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THE DYADIC GREEN'S FUNCTION FOR AN INFINITE MOVING MEDIUM

by

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ABSTRACT

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The Dyadic Green's Function pertaining to the electromagnetic field in an infinite moving medium is derived. The derivation is based on Minkowski's theory and the method of Fourier analysis is used. Also, a second derivation of the same result is given, which clearly shows the connection between the Green's functions for a moving and a stationary medium.

CONTENTS

THE DYADIC GREEN'S FUNCTION FOR AN INFINITE MOVING MEDIUM

INTRODUCTION

In this report the Dyadic Green's Function pertaining to the electromagnetic field in a moving medium is found. The medium is assumed to be of infinite extent in all directions and to be isotropic and homogeneous. It moves with a constant velocity, which is assumed to be much smaller than the velocity of light. The wave equation for the electric field is derived for harmonic time dependence and is solved using an operational method the same as has been used by Bunkin[1].

The problem of the electrodynamics of moving media was first solved exactly by Minkowski[2] in 1908, and an excellent discussion of his work has been given by Sommerfeld[3]. More recently, a review of Minkowski's theory and a discussion of several current writings on this subject have been given by Tai[4]. In regard to the construction of the dyadic function, two other works worth mentioning are those by Arbel[5] and Wu[6] on the related problem of radiation in anisotropic media.

DERIVATION OF THE GREEN'S FUNCTION

Consider a homogeneous and isotropic medium of infinite extent in all directions. Assume the medium moves with a constant linear velocity, $\overline{v}$, with respect to a fixed coordinate system. We consider only the case where the velocity $|\overline{v}|$ is much less than the speed of light c, so that $(|\overline{v}|/c)^2 << 1$. In this case the differential equations governing the electromagnetic fields in the medium are [7, 8]

(1) $$\nabla \times \overline{E} = -\frac{\partial}{\partial t}\left[\mu\overline{H} - (\epsilon\mu - \epsilon_0\mu_0)\,\overline{v}\times\overline{E}\right],$$

and

(2) $$\nabla \times \overline{H} = \frac{\partial}{\partial t}\left[\epsilon\overline{E} + (\epsilon\mu - \epsilon_0\mu_0)\,\overline{v}\times\overline{H}\right] + \overline{J},$$

where

$\overline{E}, \overline{H}$ = the electric and magnetic fields,

ϵ, μ = the permittivity and permeability of the medium when at rest,

ϵ_o, μ_o = the permittivity and permeability of free-space, and

$\overline{J}$ = the source current density, assumed to be known.

In Eqs. (1) and (2), all quantities are measured in the fixed coordinate system. (MKS units are used.)

Equations (1) and (2) may be written

(3) $$\left(\nabla - \overline{V} \frac{\partial}{\partial t}\right) \times \overline{E} = - \frac{\partial}{\partial t} (\mu \overline{H}),$$

and

(4) $$\left(\nabla - \overline{V} \frac{\partial}{\partial t}\right) \times \overline{H} = \frac{\partial}{\partial t} (\epsilon \overline{E}) + \overline{J},$$

where

(5) $$\overline{V} = (\epsilon\mu - \epsilon_o\mu_o) \overline{v} ;$$

and then Eqs. (3) and (4) may be combined to yield the following wave equation for $\overline{E}$:

(6) $$\nabla \times \nabla \times \overline{E} - \overline{V} \times \nabla \times \frac{\partial \overline{E}}{\partial t} - \nabla \times \left(\overline{V} \times \frac{\partial \overline{E}}{\partial t}\right) + \overline{V} \times \left(\overline{V} \times \frac{\partial^2 \overline{E}}{\partial t^2}\right)$$
$$+ \mu \epsilon \frac{\partial^2 \overline{E}}{\partial t^2} = - \mu \frac{\partial \overline{J}}{\partial t} .$$

It will be assumed that all field quantities have time dependence $e^{+j\omega t}$, which reduces Eq. (6) to

