_3 + B_6 \xi_3 \xi_1 + B_7 \xi_1^2 + B_8 \xi_2^2 + B_9 \xi_3^2 \\
&+ B_{10} \xi_1 \xi_2 \xi_3 + B_{11} \xi_1 \xi_2^2 + B_{12} \xi_1 \xi_3^2 + B_{13} \xi_1^2 \xi_2 \\
&+ B_{14} \xi_2 \xi_3^2 + \ldots
\end{aligned}
\tag{23}
$$

The first and second derivatives of $f\ (\alpha_1, \alpha_2, \alpha_3)$ with respect to α_1, α_2, α_3 are obtained from Equation (23) according to

$$
\begin{aligned}
\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} &= \left.\frac{\partial f}{\partial \xi_1}\right)_{(o,o,o)} = B_1, & \left.\frac{\partial^2 f}{\partial \alpha_1\, \partial \alpha_2}\right)_{i,j,k} &= B_4, & \left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} &= 2\ B_7 \\
\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} &= \left.\frac{\partial f}{\partial \xi_2}\right)_{(o,o,o)} = B_2, & \left.\frac{\partial^2 f}{\partial \alpha_2\, \partial \alpha_3}\right)_{i,j,k} &= B_5, & \left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} &= 2\ B_8 \\
\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} &= \left.\frac{\partial f}{\partial \xi_3}\right)_{(o,o,o)} = B_3, & \left.\frac{\partial^2 f}{\partial \alpha_3\, \partial \alpha_1}\right)_{i,j,k} &= B_6, & \left.\frac{\partial^2 f}{\partial \alpha_3^2}\right)_{i,j,k} &= 2\ B_9
\end{aligned}
\tag{24}
$$

By considering the values of $f\ (\xi_1, \xi_2, \xi_3)$ at the twelve nodes adjacent to i,j,k, the constants B_i are evaluated in terms of the function at these nodes and the grid spacings as shown in Figure 3.



Node	Coordinate
0	(i,j,k)
1	(i+1,j,k)
2	(i+2,j,k)
3	(i-1,j,k)
4	(i-2,j,k)
5	(i,j+1,k)
6	(i,j+2,k)
7	(i,j-1,k)
8	(i,j-2,k)
9	(i,j,k+1)
10	(i,j,k+2)
11	(i,j,k-1)
12	(i,j,k-2)

Figure 3. Coordinates of Irregular Mesh Intervals

Note that the grid spacing increments h_{ij} do not, in general, have the dimensions of length but have the dimensions of α_1, α_2 and α_3.

At points 1 and 3, Equation (23) becomes

$$f\,(h_{11},0,0) = f_{i,j,k} + B_1\,h_{11} + B_7\,h_{11}^2$$
$$f\,(-h_{13},0,0) = f_{i,j,k} - B_1\,h_{13} + B_7\,h_{13}^2 \qquad (25)$$

where terms of higher order are deleted. Solving for B_1 and B_7 from Equation (25) gives for the first and second irregular central derivative with respect to α_1:

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{h_{13}^2\,f_{i+1,j,k} + (h_{11}^2 - h_{13}^2)\,f_{i,j,k} - h_{11}^2\,f_{i-1,j,k}}{h_{11}\,h_{13}\,(h_{11} + h_{13})}$$
$$\left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} = 2\left[\frac{h_{13}\,f_{i+1,j,k} - (h_{11} + h_{13})\,f_{i,j,k} + h_{11}\,f_{i-1,j,k}}{h_{11}\,h_{13}\,(h_{11} + h_{13})}\right] \qquad (26)$$

Substituting $h_{11} = h_{13} = h_1$ into Equation (26) gives for the first and second regular central derivatives with respect to α_1

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{f_{i+1,j,k} - f_{i-1,j,k}}{2h_1}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} = \frac{f_{i+1,j,k} - 2f_{i,j,k} + f_{i-1,j,k}}{h_1^2} \qquad (27)$$

By a similar procedure the following first and second regular and irregular central derivatives are obtained with respect to the coordinates α_2 and α_3:

<u>First Regular Central Derivatives</u> ($h_2 = h_{21} = h_{23}$, $h_3 = h_{31} = h_{33}$)

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{f_{i,j+1,k} - f_{i,j-1,k}}{2h_2} \qquad (28)$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{f_{i,j,k+1} - f_{i,j,k-1}}{2h_3} \qquad (29)$$

<u>First Irregular Central Derivatives</u>

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{h_{23}^2 f_{i,j+1,k} + (h_{21}^2 - h_{23}^2) f_{i,j,k} - h_{21}^2 f_{i,j-1,k}}{h_{21} h_{23} (h_{21} + h_{23})} \qquad (30)$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{h_{33}^2 f_{i,j,k+1} + (h_{31}^2 - h_{33}^2) f_{i,j,k} - h_{31}^2 f_{i,j,k-1}}{h_{31} h_{33} (h_{31} + h_{33})} \qquad (31)$$

<u>Second Regular Central Derivatives</u> ($h_2 = h_{21} = h_{23}$, $h_3 = h_{31} = h_{33}$)

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} = \frac{f_{i,j+1,k} - 2f_{i,j,k} + f_{i,j-1,k}}{h_2^2} \qquad (32)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right)_{i,j,k} = \frac{f_{i,j,k+1} - 2f_{i,j,k} + f_{i,j,k-1}}{h_3^2} \qquad (33)$$

<u>Second Irregular Central Derivatives</u>

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} = \frac{2\left[h_{23} f_{i,j+1,k} - (h_{21} + h_{23}) f_{i,j,k} + h_{21} f_{i,j-1,k}\right]}{h_{21} h_{23} (h_{21} + h_{23})} \qquad (34)$$

Derivation of Equilibrium Equations in Terms of
Displacements in Spherical and Toroidal Coordinates (cont.)

