View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by NASA Technical Reports Server

N65 23740
(ACCESSION NUMBER) 28
(PAGES)
CR-62712
(NASA CR OR TMX OR AD NUMBER)
(THRU) 1
(CODE)
24
(CATEGORY)
FACILITY FORM 602

Technical Report No. 32-686

A New Approach to the Homogenization of Heterogeneous Media for Neutron Diffusion Calculations

Alan M. Jacobs

GPO PRICE $ ______

OTS PRICE(S) $ ______

Hard copy (HC) $2.00

Microfiche (MF) .50

jpl
JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

April 15, 1965

Copyright © 1965
Jet Propulsion Laboratory
California Institute of Technology

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

Technical Report No. 32-686

A New Approach to the Homogenization of Heterogeneous Media for Neutron Diffusion Calculations

Alan M. Jacobs

R. J. Mackin, Jr.

R. J. Mackin, Jr., Manager
Physics Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

April 15, 1965

CONTENTS

FIGURES

FOREWORD

Alan M. Jacobs is a member of the faculty of the Nuclear Engineering Department, College of Engineering, Pennsylvania State University. During July and August of 1964, Professor Jacobs held a summer appointment as Senior Scientist at the Jet Propulsion Laboratory, California Institute of Technology. This report covers the work which he carried out at that time.

ABSTRACT

23740

The mathematical formulation of a new approach to the homogenization of certain types of heterogeneous media (such as a regular array of holes in a scattering media) for the purpose of neutron diffusion calculations is developed in this Report. The new method is based on the inclusion of an angular-dependent mean free path in the theory of neutron transport. In the present effort, calculations are restricted to media with plane symmetry and monoenergetic neutron theory is employed.

It is found that a neutron-flux based theory and a collision-density based theory can lead to significantly different results when low-order approximations, such as diffusion theory, are employed in the solution of the transport equation. For the case of isotropic scattering, the normal mode technique is found applicable and closed-form, exact solutions are determined.

There is little existing experimental work in the description of neutron distribution in such anisotropic media. It is therefore difficult to evaluate the results of the new theory. The interpretation of physical and numerical (Monte Carlo) experiments could be approached directly with the new techniques.

Author

I. INTRODUCTION

This Report develops the mathematical formulation of a new method of treating neutron diffusion in certain types of heterogeneous media. The heterogeneity of immediate concern, and toward which this work is directed, is that of a regular array of vacuum channels (such as a square lattice of cylindrical holes) in an otherwise homogeneous medium. The general procedure, with modification of details, should be applicable to other types of heterogeneity. However, a requirement which should be imposed is that the heterogeneity results in two characteristic directions. For example, in the case of a regular array of vacuum channels, the two characteristic directions are parallel and perpendicular to the channel axis.

Probably the most important considerations of neutron diffusion in media with holes are those of Behrens (Ref. 1). However, all previous work, including that in Ref. 1, is devoted to the determination of the effects of heterogeneity on specific parameters relevant to neutron diffusion, rather than to a formulation of a general method from which various descriptions of the neutron distribution can be determined. The general problem may be stated along these lines: in media with heterogeneity, such as the type considered here, it is clear that a "simple homogenization" of the medium for neutron diffusion calculations is not a valid representation. A simple homogenization is defined as the process of reducing the medium cross sections merely by the ratio of material volume to material-plus-vacuum volume. The streaming of neutrons in the vacuum channels leads to a spreading of the neutrons in the longitudinal direction (parallel to channel axis) which is larger than that in the transverse direction (perpendicular to channel axis). Thus, not only is the simple homogenization questionable for omnidirectional parameter calculation, but also the anisotropic effects are completely subdued. Of course, the reason for considering homogenization of the medium is the existence of an "arsenal" of possible mathematical attacks for such problems.

The new approach to homogenization, which will be developed here, is based on the reasoning that neutrons traveling with a large component of their velocity in the longitudinal direction probably travel further between collisions, on the average, than those traveling with a large component of velocity in the transverse direction. Let this effect be introduced into the neutron transport equation for the homogenized medium by allowing the mean free path to be angular-dependent. Clearly, by so doing, a mathematical fiction will be utilized since the mean free path, as used in neutron transport calculations, is a local parameter. This concept is certainly no more confusing than the idea of medium homogenization. It should be stressed that a total cross section which varies according to the direction of neutron travel, and not merely the usual scattering directional dependence, is being imposed.

All considerations will be based on the idealization of monoenergetic neutron transport. Although time-dependent equations will be developed, most of the analysis will be devoted to the calculation of stationary states. For the purpose of completeness, the mathematical formulation will be developed along several lines. Specifically: both the neutron flux and collison density will be considered as dependent variables; the familiar P_N and double-P_N approximations, as well as the moment decomposition will be applied; and it will be demonstrated that, for the case of isotropic scattering, the normal mode procedure, recently used for the solution of several types of neutron transport problems, is applicable and yields exact, closed-form solutions.

The major part of this work is directed toward the mathematical formulation and physical interpretation of the new theory. A section, however, will be devoted to brief remarks relevant to application. There is meager experimental data available for lattices of the type considered. It will therefore be difficult to evaluate the present theory. These considerations will yield a well-defined, albeit not well-substantiated, route to solution of problems involving neutron diffusion in media pierced by vacuum channels.

II. MATHEMATICAL FORMULATION

It would seem that application of these ideas to finite media dictates the use of non-separable position-angle-dependent mean free paths. The inclusion of position-dependence leads to gross difficulties which we have as yet not resolved. It will be assumed that the mean free path in the homogenized medium depends only on the angle between the neutron velocity and the direction of the position variable. In all calculations plane symmetry is assumed, with the position variable either along the longitudinal or transverse direction.

A. The Neutron-Flux Equation

The monoenergetic neutron transport equation for homogeneous media with plane symmetry may be written as

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(x,\mu,t)+\mu\frac{\partial}{\partial x}\psi(x,\mu,t)+\sigma(\mu)\psi(x,\mu,t)$$
$$=c\int\sigma(\mu')f(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\psi(x,\mu',t)\,d\boldsymbol{\Omega}'+S(x,\mu,t) \qquad (1)$$

where $\psi(x, \mu, t)$ is the neutron flux distribution as a function of position x, direction cosine of neutron travel relative to x-direction μ, and time t; v is neutron speed; $\sigma(\mu)$ is the angular-dependent total cross section; c is the mean number of secondary neutrons which emanate from a neutron–nucleus collision; $f(\mathbf{\Omega} \cdot \mathbf{\Omega}')$ is the normalized distribution in $\mathbf{\Omega}$, the neutron post-collision direction of travel, of secondary neutrons produced by collision of a primary neutron with pre-collision direction $\mathbf{\Omega}'$; and $S(x, \mu, t)$ is the rate of neutron introduction from sources which are independent of the neutron distribution. Although c and f are not necessarily descriptive of a nonmultiplying medium, the terms scattering probability for c and scattering distribution for f will be used to avoid stilted discourse. An expansion of the scattering distribution in terms of Legendre polynomials $\{P_n(\mathbf{\Omega} \cdot \mathbf{\Omega}')\}$ will be employed, i.e.,

$$f(\mathbf{\Omega} \cdot \mathbf{\Omega}') = \sum_n \frac{2n+1}{4\pi} f_n P_n(\mathbf{\Omega} \cdot \mathbf{\Omega}') \tag{2}$$

and the spherical harmonics addition theorem (Ref. 2, p. 143) to eliminate the azimuthal direction dependence appearing in the integral in Eq. (1). The result is

$$\begin{aligned} &\frac{1}{v}\frac{\partial}{\partial t}\psi(x,\mu,t) + \mu\frac{\partial}{\partial x}\psi(x,\mu,t) + \sigma(\mu)\psi(x,\mu,t) \\ &\quad = \frac{c}{2}\sum_n (2n+1) f_n P_n(\mu) \int_{-1}^{+1} P_n(\mu')\,\sigma(\mu')\,\psi(x,\mu',t)\,d\mu' \\ &\quad + S(x,\mu,t) \end{aligned} \tag{3}$$

In order to proceed with a general discussion of the properties of Eq. (3) a more specific representation of $\sigma(\mu)$ is required. It is supposed, with little loss in relevant generality, that

$$\sigma(\mu) = \sum_n \sigma_n P_n(\mu) \tag{4}$$

The sets $\{\psi_n(x,t)\}$ and $\{S_n(x,t)\}$ are further defined by

$$\psi_n(x,t) = \int_{-1}^{+1} P_n(\mu)\,\psi(x,\mu,t)\,d\mu \tag{5a}$$

$$S_n(x,t) = \int_{-1}^{+1} P_n(\mu)\,S(x,\mu,t)\,d\mu \tag{5b}$$

Equations (5a and b) specify the respective expansion coefficients in Legendre polynomial expansions of the neutron flux and source density, e.g.

$$\psi(x,\mu,t) = \sum_n \frac{2n+1}{2}\psi_n(x,t)\,P_n(\mu) \tag{6}$$

In these terms, integration of Eq. (3) over the interval $\mu \in (-1, +1)$ yields the relation

$$\frac{1}{v}\frac{\partial \psi_0}{\partial t} + \frac{\partial \psi_1}{\partial x} + (1-c)\sum_n \sigma_n \psi_n = S_0 \tag{7}$$

Eq. (7) is the continuity equation for neutron motion. The only term which appears in an unfamiliar form is that which expresses the total neutron interaction rate. Clearly,

$$\sum_n \sigma_n \psi_n(x,t) = \int_{-1}^{+1} \sigma(\mu)\,\psi(x,\mu,t)\,d\mu \tag{8}$$

A further, familiar, reduction of the transport equation can be made in terms of the sets $\{\psi_n\}$, $\{S_n\}$, and $\{\sigma_n\}$. Using the recurrence relation for Legendre polynomials (Ref. 2, p. 32),

$$(2n+1)\,\mu P_n(\mu) = n P_{n-1}(\mu) + (n+1) P_{n+1}(\mu) \tag{9}$$

yields the set of coupled differential equations

$$\begin{aligned} &\frac{1}{v}\frac{\partial}{\partial t}\psi_n(x,t) + \frac{n}{2n+1}\frac{\partial}{\partial x}\psi_{n-1}(x,t) + \frac{n+1}{2n+1}\frac{\partial}{\partial x}\psi_{n+1}(x,t) \\ &\quad + \sum_{l,m} (2l+1)\,\sigma_m A_{lmn}(1 - cf_n)\,\psi_l(x,t) \\ &\quad = S_n(x,t) \end{aligned} \tag{10}$$

The set $\{A_{lmn}\}$ is defined by

$$A_{lmn} = \frac{1}{2}\int_{-1}^{+1} P_l(\mu)\,P_m(\mu)\,P_n(\mu)\,d\mu \tag{11}$$

and has the following properties (Ref. 2, p. 87):

1. The order of the indices is unimportant.
2. $A_{lmn} = 0$ if the sum of any two of the indices is less than the third.
3. $A_{lmn} = 0$ if $l + m + n$ is odd.
4. When $A_{lmn} \neq 0$, i.e., avoiding (2) and (3),

$$\begin{aligned} A_{lmn} = &\left(\frac{1}{l+m+n+1}\right)\left(\frac{(1)(3)\cdots(l+m-n-1)}{(2)(4)\cdots(l+m-n)}\right) \\ &\times\left(\frac{(1)(3)\cdots(l+n-m-1)}{(2)(4)\cdots(l+n-m)}\right) \\ &\times\left(\frac{(1)(3)\cdots(n+m-l-1)}{(2)(4)\cdots(n+m-l)}\right) \\ &\times\left(\frac{(2)(4)\cdots(l+m+n)}{(1)(3)\cdots(l+m+n-1)}\right) \end{aligned} \tag{12}$$

