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SUBJECT: Note on the Mean Error and Standard Deviation in the Output of a Least Square Quadratic Filter - Case 20061-1

DATE: July 13, 1964

FROM: J. G. Kreer

ABSTRACT

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The statistics of the output of a "Least Squares Quadratic Filter" are discussed and expressions derived for the means and standard deviations of the estimates of position, velocity, and acceleration, including effects of correlated noise. It is shown that the errors of the mean are of the form of a "Dynamic Tracking Error" proportional to the third and fourth time derivative and to the square of the smoothing interval for the velocity and acceleration estimates and to its fourth power for the position estimates. The standard deviations are shown to vary inversely as the square root of the number of samples as would be expected and in addition the standard deviation of the velocity estimate is inversely proportional to the smoothing period and that of the acceleration to the square of the smoothing period. All of these assuming uncorrelated noise samples. Correlation introduces additional terms whose variation cannot be expressed as simply as above but depends upon the autocorrelation function of the noise.

Author

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MEMORANDUM FOR FILE

In many cases the rate at which radars generate data exceeds the rate at which the associated transmission lines or data processing equipment can accept it. Many subterfuges are used to match the two rates, ranging all the way from merely dropping all of the excessive pieces of data generated by the radar to very sophisticated statistical procedures. Which one is to be used in any particular problem will depend upon the circumstances and constraints of the problem. For example if the errors of the radar data are very small compared with the accuracy required by the problem then the simple discarding of the superfluous data is a quite satisfactory solution of the problem. On the other hand if the radar noise is large compared to the required accuracy then it will be necessary to use all of the available data with the most efficient statistical processing.

To determine rationally what procedures to use requires a knowledge of the effect of the procedure upon the mean error, standard deviation and possibly other statistics of the observed data. It is the purpose of this memorandum to present such information for a procedure sometimes called a "Least Square Quadratic Filter". This procedure consists in grouping a number, M, of the data points together, fitting them by least squares methods to a quadratic function of time and then assuming the value of the function at the mid-point of the time interval covered by the group of points to be the value which would have been observed at that time by a radar perturbed by very much smaller noise. Sometimes when these quantities are of interest the slope and curvature of the quadratic function are also assumed to equal the rate of change and second derivative of the observed data. Consequently we will also present the errors to be expected in these quantities.

To state the problem exactly we assume that some characteristic of the satellite or other object being tracked is given by a function X(t) which can be expanded in a Taylor series convergent over the time interval of interest or at least is continuously differentiable four times over that interval. X(t) may be a single observable such as distance, range rate, azimuth, or elevation or it may be a vector whose components are such observable quantities. This is measured or observed at a sequence of times $t_{i,j} = t_{o,j} + ih$ where h is the interval between the equally spaced instants and

$t_{o,j}$ is the instant at which we choose to start counting the sequence in a particular run designated by the subscript j. The results of these observations is a sequence of values $\tilde{X}_j(t_i)$ equal to the true value $X(t_i)$ plus a perturbing noise $N_j(t_i)$. We will determine estimates $\hat{X}_j(t_o)$, $\hat{\dot{X}}_j(t_o)$, and $\hat{\ddot{X}}_j(t_o)$ by fitting the observed quantities to a quadratic relation by least squares.

In summary we have an observable characteristic of a tracked target $X(t)$ for which we <u>observe</u> at equally spaced time intervals, h, a set of values $\tilde{X}_j(t_i)$ equal to the true values $X(t_i)$ contaminated by noise samples $N_j(t_i)$, not necessarily uncorrelated, from these we make <u>estimates</u> $\hat{X}_j(t_o)$, $\hat{\dot{X}}_j(t_o)$, and $\hat{\ddot{X}}_j(t_o)$ of the observable and its first two time derivatives at the mid-point of the smoothing interval, $(M-1)h$, by fitting the observations in the least squares sense to a quadratic equation in time. Finally to complete this statement of nomenclature we will derive expressions for the <u>means</u> of the populations of estimates $\overline{\hat{X}}(t_o)$, $\overline{\hat{\dot{X}}}(t_o)$, and $\overline{\hat{\ddot{X}}}(t_o)$ together with the standard deviations of these same populations designated by $\sigma_{\hat{X}}$, $\sigma_{\hat{\dot{X}}}$, and $\sigma_{\hat{\ddot{X}}}$. The details of these derivations are given in the appendix.