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right)_{i,j,k} = \frac{2\left[h_{32}\ f_{i,j,k+1} - (h_{31} + h_{32})\ f_{i,j,k} + h_{31}\ f_{i,j,k-1}\right]}{h_{31}\ h_{32}\ (h_{31} + h_{32})} \qquad (35)$$

Forward and Backward Derivatives

By applying the same procedure as above with respect to two nodes located either forward or backward from the origin (i,j,k), the first and second regular and irregular derivatives are obtained in terms of the function $f(\alpha_1, \alpha_2, \alpha_3)$ evaluated at these nodes. The results are summarized below for the three coordinate directions:

First Irregular Forward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{-(h_{12}^2 - h_{11}^2)\ f_{i,j,k} + h_{12}^2\ f_{i+1,j,k} - h_{11}^2\ f_{i+2,j,k}}{h_{11}\ h_{12}\ (h_{12} - h_{11})} \qquad (36)$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{-(h_{22}^2 - h_{21}^2)\ f_{i,j,k} + h_{22}^2\ f_{i,j+1,k} - h_{21}^2\ f_{i,j+2,k}}{h_{21}\ h_{22}\ (h_{22} - h_{21})} \qquad (37)$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{-(h_{32}^2 - h_{31}^2)\ f_{i,j,k} + h_{32}^2\ f_{i,j,k+1} - h_{31}^2\ f_{i,j,k+2}}{h_{31}\ h_{32}\ (h_{32} - h_{31})} \qquad (38)$$

First Regular Forward Derivatives

For equal grid spacings in each of the three coordinate directions, defined according to

$$\left.\begin{aligned} h_{11} &= h_{12}/2 \equiv h_1 \\ h_{21} &= h_{22}/2 \equiv h_2 \\ h_{31} &= h_{32}/2 \equiv h_3 \end{aligned}\right\} \qquad (39)$$

Equations (36) - (38) reduce to

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{-3\ f_{i,j,k} + 4\ f_{i+1,j,k} - f_{i+2,j,k}}{2h_1} \qquad (40)$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{-3\ f_{i,j,k} + 4\ f_{i,j+1,k} - f_{i,j+2,k}}{2h_2} \qquad (41)$$

Derivation of Equilibrium Equations in Terms of Displacements in Spherical and Toroidal Coordinates (cont.)

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{-3\, f_{i,j,k} + 4\, f_{i,j,k+1} - f_{i,j,k+2}}{2h_3} \qquad (42)$$

Second Irregular Forward Derivatives

$$\left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} = 2\left[\frac{-h_{12}\, f_{i+1,j,k} + (h_{12} - h_{11})\, f_{i,j,k} + h_{11}\, f_{i+2,j,k}}{h_{11}\, h_{12}\, (h_{12} - h_{11})}\right] \qquad (43)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} = 2\left[\frac{-h_{22}\, f_{i,j+1,k} + (h_{22} - h_{21})\, f_{i,j,k} + h_{21}\, f_{i,j+2,k}}{h_{21}\, h_{22}\, (h_{22} - h_{21})}\right] \qquad (44)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right)_{i,j,k} = 2\left[\frac{-h_{32}\, f_{i,j,k+1} + (h_{32} - h_{31})\, f_{i,j,k} + h_{31}\, f_{i,j,k+2}}{h_{31}\, h_{32}\, (h_{32} - h_{31})}\right] \qquad (45)$$

Second Regular Forward Derivatives

With equal grid spacing, according to Equation (39), Equations (43) - (45) reduce to

$$\left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} = \frac{-2\, f_{i+1,j,k} + f_{i,j,k} + f_{i+2,j,k}}{h_1^2} \qquad (46)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} = \frac{-2\, f_{i,j+1,k} + f_{i,j,k} + f_{i,j+2,k}}{h_2^2} \qquad (47)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right)_{i,j,k} = \frac{-2\, f_{i,j,k+1} + f_{i,j,k} + f_{i,j,k+2}}{h_3^2} \qquad (48)$$

First Irregular Backward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{h_{13}^2\, f_{i-2,j,k} + (h_{14}^2 - h_{13}^2)\, f_{i,j,k} - h_{14}^2\, f_{i-1,j,k}}{h_{13}\, h_{14}\, (h_{14} - h_{13})} \qquad (49)$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{h_{23}^2\, f_{i,j-2,k} + (h_{24}^2 - h_{23}^2)\, f_{i,j,k} - h_{24}^2\, f_{i,j-1,k}}{h_{23}\, h_{24}\, (h_{24} - h_{23})} \qquad (50)$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{h_{33}^2\, f_{i,j,k-2} + (h_{34}^2 - h_{33}^2)\, f_{i,j,k} - h_{34}^2\, f_{i,j,k-1}}{h_{33}\, h_{34}\, (h_{34} - h_{33})} \qquad (51)$$

Derivation of Equilibrium Equations in Terms of Displacements in Spherical and Toroidal Coordinates (cont.)

First Regular Backward Derivatives ($h_{13} = \frac{h_{14}}{2} = h_1$, etc.)

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{f_{i-2,j,k} + 3\, f_{i,j,k} - 4\, f_{i-1,j,k}}{2h_1} \qquad (52)$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{f_{i,j-2,k} + 3\, f_{i,j,k} - 4\, f_{i,j-1,k}}{2h_2} \qquad (53)$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{f_{i,j,k-2} + 3\, f_{i,j,k} - 4\, f_{i,j,k-1}}{2h_3} \qquad (54)$$

Second Irregular Backward Derivatives

$$\left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} = 2\left[\frac{h_{13}\, f_{i-2,j,k} + (h_{14} - h_{13})\, f_{i,j,k} - h_{14}\, f_{i-1,j,k}}{h_{13}\, h_{14}\, (h_{14} - h_{13})}\right] \qquad (55)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} = 2\left[\frac{h_{23}\, f_{i,j-2,k} + (h_{24} - h_{23})\, f_{i,j,k} - h_{24}\, f_{i,j-1,k}}{h_{23}\, h_{24}\, (h_{24} - h_{23})}\right] \qquad (56)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right)_{i,j,k} = 2\left[\frac{h_{33}\, f_{i,j,k-2} + (h_{34} - h_{33})\, f_{i,j,k} - h_{34}\, f_{i,j,k-1}}{h_{33}\, h_{34}\, (h_{34} - h_{33})}\right] \qquad (57)$$

Second Regular Backward Derivatives

$$\left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} = \frac{f_{i-2,j,k} + f_{i,j,k} - 2\, f_{i-1,j,k}}{h_1^2} \qquad (58)$$