It is from Eq. (10) that the various approximations to the transport equation are derived. Before proceeding with detailed examination of some approximations, re-examine Eq. (3). A simpler integral term is obtained if the dependent variable is changed to the neutron collision density

$$F(x, \mu, t) = \sigma(\mu)\,\psi(x, \mu, t).$$

B. The Collision-Density Equation

Reformulating Eq. (3) in terms of the collision density, $F(x, \mu, t)$, and the mean free path, $\lambda(\mu) = 1/\sigma(\mu)$,

$$\frac{\lambda(\mu)}{v}\frac{\partial}{\partial t}F(x, \mu, t) + \mu\lambda(\mu)\frac{\partial}{\partial x}F(x, \mu, t) + F(x, \mu, t) = \frac{c}{2}\sum_n (2n+1) f_n P_n(\mu) \times \int_{-1}^{+1} P_n(\mu')\,F(x, \mu', t)\,d\mu' + S(x, \mu, t) \tag{13}$$

With the sets $\{F_n(x, t)\}$ and $\{\lambda_n\}$ defined by

$$F(x, \mu, t) = \sum_n \frac{2n+1}{2} F_n(x, t)\,P_n(\mu) \tag{14a}$$

$$\lambda(\mu) = \sum_n \lambda_n P_n(\mu) \tag{14b}$$

the set of coupled differential equations [cf., Eq. (10)] is obtained,

$$\sum_{l,m} (2l+1)\,A_{lmn}\frac{\lambda_m}{v}\frac{\partial}{\partial t}F_l(x, t) + \frac{n}{2n+1}\sum_{l,m}(2l+1)\,A_{lm,\,n-1}\lambda_m\frac{\partial}{\partial x}F_l(x, t) + \frac{n+1}{2n+1}\sum_{l,m}(2l+1)\,A_{lm,\,n+1}\,\lambda_m\frac{\partial}{\partial x}F_l(x, t) + (1 - cf_n)\,F_n(x, t) = S_n(x, t) \tag{15}$$

Note that if $\lambda_n = 0$ for $n > 0$, i.e., the familiar case of an angle-independent mean free path, then Eq. (15), with the aid of the properties of $\{A_{lmn}\}$, reduces to

$$\frac{\lambda_0}{v}\frac{\partial F_n}{\partial t} + \frac{n\lambda_0}{2n+1}\frac{\partial F_{n-1}}{\partial t} + \frac{(n+1)\,\lambda_0}{2n+1}\frac{\partial F_{n+1}}{\partial t} + (1 - cf_n)\,F_n = S_n \tag{16}$$

which is the expected result. Note further that the $n = 0$ member of Eq. (15) yields the continuity equation

$$\sum_m \lambda_m\frac{\partial F_m}{\partial t} + \sum_{l,m}(2l+1)\,\lambda_m A_{lm1}\frac{\partial F_l}{\partial x} + (1-c)\,F_0 = S_0 \tag{17}$$

In recognizing Eq. (17) as the continuity equation, use is made of the easily derived relations

$$\psi_0(x, t) = \sum_n \lambda_n F_n(x, t) \tag{18a}$$

$$\psi_1(x, t) = \sum_{m,n}(2n+1)\,\lambda_m A_{nm1} F_n(x, t) \tag{18b}$$

In the remainder of this work the case of a stationary state is assumed. In most cases this assumption merely leads to simpler algebra and notation and is actually not a requirement for the determination of a solution. The P_N-approximation and double-P_N-approximation, as applied to both flux and collision density expansions, will be discussed and the moment decomposition for both flux and collision density will be considered. Finally, in greater depth, the case of isotropic scattering using the collision density as dependent variable will be covered.

C. The P_N-Approximation

The P_N-approximation based on a flux expansion, or collision density expansion, is defined by the requirements that in Eq. (10), or Eq. (15), $\psi_n(x) = 0$ for $n > N$, or $F_n(x) = 0$ for $n > N$, and the equations labelled by $n > N$ are discarded. Thus, in a P_N-approximation, $N + 1$ coupled differential equations are obtained with the $N + 1$ dependent variables $\psi_n(x)$, $n = 0, 1, \cdots, N$, or $F_n(x)$, $n = 0, 1, \cdots, N$. For example, the P_0-approximation based on the flux expansion gives the single relation

$$\sigma_0(1-c)\,\psi_0(x) = 0 \tag{19}$$

which has the implication $c = 1$ for non-trivial ψ_0. This is a familiar result. The P_0-approximation based on a collision density expansion gives the unusual relation

$$\frac{\lambda_1}{3}\frac{dF_0}{dx} + (1-c)\,F_0(x) = 0 \tag{20}$$

After this present brief comment considerations will be restricted to the case of $\lambda(\mu)$ a symmetric function of μ on the interval $\mu \in (-1, +1)$, and, in that case $\lambda_1 = 0$. For the moment, suppose that $\lambda_1 \neq 0$. To be specific, suppose that $\lambda_1 > 0$. According to Eq. (20), the implication is that

$dF_0/dx < 0$ which indicates a neutron flow in the $+x$-direction. The question which must be posed is: can F be μ-independent (consistent with P_0-approximation) and yet have $\psi(x, \mu)$ represent a neutron flow in $+x$-direction (i.e., ψ increase with μ)? Clearly the answer is in the affirmative since, if $\lambda(\mu)$ increases with μ (implied by $\lambda_1 > 0$), then the ratio ψ/λ can be μ-independent if ψ also increases with μ. In all following considerations assume that $\lambda(\mu)$, and thus $\sigma(\mu)$, is a symmetric function of μ. Thus it will always be required that $\lambda_n = 0$, and $\sigma_n = 0$, for n odd.

The P_1-approximation (i.e., diffusion theory) based on a flux expansion gives the two equations

$$\frac{d\psi_1}{dx} + (1 - c)\,\sigma_0\psi_0(x) = S_0(x) \tag{21a}$$

$$\frac{1}{3}\frac{d\psi_0}{dx} + (1 - cf_1)\left(\sigma_0 + \frac{2}{5}\sigma_2\right)\psi_1(x) = S_1(x) \tag{21b}$$

If it is further assumed that all neutron sources are isotropic such that $S_n(x) = 0$ for $n > 0$, then Eqs. (21a and b) combine to give the usual diffusion theory relations

$$-D\frac{d^2\psi_0}{dx^2} + \sigma'\psi_0(x) = S_0(x) \tag{22a}$$

$$\psi_1(x) = -D\frac{d\psi_0}{dx} \tag{22b}$$

with diffusion coefficient D, and "absorption" cross section σ', given by

$$D = \frac{1}{3(1 - cf_1)\left(\sigma_0 + \frac{2\sigma_2}{5}\right)} \tag{23a}$$

$$\sigma' = (1 - c)\,\sigma_0 \tag{23b}$$

It should be noted that only σ_0 and σ_2 enter into the diffusion theory parameters. The $\sigma(\mu)$ expansion was not truncated at a quadratic in order to obtain Eqs. (23a and b). In passing, it is also worth noting, had time-dependence been included, that the P_1-approximation would have given the "telegraphist's equation." The added assumption that $\partial\psi_1/v\partial t << \partial\psi_0/\partial x$ results in the form of the familiar time-dependent diffusion theory with D and σ' again given by Eqs. (23a and b).

The P_1-approximation based on a collision density expansion gives the two equations

$$\left(\lambda_0 + \frac{2}{5}\lambda_2\right)\frac{dF_1}{dx} + (1 - c)\,F_0(x) = S_0(x) \tag{24a}$$

$$\frac{1}{3}\left(\lambda_0 + \frac{2}{5}\lambda_2\right)\frac{dF_0}{dx} + (1 - cf_1)\,F_1(x) = S_1(x) \tag{24b}$$

In the case of isotropic sources, Eqs. (24a and b) combine to give

$$-D'\frac{d^2F_0}{dx^2} + (1 - c)\,F_0(x) = S_0(x) \tag{25a}$$

$$F_1(x) = -\frac{D'}{\left(\lambda_0 + \frac{2\lambda_2}{5}\right)}\frac{dF_0}{dx} \tag{25b}$$

where the "diffusion coefficient" is

$$D' = \frac{\left(\lambda_0 + \frac{2\lambda_2}{5}\right)^2}{3(1 - cf_1)} \tag{26}$$

As is the case in the flux-based diffusion theory, a quadratic truncation of $\lambda(\mu)$ is not necessary to obtain the results given in Eqs. (25 and 26).

The usual result of exponential spatial decline away from sources in infinite media is found for flux and collision density. The "period" of the exponential, the diffusion length L, is given by

$$L_\psi^2 = \frac{1}{3\sigma_0(1 - c)(1 - cf_1)\left(\sigma_0 + \frac{2\sigma_2}{5}\right)} \tag{27}$$

based on a flux expansion, and

$$L_F^2 = \frac{\left(\lambda_0 + \frac{2\lambda_2}{5}\right)^2}{3(1 - c)(1 - cf_1)} \tag{28}$$

based on a collision density expansion. The dissimilar results obtained from a P_0-approximation based on flux and collision density expansions have been pointed out. The divergence of results for the P_1-approximation can be illustrated by considering the ratio L_ψ/L_F for a given problem. Suppose that $\lambda(\mu)$ is actually an even quadratic, i.e., $\lambda_n = 0$ for $n = 1$ and $n > 2$. The corresponding

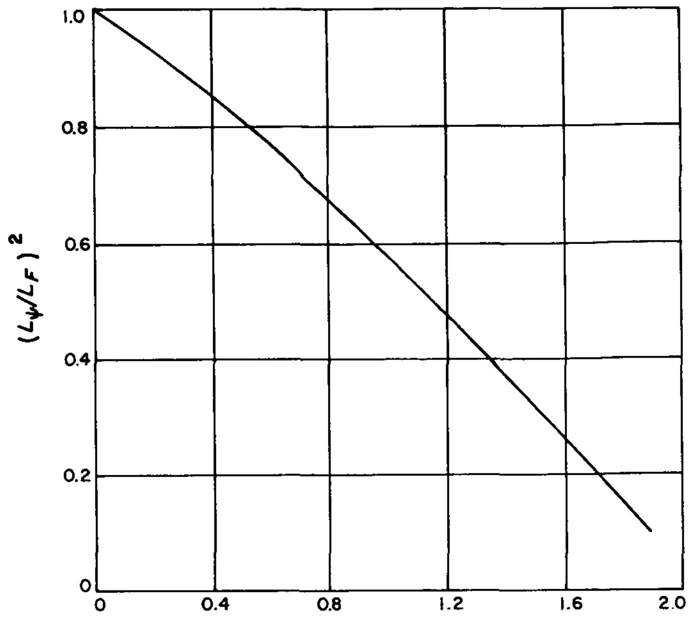

Fig. 1. The ratio of diffusion lengths as calculated by a neutron flux based and collision density based P_1-approximation for the case of an even quadratic mean free path, $\lambda(\mu) = \lambda_0 + \lambda_2 P_2(\mu)$

$\sigma(\mu)$ is not a quadratic, but only σ_0 and σ_2 need be computed since these coefficients determine L_ψ. In Fig. 1 the results are displayed for the range $\lambda_2/\lambda_0 \in (0, 2)$. Clearly, the flux and collision density based expansions can lead to significantly different results.

It should be expected that the two diffusion theories give different results. Note that whereas the ψ-approximation yields an accurate representation for neutron current $\psi_1(x)$ but an inaccurate (truncated) representation for collision density, the F-approximation results in an accurate representation for total interaction rate $F_0(x)$ but an inaccurate representation for current. For several reasons it will seem that the collision density expansion is favored in this work. However, these reasons are mainly of an algebraic character and it should not be construed that the F-formulation is, in all cases, superior.

Approximations of higher order than P_1 are accomplished following the usual general prescriptions. The added complications due to the angular-dependence of the mean free path place no restrictions on the formalism. Higher order approximations lessen the differences exhibited by the ψ and F-formulations.