The resulting expressions for the estimates taken from there are:

$$\hat{X}_j(t_o) = \frac{3(3M^2-7)}{4M(M^2-4)} \sum_0^{M-1} \tilde{X}_j(t_i) - \frac{15}{M(M^2-4)} \sum_0^{M-1} \tilde{X}_j(t_i)\left(i - \frac{M-1}{2}\right)^2$$

$$\hat{\dot{X}}_j(t_o) = \frac{12}{M(M^2-1)h} \sum_0^{M-1} \tilde{X}_j(t_i)\left(i - \frac{M-1}{2}\right)$$

$$\hat{\ddot{X}}_j(t_o) = \frac{360}{M(M^2-1)(M^2-4)h^2} \sum_0^{M-1} \tilde{X}_j(t_i)\left(i - \frac{M-1}{2}\right)^2 - \frac{30}{M(M^2-4)h^2} \sum_0^{M-1} \tilde{X}_j(t_i)$$

in which we have also replaced the sums of powers of integers by their well known values in terms of Bernoulli numbers. For the first few of these relations one may see, for example, "Mathematical Tables from Handbook of Chemistry and Physics", eighth edition, page 249.

To carry the analysis further we note that $\tilde{X}$ can be expressed as the sum of X and N where X will be independent of the particular trial j and N will be a stochastic sample dependent on the trial but chosen from a population whose statistics are independent of both the trial and the place in the sequence, although the correlation between two samples of N will generally depend upon the distance in the sequence separating the two samples. Under the limitations placed earlier on the function X(t) we may write:

$$X_j(t_1) = X(t_o) + \dot{X}(t_o)[t_1-t_o] + \frac{1}{2}\ddot{X}(t_o)[t_1-t_o]^2 + \frac{1}{6}\dddot{X}(t_o)[t_1-t_o]^3$$

$$+ \frac{1}{24}\ddddot{X}(t)[t_1-t_o]^4 + 0\left([t_1-t_o]^5\right)$$

If we choose t_o to be midway between the first and last of the M instants this becomes:

$$X_j(t_1) = X(t_o) + \dot{X}(t_o)h\left(1 - \frac{M-1}{2}\right) + \frac{1}{2}\ddot{X}(t_o)h^2\left(1 - \frac{M-1}{2}\right)^2 +$$

$$\frac{1}{6}\dddot{X}(t_o)h^3\left(1 - \frac{M-1}{2}\right)^3 + \frac{1}{24}\ddddot{X}(t_o)h^4\left(1 - \frac{M-1}{2}\right)^4 +$$

$$0\left(h^5\left[1 - \frac{M-1}{2}\right]^5\right)$$

and by adding $N_j(t_1)$ to this and substituting the results into the expressions for the estimates in place of $\tilde{X}(t_1)$ we obtain expressions which can be averaged over a large number of j values to obtain the mean of the estimate populations. The next step is to subtract these means from the appropriate expression for an individual estimate, square the difference and again average over j. Again these operations are carried out in detail in the appendix to this memorandum. The results are:

$$\overline{\hat{X}}(t_o) = X(t_o) + \overline{N} - \frac{(M^2-1)(M^2-9)}{4480} h^4 \overset{....}{X}(t_o)$$

$$\overline{\hat{\dot{X}}}(t_o) = \dot{X}(t_o) + \frac{3M^2-7}{120} h^2 \dddot{X}(t_o)$$

$$\overline{\hat{\ddot{X}}}(t_o) = \ddot{X}(t_o) + \frac{3M^2-13}{168} h^2 \overset{....}{X}(t_o)$$

and for the standard deviations due to noise:

$$\sigma^2_{\hat{X}} = \frac{3(3M^2-7)}{4(M^2-1)M}\left[\overline{N^2}-\overline{N}^2\right] + \frac{450}{M^2(M^2-4)2}\sum_{1}^{M-1} r\ [\overline{N(T)N(t+rh)}-\overline{N}^2][M-r] \times$$

$$\left[\frac{(3M^2-7)^2}{400} - \frac{(28M^2-20Mr-5r^2-52)r^2}{40} - \frac{(3M^2-7+6Mr-3r^2)(M-r-1)(M-r+1)}{240}\right]$$

$$\sigma^2_{\hat{\dot{X}}} = \frac{12}{h^2M(M^2-1)}\left[\overline{N^2}-\overline{N}^2\right] +$$