(7) $$\nabla \times \nabla \times \overline{E} - j\omega \overline{V} \times \nabla \times \overline{E} - j\omega \nabla \times (\overline{V} \times \overline{E}) - \omega^2 \overline{V} \times (\overline{V} \times \overline{E}) - k^2 \overline{E} = -j\omega\mu \overline{J},$$

where

(8) $$k^2 = \omega^2 \mu \epsilon .$$

Equation (7) may be solved for $\overline{E}$ using a method employed by Bunkin[9], and subsequently by Chow[10], for the case of radiation in an anisotropic medium. In a rectangular cartesian coordinate system with axes x_1, x_2, x_3, Eq. (7) may be written

(9) $$\sum_{j=1}^{3} q_{ij} E_j = -j\omega\mu J_i, \qquad i = 1, 2, 3,$$

where q_{ij} is the differential operator;

(10) $$q_{ij} = \frac{\partial^2}{\partial x_i \, \partial x_j} - j\omega \left(V_i \frac{\partial}{\partial x_j} + V_j \frac{\partial}{\partial x_i} \right) - \omega^2 V_i V_j - \delta_{ij} \nabla^2 + 2j\omega \, \delta_{ij} \overline{V} \cdot \nabla + \omega^2 V^2 \delta_{ij} - k^2 \delta_{ij} ,$$

with $V^2 = V_1^2 + V_2^2 + V_3^2$ and δ_{ij}, the Kronecker delta, defined by

(11) $$\delta_{ij} = \begin{cases} 1 : & i = j \\ 0 : & i \neq j \end{cases} .$$

A solution for Eq. (9) may be obtained by putting

(12) $$E_j = - j\omega\mu \sum_{k=1}^{3} \iiint_{\tau'} T_{jk}(\overline{R} \mid \overline{R}') \, J_k(\overline{R}') \, d\tau' ,$$

where τ' indicates the volume occupied by the source $\overline{J}$. On substituting Eq. (12) into Eq. (9), and making use of the relation

$$(13) \qquad J_i = \sum_{k=1}^{3} \iiint_{\tau'} \delta_{ik}\, \delta(\overline{R} \mid \overline{R}')\, J_k(\overline{R}')\, d\tau' \quad ,$$

where $\delta(\overline{R} \mid \overline{R}')$ is the Dirac Delta Function, there results

$$(14) \qquad \sum_{j=1}^{3} \sum_{k=1}^{3} \iiint_{\tau'} q_{ij}\, T_{jk}(\overline{R} \mid \overline{R}')\, J_k(\overline{R}') d\tau'$$

$$= \sum_{k=1}^{3} \iiint_{\tau'} \delta_{ik}\, \delta(\overline{R} \mid \overline{R}')\, J_k(\overline{R}')\, d\tau' \quad .$$

Since Eq. (14) must hold for arbitrary $\overline{J}$, it follows that

$$(15) \qquad \sum_{j=1}^{3} q_{ij}\, T_{jk}(\overline{R} \mid \overline{R}') = \delta_{ik}\, \delta(\overline{R} \mid \overline{R}') \; .$$

Equation (15) may be solved by setting

$$(16) \qquad T_{jk}(\overline{R} \mid \overline{R}') = D_{jk}\, G(\overline{R} \mid \overline{R}') \; ,$$

where D_{jk} is the differential operator defined by

$$(17) \qquad D_{jk} = \text{cofactor of } q_{kj} \; ,$$

and $G(\overline{R} \mid \overline{R}')$ is a scalar function. Then

$$(18) \qquad \sum_{j=1}^{3} q_{ij}\, D_{jk} = D\, \delta_{ik} \; ,$$

with D the determinant of the matrix $\{q_{ij}\}$,

(19) $$D = \det \{q_{ij}\} ,$$

and the scalar function $G(\overline{R}\,|\overline{R}')$ must satisfy

(20) $$D\, G(\overline{R}\,|\overline{R}') = \delta(\overline{R}\,|\overline{R}') .$$

A function satisfying Eq. (20) may be constructed as follows: First define

(21) $$\overline{p} = p_1 \hat{x}_1 + p_2 \hat{x}_2 + p_3 \hat{x}_3 .$$

and

(22) $$\overline{r} = \overline{R} - \overline{R}' .$$

From Eq. (10), it may be seen that

(23) $$q_{ij} \exp(-j\, \overline{p} \cdot \overline{r}) = P_{ij} \exp(-j\, \overline{p} \cdot \overline{r}) ,$$

where

(24) $$P_{ij} = - p_i p_j + \delta_{ij}\, \overline{p} \cdot \overline{p} - \omega(V_i p_j + V_j p_i) + 2\omega\delta_{ij}\, \overline{V} \cdot \overline{p} - \omega^2 V_i V_j + \omega^2 \delta_{ij}\, \overline{V} \cdot \overline{V} - k^2 \delta_{ij} .$$