D. The Double-P_N-Approximation

The double-P_N-approximation is derived from Yvon's method whereby the angular-dependence of the neutron flux, or collision density, is decomposed into contributions from $+x$-directed and from $-x$-directed neutrons. In order to simplify notation, let us consider the case of a stationary state in a medium characterized by isotropic scattering. The plane symmetry, monoenergetic neutron transport equation, Eq. (3), is then

$$\mu \frac{\partial}{\partial x} \psi(x, \mu) + \sigma(\mu) \psi(x, \mu) = \frac{c}{2} \times \int_{-1}^{+1} \sigma(\mu') \psi(x, \mu') \, d\mu' + S(x, \mu) \tag{29}$$

Use the half-angle-range expansions

$$\psi(x, \mu) = \sum_n (2n + 1) \psi_n^+(x) P_n(2\mu - 1), \mu > 0 \tag{30a}$$

$$= \sum_n (2n + 1) \psi_n^-(x) P_n(2\mu + 1), \mu < 0 \tag{30b}$$

$$S(x, \mu) = \sum_n (2n + 1) S_n^+(x) P_n(2\mu - 1), \mu > 0 \tag{30c}$$

$$= \sum_n (2n + 1) S_n^-(x) P_n(2\mu + 1), \mu < 0 \tag{30d}$$

$$\sigma(\mu) = \sum_n \sigma_n^+ P_n(2\mu - 1), \mu > 0 \tag{30e}$$

$$= \sum_n \sigma_n^- P_n(2\mu + 1), \mu < 0 \tag{30f}$$

to obtain the set of coupled differential equations

$$\frac{n}{2n+1} \frac{d\psi_{n-1}^{\pm}}{dx} + \frac{n+1}{2n+1} \frac{d\psi_{n+1}^{\pm}}{dx} \pm \frac{d\psi_n^{\pm}}{dx} + 2 \sum_{l,m} (2l + 1) A_{lmn} \sigma_m^{\pm} \psi_l^{\pm} = c\delta_{no} \sum_l (\sigma_l^- \psi_l^- + \sigma_l^+ \psi_l^+) + 2S_n^{\pm} \tag{31}$$

The double-P_N-approximation is defined by the requirement $\psi_n^{\pm}(x) = 0$ for $n > N$ and the equations labelled by $n > N$ are discarded.

The same analysis is followed for the collision density based approximation. Thus, with the definitions

$$F(x,\mu) = \sum_n (2n+1)\, F_n^+(x)\, P_n(2\mu - 1), \mu > 0 \tag{32a}$$

$$= \sum_n (2n+1)\, F_n^-(x)\, P_n(2\mu + 1), \mu < 0 \tag{32b}$$

$$\lambda(\mu) = \sum_n \lambda_n^+ P_n(2\mu - 1), \mu > 0 \tag{32c}$$

$$= \sum_n \lambda_n^- P_n(2\mu + 1), \mu < 0 \qquad (32a)$$

the transport equation is obtained in the form

$$\frac{n}{2n+1} \sum_{l,m} (2l+1)\, \lambda_m^{\pm} A_{lm,n-1} \frac{dF_l^{\pm}}{dx} + \frac{n+1}{2n+1} \sum_{l,m} (2l+1)\, \lambda_m^{\pm} A_{m,n+1} \frac{dF_l^{\pm}}{dx} \pm \sum_{l,m} (2l+1)\, \lambda_m^{\pm} A_{lmn} \frac{dF_l^{\pm}}{dx} + 2F_n^{\pm} = c\delta_{n_0} (F_0^+ + F_0^-) + 2S_n^{\pm} \tag{33}$$

The collision density double-P_N-approximation is defined as in the ψ-formulation.

In addition to the usefulness of Yvon's method in the solution of problems where accurate representation of source or free boundaries are required, there is an added flexibility for the angular-dependence of the mean free path. Thus, $\lambda(\mu)$ may have one form for $\mu > 0$ determined by the set $\{\lambda_n^+\}$, or $\{\sigma_n^+\}$, and another form for $\mu < 0$ following the set $\{\lambda_n^-\}$, or $\{\sigma_n^-\}$. For example, by using these methods a symmetric $\lambda(\mu)$ which varies linearly with μ for $\mu > 0$ and $\mu < 0$ can be expressed by using only two terms in each $\lambda(\mu)$ expansion, i.e., set $\lambda_0^+ = \lambda_0^-, \lambda_1^+ = -\lambda_1^-$, and $\lambda_n^{\pm} = 0$ for n > 1.

E. A Moment Decomposition

In the usual theory of neutron transport through homogeneous media, it is well-know that any space-angle moment of the neutron distribution can be found even though the distribution itself is unknown. In fact, an important method of determining the neutron distribution is the construction of a likely flux shape from a finite set of moments. Let us now consider a moment decomposition for the case of an angular-dependent mean free path where, as in our previous considerations, two formulations will be examined, i.e., with ψ and F as dependent variable.

We define the neutron flux and source moments by

$$\psi_n^j = \int_{-\infty}^{+\infty} x^j \psi_n(x)\, dx \tag{34a}$$

$$S_n^j = \int_{-\infty}^{+\infty} x^j S_n(x)\, dx \tag{34b}$$

Assume that the medium is of infinite extent. Multiplication of the stationary state form of Eq. (10) by x^j and integration over $x \epsilon (-\infty, +\infty)$ yields the set of algebraic moment relations

$$\sum_{l,m} (2l+1)\, \sigma_m A_{lmn} (1 - cf_n)\, \psi_l^j = S_n^j + \frac{j}{2n+1} \times \left[n\psi_{n-1}^{j-1} + (n+1)\, \psi_{n+1}^{j-1} \right] \tag{35}$$

With the collision density moments similarly defined, i.e.,

$$F_n^j = \int_{-\infty}^{+\infty} x^j F_n(x)\, dx \tag{36}$$

and performing similar operations on the stationary state form of Eq. (15), the result is a set of algebraic equations relating the collision density moments, i.e.,

$$(1 - cf_n)\, F_n^j = S_n^j + \frac{jn}{2n+1} \sum_{l,m} (2l+1)\, \lambda_m A_{lm,n-1} F_l^{j-1} + \frac{j(n+1)}{2n+1} \sum_{l,m} (2l+1)\, \lambda_m A_{lm,n+1} F_l^{j-1} \tag{37}$$

The moments of the neutron distribution resulting from a unit, plane, isotropic source (at $x = 0$) are easily interpretable in terms of important macroscopic parameters. For this source, $S_n^j = \delta_{n0}\delta_{j0}$. Consider calculation of flux moments first and then contrast this result with the method applied to collision density moments. For the sake of definiteness, assume that $\sigma(\mu)$ is an even M degree polynomial in μ. With $\sigma(\mu)$ an even function, it is clear that $\psi_n^j = 0$ for $j + n$ odd. An examination of Eq. (35), using the properties of the set $\{A_{lmn}\}$, indicates that with $M \neq 0$ and $n + j$ even, ψ_n^j depends on ψ_{n-1}^{j-1}, ψ_{n+1}^{j-1}, $\psi_{|n-M|}^j$, $\psi_{|n-M|+2}^j$, $\cdots$, ψ_{n+M-2}^j, ψ_{n+M}^j. Thus, ψ_n^j can not be determined without employing a truncation on the set $\{\psi_n(x)\}$. This is, of course, the simplification used in a P_N-approximation and would not yield exact moments. The familiar case of $M = 0$ poses no such difficulties and one can readily find the exact moments.

In considering the collision density moments, assume that $\lambda(\mu)$ is an even M degree polynomial in μ. It follows that $F_n^j = 0$ for $n + j$ odd. It is also possible to show, from Eq. (37) and the properties of $\{A_{lmn}\}$, that $F_n^j = 0$ for $n > j(M + 1)$ and therefore, with finite M, any collision density moment can be calculated exactly. For example, for the case of $M = 2$

$$\langle x^2 \rangle = \frac{F_0^2}{F_0^0} = 2L_F^2 + \frac{18}{175}\frac{\lambda_2^2}{(1-c)(1-cf_3)} \tag{38}$$

where $\langle x^2 \rangle$ is the mean value of x^2 and L_F is the diffusion length as calculated by a collision density based P_1-approximation. Note that the result of Eq. (38) is unlike that found in the angle-independent mean free path case. In that case $M = 0$ and the second spatial moment is given correctly by diffusion theory, i.e., $\langle x^2 \rangle = 2L^2$ both in the exact calculation and in diffusion theory.

In passing, note that the set $\{\psi_n^j\}$ can be found from the set $\{F_n^j\}$ via the easily derived relation

$$\psi_n^j = \sum_{l,m} (2l+1)\lambda_m A_{lmn} F_l^j \tag{39}$$

The result of Eq. (39) does not contradict the earlier assertion regarding the problem of finding the flux moments. When the determination of the $\{\psi_n^j\}$ is approached directly by Eq. (35), it is the total cross section which is considered to be an M-degree polynomial in μ. When $\{\psi_n^j\}$ is found using $\{F_n^j\}$ as in Eq. (39), the mean free path is assumed to be an M-degree polynomial in μ.

F. The Case of Isotropic Scattering

The stationary state form of Eq. (13) with the additional assumption of isotropic scattering gives the collision density equation

$$\mu\lambda(\mu)\frac{\partial}{\partial x}F(x,\mu) + F(x,\mu) = \frac{c}{2}\int_{-1}^{+1} F(x,\mu')\,d\mu' + S(x,\mu) \tag{40}$$

In this case of isotropic scattering an angle variable change is suggested. Specifically, define the angle variable $u = \mu\lambda(\mu)/\lambda(1)$, measure x in units of $\lambda(1)$, and change dependent variable to $F(x,u) = g(u)F(x,\mu)$ with $g(u) = |d\mu/du|$. With the source density change $S(x,u) = g(u)S(x,\mu)$, Eq. (40) takes the form

$$u\frac{\partial}{\partial x}F(x,u) + F(x,u) = \frac{c}{2}g(u)\int_{-1}^{+1} F(x,u')\,du' + S(x,u) \tag{41}$$

Various aspects of the solution of Eq. (41) will be considered.

1. Legendre Polynomial Expansion

Following the procedure used in the derivation of Eq. (10), the coupled differential equation form of the transport equation

$$\frac{n}{2n+1}\frac{dF_{n-1}}{dx} + \frac{n+1}{2n+1}\frac{dF_{n+1}}{dx} + F_n(x) = \frac{cg_n}{2n+1}F_0(x) + S_n(x) \tag{42}$$

is obtained where the Legendre polynomial expansions are used:

$$F(x,u) = \sum_n \frac{2n+1}{2}F_n(x)P_n(u) \tag{43a}$$

$$S(x,u) = \sum_n \frac{2n+1}{2}S_n(x)P_n(u) \tag{43b}$$

$$g(u) = \sum_n g_n P_n(u) \tag{43c}$$

It should be noted that the requirement of a symmetric $\lambda(\mu)$ imposes the condition that $g_n = 0$ for n odd.

The idea of a P_N-approximation is equally well-applied here. For example, the P_1-approximation (diffusion theory) takes the usual form [cf., Eq. (25)]

$$-\frac{1}{3}\frac{d^2F_0}{dx^2} + (1 - cg_0)F_0 = S_0 \tag{44a}$$

$$F_1 = -\frac{1}{3}\frac{dF_0}{dx} \tag{44b}$$

where only isotropic sources are allowed.

2. Moment Decomposition

With the collision density and source moments defined in the usual manner [i.e., by Eq. (34)], Eq. (42) can be transformed to the algebraic set

$$F_n^j = S_n^j + \frac{cg_n}{2n+1}F_0^j + \frac{j}{2n+1}\left[nF_{n-1}^{j-1} + (n+1)F_{n+1}^{j-1}\right] \tag{45}$$

It has been assumed that $\lambda(\mu)$ is symmetric which implies that $g(u)$ is symmetric. We also find that $F_n^j = 0$ for

$j + n$ odd, and, for odd n, F_n^j depends only on F_{n-1}^{j-1} and F_{n+1}^{j-1}. Furthermore there is the interesting property that the spatial moment F_0^j depends only on the set of moments $\{F_n^i, n + i \leqq j\}$. Therefore, the calculations of a low-order spatial moment require the specification of a small number of the g_n and the prior determination of a small number of other moments.