$$\frac{24}{h^2M^2(M^2-1)^2}\sum_{1}^{M-1} r\ [\overline{N(t)N(t+rh)}-\overline{N}^2][(M-r)^3-(1+3r^2)(M-r)]$$

$$\sigma^2_{\hat{\ddot{X}}} = \frac{720}{h^4 M(M^2-1)(M^2-4)}\left[\overline{N^2}-\overline{N}^2\right] + \frac{1800}{h^2 M^2(M^2-1)^2(M^2-4)^2}\sum_{1}^{M-1} r\,[\overline{N(t)N(t+rh)}-\overline{N}^2][M-r]\times$$

$$\Big[\tfrac{3}{5}(M-r-1)(M-r+1)(3[M-r]^2-7) - (M-r-1)(M-r+1)(2M^2+6r^2-3) -$$

$$(M^2-1)(M^2-6r^2-1) + 9r^4\Big]$$

these expressions for the standard deviations simplify greatly if we make the assumption that the successive samples of noise are uncorrelated. In that case we have:

$$\sigma^2_{\hat{X}} = \frac{3(3M^2-7)}{4M(M^2-4)}\left[\overline{N^2}-\overline{N}^2\right]$$

$$\sigma^2_{\hat{\dot{X}}} = \frac{12}{h^2 M(M^2-1)}\left[\overline{N^2}-\overline{N}^2\right]$$

$$\sigma^2_{\hat{\ddot{X}}} = \frac{720}{h^4 M(M^2-1)(M^2-4)}\left[\overline{N^2}-\overline{N}^2\right]$$

Whether or not this is a valid approximation depends upon the power spectrum of the disturbing noise N(t) and on the sampling interval h. Rice has stated* that:

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*"Mathematical Analysis of Random Noise", S. O. Rice, BSTJ, Vol. 23, pp. 285.

$$\overline{N(t)N(t+rh)}-\overline{N}^2 = \int_0^\infty w(f) \cos 2\pi frhdf$$

where w(f) is the power spectrum of N(t). This may be written as:

$$\overline{N(t)N(t+rh)}-\overline{N}^2 = \frac{1}{2\pi rh} \int_0^\infty w(X) \cos XdX$$

and since the integral has finite value for all values of h the correlation function must eventually decrease at least as fast as inversely with h and hence can be made as small as may be desirable by taking h large enough. This is not necessarily the desirable thing to do however since increasing h inevitably decreases M if we have a bounded smoothing time and hence may actually increase σ^2. The optimum size of h will therefore have to be investigated for a particular problem after the power spectrum of the disturbing noise is known.

Referring now to the population means we see that if X(t) is truly a quadratic function and $\overline{N}$ is zero then there is no error in the mean but if X(t) is not exactly a quadratic function of time but on the contrary has nonzero third and fourth derivatives there will be a so called "dynamic error" or "tracking error" which is proportional to the fourth time derivative of X(t) for the error in the position and acceleration estimates and to the third derivative for the error of the velocity estimate. Further if we call (M-1)h the smoothing interval T then the "dynamic error" in position is proportional to T^4 while that in velocity and acceleration is proportional to T^2.

To summarize we have derived expressions for the estimates of position, velocity, and acceleration as obtained from this method. These expressions show three types of error; a. an error in the mean due to the fact that the actual variation of the observable is not truly a quadratic; b. a variance due to the variance in the contaminating noise which can never be completely eliminated with a finite sample; and c. another component of the variance due to correlation of the contaminating noise samples. In addition, we have discussed at least three parameters of the system which are wholly or partly under control and can be adjusted to optimize the overall result in some desired sense. These parameters are the sampling interval, the smoothing interval, and the power spectrum of the contaminating noise.

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Appendix

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APPENDIX

Derivation of Means and Standard Deviation of Estimates of Position, Velocity, and Acceleration as Obtained by a "Least Squares Quadratic Filter".