Hence

(25) $$D \exp(-j\, \overline{p} \cdot \overline{r}) = \det \{P_{ij}\} \exp(-j\, \overline{p} \cdot \overline{R}) ,$$

and therefore a solution for $G(\overline{R}\,|\overline{R}')$ is given by

$$(26) \qquad G(\bar{R} \,|\, \bar{R}') = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\exp(-j\,\bar{p}\cdot\bar{r})}{\det\{P_{ij}\}}\, dp_1\, dp_2\, dp_3 \quad .$$

Next, it will be supposed that $\bar{V}$ lies entirely in the $\hat{x}_3$-direction. That is,

$$(27) \qquad \bar{V} = V_3\, \hat{x}_3 \quad .$$

Since the orientation of the coordinate system is arbitrary up to this point, this assumption involves no loss of generality. With this simplification, it is found from Eq. (24), after considerable algebra,

$$(28) \qquad \det\{P_{ij}\} = -k^2 (p_3 + \omega V_3 + \sqrt{k^2 - p_1^2 - p_2^2}\,)^2 \; (p_3 + \omega V_3 - \sqrt{k^2 - p_1^2 - p_2^2}\,)^2 \; .$$

With Eq. (28) substituted in Eq. (26), the integration on p_3 is easily done by Cauchy's Residue Theorem. For $x_3 > x_3'$, the contour of integration may be closed on an infinite semi-circle in the lower-half p_3-plane. For $x_3 < x_3'$, it may be closed in the upper-half p_3-plane. Since the time convention is $e^{+j\omega t}$, the (double-order) pole at $p_3 = -\omega V_3 + \sqrt{k^2 - p_1^2 - p_2^2}$ may be considered as lying slightly below the real p_3-axis, corresponding to a small amount of conduction loss in the dielectric medium. Similarly, the pole at $p_3 = -\omega V_3 - \sqrt{k^2 - p_1^2 - p_2^2}$ is considered as lying slightly above the real p_3-axis. The residue of the pole at $p_3 = -\omega V_3 + \sqrt{k^2 - p_1^2 - p_2^2}$ is

$$(29) \qquad \mathrm{Res}(-\omega V_3 + \sqrt{k^2 - p_1^2 - p_2^2}\,) = +\frac{1}{k^2} \left[\frac{+j(x_3 - x_3')}{4(k^2 - p_1^2 - p_2^2)} + \frac{1}{4(k^2 - p_1^2 - p_2^2)^{3/2}} \right]$$

$$e^{-j(x_3 - x_3')(-\omega V_3 + \sqrt{k^2 - p_1^2 - p_2^2}\,)} \; e^{-jp_2 (x_2 - x_2')} \; e^{-jp_1 (x_1 - x_1')} ,$$

so that for $x_3 > x_3'$,

(30) $$G(\overline{R}\,|\overline{R}') = \frac{e^{j\omega V_3 (x_3 - x_3')}}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{1}{k^2}\left[\frac{(x_3 - x_3')}{4(k^2 - p_1^2 - p_2^2)} - \frac{j}{4(k^2 - p_1^2 - p_2^2)^{3/2}}\right]$$

$$e^{-j(x_3 - x_3')\sqrt{k^2 - p_1^2 - p_2^2}}\; e^{-jp_2 (x_2 - x_2')}\; e^{-jp_1 (x_1 - x_1')}\, dp_1\, dp_2 \;.$$