As an example, calculate by these methods the second spatial moment of the neutron distribution resulting from a unit, plane, isotropic source (at $x = 0$). In this case $S_n^j = \delta_{n0}\delta_{j0}$. The moments F_0^0 F_2^0, and F_1^1 are easily determined and are the only values required in the calculation of F_0^2. For the normalized second spatial moment [cf., Eq. (38)],

$$\langle x^2 \rangle = \frac{F_0^2}{F_0^0} = \frac{2}{3} \frac{1 + \dfrac{2cg_2}{5}}{1 - cg_0} \tag{46}$$

3. Normal Mode Expansion

The recently developed normal mode technique (Ref. 3) will be applied to the problem of determining the exact and asymptotic solution of Eq. (41). In so doing there is generated an interesting mathematical problem the details of which are considered in Appendix B. Consider the homogeneous form of Eq. (41), i.e., $S = 0$. Translational invariance suggests the "ansatz"

$$F(x, u) = \phi(\nu, u) \exp\left(\frac{-x}{\nu}\right) \tag{47}$$

where the separation variable ν is allowed to be complex. The integral equation

$$(\nu - u)\,\phi(\nu, u) = \frac{c}{2}\,\nu\, g(u) \int_{-1}^{+1} \phi(\nu, u')\, du' \tag{48}$$

is thereby obtained. Adopting the usual normalization,

$$\int_{-1}^{+1} \phi(\nu, u)\, du = 1 \tag{49}$$

If solutions of Eq. (48) are allowed to be distributions (in the sense of L. Schwartz, Ref. 4), then

$$\phi(\nu, u) = \frac{c}{2} \frac{\nu g(u)}{\nu - u} + \Lambda(\nu)\,\delta(\nu - u) \tag{50}$$

Assume that $g(u)$ satisfies a Holder condition (Ref. 5, p. 11) on the interval of the real line $u \in (-1, +1)$. Any singular integrals which might appear are then of the Cauchy type and their evaluation is defined as the Cauchy principal value (Ref. 5, p. 26).

The normalization required of $\phi(\nu, u)$, i.e., Eq. (49), leads to a specification of allowed discrete values of ν in the region of the ν-complex-plane excluding the line $(-1, +1)$, and to a specification of the function $\Lambda(\nu)$ for $\nu \in (-1, +1)$. To aid in the analysis of these results, define the Cauchy integral, $G(\nu)$, by

$$G(\nu) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{g(u)}{u - \nu}\, du \tag{51}$$

With $\nu \notin (-1, +1)$, Eq. (49) gives

$$1 + i\pi c\nu G(\nu) = 0 \tag{52}$$

which has a set of roots which are distinct. With $\nu \in (-1, +1)$, Eq. (49) yields an explicit formula for the function $\Lambda(\nu)$ and no restrictions are placed on allowed values of ν. We find

$$\Lambda(\nu) = 1 + i\pi c\nu G(\nu) \tag{53}$$

Thus, if the definition of $\Lambda(\nu)$ is extended, as expressed in Eq. (53), to the entire ν-plane, the zeroes of $\Lambda(\nu)$ determine the set of allowed distinct values of ν. Since $g(u)$ is symmetric, $G(-\nu) = -G(\nu)$, whence, $\Lambda(\nu)$ is an even function of ν. The zeroes of $\Lambda(\nu)$, therefore, appear in pairs which are labeled $\pm\nu_j$.

A set of functions of the angle variable u indexed by ν, $\{\phi(\nu, u)\}$ have been found. There is a discrete indexed set with $\nu \notin (-1, +1)$ and members characterized by

$$\phi(\pm\nu_j, u) = \frac{c}{2} \frac{\nu_j g(u)}{\nu_j \mp u} \tag{54}$$

and, a continuous indexed set with $\nu \in (-1, +1)$ and of form given by Eq. (50). The function $\Lambda(\nu)$, which appears in $\phi(\nu, u)$ for $\nu \in (-1, +1)$, is given by Eq. (53). Furthermore, the zeroes of $\Lambda(\nu)$ for $\nu \notin (-1, +1)$ establish the set of discrete indices $\{\pm\nu_j\}$.

If we assume that $g(u) \neq 0$ for $u \in (-1, +1)$, Eq. (48), with the normalization of Eq. (49), may be written in the form

$$\left[1 - \frac{u}{\nu}\right] \frac{\phi(\nu, u)}{g(u)} = \frac{c}{2} \tag{55}$$

Multiply Eq. (55) with index ν by $\phi(\nu', u)$ and subtract the result from Eq. (55) with index ν' multiplied by $\phi(\nu, u)$. Employing Eq. (49) and integrating over $u \epsilon (-1, +1)$

$$\left[\frac{1}{\nu} - \frac{1}{\nu'}\right] \int_{-1}^{+1} \frac{u}{g(u)} \phi(\nu, u) \phi(\nu', u) \, du = 0 \tag{56}$$

There is clearly no degeneracy and thus Eq. (56) may be rewritten as the orthogonality relation

$$\int_{-1}^{+1} \frac{u}{g(u)} \phi(\nu, u) \phi(\nu', u) \, du = 0 \text{ for } \nu \neq \nu' \tag{57}$$

The nature of the orthogonality relation including the case $\nu = \nu'$ depends on whether ν is a member of the discrete index set or belongs to the continuum. If ν is a discrete index, then

$$\int_{-1}^{+1} \frac{u}{g(u)} \phi(\pm\nu_j, u) \phi(\pm\nu_i, u) \, du = I(\pm\nu_j) \delta_{ji} \tag{58a}$$

$$I(\pm\nu_j) = \frac{c^2}{4} \nu_j^3 \int_{-1}^{+1} \frac{ug(u)}{(\nu_j \pm u)^2} du \tag{58b}$$

If ν belongs to the continuum, then

$$\int_{-1}^{+1} \frac{u}{g(u)} \phi(\nu, u) \phi(\nu', u) \, du = \frac{\nu \Lambda^2(\nu)}{g(\nu)} \delta(\nu - \nu') \tag{59}$$

It was found that the set of normal modes $\{\phi(\nu, u)\}$ is orthogonal, with weight function $u/g(u)$, on the interval $u \epsilon (-1, +1)$. For the remainder of this section assume that the normal modes are also complete in the space of functions which satisfy a Holder condition on the interval $u \epsilon (-1, +1)$. Appendix B, in measure, substantiates this hypothesis by demonstrating the existence of the modal expansion coefficients. In so doing, the interval of completeness is generalized to include all physically relevant cases.

Assuming that the normal modes form a complete set on the interval $u \epsilon (-1, +1)$, the general solution of Eq. (41) is in the form

$$F(x, u) = \sum_{\nu} a(\nu) \phi(\nu, u) \exp\left(\frac{-x}{\nu}\right) \tag{60}$$

where the summation indicates integration over continuous spectra when applicable. In many problems there are boundary conditions which can be formulated as

$$F(0, u) = \phi(u) = \sum_{\nu} a(\nu) \phi(\nu, u) \text{ for } u \epsilon (-1, +1) \tag{61}$$

and the orthogonality relations can be used to determine the expansion coefficients, $a(\nu)$. In detail, Eq. (61) is rewritten as

$$\phi(u) = \sum_j a(+\nu_j) \phi(+\nu_j, u) + \sum_j a(-\nu_j) \phi(-\nu_j, u) + \int_{-1}^{+1} a(\nu) \phi(\nu, u) \, d\nu \text{ for } u \epsilon (-1, +1) \tag{62}$$

Direct application of the discrete index orthogonality relation, Eq. (58), yields the discrete indexed expansion coefficients,

$$a(\pm\nu_j) = \frac{1}{I(\pm\nu_j)} \int_{-1}^{+1} \frac{u}{g(u)} \phi(u) \phi(\pm\nu_j, u) \, du \tag{63}$$

Using Eq. (57) in Eq. (62) gives the result

$$\int_{-1}^{+1} \frac{u}{g(u)} \phi(u) \phi(\nu, u) \, du = \int_{-1}^{+1} du \frac{u}{g(u)} \phi(\nu, u) \times \int_{-1}^{+1} a(\nu') \phi(\nu', u) \, d\nu' \tag{64}$$

There appears a doubly singular Cauchy integral and thus the order of integration in Eq. (64) may not be reversed without due caution. The doubly singular term appears as

$$\frac{c^2}{4} \nu \int_{-1}^{+1} du \frac{ug(u)}{\nu - u} \int_{-1}^{+1} \frac{\nu' a(\nu')}{\nu' - u} d\nu'$$

Assume that $a(\nu)$ satisfies a Holder condition for $\nu \epsilon (-1, +1)$ and follow the dictates of the Poincaré-Bertrand formula for inverting the integration order (Ref. 5, p. 57). Then

$$\frac{C^2}{4} \nu \int_{-1}^{+1} du \frac{ug(u)}{\nu - u} \int_{-1}^{+1} \frac{\nu' a(\nu')}{\nu' - u} d\nu' = \pi^2 \frac{c^2}{4} \nu^3 g(\nu) a(\nu) + \frac{c^2}{4} \nu \int_{-1}^{+1} d\nu' \frac{\nu' a(\nu')}{\nu' - u} \int_{-1}^{+1} \frac{ug(u)}{\nu - u} du \tag{65}$$

Using Eqs. (59) and (65) results in the more useful "orthogonality relation,"

$$\int_{-1}^{+1} du \frac{u}{g(u)} \phi(\nu, u) \int_{-1}^{+1} a(\nu') \phi(\nu', u) \, d\nu' = I(\nu) a(\nu) \text{ for } \nu \epsilon (-1, +1) \tag{66a}$$

$$I(\nu) = \nu g(\nu) \left[\left(\frac{\Lambda(\nu)}{g(\nu)}\right)^2 + \left(\frac{\pi c \nu}{2}\right)^2\right] \tag{66b}$$

Now, applying Eq. (66) to the problem of finding the continuum expansion coefficients in Eq. (62),

$$a(\nu) = \frac{1}{I(\nu)} \int_{-1}^{+1} \frac{u}{g(u)} \phi(u) \phi(\nu, u)\, du \tag{67}$$

for $\nu \epsilon (-1, +1)$

G. The Green's Function for the Case of Isotropic Scattering

As a specific example of the use of the relations just developed, consider the problem of finding the infinite medium Green's function for isotropic, plane sources. In this case, the source density, $S(x, u)$, of Eq. (41) represents a unit, plane, isotropic emission of neutrons at a position which is designated $x = 0$, i.e., $S(x, u) = g(u)\,\delta(x)/2$. Integration of Eq. (41) over a vanishing interval about $x = 0$ yields the boundary condition

$$u\,[F(0^+, u) - F(0^-, u)] = \frac{g(u)}{2} \text{ for } u \,\epsilon\, (-1, +1) \tag{68}$$

The additional condition that as $|x| \to \infty$, $F(x, u) \to 0$ is imposed and the solution is expressed in the form

$$F(x. u) = \sum_j a(+\nu_j)\, \phi(+\nu_j, u) \exp\left(\frac{-x}{\nu_j}\right) + \int_0^{+1} a(\nu)\, \phi(\nu, u) \exp\left(\frac{-x}{\nu}\right) d\nu \text{ for } x > 0 \tag{69a}$$

$$F(x, u) = -\sum_j a(-\nu_j)\, \phi(-\nu_j,\ u) \exp\left(\frac{x}{\nu_j}\right) - \int_{-1}^{0} a(\nu)\, \phi(\nu, u) \exp\left(\frac{-x}{\nu}\right) d\nu \text{ for } x < 0 \tag{69b}$$

The source condition, Eq. (68), then takes the form of the general boundary condition, Eq. (62). Specifically

$$\frac{u}{g(u)} \sum_j a(+\nu_j)\, \phi(+\nu_j, u) + \frac{u}{g(u)} \sum_j a(-\nu_j)\, \phi(-\nu_j, u) + \frac{u}{g(u)} \int_{-1}^{+1} a(\nu)\, \phi(\nu, u)\, d\nu = \frac{1}{2} \tag{70}$$

Whence, employing the normalization expressed by Eq. (49),

$$a(\pm\nu_j) = \frac{1}{2I(\pm\nu_j)} \tag{71a}$$

$$a(\nu) = \frac{1}{2I(\nu)} \text{ for } \nu \,\epsilon\, (-1, +) \tag{71b}$$

and the solution of the Green's function has been completed.