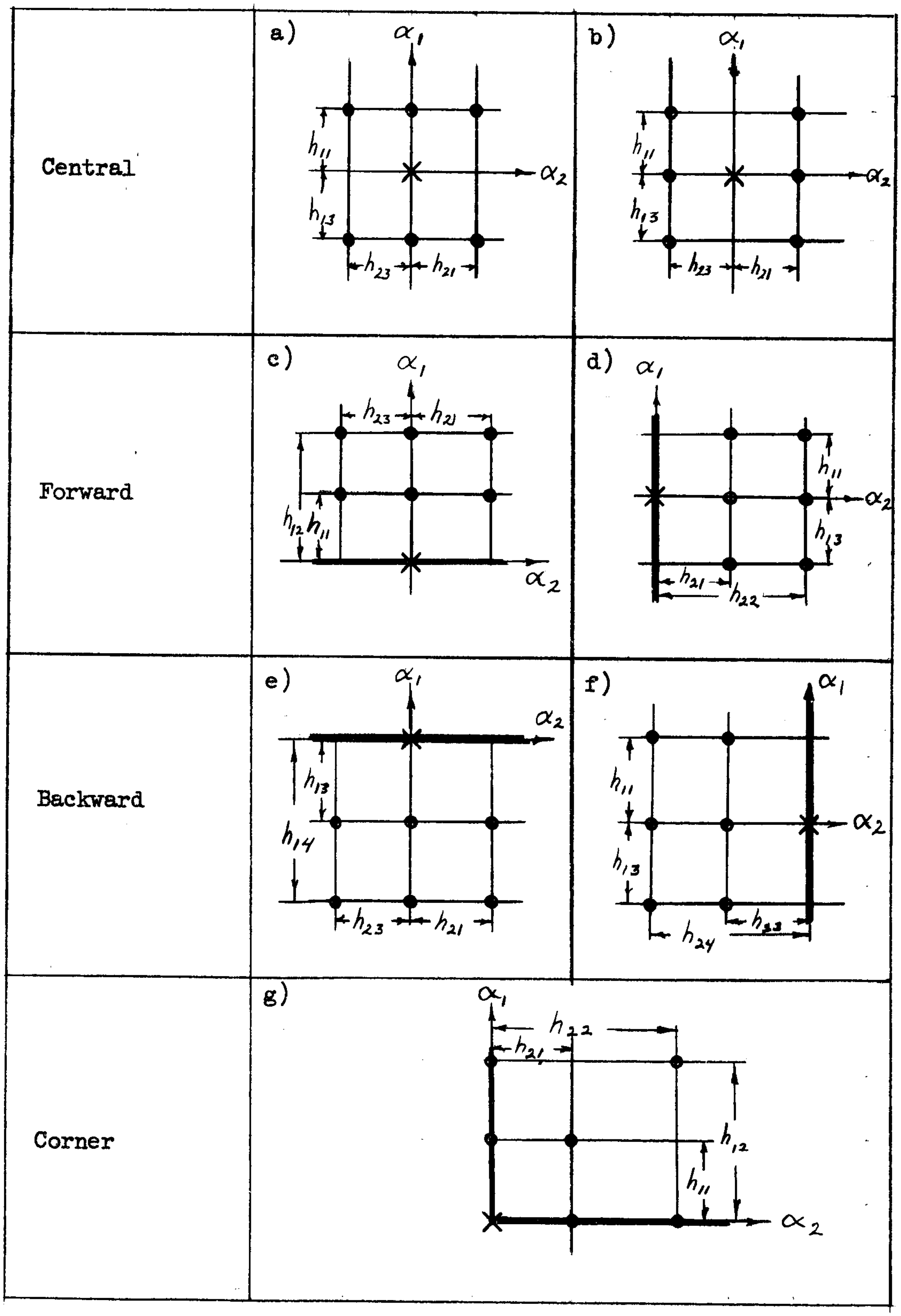
$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} = \frac{f_{i,j-2,k} + f_{i,j,k} - 2\, f_{i,j-1,k}}{h_2^2} \qquad (59)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right)_{i,j,k} = \frac{f_{i,j,k-2} + f_{i,j,k} - 2\, f_{i,j,k-1}}{h_3^2} \qquad (60)$$

Mixed Derivatives

It can be shown from Equation (23) that mixed derivatives require values of the function at any six nodes in the vicinity of the point under consideration. Figure 3 shows various combinations of mixed derivatives with respect to the

Figure 3. Irregular Mesh Intervals for Mixed Central, Forward Backward and Corner Derivatives



coordinate axes α_1 and α_2. It is noted that the mixed central derivatives involve the four corner nodes as well as two adjacent nodes in either of the two coordinate directions. The various combinations shown in Figure 3 are summarized below for the coordinate directions α_1 and α_2:

<u>Second Mixed Irregular Central Derivative With Respect to α_1 and α_2</u>

a) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{21}\;h_{23}\;(h_{11}+h_{13})\;(h_{21}+h_{23})}$$

$$\Big[h_{23}^2\left(f_{i+1,j+1,k}-f_{i-1,j+1,k}\right)-\left(h_{23}^2-h_{21}^2\right)\left(f_{i+1,j,k}-f_{i-1,j,k}\right)$$

$$-\,h_{21}^2\left(f_{i+1,j-1,k}-f_{i-1,j-1,k}\right)\Big] \qquad (61)$$

b) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{11}\;h_{13}\;(h_{11}+h_{13})\;(h_{21}+h_{23})}$$

$$\Big[h_{13}^2\left(f_{i+1,j+1,k}-f_{i+1,j-1,k}\right)-\left(h_{13}^2-h_{11}^2\right)\left(f_{i,j+1,k}-f_{i,j-1,k}\right)$$

$$-\,h_{11}^2\left(f_{i-1,j+1,k}-f_{i-1,j-1,k}\right)\Big] \qquad (62)$$

<u>Second Mixed Irregular Forward Derivative with Respect to α_1 and α_2</u>

c) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{21}\;h_{23}\;(h_{11}-h_{12})\;(h_{21}+h_{23})}$$

$$\Big[h_{23}^2\left(f_{i+1,j+1,k}-f_{i+1,j,k}-f_{i+2,j+1,k}+f_{i+2,j,k}\right) \qquad (63)$$

$$-\,h_{21}^2\left(f_{i+1,j-1,k}-f_{i+1,j,k}-f_{i+2,j-1,k}+f_{i+2,j,k}\right)\Big]$$

d) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{11}\;h_{13}\;(h_{21} - h_{22})\;(h_{11} + h_{13})}$$

$$\Big[h_{13}^{\,2}\left(f_{i+1,j+1,k} - f_{i,j+1,k} - f_{i+1,j+2,k} + f_{i,j+2,k}\right) \tag{64}$$

$$- h_{11}^{\,2}\left(f_{i-1,j+1,k} - f_{i,j+1,k} - f_{i-1,j+2,k} + f_{i,j+2,k}\right)\Big]$$