The residue of the pole at $p_3 = -\omega V_3 - \sqrt{k^2 - p_1^2 - p_2^2}$ is

(31) $$\mathrm{Res}(-\omega V_3 - \sqrt{k^2 - p_1^2 - p_2^2}) = + \frac{1}{k^2}\left[\frac{+j(x_3 - x_3')}{4(k^2 - p_1^2 - p_2^2)} + \frac{-1}{4(k^2 - p_1^2 - p_2^2)^{3/2}}\right]$$

$$e^{-j(x_3 - x_3')(-\omega V_3 - \sqrt{k^2 - p_1^2 - p_2^2})}\; e^{-jp_2 (x_2 - x_2')}\; e^{-jp_1 (x_1 - x_1')} \;,$$

and thus for $x_3 < x_3'$,

(32) $$G(\overline{R}\,|\overline{R}') = \frac{e^{j\omega V_3 (x_3 - x_3')}}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{1}{k^2}\left[\frac{-(x_3 - x_3')}{4(k^2 - p_1^2 - p_2^2)} - \frac{j}{4(k^2 - p_1^2 - p_2^2)^{3/2}}\right]$$

$$e^{+j(x_3 - x_3')\sqrt{k^2 - p_1^2 - p_2^2}}\; e^{-jp_2 (x_2 - x_2')}\; e^{-jp_1 (x_1 - x_1')}\, dp_1\, dp_2 \;.$$

Combining Eqs. (30) and (32) yields for any x_3, x_3',

(33) $$G(\overline{R}\,|\overline{R}') = \frac{e^{+j\omega V_3 (x_3 - x_3')}}{4(2\pi)^2\, k^2} \iint_{-\infty}^{\infty} \left[\frac{|x_3 - x_3'|}{k^2 - p_1^2 - p_2^2} - \frac{j}{(k^2 - p_1^2 - p_2^2)^{3/2}}\right]$$

$$e^{-j|x_3 - x_3'|\sqrt{k^2 - p_1^2 - p_2^2}}\; e^{-jp_2 (x_2 - x_2')}\; e^{-jp_1 (x_1 - x_1')}\, dp_1\, dp_2 \,,$$

which may be written

(34) $$G(\overline{R}\,|\overline{R}') = \frac{j e^{+j\omega V_3 (x_3 - x_3')}}{4(2\pi)^2\, k^3} \frac{\partial}{\partial k} \iint_{-\infty}^{\infty} \frac{e^{-j|x_3 - x_3'|\sqrt{k^2 - p_1^2 - p_2^2}}}{\sqrt{k^2 - p_1^2 - p_2^2}}$$

$$e^{-jp_2 (x_2 - x_2')}\, e^{-jp_1 (x_1 - x_1')}\, dp_1\, dp_2 \;.$$

Next, the change of variables

(35) $$x_1 - x_1' = r\cos\theta ,$$

(36) $$x_2 - x_2' = r\sin\theta ,$$

(37) $$p_1 = p\cos\alpha ,$$

(38) $$p_2 = p\sin\alpha , \quad \text{and}$$

(39) $$dp_1\, dp_2 = p\,dp\, d\alpha$$

is made, giving

(40) $$G(\overline{R}\,|\overline{R}') = \frac{je^{+j\omega V_3 (x_3 - x_3')}}{4(2\pi)^2\, k^3} \frac{\partial}{\partial k} \int_{p=0}^{\infty} \int_{\alpha=0}^{2\pi} \frac{e^{-j|x_3 - x_3'|\sqrt{k^2 - p^2}}}{\sqrt{k^2 - p^2}}\, e^{-jpr\cos(\alpha - \theta)}\, p\, dp\, d\alpha \ .$$

The integral on α is easily done[11], with the result

(41) $$G(\overline{R}\,|\overline{R}') = \frac{je^{j\omega V_3 (x_3 - x_3')}}{8\pi k^3} \frac{\partial}{\partial k} \int_0^{\infty} \frac{e^{-j|x_3 - x_3'|\sqrt{k^2 - p^2}}}{\sqrt{k^2 - p^2}}\, J_o(pr)\, p\, dp \ ;$$

or equivalently

(42) $$G(\overline{R}\,|\overline{R}') = - \frac{e^{+j\omega V_3 (x_3 - x_3')}}{8\pi k^3} \frac{\partial}{\partial k} \int_0^{\infty} \frac{e^{-|x_3 - x_3'|\sqrt{p^2 - k^2}}}{\sqrt{p^2 - k^2}}\, J_o(pr)\, p\, dp \ .$$