Some aspects of the Green's function will be examined. For simplicity, assume that there is only one pair of zeroes of $\Lambda(\nu)$, $\pm\nu_0$. A sufficient condition for this property is developed in Appendix A. The Green's function is then given by

$$F(x, u) = \frac{\phi(+\nu_0, u)}{2I(+\nu_0)} \exp\left(\frac{-x}{\nu_0}\right) + \int_0^{+1} \frac{\phi(\nu, u)}{2I(\nu)} \exp\left(\frac{-x}{\nu}\right) d\nu \text{ for } x > 0 \tag{72a}$$

$$F(x, u) = -\frac{\phi(-\nu_0, u)}{2I(-\nu_0)} \exp\left(\frac{x}{\nu_0}\right) - \int_{-1}^{0} \frac{\phi(\nu, u)}{2I(\nu)} \exp\left(\frac{-x}{\nu}\right) d\nu \text{ for } x < 0 \tag{72b}$$

With the definition

$$\phi_n(\nu) = \int_{-1}^{+1} \phi(\nu, u)\, P_n(u)\, du \tag{73}$$

and the easily derived symmetry properties,

$$I(-\nu_0) = -I(+\nu_0) \text{ and } I(-\nu) = -I(\nu),$$

the collision density moments for the neutron distribution from a unit, plane, isotropic source, are

$$F_n^j = \frac{j!}{2}\,[\phi_n(+\nu_0) + (-1)^j \phi_n(-\nu_0)]\, \frac{\nu_0^{j+1}}{I(+\nu_0)} + \frac{j!}{2} \int_0^{+1} [\phi_n(\nu) + (-1)^j \phi_n(-\nu)]\, \frac{\nu^{j+1}}{I(\nu)}\, d\nu \tag{74}$$

Using Eqs. (9), (43c), (48), and (49), a recurrence relation is obtained for the set $\{\phi_n(\nu)\}$,

$$(2n+1)\,\nu\,\phi_n(\nu) = n\phi_{n-1}(\nu) + (n+1)\,\phi_{n+1}(\nu) + c\nu g_n \tag{75}$$

and the normalization $\phi_0(\nu) = 1$. Note that Eq. (75) implies that $\phi_n(\nu)$ is an even, or odd, polynomial in ν of degree n. Therefore, $\phi_n(-\nu) = (-1)^n \phi_n(+\nu)$, and Eq. (74) reduces to

$$F_n^j = j!\left[\frac{\nu_0^{j+1}}{I(+\nu_0)}\phi_n(+\nu_0) + \int_0^{+1}\frac{\nu^{j+1}}{I(\nu)}\phi_n(\nu)\,d\nu\right] \qquad \text{if } j+n \text{ is even} \tag{76a}$$

$$F_n^j = 0 \qquad \text{if } j+n \text{ is odd} \tag{76b}$$

Already considered is the moments set $\{F_n^j\}$. For this particular case the $\{F_n^j\}$ is determined from Eq. (45) and the source condition $S_n^j = \delta_{n0}\delta_{j0}$. In passing, note that the consistency of Eq. (45) and (76) is easily demonstrated via the recurrence relation Eq. (75). Moreover, equating the F_0^0 and F_0^2 moments as derived by the two relations gives

$$\frac{\nu_0}{I(+\nu_0)} + \int_0^{+1}\frac{\nu}{I(\nu)}\,d\nu = \frac{1}{1-cg_0} \tag{77a}$$

$$\frac{\nu_0^3}{I(+\nu_0)} + \int_0^{+1}\frac{\nu^3}{I(\nu)}\,d\nu = \frac{1}{3}\,\frac{1+\dfrac{2cg_2}{5}}{(1-cg_0)^2} \tag{77b}$$

From Eqs. (77a and b) an explicit expression for the discrete index ν_0 is obtained:

$$\nu_0^2 = \frac{\dfrac{1}{3}\,\dfrac{\left(1+\dfrac{2cg_2}{5}\right)}{(1-cg_0)^2} - \displaystyle\int_0^{+1}\frac{\nu^3}{I(\nu)}\,d\nu}{\dfrac{1}{1-cg_0} - \displaystyle\int_0^{+1}\frac{\nu}{I(\nu)}\,d\nu} \tag{78}$$

For $c < 1$, ν_0 is real and is interpreted as the exact asymptotic diffusion length [here, measured in units of $\lambda(1)$]. It should be noted that the integral terms in Eq. (78) depend on c and $\{g_n\}$ via the dependence of $I(\nu)$ on these parameters [cf., Eq. (66b)].

III. REMARKS REGARDING APPLICATION OF THE THEORY

A limited number of considerations relevant to the application of the theory presented in Section II will be developed as a brief illustration of possible methods of application of the present theory to physical problems. Many interesting calculations are possible, and with the accomplishment of experimental measurements of neutron distributions in the types of media under discussion, many comparisons of theoretical and experimental results would be profitable.

Methods of determining the proper variation of mean free path are certainly required if these mathematical formulations are to be applied to physical problems. In this section the general types of heterogeneity, toward which the current theory applies, will be discussed. A simple method will be detailed, using known diffusion lengths, to specify the angular dependence of the mean free path for a particular type of heterogeneity.

A. Types of Heterogeneity

The motivation of the present effort, as mentioned earlier, is the establishment of a method of homogenization of regular arrays of vacuum channels for the purpose of neutron diffusion calculations. Also imposed was the necessary restriction that, in general, the type of heterogeneity considered should yield two characteristic orthogonal directions. As an example of the caution which must be exercised in application of the theory, consider a type of heterogeneity which, at first approach, appears to satisfy the necessary requirements, but which actually is unsuitable for these methods. Specifically, examine the case of a periodic slab array of scatterer and vacuum. This heterogeneity exhibits two characteristic orthogonal directions; perpendicular to slab, and the directions in the plane of the slab. Moreover, the direction perpendicular to the slabs (transverse to slab "channels") yields considerations which are algebraically easily accomplished.

If $\lambda(\mu, x)$ represents the mean distance traveled to a collision by a neutron located at a position x to the left of the right-hand-face of a slab of scatterer, traveling with direction cosine μ relative to the slab perpendicular direction, gives

$$\lambda(\mu, x) = \lambda_s + \frac{\dfrac{T_v}{\mu} \exp\left(\dfrac{-x}{\lambda_s \mu}\right)}{1 - \exp\left(\dfrac{-T_s}{\lambda_s \mu}\right)} \text{ for } x \leqq T_s, \mu > 0 \tag{79}$$

In Eq. (79), λ_s is the mean free path in the scatterer material which has slab thickness T_s, and T_v is the vacuum slab thickness. For the homogenized medium we require a function $\lambda(\mu)$ which, it would seem, should be a "suitable" average of $\lambda(\mu, x)$. For the case of isotropic scattering, the average

$$\lambda(\mu) = \frac{\displaystyle\int_0^{T_s} \lambda(\mu, x)\, \psi_0(x)\, dx}{\displaystyle\int_0^{T_s} \psi_0(x)\, dx} \tag{80}$$

is clearly indicated. In Eq. (80), $\psi_0(x)$ represents the actual angular integrated neutron flux. Note that, in the present case, $\psi_0(x)$ can be found. Far from neutron sources there is $\psi_0(x) \rightarrow \exp(x/L_s)$ where L_s is the "asymptotic" diffusion length in the scatterer material. A note in passing: if $T_s/L_s << 1$, then $\psi_0(x)$ is approximately constant and Eq. (80) gives the result

$$\lambda(\mu) = \lambda_s\left(1 + \frac{T_v}{T_s}\right) \tag{81}$$

which is the "simply homogenized" parameter. Of course, the condition $T_s/L_s << 1$ should yield the homogeneous limit.

If x now represents the direction perpendicular to the slabs we have the asymptotic result $\psi_0(x) \rightarrow \exp(-x/L_s)$ when the position x falls in a scatterer slab and $\psi_0(x)$ is a constant when x falls in a vacuum slab. The "best fit" to this flux, for the homogenized medium is $\psi_0(x) \rightarrow \exp(-x/L)$ where L is the simply homogenized diffusion length, i.e., $L = L_s(1 + T_v/T_s)$. This result would be obtained if Eq. (81) were used. In this particular case, the situation is that a calculation based on an angular-dependent mean free path yields results less representative than the simply homogenized calculation. It is expected that, in the orthogonal characteristic direction (i.e., in the plane of the slab), use of an angular-dependent mean free path is indicated.

The case of a calculation in the slab perpendicular direction for a periodic slab array is certainly excluded from the present considerations. Moreover, no motivation should be felt toward developing a theory for that case since it is easily treated by a standard method, i.e., change of position variable to "optical thickness." Consider now the details of a macroscopic-parameter-based calculation for a heterogeneity for which the present methods were clearly intended, i.e., a regular array of cylindrical vacuum channels.

B. Cylindrical Channels in a Regular Array

Consider a regular array of vacuum channels of cylindrical cross section. With every vacuum channel of cross sectional area A_v a cross sectional area of scatterer material A_s is associated such that $V = A_v/A_s$ is the ratio of vacuum volume to scatterer volume characteristic of the medium. Label the axial, or longitudinal, direction with x and direction cosine μ, and the radial, or transverse direction with y and direction cosine η. Due to streaming along channels, different diffusion properties in the x- and y-directions are expected with both of these cases being different than the simply homogenized diffusion. The simply homogenized mean free path is given by

$$\lambda_h = \lambda_s(1 + V) \tag{82}$$

Before presenting a specific method for obtaining a representative $\lambda(\mu)$, some general comments will be made regarding the features of such a calculation. It is clear that only two representative directions are being considered: axial, or x-direction; and transverse, or y direction. It is also clear that in the present lattice the actual description of a straight line path in the transverse direction starting from a point in the scattering material depends upon both the azimuthal angle about the x-direction and the particular position in the scattering material relative to, say, the center of a vacuum channel. A "suitable" averaging technique must be employed. Furthermore, the same difficulty is encountered when considering a description of an axially-oriented path. Let $\lambda_x(\mu)$ and $\lambda_y(\eta)$ represent the angular-dependent mean free paths with respect to x-direction diffusion and y-direction diffusion.