We assume an observable characteristic of a target or satellite designated $X(t)$ contaminated by a noise $N_j(t)$ and the sum sampled at equally spaced instants t_i to produce samples $\tilde{X}_j(t_i) = X(t_i) + N_j(t_i)$. We select a group of M consecutive samples covering a time interval $T = (M-1)h$ where h is the sampling interval. To fit these samples in the least squares sense to a quadratic we first form the residuals $\varepsilon_{i,j}$ by subtracting from each $\tilde{X}$ a quadratic with three unknown coefficients. Further we choose for the origin of time of the quadratic function the midpoint of the interval T. This gives:

$$\varepsilon_{i,j} = \tilde{X}_j(t_i) - \hat{X}_j(t_o) - \hat{\dot{X}}_j(t_o)h\left(i - \frac{M-1}{2}\right) - \frac{1}{2}\hat{\ddot{X}}(t_o)h^2\left(i - \frac{M-1}{2}\right)^2$$

we then square this expression and sum the result over all of the M values of i. We then differentiate this sum partially with respect to $\hat{X}$, $\hat{\dot{X}}$, and $\hat{\ddot{X}}$ and equate each of the partial derivatives to zero so as to minimize the sum of the squares of the residuals. This procedure gives the following three linear equations in the undetermined coefficients:

$$M\hat{x}_j(t_o) + \frac{M(M^2-1)}{12} h^2 \frac{1}{2} \hat{\ddot{x}}_j(t_o) = \sum_{0}^{M-1} \tilde{x}_j(t_i)$$

$$\frac{M(M^2-1)}{12} h^2 \dot{x}_j(t_o) = \sum_{0}^{M-1} \tilde{x}_j(t_i) h \left(i - \frac{M-1}{2}\right)$$

$$\frac{M(M^2-1)}{12} h^2 \hat{x}_j(t_o) + \frac{M(M^2-1)(3M^2-7)}{240} h^4 \frac{1}{2} \hat{\ddot{x}}_j(t_o) = \sum_{0}^{M-1} \tilde{x}_j(t_i) h^2 \left(i - \frac{M-1}{2}\right)^2$$

in which we have used the relations

$$\sum_{0}^{M-1} \left(i - \frac{M-1}{2}\right)^2 = \frac{M(M^2-1)}{12}$$

$$\sum_{0}^{M-1} \left(i - \frac{M-1}{2}\right)^4 = \frac{M(M^2-1)(3M^2-7)}{240}$$

$$\sum_{0}^{M-1} \left(i - \frac{M-1}{2}\right) = \sum_{0}^{M-1} \left(i - \frac{M-1}{2}\right)^3 = 0$$

which can be found for example, in the "Mathematical Tables from Handbook of Chemistry and Physics" eighth edition, page 249.

Solving these equations gives

$$\hat{X}_j(t_o) = \frac{3(3M^2-7)}{4M(M^2-4)} \sum_0^{M-1} \tilde{X}_j(t_i) - \frac{15}{M(M^2-4)} \sum_0^{M-1} \tilde{X}_j(t_i)\left(i - \frac{M-1}{2}\right)^2$$

$$\hat{\dot{X}}_j(t_o) = \frac{12}{M(M^2-1)h} \sum_0^{M-1} \tilde{X}_j(t_i)\left(i - \frac{M-1}{2}\right)$$

$$\hat{\ddot{X}}_j(t_o) = \frac{360}{M(M^2-1)(M^2-4)h^2} \sum_0^{M-1} \tilde{X}_j(t_i)\left(i - \frac{M-1}{2}\right)^2 - \frac{30}{M(M^2-4)h^2} \sum_0^{M-1} \tilde{X}_j(t_i)$$

but by definition

$$\tilde{X}_j(t_i) = X(t_i) + N_j(t_i) = X(t_o) + N_j(t_i) + \dot{X}(t_o)h\left(i - \frac{M-1}{2}\right) +$$

$$\frac{1}{2}\ddot{X}(t_o)h^2\left(i - \frac{M-1}{2}\right)^2 + \frac{1}{6}\dddot{X}(t_o)h^3\left(i - \frac{M-1}{2}\right)^3 +$$

$$\frac{1}{24}\ddddot{X}(t_o)h^4\left(i - \frac{M-1}{2}\right)^4 + 0\left(h^5\left[i - \frac{M-1}{2}\right]^5\right)$$

where $0\left(h^5\left[i - \frac{M-1}{2}\right]^5\right)$ indicates a remainder which vanishes as h vanishes even though divided by h, h^2, h^3, or h^4 but approaches a finite limit when divided by h^5 and becomes infinite if divided by a higher power of h. Neglecting this remainder term, substituting the rest of the expression into the expressions for the three estimates and performing the indicated summations wherever possible gives,

$$\hat{X}_j(t_o) = \left[\frac{3(3M^2-7)}{4M(M^2-4)} M - \frac{15}{M(M^2-4)} \frac{M(M^2-1)}{12}\right] X(t_o) +$$