The integral in Eq. (42) is Sommerfeld's integral[12]. Thus Eq. (42) yields

$$(43) \qquad G(\bar{R} \mid \bar{R}') = - \frac{e^{+j\omega V_3 (x_3 - x_3')}}{8\pi k^3} \frac{\partial}{\partial k} \left[\frac{e^{-jkR}}{R} \right] ,$$

where

$$(44) \qquad R = \sqrt{r^2 + (x_3 - x_3')^2} .$$

Performing the differentiation finally gives, for $G(\bar{R} \mid \bar{R}')$,

$$(45) \qquad G(\bar{R} \mid \bar{R}') = \frac{j}{8\pi k^3} e^{+j\omega V_3 (x_3 - x_3')} e^{-jkR} .$$

From $G(\bar{R} \mid \bar{R}')$, the quantities $T_{jk}(\bar{R} \mid \bar{R}')$ in Eq. (12) may now be computed. From Eqs. (17) and (10), it is found that for the case where $V_1 = V_2 = 0$,

(46)

$$[D_{jk}] = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} + k^2 & \frac{\partial^2}{\partial x_1 \partial x_2} & \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_3} - j\omega V_3\right) \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \frac{\partial^2}{\partial x_2^2} + k^2 & \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_3} - j\omega V_3\right) \\ \left(\frac{\partial}{\partial x_3} - j\omega V_3\right)\frac{\partial}{\partial x_1} & \left(\frac{\partial}{\partial x_3} - j\omega V_3\right)\frac{\partial}{\partial x_2} & \left(\frac{\partial}{\partial x_3} - j\omega V_3\right)^2 + k^2 \end{bmatrix} .$$

$$\cdot \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \left(\frac{\partial}{\partial x_3} - j\omega V_3\right)^2 + k^2 \right] .$$

Thus, Eq. (16) gives

$$(47) \qquad T_{jk}(\bar{R} \mid \bar{R}') = \frac{e^{+j\omega V_3 (x_3 - x_3')}}{4\pi k^2} \left[\frac{\partial^2}{\partial x_j \partial x_k} + k^2 \delta_{jk} \right] \frac{e^{-jkR}}{R} ,$$

or, from Eq. (5),

$$(48) \qquad T_{jk}(\overline{R}\,|\overline{R}') = \frac{e^{+j\omega(\epsilon\mu - \epsilon_o\mu_o)\, v_3\,(x_3 - x_3')}}{4\pi\ k^2} \left[\frac{\partial^2}{\partial x_j\, \partial x_k} + k^2\,\delta_{jk}\right] \frac{e^{-jkR}}{R} \ .$$

Notice that with $v_3' = 0$, Eq. (48) yields the components of the free-space dyadic Green's function (with ϵ, μ replacing ϵ_o, μ_o). The effect of the velocity is to change the phase constant in the direction of the velocity. For example, if $\epsilon\mu > \epsilon_o\mu_o$ and $v_3 > 0$, the phase constant in the region $x_3 > x_3'$ is decreased by an amount $\omega(\epsilon\mu - \epsilon_o\mu_o)v_3$; therefore the wavelength and the (phase) velocity of propagation are both increased correspondingly.

In the derivation of Eq. (48), it was assumed that the velocity lies entirely in the x_3-direction. It is easy to surmise from the form of Eq. (48), however, that in the general situation where $\overline{v}$ is oriented arbitrarily with respect to the coordinate system

$$(49) \qquad T_{jk}(\overline{R}\,|\overline{R}') = \frac{e^{+j\omega(\epsilon\mu - \epsilon_o\mu_o)\, \overline{v}\cdot(\overline{R} - \overline{R}')}}{4\pi\ k^2} \left[\frac{\partial^2}{\partial x_j\, \partial x_k} + k^2\,\delta_{jk}\right] \frac{e^{-jkR}}{R} \ .$$

That Eq. (49) is indeed correct may be verified by substitution into Eq. (15), or by subjecting Eq. (48) to a rotation of coordinates.

It is worth remarking that once the solution for T_{jk} is known, simpler "derivations" of it can be recognized. One possible alternate derivation is included in the Appendix.

CONCLUSIONS

The Dyadic Green's Function for an infinite moving medium has been found. The solution of Eq. (7) for the electric field is given by Eq. (12), where the nine components $T_{jk}(\overline{R}\,|\overline{R}')$ are given in Eq. (49). It is noted that the effect of the velocity of the moving medium is to change the phase constant in the direction of the velocity.