Second Mixed Irregular Backward Derivative With Respect to α_1 and α_2

e) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{-1}{h_{21}\;h_{23}\;(h_{13} - h_{14})\;(h_{21} + h_{23})}$$

$$\Big[h_{23}^{\,2}\left(f_{i-1,j+1,k} - f_{i-2,j+1,k} + f_{i-2,j,k} - f_{i-1,j,k}\right) \tag{65}$$

$$- h_{21}^{\,2}\left(f_{i-1,j-1,k} - f_{i-1,j,k} - f_{i-2,j-1,k} + f_{i-2,j,k}\right)\Big]$$

f) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{-1}{h_{11}\;h_{13}\;(h_{23} - h_{24})\;(h_{11} + h_{13})}$$

$$\Big[h_{13}^{\,2}\left(f_{i+1,j-1,k} - f_{i+1,j-2,k} + f_{i,j-2,k} - f_{i,j-1,k}\right) \tag{66}$$

$$- h_{11}^{\,2}\left(f_{i-1,j-1,k} - f_{i,j-1,k} - f_{i-1,j-2,k} + f_{i,j-2,k}\right)\Big]$$

Second Mixed Irregular Corner Derivative With Respect to α_1 and α_2

g) $$\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2} = \frac{1}{h_{11}\;h_{12}\;h_{21}\;h_{22}\;(h_{22} - h_{21})}$$

$$\Big[h_{12}\;h_{22}^{\,2}\left(f_{i+1,j+1,k} - f_{i+1,j,k} - f_{i,j+1,k} + f_{i,j,k}\right) \tag{67}$$

$$- h_{11}\;h_{21}^{\,2}\left(f_{i+2,j+2,k} - f_{i+2,j,k} - f_{i,j+2,k} + f_{i,j,k}\right)\Big]$$

Second Mixed Regular Derivatives

All of the above results can be reduced to regular derivatives with respect to either α_1, α_2 or both coordinates by making the substitutions

$$h_{11} = h_{13} = \frac{h_{12}}{2} = \frac{h_{14}}{2} \equiv h_1 \qquad (68)$$

$$h_{21} = h_{23} = \frac{h_{22}}{2} = \frac{h_{24}}{2} \equiv h_2 \qquad (69)$$

$$h_1 = h_2 = h \qquad (70)$$

The various derivatives are summarized below for the case in which all grid spacings are equal (i.e., $h_1 = h_2$).

<u>Second Mixed Regular Central Derivative With Respect to α_1 and α_2</u>

a), b)

$$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right|_{i,j,k} = \frac{1}{4\,h^2}\left(f_{i+1,j+1,k} - f_{i+1,j-1,k} - f_{i-1,j+1,k} + f_{i-1,j-1,k}\right) \qquad (71)$$

<u>Second Mixed Regular Forward Derivative With Respect to α_1 and α_2</u>

c) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right|_{i,j,k} = \frac{-1}{2\,h_2}\left(f_{i+1,j+1,k} - f_{i+2,j+1,k} - f_{i+1,j-1,k} + f_{i+2,j-1,k}\right) \qquad (72)$$

d) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right|_{i,j,k} = \frac{-1}{2\,h^2}\left(f_{i+1,j+1,k} - f_{i+1,j+2,k} - f_{i-1,j+1,k} + f_{i-1,j+2,k}\right) \qquad (73)$$

<u>Second Mixed Regular Backward Derivative With Respect to α_1 and α_2</u>

e) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right|_{i,j,k} = \frac{1}{2\,h^2}\left(f_{i-1,j+1,k} - f_{i-2,j+1,k} - f_{i-1,j-1,k} + f_{i-2,j-1,k}\right) \qquad (74)$$

f) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right|_{i,j,k} = \frac{1}{2\,h^2}\left(f_{i+1,j-1,k} - f_{i+1,j-2,k} - f_{i-1,j-1,k} + f_{i-1,j-2,k}\right) \qquad (75)$$

<u>Second Mixed Regular Corner Derivative With Respect to α_1 and α_2</u>

g) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right|_{i,j,k} = \frac{1}{4\,h^2}\Big[8\left(f_{i+1,j+1,k} - f_{i+1,j,k} - f_{i,j+1,k} + f_{i,j,k}\right) \qquad (76)$$

$$- \left(f_{i+2,j+2,k} - f_{i+2,j,k} - f_{i,j+2,k} + f_{i,j,k}\right)\Big]$$

APPENDIX B

Singularities at an Axis $\varphi = 0$

The equilibrium equations in spherical coordinates in terms of displacements are written in the form

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial R^2} + \frac{\mu}{R^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\mu}{R^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} + \frac{2(\lambda+2\mu)}{R}\frac{\partial u}{\partial R} + \frac{\mu\cos\varphi}{R^2 \sin\varphi}\frac{\partial u}{\partial \varphi}$$

$$- \frac{2(\lambda+2\mu)}{R^2} u + \frac{\lambda+\mu}{R}\frac{\partial^2 v}{\partial R \partial \varphi} + \frac{(\lambda+\mu)\cos\varphi}{R \sin\varphi}\frac{\partial v}{\partial R} - \frac{\lambda+3\mu}{R^2}\frac{\partial v}{\partial \varphi} \qquad (1)$$

$$- \frac{(\lambda+3\mu)\cos\varphi}{R^2 \sin\varphi} v + \frac{\lambda+\mu}{R \sin\varphi}\frac{\partial^2 w}{\partial R \partial \theta} - \frac{\lambda+3\mu}{R^2 \sin\varphi}\frac{\partial w}{\partial \theta} = \frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{11}}}\frac{\partial T}{\partial R}$$

$$\frac{\lambda+\mu}{R}\frac{\partial^2 u}{\partial R \partial \varphi} + \frac{2(\lambda+2\mu)}{R^2}\frac{\partial u}{\partial \varphi} + \mu\frac{\partial^2 v}{\partial R^2} + \frac{\lambda+2\mu}{R^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{\mu}{R^2 \sin^2\varphi}\frac{\partial^2 v}{\partial \theta^2}$$