$$\left[\frac{3(3M^2-7)}{4M(M^2-4)} \frac{M(M^2-1)}{12} - \frac{15}{M(M^2-4)} \frac{M(M^2-1)(3M^2-7)}{240}\right] \times \frac{1}{2} \ddot{X}(t_o)h^2 +$$

$$\left[\frac{3(3M^2-7)}{4M(M^2-4)} \frac{M(M^2-1)(3M^2-7)}{240} - \frac{15}{M(M^2-4)} \frac{M(M^2-1)(3M^4-18M^2+31)}{1344}\right] \times$$

$$\frac{1}{24} \ddddot{X}(t_o)h^4 + \frac{3(3M^2-7)}{4M(M^2-4)} \sum_0^{M-1} N_j(t_\varepsilon) - \frac{15}{M(M^2-4)} \sum_0^{M-1} N_j(t_i)\left(i - \frac{M-1}{2}\right)^2$$

$$= X(t_o) - \frac{3(M^2-1)(M^2-1)}{560} \frac{1}{24} \ddddot{X}(t_o)h^2 + \frac{3(3M^2-7)}{4M(M^2-4)} \sum_0^{M-1} N_j(t_i) -$$

$$\frac{15}{M(M^2-4)} \sum_0^{M-1} N_j(t_i)\left(i - \frac{M-1}{2}\right)^2$$

$$\dot{X}_j(t_o) = \frac{12}{M(M^2-1)} \frac{M(M^2-1)}{12} \dot{X}(t_o) + \frac{12}{M(M^2-1)} \frac{M(M^2-1)(3M^2-7)}{240} \frac{1}{6} \dddot{X}(t_o)h^2 +$$

$$\frac{12}{M(M^2-1)h} \sum_0^{M-1} N_j(t_j)\left(i - \frac{M-1}{2}\right)$$

$$= \dot{X}(t_o) + \frac{3M^2-7}{120} \dddot{X}(t_i)h^2 + \frac{12}{M(M^2-1)h} \sum N_j(t_i)\left(i - \frac{M-1}{2}\right)$$

$$\hat{\ddot{X}}_j(t_o) = \left[\frac{360}{M(M^2-1)(M^2-4)h^2}\frac{M(M^2-1)}{12} - \frac{30}{M(M^2-4)h^2}M\right]X_o(t) +$$

$$\left[\frac{360}{M(M^2-1)(M^2-4)h^2}\frac{M(M^2-1)(3M^2-7)}{240} - \frac{30}{M(M^2-4)h^2}\frac{M(M^2-1)}{12}\right]\frac{1}{2}\ddot{X}(t_o)h^2 +$$

$$\left[\frac{360}{M(M^2-1)(M^2-4)h^2}\frac{M(M^2-1)(3M^4-18M^2+31)}{1344} - \frac{30}{M(M^2-4)h^2}\frac{M(M^2-1)(3M^2-7)}{240}\right]\times$$

$$\frac{1}{24}\ddot{\ddot{X}}(t_o)h^4 + \frac{360}{M(M^2-1)(M^2-4)h^2}\sum_0^{M-1} i\, N_j(t_i)\left(i - \frac{M-1}{2}\right)^2 - \frac{30}{M(M^2-4)h^2}\sum_0^{M-1} i\, N_j(t_i)$$

$$= \ddot{X}(t_o) + \frac{3M^2-13}{168}\ddot{\ddot{X}}(t_o)h^2 + \frac{360}{M(M^2-1)(M^2-4)h^2}\sum_0^{M-1} i\, N_j(t_i)\left(i - \frac{M-1}{2}\right)^2 -$$

$$\frac{30}{M(M^2-4)h^2}\sum_0^{M-1} i\, N_j(t_i)$$

The remainder of our derivation must take account of the stochastic nature of $N_j(t_i)$. Because of this character we cannot derive a complete expression for an individual trial, rather we derive statistics of the population of the results of many such trials. We will be satisfied with deriving expressions for the three means, and the three standard deviations although higher moments could be derived by the same methods.