Bunkin's method, which is systematic and straightforward, has been used to derive the Green's Function. Of course, after the answer is known, simpler and more direct methods of deriving it can be found. One such "derivation" is given in the Appendix.

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APPENDIX

AN ALTERNATE DERIVATION OF THE GREEN'S FUNCTION

Let $\overline{\overline{T}}$ be the Dyadic Green's Function for the moving medium. (The double overbar indicates a dyadic quantity; $\overline{\overline{T}}$ has nine components T_{jk}.) $\overline{\overline{T}}$ is required to satisfy the vector differential equation

(A-1) $$(\nabla - j\omega\overline{V}) \times (\nabla - j\omega\overline{V}) \times \overline{\overline{T}} - k^2 \overline{\overline{T}} = \overline{\overline{\epsilon}}\, \delta(\overline{R}\, |\overline{R}') ,$$

where $\overline{\overline{\epsilon}}$ is the idemfactor. We write $\overline{\overline{T}}$ as the produce of a scalar function ϕ and another dyadic $\overline{\overline{\Gamma}}$:

(A-2) $$\overline{\overline{T}} = \phi\, \overline{\overline{\Gamma}} ,$$

where the scalar function is to be chosen in such a way as to simplify Eq. (A-1). From the relation

(A-3) $$(\nabla - j\omega\overline{V}) \times \phi\, \overline{\overline{\Gamma}} = \phi\, \nabla \times \overline{\overline{\Gamma}} + (\nabla\phi - j\omega\phi\overline{V}) \times \overline{\overline{\Gamma}} ,$$

it is seen that if ϕ is chosen so that

(A-4) $$\nabla\phi - j\omega\phi\overline{V} = 0,$$

then Eq. (A-3) reduces to

(A-5) $$(\nabla - j\omega\overline{V}) \times \phi\overline{\overline{\Gamma}} = \phi\nabla \times \overline{\overline{\Gamma}} ;$$

and also

(A-6) $$(\nabla - j\omega\overline{V}) \times (\nabla - j\omega\overline{V}) \times \phi\overline{\overline{\Gamma}} = \phi\nabla \times \nabla \times \overline{\overline{\Gamma}} \ .$$

Equation (A-5) is easily solved. One possible solution is

(A-7) $$\phi = e^{+j\omega \overline{V} \cdot \overline{R}} \ .$$

Thus, the substitution of Eq. (A-2) into Eq. (A-1), with ϕ given by Eq. (A-7), reduces Eq. (A-1) to

(A-8) $$e^{j\omega\overline{V} \cdot \overline{R}} \ [\nabla \times \nabla \times \overline{\overline{\Gamma}} - k^2 \overline{\overline{\Gamma}}] = \overline{\overline{\epsilon}}\, \delta(\overline{R} | \overline{R}') \ ,$$

or finally

(A-9) $$\nabla \times \nabla \times \overline{\overline{\Gamma}} - k^2 \overline{\overline{\Gamma}} = \overline{\overline{\epsilon}}\, \delta(\overline{R} | \overline{R}')\, e^{-j\omega\overline{V} \cdot \overline{R}}$$

$$= \overline{\overline{\epsilon}}\, \delta(\overline{R} | \overline{R}')\, e^{-j\omega\overline{V} \cdot \overline{R}'} \ .$$

The last equality in Eq. (A-9) follows from the properties of the Delta function.

Except for the constant factor $e^{-j\omega\overline{V} \cdot \overline{R}'}$, Eq. (A-9) is the same as the equation for the free-space Dyadic Green's Function (with k replacing the free-space propagation constant k_o). Hence the solution to Eq. (A-9) is given by

(A-10) $$\overline{\overline{\Gamma}} = \frac{e^{-j\omega\overline{V} \cdot \overline{R}'}}{4\pi k^2} \ (k^2 \overline{\overline{\epsilon}} + \nabla\nabla) \ \frac{e^{-jkR}}{R} \ ;$$

and, therefore,

$$\overline{\overline{T}} = \frac{e^{j\omega\overline{V}\cdot(\overline{R}-\overline{R}')}}{4\pi\, k^2} \left(k^2\ \overline{\overline{\epsilon}} + \nabla\nabla\right) \frac{e^{-jkR}}{R} \quad . \tag{A-11}$$