$$+ \frac{2\mu}{R}\frac{\partial v}{\partial R} + \frac{(\lambda+2\mu)\cos\varphi}{R^2 \sin\varphi}\frac{\partial v}{\partial \varphi} - \frac{\lambda+2\mu}{R^2 \sin^2\varphi} v + \frac{\lambda+\mu}{R^2 \sin\varphi}\frac{\partial^2 w}{\partial \theta \partial \varphi} \qquad (2)$$

$$- \frac{(\lambda+3\mu)\cos\varphi}{R^2 \sin^2\varphi}\frac{\partial w}{\partial \theta} = \frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{22}}}\frac{\partial T}{\partial \varphi}$$

$$\frac{\lambda+\mu}{R \sin\varphi}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{2(\lambda+2\mu)}{R^2 \sin\varphi}\frac{\partial u}{\partial \theta} + \frac{\lambda+\mu}{R^2 \sin\varphi}\frac{\partial^2 v}{\partial \theta \partial \varphi} + \frac{(\lambda+3\mu)\cos\varphi}{R^2 \sin^2\varphi}\frac{\partial v}{\partial \theta}$$

$$+ \mu\frac{\partial^2 w}{\partial R^2} + \frac{\mu}{R^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\lambda+2\mu}{R^2 \sin^2\varphi}\frac{\partial^2 w}{\partial \theta^2} + \frac{2\mu}{R}\frac{\partial w}{\partial R} + \frac{\mu\cos\varphi}{R^2 \sin\varphi}\frac{\partial w}{\partial \varphi} \qquad (3)$$

$$- \frac{\mu}{R^2 \sin^2\varphi} w = \frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{33}}}\frac{\partial T}{\partial \theta}$$

The temperature terms on the right hand sides of Equations (1), (2) and (3) are equivalent to body forces defined as

$$\left.\begin{aligned}
\frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{11}}}\frac{\partial T}{\partial R} &= F_R\ (R,\varphi,\theta)/\sqrt{g_{11}} \\
\frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{22}}}\frac{\partial T}{\partial \varphi} &= F_\varphi\ (R,\varphi,\theta)/\sqrt{g_{22}} \\
\frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{33}}}\frac{\partial T}{\partial \theta} &= F_\theta\ (R,\varphi,\theta)/\sqrt{g_{33}}
\end{aligned}\right\} \qquad (4)$$

If the elastic constants are temperature dependent the equilibrium equations (1), (2) and (3) are written in the form

$$\sum_{i=1}^{10}\left[(A_{ki} + A'_{ki})\,U_i + (B_{ki} + B'_{ki})\,V_i + (C_{ki} + C'_{ki})W_i\right] = F_k,\ (k = R,\varphi,\theta) \qquad (5)$$

where

A_{ki}, B_{ki}, C_{ki} = functions of coordinates (R,φ,θ) and elastic constants λ and μ

A'_{ki}, B'_{ki}, C'_{ki} = functions of coordinates (R,φ,θ) and elastic constants $\lambda(T)$ and $\mu(T)$, where T is the heat shield temperature which is a function of the coordinate (R,φ,θ)

U_i, V_i, W_i = functions of displacements $u(R,\varphi,\theta)$, $v(R,\varphi,\theta)$ and $w(R,\varphi,\theta)$ and their respective derivatives of the first and second orders, respectively.

F_k = body force expressed as $(3\lambda + 2\mu)\ \alpha\ (T)\ \frac{\partial T}{\partial \alpha_k}$

Equation (1) may be further shortened into the form

$$\sum_{m=1}^{3}\sum_{i=1}^{10}(G_{mki} + G'_{mki})\ \Phi_{mi} = F_k,\ (k = R,\varphi,\theta) \qquad (6)$$

where

$G_{1,2,3\,ki} = A_{ki},\ B_{ki},\ C_{ki}$

$G'_{1,2,3\,ki} = A'_{ki},\ B'_{ki},\ C'_{ki}$

$\Phi_{1,2,3\,i} = U_i,\ V_i,\ W_i$

It is first considered that the elastic constants are independent of temperature. In this case, Equation (6) becomes, deleting the symbol of summation,

$$G_{mki}\ (R,\varphi,\theta)\ \Phi_{mi}\ (R,\varphi,\theta) = F_k\ (R,\varphi,\theta) \tag{7}$$

Replacing the variables R, φ, and θ of Equation (7) by $R-R'$, $\varphi-\varphi'$ and $\theta-\theta'$, respectively, and integrating the result with respect to the variables $(R-R',\ \varphi-\varphi',\ \theta-\theta')$ over a finite volume V_k gives

$$\iiint\limits_{V_k} G_{mki}\ \Phi_{mi}\ dV' = \iiint\limits_{V_k} F_k\ dV' \tag{8}$$

where $dV' = (R-R')^2\ \sin\ (\varphi-\varphi')\ d\ (R-R')\ d\ (\varphi-\varphi')\ d\ (\theta-\theta')$ and the finite volume V_k is bounded as

$$R_{K1} \le R-R' \le R_{K2}$$

$$\varphi_{K1} \le \varphi-\varphi' \le \varphi_{K2}$$

$$\theta_{K1} \le \theta-\theta' \le \theta_{K2}$$

Let

$$\left.\begin{aligned} I_R &= \int_{R_{K1}}^{R_{K2}} G_{mki}\ (R-R')^2\ d\ (R-R') \\ I_{R\varphi} &= \int_{\varphi_{K1}}^{\varphi_{K2}} I_R\ \sin\ (\varphi-\varphi')\ d\ (\varphi-\varphi') \\ I_{R\varphi\theta} &= \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi}\ d\ (\theta-\theta') \end{aligned}\right\} \tag{9}$$

Integration of the function $G_{mki}\ \Phi_{mi}$ with respect to $(R-R')$ gives

Singularities at an Axis $\varphi = 0$ (cont.)

$$\int_{R_{K1}}^{R_{K2}} G_{mki}\,\Phi_{mi}\,(R-R')^2\,d\,(R-R') = I_R\,\Phi_{mi}\Big|_{R_{K1}}^{R_{K2}} - \int_{R_{K1}}^{R_{K2}} I_R\,\frac{\partial\Phi_{mi}}{\partial(R-R')}\,d\,(R-R') \quad (10)$$