To derive the mean of the populations we simply average the above expressions over many runs noting that $X(t_o)$, $\dot{X}(t_o)$, $\ddot{X}(t_o)$, $\dddot{X}(t_o)$, and $\ddddot{X}(t_o)$ are independent of j and that

$$\frac{1}{J}\sum_{j=1}^{J} N_j(t_i) = \bar{N}$$

This gives:

$$\bar{\hat{X}}(t_o) = X(t_o) + \bar{N} - \frac{3(M^2-1)(M^2-9)}{560}\,\frac{1}{24}\,\ddddot{X}(t_o)\,h^4$$

$$\bar{\hat{\dot{X}}}(t_o) = \dot{X}(t_o) + \frac{3M^2-7}{120}\,\dddot{X}(t_o)\,h^2$$

$$\bar{\hat{\ddot{X}}}(t_o) = \ddot{X}(t_o) + \frac{3M^2-13}{168}\,\ddddot{X}(t_o)\,h^2$$

Subtracting these mean values from the corresponding individual estimates gives for the residuals;

$$\hat{X}_j(t_o)-\bar{\hat{X}}(t_o) = \frac{3(3M^2-7)}{4M(M^2-4)}\sum_{i=0}^{M-1}[N_j(t_i)-\bar{N}] - \frac{15}{M(M^2-4)}\sum_{i=0}^{M-1}[N_j(t_i)-\bar{N}]\left[i - \frac{M-1}{2}\right]^2$$

$$\hat{\dot{X}}_j(t_o)-\bar{\hat{\dot{X}}}(t_o) = \frac{12}{M(M^2-1)h}\sum_{i=0}^{M-1}[N_j(t_i)-\bar{N}]\left[i - \frac{M-1}{2}\right]$$

$$\hat{\ddot{X}}_j(t_o)-\bar{\hat{\ddot{X}}}(t_o) = \frac{360}{M(M^2-1)(M^2-4)h^2}\sum_{0}^{M-1} i\,[N_j(t_i)-\bar{N}]\left[i-\frac{M-1}{2}\right]^2 - \frac{30}{M(M^2-4)h^2}\sum_{0}^{M-1} i\,[N_j(t_i)-\bar{N}]$$

We now square these expressions obtaining:

$$\left[\hat{X}_j(t_o)-\bar{\hat{X}}(t_o)\right]^2 = \frac{9(3M^2-7)^2}{16M^2(M^2-4)^2}\sum_{0}^{M-1} i \sum_{0}^{M-1} k\,[N_j(t_i)-\bar{N}][N_j(t_k)-\bar{N}] -$$

$$\frac{45(3M^2-7)}{2M^2(M^2-4)^2}\sum_{0}^{M-1} i \sum_{0}^{M-1} k\,[N_j(t_i)-\bar{N}][N_j(t_k)-\bar{N}]\left[i-\frac{M-1}{2}\right]^2 +$$

$$\frac{225}{M^2(M^2-4)^2}\sum_{0}^{M-1} i \sum_{0}^{M-1} k\,[N_j(t_i)-\bar{N}][N_j(t_k)-\bar{N}]\left[i-\frac{M-1}{2}\right]^2\left[k-\frac{M-1}{2}\right]^2$$

$$\left[\hat{\dot{X}}_j(t_o)-\bar{\hat{\dot{X}}}(t_o)\right]^2 = \frac{144}{M^2(M^2-1)^2h^2}\sum_{0}^{M-1} i \sum_{0}^{M-1} k\,[N_j(t_i)-\bar{N}][N_j(t_k)-\bar{N}]\left[i-\frac{M-1}{2}\right]\left[k-\frac{M-1}{2}\right]$$

$$\left[\hat{\ddot{X}}_j(t_o)-\bar{\hat{\ddot{X}}}(t_o)\right]^2 = \frac{129600}{M^2(M^2-1)^2(M^2-4)^2h^4}\sum_{0}^{M-1} i \sum_{0}^{M-1} k\,[N_j(t_i)-\bar{N}][N_j(t_k)-\bar{N}]\left[i-\frac{M-1}{2}\right]^2\left[k-\frac{M-1}{2}\right]^2 -$$

$$\frac{21600}{M^2(M^2-1)(M^2-4)^2h^4}\sum_{0}^{M-1} i \sum_{0}^{M-1} k\,[N_j(t_i)-\bar{N}][N_j(t_k)-\bar{N}]\left[i-\frac{M-1}{2}\right]^2 +$$

$$\frac{900}{M^2(M^2-4)^2h^4}\sum_{0}^{M-1} i \sum_{0}^{M-1} k\,[N_j(t_i)-\bar{N}][N_j(t_k)-\bar{N}]$$