The following constraints on the "suitable" averaging technique seem intuitively reasonable:

1. The average mean free path based on axial and transverse directions should be equal, i.e.,

$$\int_{-1}^{+1} \lambda_x(\mu)\, d\mu = \int_{-1}^{+1} \lambda_y(\eta)\, d\eta \tag{83}$$

2. The axial mean free path in the transverse direction (i.e., at $\mu = 0$) should be equal to the transverse mean free path in the transverse direction (i.e., at $\eta = \pm 1$), i.e.,

$$\lambda_x(0) = \lambda_y(\pm 1) \tag{84}$$

3. Both axial and transverse mean free paths should be symmetric, i.e.,

$$\lambda_x(\mu) = \lambda_x(-\mu) \tag{85a}$$

$$\lambda_y(\eta) = \lambda_\eta(-\eta) \tag{85b}$$

A possible method of obtaining $\lambda(\mu)$ is to find, as a function of starting position in the scatterer, the mean free path length traveled in all directions. Then, upon "suitably" weighting this quantity, according to whether $\lambda_x(\mu)$ or $\lambda_y(\eta)$ is desired, an average yields the angular-dependent mean free path. In even the simplest lattice this is a geometric task of considerable magnitude. Here, for the sake of brevity, an alternate, albeit certainly less self-contained, route will be taken. Assume certain macroscopic diffusion parameters, such as diffusion length, are given and use the general constraints of Eqs. (83–85) to obtain a representation of the mean free path which yields the given parameters. To be specific, assume that $\lambda_x(\mu)$ and $\lambda_y(\eta)$ are even quadratics of the respective variables. Thus, in terms of the Legendre polynomial expansion

$$\lambda_x(\mu) = \lambda_{x0} + \lambda_{x2} P_2(\mu) \tag{86a}$$

$$\lambda_y(\eta) = \lambda_{y0} + \lambda_{y2} P_2(\mu) \tag{86b}$$

From Eq. (83) $\lambda_{x0} = \lambda_{y0}$, and this result used in Eq. (84) yields $\lambda_{y2} = -\lambda_{x2}/2$. Therefore, in terms of the two unknowns, λ_0 and λ_2, Eq. (86) may be reformulated as

$$\lambda_x(\mu) = \lambda_0 + \lambda_2 P_2(\mu) \tag{87a}$$

$$\lambda_y(\eta) = \lambda_0 - \frac{1}{2}\lambda_2 P_2(\eta) \tag{87b}$$

The arguments used here with respect to the mean free path also apply to the determination of the total cross section. Thus, the general constraints are expected:

1. $$\int_{-1}^{+1} \sigma_x(\mu)\, d\mu = \int_{-1}^{+1} \sigma_y(\eta)\, d\eta$$

2. $$\sigma_x(0) = \sigma_y(\pm 1)$$

3. $$\sigma_x(\mu) = \sigma_x(-\mu)$$

   and

$$\sigma_y(\eta) = \sigma_y(-\eta)$$

If the total cross section is assumed to be an even quadratic, then in terms of the two unknowns, σ_0 and σ_2,

$$\sigma_x(\mu) = \sigma_0 + \sigma_2 P_2(\mu)$$

$$\sigma_y(\eta) = \sigma_0 - \frac{1}{2}\sigma_2 P_2(\eta)$$

For the remainder of this discussion assume that the neutron collision density is used as dependent variable and thus the mean free path is the relevant parameter. These considerations can be equally well applied to the neutron flux and total cross section.

From Eqs. (28) and (87) results are

$$L_x^2 = \frac{\left(\lambda_0 + \dfrac{2\lambda_2}{5}\right)^2}{3(1-c)(1-cf_1)} \tag{88a}$$

$$L_y^2 = \frac{\left(\lambda_0 - \dfrac{\lambda_2}{5}\right)^2}{3(1-c)(1-cf_1)} \tag{88b}$$

Also,

$$L_s^2 = \frac{\lambda_s^2}{3(1-c)(1-cf_1)} \tag{89}$$

From Eqs. (88 and 89)

$$\frac{\lambda_0}{\lambda_s} = \frac{\langle L \rangle}{L_s} \tag{90a}$$

$$\langle L \rangle = \frac{L_x}{3} + \frac{2L_y}{3} \tag{90b}$$

$$\frac{\lambda_2}{\lambda_s} = \frac{5}{3}\left[\frac{L_x}{L_s} - \frac{L_y}{L_s}\right] \tag{90c}$$

Measured values of L_x and L_y, or other theoretical treatments, can be used to find L_x and L_y in order to determine

λ_0/λ_s and λ_2/λ_s. For example, if Behren's theoretical formulation (Ref. 1) is used,

$$\left(\frac{L_x}{L_s}\right)^2 = 1 + 2V + \frac{2RV}{\exp\left(\frac{2R}{V}\right) - 1} + 2RV \tag{91a}$$

$$\left(\frac{L_y}{L_s}\right)^2 = 1 + 2V + \frac{2RV}{\exp\left(\frac{2R}{V}\right) - 1} + RV \tag{91b}$$

where R is the ratio of the vacuum channel radius to λ_s. In Fig. 2 L_x/L_s and L_y/L_s are presented as a function of R, $R \in (0, 5)$, for the cases $V = 0.5$, 1.0 and 2.0 as determined by Eq. (91). Then in Fig. 3 for the same values of R and V, the results for λ_0/λ_s and λ_2/λ_s are based on the curves in Fig. 2.

It should be noted that what is referred to as L_x^2 and L_y^2 is Eq. (91) are actually calculated by Behrens (Ref. 1) as $\langle x^2 \rangle/2$ and $\langle y^2 \rangle/2$ and, via Eq. (38),

$$\langle x^2 \rangle = 2L_x^2 + \frac{18}{175} \frac{\lambda_2^2}{(1-c)(1-cf_3)} \tag{92a}$$

$$\langle y^2 \rangle = 2L_y^2 + \frac{18}{175} \frac{\frac{\lambda_2^2}{4}}{(1-c)(1-cf_3)} \tag{92b}$$

If $\lambda_2^2 > 1$ the validity of Fig. 3 as a relevant representation for $\lambda(\mu)$ is questionable. However, truncation of $\lambda(\mu)$ at a quadratic would, in that case, also be of questionable usefulness.

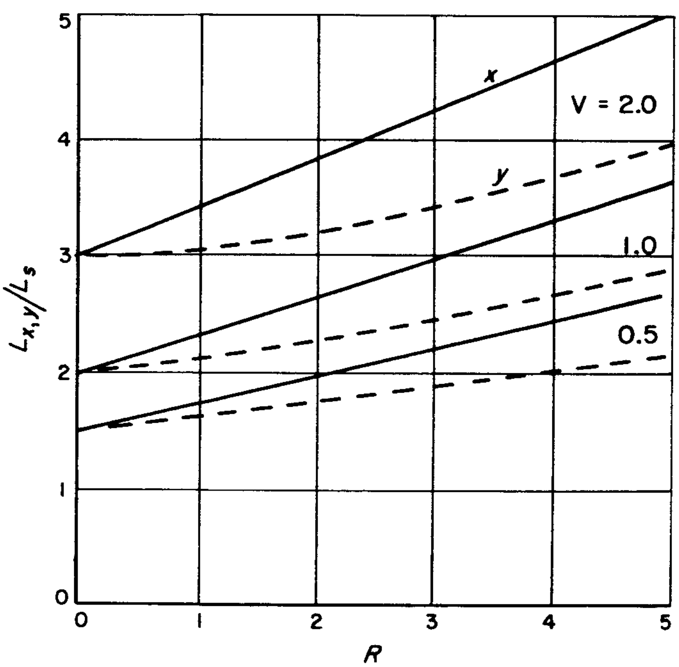

Fig. 2. Neutron diffusion lengths as found by Behren's theoretical formulation

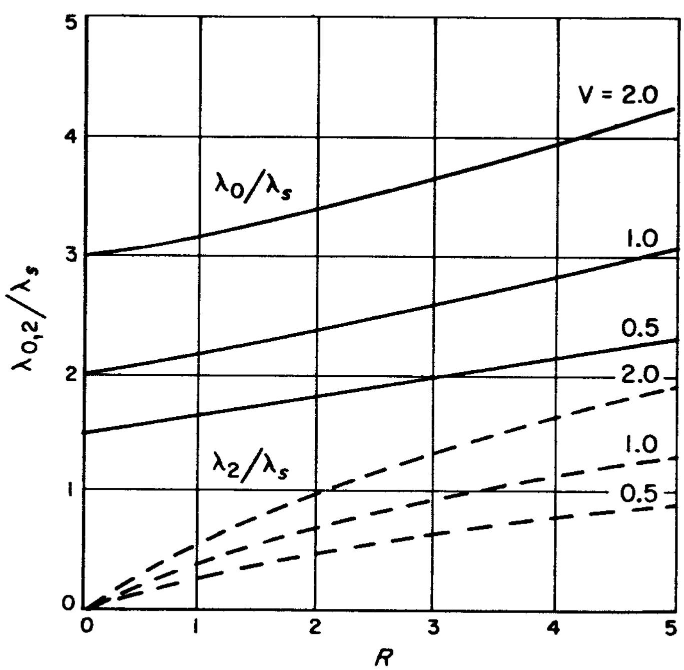

Fig. 3. Expansion coefficients for an even quadratic representation of the mean free path angular dependence

IV. SUMMARY

The mathematical formulation has been developed of a new approach to the homogenization of certain types of heterogeneous media (such as a regular array of vacuum channels) for the purpose of neutron diffusion calculations.

The new method is based on the inclusion of an angular-dependent mean free path in the theory of neutron transport. In the present effort, calculations are restricted to media with plane symmetry and monoenergetic neutron theory is employed. Extension to energy-dependent theory and to other symmetries would probably follow the general lines for the familiar, angular-independent case without significant additional complication. However, it seems clear that the requirement of the existence of two orthogonal characteristic directions in the development of the angular dependence of the mean free path must be imposed.

It was found in this Report that a neutron flux based theory and a collision density based theory can lead to significantly different results when low-order approximations, such as diffusion theory, are employed in the solution of the transport equation. For the case of isotropic scattering, the normal mode technique is applicable, and exact, closed-form solutions can be determined.

Evaluating the results implied by the present theory with respect to measurements is impossible. There is a current lack of pertinent experimental results for neutron distribution description in the relevant type of media.

APPENDIX A

The Function $\Lambda(\nu)$

It has been found previously that the zeroes of $\Lambda(\nu)$ for $\nu \notin (-1, +1)$ are the discrete set of normal mode indices and that they appear in pairs, $\pm\nu_j$. The number of these allowed discrete indices will be discussed. To this end, and for relationships which are useful in Section II G, there follows a brief study of the general properties of the function $\Lambda(\nu)$ as defined by Eqs. (51) and (53), i.e.,

$$\Lambda(\nu) = 1 + i\pi c\nu G(\nu)$$

$$G(\nu) = \frac{1}{2\pi i}\int_{-1}^{+1} \frac{g(u)}{u-\nu}\,du$$

In terms of the set $\{g_n\}$, as defined in Eq. (43c), $\Lambda(\nu)$ may be rewritten

$$\Lambda(\nu) = 1 - c\nu \sum_n g_n Q_n(\nu) \qquad \text{(A-1)}$$

where $Q_n(\nu)$ is a Legendre function of the second kind defined for the entire ν-plane by an extension of the Neumann formula (Ref. 2, p. 51) to include $\nu \notin (-1, +1)$, i.e.,

$$Q_n(\nu) = \frac{1}{2}\int_{-1}^{+1} \frac{P_n(u)}{\nu-u}\,du \qquad \text{(A-2)}$$

with singular integrals evaluated as the Cauchy principal value. For large ν, $\nu Q_n(\nu)$ varies as ν^{-n}. Thus, $\Lambda(\nu)$ is bounded for large ν. Furthermore, the $Q_n(\nu)$ are analytic in the ν-plane excluding $\nu \in (-1, +1)$ and, therefore, $\Lambda(\nu)$ is analytic in this same region. The contour illustrated in Fig. A-1 and the argument theorem (Ref. 6, p. 116) establishes the number of zeroes of $\Lambda(\nu)$ in the region $\nu \notin (-1, +1)$. Since the zeroes of $\Lambda(\nu)$ appear in pairs, the number of zeroes are denoted by $2J$. The argument theorem applied here yields

$$\begin{aligned} 4\pi J = {} & \text{change in arg } \Lambda^{+}(u) \text{ on } C_{+} \\ & + \text{change in arg } \Lambda^{-}(u) \text{ on } C_{-} \qquad \text{(A-3)} \end{aligned}$$