Integration of Equation (10) with respect to $(\varphi-\varphi')$, after multiplying both sides by $\sin(\varphi-\varphi')$, gives

$$\int_{\varphi_{K1}}^{\varphi_{K2}} I_R\,\Phi_{mi}\Big|_{R_{K1}}^{R_{K2}} \sin(\varphi-\varphi')\,d\,(\varphi-\varphi') -$$

$$- \int_{\varphi_{K1}}^{\varphi_{K2}}\int_{R_{K1}}^{R_{K2}} I_R\,\frac{\partial\Phi_{mi}}{\partial(R-R')}\,\sin(\varphi-\varphi')\,d\,(R-R')\,d\,(\varphi-\varphi') = I_{R\varphi}\,\Phi_{mi}\Big|_{\substack{\varphi_{K1}\\ R_{K1}}}^{\substack{\varphi_{K2}\\ R_{K2}}} \quad (11)$$

$$- \int_{\varphi_{K1}}^{\varphi_{K2}} I_{R\varphi}\,\frac{\partial\Phi_{mi}}{\partial(\varphi-\varphi')}\,d\,(\varphi-\varphi') - \int_{\varphi_{K1}}^{\varphi_{K2}}\int_{R_{K1}}^{R_{K2}} I_R\,\frac{\partial\Phi_{mi}}{\partial(R-R')}\,\sin(\varphi-\varphi')\,d\,(R-R')\,d\,(\varphi-\varphi')$$

Integration of Equation (11) with respect to $(\theta-\theta')$ gives

$$\iiint G_{mki}\,\Phi_{mi}\,(R-R')^2\,\sin(\varphi-\varphi')\,d\,(R-R')\,d\,(\varphi-\varphi')\,d\,(\theta-\theta')$$

$$= I_{R\varphi\theta}\,\Phi_{mi}\Big|_{\substack{\theta_{K1}\\ \varphi_{K1}\\ R_{K1}}}^{\substack{\theta_{K2}\\ \varphi_{K2}\\ R_{K2}}} - \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi\theta}\,\frac{\partial\Phi_{mi}}{\partial(\theta-\theta')}\,d\,(\theta-\theta') \quad (12)$$

$$- \int_{\varphi_{K1}}^{\varphi_{K2}} \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi\theta} \frac{\partial \Phi_{mi}}{\partial(\theta-\theta')} d(\varphi-\varphi') \, d(\theta-\theta')$$

$$- \int_{R_{K1}}^{R_{K2}} \int_{\varphi_{K1}}^{\varphi_{K2}} \int_{\theta_{K1}}^{\theta_{K2}} I_R \frac{\partial \Phi_{mi}}{\partial(R-R')} \sin(\varphi-\varphi') \, d(R-R') \, d(\varphi-\varphi') \, d(\theta-\theta') \tag{12}$$

At a point $(R_{KC}, \varphi_{KC}, \theta_{KC})$, Equation (12) becomes

$$\iiint \dot{G}_{mki} \, \Phi_{mi} \, (R_{KC}-R')^2 \sin(\varphi_{KC}-\varphi') \, d(R_{KC}-R') \, d(\varphi_{KC}-\varphi') \, d(\theta_{KC}-\theta')$$

$$= I_{R\varphi\theta} \, \Phi_{mi} \Bigg|_{\substack{\theta_{K1} \\ \varphi_{K1} \\ R_{K1}}}^{\substack{\theta_{K2} \\ \varphi_{K2} \\ R_{K2}}} - \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi\theta} \frac{\partial \Phi_{mi}}{\partial \theta'} d\theta' + \int_{\varphi_{K1}}^{\varphi_{K2}} \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi} \frac{\partial \Phi_{mi}}{\partial \varphi'} d\varphi' \, d\theta' \tag{13}$$

$$- \int_{R_{K1}}^{R_{K2}} \int_{\varphi_{K1}}^{\varphi_{K2}} \int_{\theta_{K1}}^{\theta_{K2}} I_R \frac{\partial \Phi_{mi}}{\partial R'} \sin(\varphi_{KC}-\varphi') \, dR' d\varphi' d\theta'$$

The right hand side of Equation (8) becomes

$$- \int_{R_{K1}}^{R_{K2}} \int_{\varphi_{K1}}^{\varphi_{K2}} \int_{\theta_{K1}}^{\theta_{K2}} F_K (R_{KC}-R', \varphi_{KC}-\varphi', \theta_{KC}-\theta') (R_{KC}-R')^2 \sin(\varphi_{KC}-\varphi') \, dR' d\varphi' d\theta' \tag{13a}$$

By the definition of Kelvin's point force, diminishing the force field V_K' indefinitely always including a point $(R' = 0, \varphi' = 0, \theta = 0)$ gives

$$\left.\begin{aligned}
&\lim_{V_K' \to 0} \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi\theta} \frac{\partial \Phi_{mi}}{\partial \theta'} \, d\theta' \approx 0 \\
&\lim_{V_K' \to 0} \int_{\varphi_{K1}}^{\varphi_{K2}} \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi} \frac{\partial \Phi_{mi}}{\partial \varphi'} \, d\varphi' d\theta' \approx 0 \\
&\lim_{V_K' \to 0} \iiint I_R \frac{\partial \Phi_{mi}}{\partial R'} \sin (\varphi_{KC} - \varphi') \, dR' d\varphi' d\theta' \approx 0 \\
&\lim_{V_K' \to 0} \iiint F_K \, (R_{KC} - R')^2 \sin (\varphi_{KC} - \varphi') \, dR' d\varphi' d\theta' = {}^{o}F_{KC}
\end{aligned}\right\} \qquad (14)$$

But

$$\iiint_{V_K} F_K \, dV' = F_K \, V_K \qquad (15)$$

and, hence,

$${}^{o}F_{KC} = F_K \, V_K \qquad (16)$$

Hence, from Equations (13), (13a), (14) and (16), Equation (8) becomes

$$\frac{I_{R\varphi\theta} \, \Phi_{mi}}{V_K} \Bigg|_{\substack{\theta_{K1} \\ \varphi_{K1} \\ R_{K1}}}^{\substack{\theta_{K2} \\ \varphi_{K2} \\ R_{K2}}} = F_K \Bigg|_{\substack{R_{KC} \\ \varphi_{KC} \\ \theta_{KC}}} \qquad (17)$$

Singularities at an Axis $\varphi = 0$ (cont.)

$$\text{Where } \Phi_{mi}\Bigg|_{\theta_{K1},\ \varphi_{K1},\ R_{K1}}^{\theta_{K2},\ \varphi_{K2},\ R_{K1}}$$

may be found by taking the average of eight surrounding points.