The next step is to average these expressions over the population of runs i.e. over all j. To do this we note that the only factors in the expressions which depend upon j are the factors involving the noise and that the averaging can be performed before the summation. Therefore the quantity to be averaged is $[N_j(t_i)-\overline{N}][N_j(t_k)-\overline{N}]$. We first perform the indicated multiplication. This gives:

$$[N_j(t_i)-\overline{N}][N_j(t_k)-\overline{N}] = [N_j(t_i)N_j(t_k)-\overline{N}N_j(t_i)-\overline{N}N_j(t_k)+\overline{N}^2]$$

The averaging can then be performed term by term. Thus:

$$\text{Ave}\ \ N_j(t_i)N_j(t_k) = \overline{N(t_i)N(t_k)}$$

$$\text{Ave}\ \ \overline{N}N_j(t_i) = \text{Ave}\ \ \overline{N}N_j(t_k) = \overline{N}^2$$

$$\text{Ave}\ \ \overline{N}^2 = \overline{N}^2$$

Hence

$$\text{Ave}\ [N_j(t_i)-\overline{N}][N_j(t_k)-\overline{N}] = [\overline{N(t_i)N(t_k)}-\overline{N}^2]$$

Substituting this into the expressions already obtained for the square of the deviations we obtain for the square of the standard deviations:

$$\sigma^2_{\hat{X}} = \overline{\left[\hat{X}_j(t_o)-\bar{\hat{X}}(t_o)\right]^2} = \frac{9(3M^2-7)^2}{16M^2(M^2-4)^2}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\left[\overline{N(t_i)N(t_k)}-\bar{N}^2\right] -$$

$$\frac{45(3M^2-7)}{2M^2(M^2-4)^2}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\left[\overline{N(t_i)N(t_k)}-\bar{N}^2\right]\left[i-\frac{M-1}{2}\right]^2 +$$

$$\frac{225}{M^2(M^2-4)^2}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\left[\overline{N(t_i)N(t_k)}-\bar{N}^2\right]\left[i-\frac{M-1}{2}\right]^2\left[k-\frac{M-1}{2}\right]^2$$

$$\sigma^2_{\dot{\hat{X}}} = \overline{\left[\dot{\hat{X}}_j(t_o)-\bar{\dot{\hat{X}}}(t_o)\right]^2} = \frac{144}{M^2(M^2-1)^2h^2}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\left[\overline{N(t_i)N(t_k)}-\bar{N}^2\right]\left[i-\frac{M-1}{2}\right]\left[k-\frac{M-1}{2}\right]$$

$$\sigma^2_{\ddot{\hat{X}}} = \overline{\left[\ddot{\hat{X}}_j(t_o)-\ddot{\hat{X}}(t_o)\right]^2} = \frac{129600}{M^2(M^2-1)^2(M^2-4)^2h^4}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\left[\overline{N(t_i)N(t_k)}-\bar{N}^2\right]\left[i-\frac{M-1}{2}\right]^2\left[k-\frac{M-1}{2}\right]^2 -$$

$$\frac{21600}{M^2(M^2-1)(M^2-4)^2h^4}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\left[\overline{N(t_i)N(t_k)}-\bar{N}^2\right]\left[i-\frac{M-1}{2}\right]^2 +$$

$$\frac{900}{M^2(M^2-4)^2h^4}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\left[N(t_i)N(t_k)-\bar{N}^2\right]$$

Up to this point the only assumption we have made about $N(t)$ is that it is a stochastic process. We have gone as far as seems possible without making further restrictions on the noise. The least stringent restriction we can make at this point is that $N(t)$ is stationary, i.e. that any statistics which we determine for the process will be independent of the time of their occurrence. Under this assumption the correlation function $\overline{N(t_1)N(t_k)}-\overline{N}^2$ will depend only upon the absolute value of the difference between t_1 and t_k and upon the power spectrum of the noise. If we make this more stringent assumption then by changing the order of summation and summing first over terms for which the absolute value of the difference between t_1 and t_k is constant and then over the possible values of this difference we obtain the following somewhat simpler expressions.