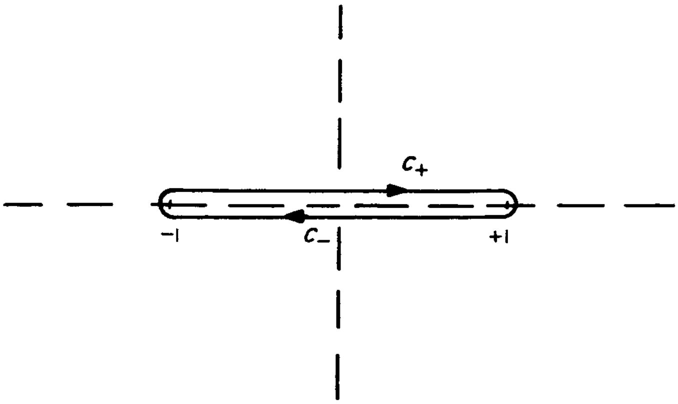

Fig. A-1. Contour in ν-plane used in determination of the number of zeroes of the function $\Lambda(\nu)$

It was assumed that $g(u)$ satisfies a Holder condition on $u \epsilon (-1, +1)$ and therefore $G(\nu)$ is a Cauchy integral. The Plemelj formulae (Ref. 5, p. 43) are applied to find the limit values $G^{\pm}(u)$. It is found that

$$G^{\pm}(u) = G(u) \pm \frac{1}{2} g(u) \qquad \text{(A-4)}$$

where $G^{+}(u)$ and $G^{-}(u)$ refer to the limit values of $G(\nu)$ as ν approaches u from above and below the real line respectively. From Eq. (A-4) the limit values

$$\Lambda^{\pm}(u) = \Lambda(u) \pm \frac{i\pi}{2} cug(u) \qquad \text{(A-5)}$$

Now, $\Lambda(u)$ with $u \epsilon (-1, +1)$ is a real function (with singularities at $u = \pm 1$), so that $\Lambda(0) = 1$ and $g(u)$ is a symmetric function. Whence, the relations

$$\arg \Lambda^{+}(u) = -\arg \Lambda^{-}(u) \qquad \text{(A-6a)}$$

$$\arg \Lambda^{+}(0) = \arg \Lambda^{-}(0) = 0 \qquad \text{(A-6b)}$$

$$\Lambda^{+}(u) = \Lambda^{-}(-u) \qquad \text{(A-6c)}$$

These results used in Eq. (A-3) yield the number of pairs of zeroes J in terms of the single angle arg $\Lambda^{+}(+1)$, i.e.,

$$J = \frac{1}{\pi} \arg \Lambda^{+}(+1) \qquad \text{(A-7)}$$

It should be noted that Eq. (A-3) contains the implicit requirement that $\Lambda^{+}(u) \neq 0$ for $u \epsilon (-1, +1)$. This assumption is not completely necessary; however, it probably applies to most cases of physical interest and its application greatly simplifies these considerations.

A sufficient condition for $J = 1$ will be developed in the case that $g(u)$ is an N-degree polynomial in u, i.e.,

$$g(u) = \sum_{n=0}^{N} g_n P_n(u) \qquad \text{(A-8)}$$

Note that the Legendre functions, $Q_n(\nu)$, can be expressed as

$$Q_n(\nu) = P_n(\nu) Q_0(\nu) - W_{n-1}(\nu) \qquad \text{(A-9a)}$$

$$Q_0(\nu) = \text{arc tanh}\, \nu, \nu \epsilon (-1, +1)$$

$$= \text{arc tanh} \frac{1}{\nu}, \nu \notin (-1, +1) \qquad \text{(A-9b)}$$

where $W_{n-1}(\nu)$ is an even, or odd, polynomial in ν of degree $n - 1$ (Ref. 2, p. 51). In these terms $\Lambda(\nu)$ is rewritten as

$$\Lambda(\nu) = 1 - c\nu Q_0(\nu) \sum_{n=0}^{N} g_n P_n(\nu) + \sum_{n=0}^{N} g_n W_{n-1}(\nu) \qquad \text{(A-10)}$$

Also, as $u \to +1$, $Q_0(u) \to +\infty$ and $P_n(+1) = 1$. Clearly, $W_n(+1)$ is bounded, and thus, if

$$\sum_{n=0}^{N} g_n > 0 \qquad \text{(A-11)}$$

then as $u \to +1$, $\Lambda(\nu) \to -\infty$. From Eq. (A-5), in the present case,

$$\Lambda^{+}(u) = \Lambda(u) + \frac{i\pi}{2} cu \sum_{n=0}^{N} g_n P_n(u) \qquad \text{(A-12)}$$

Therefore, the following may be concluded: if Eq. (A-11) holds and, in the range $u \epsilon (0, +1)$,

$$\sum_{n=0}^{N} g_n P_n(u) > 0 \qquad \text{(A-13)}$$

then arg $\Lambda^{+}(+1) = \pi$ and we have the desired result, $J = 1$. It should be stressed that Eqs. (A-11) and (A-13) give a sufficient, not necessary, condition for the number of pairs of discrete indexed normal modes to be unity.

APPENDIX B

A Relevant Hilbert Problem

In Section II F the existence of the modal expansion coefficients $\{a(\pm\nu_j), j = 1, 2, \cdots, J, a(\nu), \nu \epsilon (-1, +1)\}$ was assumed. Moreover, the orthogonality relations are based on the whole angle range $u \epsilon (-1, +1)$ and thus only provide a means of determining expansion coefficients for the case of a boundary condition given over all angles. It is found, by reducing the problem of finding expansion coefficients to the solution of an inhomogeneous Hilbert problem, that the existence of expansion coefficients can be demonstrated, and a method prescribed for determining the value of the coefficients for problems involving all physically relevant boundary conditions. The techniques elegantly described by Muskhelishvili (Ref. 5), are followed closely.

I. REDUCTION OF TRANSPORT PROBLEM TO AN INHOMOGENEOUS HILBERT PROBLEM

Transport problem boundary conditions will be encountered, in general, of the form

$$\phi(u) = \sum_{j=1}^{J} a(+\nu_j)\phi(+\nu_j, u) + \sum_{j=1}^{J} a(-\nu_j, u)\phi(-\nu_j, u) + \int_\alpha^\beta a(\nu)\phi(\nu, u)\,d\nu \text{ for } u \epsilon (\alpha, \beta) \tag{B-1}$$

where $-1 \leqq \alpha < \beta \leqq +1$. If it were possible by some method to determine the set of discrete indexed coefficients $\{a(\pm\nu_j), j = 1, 2, \cdots, J\}$, and define

$$\phi'(u) = \phi(u) - \sum_{j=1}^{J} a(+\nu_j)\phi(+\nu_j, u) - \sum_{j=1}^{J} a(-\nu_j)\phi(-\nu_j, u) \tag{B-2}$$

then there would be an integral equation for $a(\nu)$, $\nu \epsilon (\alpha, \beta)$, i.e.,

$$\int_\alpha^\beta a(\nu)\phi(\nu, u)\,d\nu = \phi'(u) \text{ for } u \epsilon (\alpha, \beta) \tag{B-3}$$

Using the derived form of $\phi(\nu, u)$ Eq. (50)

$$\Lambda(u)a(u) + \frac{c}{2}g(u)\int_\alpha^\beta \frac{\nu a(\nu)}{\nu - u}\,d\nu = \phi'(u) \text{ for } u \epsilon (\alpha, \beta) \tag{B-4}$$

From Eq. (A-5)

$$\frac{c}{2}ug(u) = \frac{1}{2\pi i}[\Lambda^+(u) - \Lambda^-(u)] \tag{B-5a}$$

$$\Lambda(u) = \frac{1}{2}[\Lambda^+(u) + \Lambda^-(u)] \tag{B-5b}$$

and therefore Eq. (B-4) may be rewritten as

$$\frac{1}{2}[\Lambda^+(u) + \Lambda^-(u)]ua(u) + [\Lambda^+(u) - \Lambda^-(u)]A(u) = u\phi'(u) \tag{B-6a}$$

$$A(u) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{\nu a(\nu)}{\nu - u}\,d\nu \text{ for } u \epsilon (\alpha, \beta) \tag{B-6b}$$

There is assurance that $A(u)$, as defined in Eq. (B-6b), exists if $a(u)$ satisfies a Holder condition on $u \epsilon (\alpha, \beta)$. For the moment, assume that this condition is fulfilled and define the Cauchy integral, $A(\nu)$, over the entire ν-plane,

$$A(\nu) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{ua(u)}{u - \nu}\,du \tag{B-7}$$

The Plemelj formulae yield the limit relations on the line $u \epsilon (\alpha, \beta)$,

$$A^+(u) - A^-(u) = ua(u) \tag{B-8a}$$

$$A^+(u) + A^-(u) = 2A(u) \tag{B-8b}$$

The results of Eqs. (B-5) and (B-8) applied to Eq. (B-6) give the alternate form

$$\Lambda^+(u)\,A^+(u) - \Lambda^-(u)\,A^-(u) = u\,\phi'(u) \text{ for } u \in (\alpha, \beta) \tag{B-9}$$

It was assumed that $\Lambda^{\pm}(u) \neq 0$ for $u \in (\alpha, \beta)$. With this restriction Eq. (B-9) can be easily transformed to the form of a boundary condition for an inhomogeneous Hilbert problem on an arc (Ref. 5, chapter 10). The problem of determining $a(u)$ restated in these terms is: Find the sectionally analytic function, $A(\nu)$, vanishing at infinity, with boundary condition on the line $u \in (\alpha, \beta)$,

$$A^+(u) = \frac{\Lambda^-(u)}{\Lambda^+(u)} A^-(u) + \frac{u\phi'(u)}{\Lambda^+(u)} \tag{B-10}$$

Note that the assumptions on $g(u)$ and $\Lambda^{\pm}(u)$ imply that $\Lambda^-(u)/\Lambda^+(u)$ is a function satisfying a Holder condition and not vanishing on $u \in (\alpha, \beta)$, and, if it is assumed that the angle boundary condition, $\phi(u)$, satisfies a Holder condition and $a(\pm\nu_i)$ exist, then $u\phi'(u)/\Lambda^+(u)$ satisfies a Holder condition on $u \in (\alpha, \beta)$.

Summarizing will help clarify the procedure. If it is assumed (what is to be proved) that $a(u)$ satisfies a Holder condition on $u \in (\alpha, \beta)$, then the integral $A(\nu)$, defined by Eq. (B-7), is of the Cauchy type. Now, Cauchy integrals are sectionally analytic functions with boundary the line of integration. Specifically, if (α, β) is the line of integration:

1. $A(\nu)$ is analytic in ν-plane excluding (α, β).
2. $A(\nu)$ approaches well-defined limits as $u \in (\alpha, \beta)$ is approached from either side of (α, β) with possible exception of the end points, α and/or β.
3. Near the end points, $A(\nu)$ satisfies the conditions

$$|A(\nu)| \leq \frac{A}{|\nu - \alpha|^a} \text{ as } \nu \to \alpha$$

$$|A(\nu)| \leq \frac{B}{|\nu - \beta|^b} \text{ as } \nu \to \beta$$

where a, b, A and B are real constants, and $a < 1$ and $b < 1$.

Moreover, $A(\nu)$ vanishes as $|\nu| \to \infty$. The integral equation for $a(u)$ has been transformed into the boundary condition Eq. (B-10) which is the form of an inhomogeneous Hilbert problem boundary condition. Thus, the original transport problem has been reduced to an inhomogeneous Hilbert problem. If a solution $A(\nu)$ which introduces no physical ambiguity can be found, then the assumption of the existence of $a(u)$, $u \in (\alpha, \beta)$, will be substantiated.