APPENDIX C

Thin-Shell Interface Conditions for Stress Analysis of Thick Laminate Structures

Equations are derived for treating the stresses in a thick-shell laminate structure in the neighborhood of a thin layer which, by itself, can satisfy the Kirchhoff bending hypothesis for thin shells. It is shown that the thin layer can be treated by an equivalent interface condition which relates the displacements of the median surface of the shell to the discontinuous normal and shear stresses on the adjoining surfaces. From continuity of displacements across the thin layer the interface stresses can be climated to yield three simultaneous partial differential equations for the three displacement components at the interface. The analysis is presented for a flat plate using a system of Cartesian coordinates and will be generalized later to the curvilinear coordinate systems of interest in the heat shield analysis.

Consider a thin plate of thickness b with its median surface lying in the x-y plane and the distance z measured from the median surface. The temperature and, consequently, the coefficient of thermal expansion and modulus of elasticity will be allowed to vary through the plate thickness so that the median surface will not, in general, bisect the plate thickness. With this generality, the thin plate itself can consist of a laminate of different materials. According to Kirchhoff's bending hypothesis the strain-displacement relations for a point (x,y,z) in the plate are given by Reference (1).

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= \frac{\partial v}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\,\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \qquad (1)$$

Where u,v, and w are displacements of a point (x,y) on the median surface and ϵ_x, ϵ_y and γ_{xy} are the normal strains and shear strain, respectively in the x-y plane. The stress-strain relations are given by

$$\left.\begin{aligned}
\sigma_x &= \frac{E(x,y,z)}{1-\nu^2}\left[(\epsilon_x + \nu\,\epsilon_y) - (1+\nu)\,\alpha\,(x,y,z)\,T\,(x,y,z)\right]\\
\sigma_y &= \frac{E(x,y,z)}{1-\nu^2}\left[(\epsilon_y + \nu\,\epsilon_x) - (1+\nu)\,\alpha\,(x,y,z)\,T\,(x,y,z)\right]\\
\tau_{xy} &= \frac{E(x,y,z)}{2(1+\nu)}\,\gamma_{xy}
\end{aligned}\right\} \qquad (2)$$

Where σ_x and σ_y are normal stresses and τ_{xy} is the shear stress in the x-y plane. The normal stress σ_z and shear stresses τ_{xz} are usually small in comparison with the stress components of Equation (2) and are neglected in thin shell theory. For the problem under consideration, however, the thin shell will be subjected to both normal and shear stresses over its lateral surfaces and it is desired to relate the difference or discontinuity of these stresses across the shell to the displacements of the median surface. These relationships may be obtained from the equations of equilibrium expressed in terms of displacements using the Kirchhoff bending hypothesis of Equation (1). The equilibrium equations in terms of stresses are given by

$$\left.\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} &= 0\\
\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} &= 0\\
\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} &= 0
\end{aligned}\right\} \qquad (3)$$

Writing the stresses of Equation (2) in terms of displacements using Equations (1) and substituting the results in Equation (3), the equilibrium equations in terms of displacements become

$$\left.\begin{aligned}
&\frac{E(x,y,z)}{1-\nu^2} f_1(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} f_2(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} f_3(x,y) \\
&- \frac{zE(x,y,z)}{1-\nu^2} g_1(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} g_2(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} g_3(x,y) \\
&+ \frac{\partial \tau_{xz}}{\partial z} = \frac{E(x,y,z)}{1-\nu}\frac{\partial}{\partial x}\left[\alpha(x,y,z)\, T(x,y,z)\right] \\
&+ \frac{1}{1-\nu}\frac{\partial E(x,y,z)}{\partial x}\alpha(x,y,z)\, T(x,y,z) \\
&\\
&\frac{E(x,y,z)}{1-\nu^2} f_1'(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} f_2'(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} f_3'(x,y) \\
&- \frac{zE(x,y,z)}{1-\nu^2} g_1'(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} g_2'(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} g_3'(x,y) \\
&+ \frac{\partial \tau_{yz}}{\partial z} = \frac{E(x,y,z)}{1-\nu}\frac{\partial}{\partial y}\left[\alpha(x,y,z)\, T(x,y,z)\right] \\
&+ \frac{1}{1-\nu}\frac{\partial E(x,y,z)}{\partial y}\alpha(x,y,z)\, T(x,y,z) \\
&\\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0
\end{aligned}\right\} \quad (4)$$

Where

$$\left.\begin{aligned}
f_1(x,y) &= \frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 v}{\partial x \partial y} + \frac{1-\nu}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) \\
f_2(x,y) &= \frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} \\
f_3(x,y) &= \frac{1-\nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
g_1(x,y) &= \frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2} + (1-\nu)\frac{\partial^3 w}{\partial x \partial y^2} = \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}
\end{aligned}\right\} \quad (5)$$

$$
\left.
\begin{aligned}
g_2(x,y) &= \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \\
g_3(x,y) &= (1-\nu) \frac{\partial^2 w}{\partial x \partial y} \\
f_1{}'(x,y) &= \frac{\partial^2 v}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x \partial y} + \frac{1-\nu}{2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) \\
f_2{}'(x,y) &= \frac{1-\nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
f_3{}'(x,y) &= \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \\
g_1{}'(x,y) &= \frac{\partial^3 w}{\partial y^3} + \nu \frac{\partial^3 w}{\partial x^2 \partial y} + (1-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y} \\
g_2{}'(x,y) &= (1-\nu) \frac{\partial^2 w}{\partial x \partial y} \\
g_3{}'(x,y) &= \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}
\end{aligned}
\right\} \quad (5)
$$

If the first two of Equations (4) are integrated across the plate thickness there results

$$
\left.
\begin{aligned}
& f_1(x,y)\, D_0 + f_2(x,y) \frac{\partial D_0}{\partial x} + f_3(x,y) \frac{\partial D_0}{\partial y} - g_1(x,y)\, D_1 - g_2(x,y) \frac{\partial D_1}{\partial x} \\
& - g_3(x,y) \frac{\partial D_1}{\partial y} + \tau_{xz}\Big|_2 - \tau_{xz}\Big|_1 = \frac{\partial N_T}{\partial x} \\
& f_1{}'(x,y)\, D_0 + f_2{}'(x,y) \frac{\partial D_0}{\partial x} + f_3{}'(x,y) \frac{\partial D_0}{\partial y} - g_1{}'(x,y)\, D_1 - g_2{}'(x,y) \frac{\partial D_1}{\partial x} \\
& - g_3{}'(x,y) \frac{\partial D_1}{\partial y} + \tau_{yz}\Big|_2 - \tau_{yz}\Big|_1 = \frac{\partial N_T}{\partial y}
\end{aligned}
\right\} \quad (6)
$$

Where the quantities D_0, D_1 and N_T are defined by

$$
\left.
\begin{aligned}
D_0 &= \frac{1}{1-\nu^2} \int E(x,y,z)\, dz \\
D_1 &= \frac{1}{1-\nu^2} \int z\, E(x,y,z)\, dz \\
N_T &= \frac{1}{1-\nu} \int E(x,y,z)\, \alpha(x,y,z)\, T(x,y,z)\, dz
\end{aligned}
\right\} \quad (7)
$$

and $\tau_{xz}\big|_1$, $\tau_{xz}\big|_2$, etc. are the respective shear stresses on the two surfaces of the plate. If the median surface is determined such that

$$D_1 = 0 \, , \tag{8}$$

which is, in fact, the condition defining the median or "neutral" surface, then Equations (6) reduce to two expressions for the shear stress discontinuities across the thin plate in terms of the median surface displacements; i.e.,

$$\left.\begin{aligned} \tau_{xz}\big|_2 - \tau_{xz}\big|_1 &= \frac{\partial N_T}{\partial x} - f_1(x,y)\, D_0 - f_2(x,y)\,\frac{\partial D_0}{\partial x} - f_3(x,y)\,\frac{\partial D_0}{\partial y} \\ \tau_{yz}\big|_2 - \tau_{yz}\big|_1 &= \frac{\partial N_T}{\partial y} - f_1'(x,y)\, D_0 - f_2'(x,y)\,\frac{\partial D_0}{\partial x} - f_3'(x,y)\,\frac{\partial D_0}{\partial y} \end{aligned}\right\} \tag{9}$$

A third equation, which is necessary to define the three displacement components u, v and w at the median surface, is obtained from a consideration of equilibrium of forces normal to the plane of the plate. It is shown in Reference (2) that this expression of equilibrium can be written as

$$\frac{\partial^2 M_x}{\partial x^2} - 2\,\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\,p - N_x\,\frac{\partial^2 w}{\partial x^2} - 2\,N_{xy}\,\frac{\partial^2 w}{\partial x \partial y} - N_y\,\frac{\partial^2 w}{\partial y^2} \tag{10}$$

where p is the lateral pressure loading on the plate and the N's and M's are sectional forces and moments defined by

$$\left.\begin{aligned} N_x &= \int \sigma_x \, dz \, , \quad N_y = \int \sigma_y \, dz \, , \quad N_{xy} = \int \tau_{xy} \, dz \\ M_x &= \int z\, \sigma_x \, dz \, , \quad M_y = \int z\, \sigma_y \, dz \, , \quad M_{xy} = -\int z\, \tau_{xy} \, dz \end{aligned}\right\} \tag{11}$$

Substituting for σ_x, σ_y and τ_{xz} from Equations (2), with the definition, Equation (8), of the median surface, the sectional quantities of Equation (11) become

$$\left.\begin{aligned} N_x &= D_0 \left(\frac{\partial u}{\partial x} + \nu\,\frac{\partial v}{\partial y} \right) - N_T \\ N_y &= D_0 \left(\frac{\partial v}{\partial y} + \nu\,\frac{\partial u}{\partial x} \right) - N_T \end{aligned}\right\} \tag{12}$$

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$$\left.\begin{aligned}
N_{xy} &= \frac{1-\nu}{2} D_0 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\
M_x &= - D_2 \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) - M_T \\
M_y &= - D_2 \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) - M_T \\
M_{xy} &= (1-\nu) D_2 \frac{\partial^2 w}{\partial x \partial y}
\end{aligned}\right\} \qquad (12)$$

Where

$$\left.\begin{aligned}
D_2 &= \frac{1}{1-\nu^2} \int z^2 \, E(x,y,z) \, dz \\
M_T &= \frac{1}{1-\nu} \int z \, E(x,y,z) \, \alpha(x,y,z) \, T(x,y,z) \, dz
\end{aligned}\right\} \qquad (13)$$

The lateral pressure, p, acting on the thin plate is simply the difference between the normal stresses $\sigma_z\big|_1$ and $\sigma_z\big|_2$ acting on the two surfaces; i.e.,

$$p = \sigma_z\big|_2 - \sigma_z\big|_1 \qquad (14)$$

Hence, on substituting the sectional forces and moments defined by Equations (12) in Equation (10), an expression is obtained analogous to Equations (9) for the discontinuity of normal stresses across the thin plate in terms of the three displacement components at the median surface. This equation is found to be

$$\begin{aligned}
\sigma_z\big|_2 - \sigma_z\big|_1 = {} & \frac{\partial^2}{\partial x^2} \left[D_2 \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right] + 2(1-\nu) \frac{\partial^2}{\partial x \partial y} \left(D_2 \frac{\partial^2 w}{\partial x \partial y} \right) \\
& + \frac{\partial^2}{\partial y^2} \left[D_2 \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right] - \frac{\partial^2 w}{\partial x^2} \left[D_0 \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) - N_T \right] \\
& - 2 \frac{\partial^2 w}{\partial x \partial y} \left[\frac{1-\nu}{2} D_0 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] - \frac{\partial^2 w}{\partial y^2} \left[D_0 \left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right) - N_T \right] + \nabla^2 M_T
\end{aligned} \qquad (15)$$

If it is assumed that the displacements at the surfaces of the two media in contact with the thin layer under consideration are equal to the displacements in this layer at the median surface, then the surface stresses may be expressed in terms of these displacements using Hooke's law with the respective material properties of the two adjoining media. Thus, Equations (9) and (15) become three partial differential equations in the three displacement components u, v and w at the thin-shell interface. These equations will replace the general three-dimensional equations at the "interface" nodes resulting in only one node at each such interface through the thick laminate structure. Once the three displacement components in the interface plane are determined, the stress distributions throughout the thin layer are obtained from the foregoing thin-shell analysis.

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