II. SOLUTION OF THE HILBERT PROBLEM

In terms of

$$\theta(u) = \arg \Lambda^+(u), \quad \Lambda^-(u)/\Lambda^+(u) = \exp[-2i\theta(u)]$$

the Hilbert problem boundary condition [cf., Eq. (B-10)] becomes

$$A^+(u) = \exp[-2i\,\theta(u)]\,A^-(u) + \frac{u\phi'(u)}{\Lambda^+(u)} \text{ for } u \in (\alpha, \beta) \tag{B-11}$$

Since $A(\nu)$ must also vanish $|\nu| \to \infty$, the solution (Ref. 5) is

$$A(\nu) = \frac{H(\nu)}{2\pi i} \int_\alpha^\beta \frac{u\phi'(u)}{(u - \nu)\,\Lambda^+(u)\,H^+(u)}\, du \tag{B-12}$$

where $H(\nu)$ is the fundamental solution of the associated homogeneous Hilbert problem and is given by

$$H(\nu) = (\alpha - \nu)^{-\theta(\alpha)/\pi} (\beta - \nu)^{\theta(\beta)/\pi} e^{\Theta(\nu)} \tag{B-13}$$

The Cauchy integral $\Theta(\nu)$ is defined by

$$\Theta(\nu) = -\frac{1}{\pi} \int_\alpha^\beta \frac{\theta(u)}{u - \nu}\, du \tag{B-14}$$

Providing $\kappa = \theta(\beta)/\pi - \theta(\alpha)/\pi$ is a positive integer, there are the κ additional requirements

$$\int_\alpha^\beta \frac{u^{n+1}\phi'(u)}{\Lambda^+(u)\,H^+(u)}\, du = 0 \qquad \text{for } n = 0, 1, \cdots, \kappa - 1 \tag{B-15}$$

These additional requirements are a necessary feature of the solution. It should be recalled that the function $\phi'(u)$, $u \in (\alpha, \beta)$, is not completely specified, i.e., the discrete indexed expansion coefficients, $a(\pm\nu_j)$, in Eq. (B-2) are, as yet, unknown. For the general problems considered later, it will be demonstrated that, in each case, the κ requirements are necessary and sufficient for the complete specification of all discrete and continuum expansion coefficients.

III. APPLICATION OF THE HILBERT PROBLEM SOLUTION

Plane symmetry transport problems fall into two general categories:

1. Infinite media problems with full-angle-range boundary conditions (such as the Green's function solved in Section II G).
2. Half-space media problems with half-angle-range boundary conditions (such as albedo or Milne type problems).

Combinations of the solutions of these type problems lead to the solution of cases with finite media (slabs). For full-range boundary conditions, the orthogonality of the normal modes provides a direct method for determining expansion coefficients. The solution of the Hilbert problem in these cases demonstrates the existence of the coefficients and thus partially supports the completeness hypothesis. For half-range boundary problems, there are no apparent orthogonality properties of the normal modes. In these cases, the solution of the Hilbert problem not only provides proof of existence, but also gives a well-defined prescription for the determination of expansion coefficients. The application of the Hilbert problem solution will now be outlined to the categories of full-range and half-range boundary conditions.

In the case of an infinite medium, full-range boundary condition problem, a source condition is usually given at some position, which is chosen to be designated $x = 0$. For $c < 1$, it follows that $F(x,u)$ should vanish as $|x| \to \infty$. Thus, the general form of solution is that given in Eq. (69). The source condition can be formulated as

$$\phi(u) = \sum_{j=1}^{J} a(+\nu_j)\,\phi(+\nu_j,u) + \sum_{j=1}^{J} a(-\nu_j)\,\phi(-\nu_j,u) + \int_{-1}^{+1} a(\nu)\,\phi(\nu,u)\,d\nu \qquad \text{for } u \in (-1,+1) \tag{B-16}$$

Instead of using the obviously indicated orthogonality properties, consider the coefficient evaluation by the route prescribed in the Hilbert problem solution. Note that $\alpha = -1$ and $\beta = +1$. From Eqs. (A-6) and (A-7), the results are $\theta(-1) = -J\pi$ and $\theta(+1) = J\pi$. Therefore, in this case, $\kappa = 2J$ and there are $2J$ requirements of the form of Eq. (B-15). Specifically,

$$\int_{-1}^{+1} \frac{u^{n+1}\phi'(u)}{\Lambda^{+}(u)\,H^{+}(u)}\,du = 0 \qquad \text{for } n = 0, 1, \cdots, 2J-1 \tag{B-17}$$

Equation (B-17) provides a sufficient number of equations to find the discrete indexed expansion coefficients, $a(\pm\nu_j)$, $j = 1, 2, \cdots, J$. The fundamental solution $H(\nu)$, [cf., Eq. (B-13)], is given by

$$H(\nu) = (-1-\nu)^{J}\,(1-\nu)^{J} e^{\Theta(\nu)} \tag{B-18a}$$

$$\Theta(\nu) = -\frac{1}{\pi}\int_{-1}^{+1} \frac{\theta(u)}{u-\nu}\,du \tag{B-18b}$$

Thus, $A(\nu)$ is determined [by Eq. (B-12)] and $a(u)$ for $u \in (-1, +1)$ can be found from the limit relation [cf., Eq. (B-8a)]

$$ua(u) = A^{+}(u) - A^{-}(u) \qquad \text{for } u \in (-1,+1) \tag{B-19}$$

Since the problem has been completely and unambiguously solved, it is clear that the supposition that $a(u)$ satisfies a Holder condition is substantiated and the existence of the expansion coefficients has been demonstrated.

For half-space media, consider two types of problems. An "albedo problem" is described by a boundary condition at the medium surface ($x = 0$ with medium occupying $x > 0$) specified for $u \in (0, +1)$ and the condition that $F(x, u)$ vanish as $x \rightarrow \infty$. A "Milne problem" is described by a similar boundary condition at $x = 0$, but with $F(x, u) \rightarrow \phi(-\nu, u) \exp(x/\nu)$ with $\nu = \nu_j$, $j = 1, 2, \cdots, J$, or $\nu \in (0, +1)$, as $x \rightarrow \infty$. These problems have been specified as boundary conditions on the half-range $u \in (0, +1)$. With obvious modifications, the procedure is easily applied to half-space media occupying $x < 0$ and boundary conditions on $u \in (-1, 0)$. With the half-space occupying $x > 0$, the general solution of an albedo problem is

$$F(x, u) = \sum_{j=1}^{J} a(+\nu_j)\phi(+\nu_j, u)\exp\left(\frac{-x}{\nu_j}\right) + \int_0^{+1} a(\nu)\phi(\nu, u)\exp\left(\frac{-x}{\nu}\right)d\nu \text{ for } x > 0 \tag{B-20}$$

and for a Milne problem,

$$F(x, u) = A\phi(-\nu, u)\exp\left(\frac{x}{\nu}\right) + \sum_{j=1}^{J} a(+\nu_j)\phi(+\nu_j, u)\exp\left(\frac{-x}{\nu_j}\right) + \int_0^{+1} a(\nu)\phi(\nu, u)\exp\left(\frac{-x}{\nu}\right)d\nu \text{ for } x > 0 \tag{B-21}$$

In both cases, the boundary condition at $x = 0$ can be expressed in the form of Eq. (B-1), i.e.,

$$\phi(u) = \sum_{j=1}^{J} a(+\nu_j)\phi(+\nu_j, u) + \int_0^{+1} a(\nu)\phi(\nu, u)\,d\nu \quad \text{for } u \in (0, +1) \tag{B-22}$$

Now, $\alpha = 0$ and $\beta = +1$ and, from Eqs. (A-6) and (A-7), $\theta(0) = 0$ and $\theta(+1) = J\pi$. Thus, $\kappa = J$ and we have the J requirements

$$\int_0^{+1} \frac{u^{n+1}\phi'(u)}{\Lambda^+(u)H^+(u)}\,du = 0 \qquad \text{for } n = 0, 1, \cdots, J-1 \tag{B-23}$$

These are sufficient to determine the discrete indexed coefficients, $a(+\nu_j)$, $j = 1, 2, \cdots, J$. The fundamental solution takes the form

$$H(\nu) = (1-\nu)^J \exp(\Theta(\nu)) \tag{B-24a}$$

$$\Theta(\nu) = -\frac{1}{\pi}\int_0^{+1} \frac{\theta(u)}{u-\nu}\,du \tag{B-24b}$$

The Hilbert problem solution, $A(\nu)$, $\nu \in (0, +1)$, and the continuum expansion coefficients, $a(u)$, $u \in (0, +1)$, are found as in the case of a full-range boundary problem. Again, substantiation is found for the supposition of the existence of the relevant members of $\{a(\nu)\}$. Moreover, a prescription is found for calculating the expansion coefficients when the use of orthogonality conditions is impossible.

NOMENCLATURE

- a modal expansion coefficients of F
- A Cauchy integral with density a
- A_s cross-sectional area of scatterer material per lattice cell
- A_v cross-sectional area of vacuum per lattice cell
- A_{lmn} triple Legendre polynomial integral
- c neutron scattering probability
- D diffusion coefficient
- f neutron scattering distribution
- f_n Legendre polynomial expansion coefficients of f
- F neutron collision density
- F_n Legendre polynomial expansion coefficients of F
- F^j_n angle-space-moments of F
- g Jacobian of angle-variable change, i.e. $g(u) = |d\mu/du|$
- g_n Legendre polynomial expansion coefficients of g

NOMENCLATURE (Cont'd)

G Cauchy integral with density g

H fundamental solution of Hilbert problem

I orthogonality integral value

J half of the number of discrete indexed eigenmodes

L_F neutron diffusion length based on an F-expansion

L_ψ neutron diffusion length based on a ψ-expansion

L_s asymptotic diffusion length in scatterer material

L_x diffusion length with respect to x-direction

L_y diffusion length with respect to y-direction

P_n Legendre polynomial

Q_n Legendre functions of the second kind

R ratio of vacuum channel radius to λ_s

S neutron source distribution

S_n Legendre polynomial expansion coefficients of S

S_n^j angle-space-moments of S

t time variable

T_s slab thickness of scatterer material

T_v slab thickness of vacuum material

u angle variable defined by $u = \mu\lambda(\mu)/\lambda(1)$

v neutron speed

V ratio of vacuum volume to scatterer volume, i.e. $V = A_v/A_s$

W_n regular part of Legendre functions of the second kind

x position variable

$\langle x^2 \rangle$ mean value of x^2

y position variable perpendicular to x-direction

δ Dirac δ-distribution

δ_{mn} Kronecker δ-distribution

ϵ included in the interval

η direction cosine of neutron travel relative to y-direction

θ argument of the function Λ

Θ Cauchy integral with density θ

κ index of the Hilbert problem

λ neutron mean free path

λ_h simply homogenized medium mean free path

λ_n Legendre polynomial expansion coefficients of λ

λ_x mean free path with respect to x-direction

λ_y mean free path with respect to y-direction

Λ coefficient of δ-distribution in angle eigenmode

μ direction cosine of neutron travel relative to x-direction

ν angular flux eigenvalues

ν_j discrete indexed angular flux eigenvalues

σ total cross section

σ' absorption cross section

σ_n Legendre polynomial expansion coefficients of σ

σ_x total cross section with respect to x-direction

σ_y total cross section with respect to y-direction

ϕ angular flux eigenmodes (and angular boundary conditions)

ϕ_n Legendre polynomial expansion coefficients of ϕ

ψ neutron flux distribution

ψ_n Legendre polynomial expansion coefficients of ψ

ψ_n^j angle-space-moments of ψ

$\mathbf{\Omega}$ unit vector in direction of neutron travel

REFERENCES

1. Behrens, D. J., *Proceedings of the Physical Society*, (London), Section A, Vol. 62, p. 607, 1949.
2. Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea Publishing Co., New York, 1955.
3. Case, K. M., *Annals of Physics*, (New York), Vol. 9, pp. 1–23, 1960.
4. Schwartz, L., *Theorie des Distributions*, Vol. 1 and 2, Hermann et Cie, Paris, 1951.
5. Muskhelishvili, N. I., *Singular Integral Equations*, P. Noordhoff, Ltd., Groningen, Nederland, 1953.
6. Titchmarsh, E. C., *The Theory of Functions*, 2nd ed., Oxford University Press, London, 1939.