$$\sigma^2_{\hat{X}} = \frac{9(3M^2-7)^2}{16M^2(M^2-4)^2} M(\overline{N^2}-\overline{N}^2) + \frac{9(3M^2-7)^2}{8M^2(M^2-4)^2} \sum_{r=1}^{M-1} [\overline{N(t)N(t+rh)}-\overline{N}^2][M-r] -$$

$$\frac{45(3M^2-7)}{2M^2(M^2-4)^2} \frac{M(M^2-1)}{12} (\overline{N^2}-\overline{N}^2) + \frac{45(3M^2-7)}{M^2(M^2-4)^2} \sum_{r=1}^{M-1} [\overline{N(t)N(t+rh)}-\overline{N}^2] \sum_{i=0}^{M-r-1} \left[i - \frac{M-1}{2}\right]^2 +$$

$$\frac{225}{M^2(M^2-4)^2} \frac{M(M^2-1)(3M^2-7)}{240} (\overline{N^2}-\overline{N}^2) + \frac{450}{M^2(M^2-4)^2} \sum_{r=1}^{M-1} [\overline{N(t)N(t+rh)}-\overline{N}^2] \sum_{i=0}^{M-r-1} \left[i \quad \frac{M-1}{2}\right]^2 \times$$

$$\left[i+r - \frac{M-1}{2}\right]^2$$

$$\sigma^2_{\hat{\dot{X}}} = \frac{144}{M^2(M^2-1)^2h^2} \frac{M(M^2-1)}{12} (\overline{N^2}-\overline{N}^2) + \frac{288}{M^2(M^2-4)^2h^2} \sum_{1}^{M-1} r \, [\overline{N(t)N(t+rh)}-\overline{N}^2]$$

$$\sum_{0}^{M-r-1} i \left[i - \frac{M-1}{2}\right] \left[i+r - \frac{M-1}{2}\right]$$

$$\sigma^2_{\hat{\ddot{X}}} = \frac{129600}{M^2(M^2-1)^2(M^2-4)^2h^4} \frac{M(M^2-1)(3M^2-7)}{240} (\overline{N^2}-\overline{N}^2) +$$

$$\frac{259200}{M^2(M^2-1)^2(M^2-4)^2h^4} \sum_{1}^{M-1} r \, [\overline{N(t)N(t+rh)}-\overline{N}^2) \sum_{0}^{M-r-1} t \left[i - \frac{M-1}{2}\right]^2 \left[i+r - \frac{M-1}{2}\right]^2 -$$

$$\frac{21600}{M^2(M^2-1)(M^2-4)^2h^4} \frac{M(M^2-1)}{12} (\overline{N^2}-\overline{N}^2) -$$

$$\frac{43200}{M^2(M^2-1)(M^2-4)^2h^4} \sum_{1}^{M-1} r \, [\overline{N(t)N(t+rh)}-\overline{N}^2] \sum_{0}^{M-r-1} i \left[i - \frac{M-1}{2}\right]^2 +$$

$$\frac{900}{M^2(M^2-4)^2h^4} M \, [\overline{N^2}-\overline{N}^2] + \frac{1800}{M^2(M^2-4)^2h^4} \sum_{1}^{M-1} r \, [\overline{N(t)N(t+rh)}-\overline{N}^2] \, [M-r-1]$$

To simplify these expressions we must evaluate certain sums which are not in the standard form which we have been using but require a certain amount of algebraic manipulation to put them in that form. They are:

$$\sum_{0}^{M-r-1} r\left(i-\frac{M-1}{2}\right)^2 = \sum_{0}^{M-r-1} i\left(i-\frac{M-r-1}{2}-\frac{r}{2}\right)^2$$

$$= \sum_{0}^{M-r-1} i\left(i-\frac{M-r-1}{2}\right)^2 - r\sum_{0}^{M-r-1} i\left(i-\frac{M-r-1}{2}+\frac{r^2}{4}\,[M-r]\right)$$

$$= \frac{(M-r-1)(M-r)(M-r+1)}{12} + \frac{r^2[M-r]}{4}$$

$$\sum_{0}^{M-r-1} i\left(i-\frac{M-1}{2}\right)^2\left(i+r-\frac{M-1}{2}\right)^2 = \sum_{0}^{M-r-1} i\left(i-\frac{M-r-1}{2}-\frac{r}{2}\right)^2\left(i-\frac{M-r-1}{2}+\frac{r}{2}\right)^2$$

$$= \sum_{0}^{M-r-1} i\left[\left(i\quad\frac{M-r-1}{2}\right)^2-\frac{r^2}{4}\right]^2$$

$$= \sum_{0}^{M-r-1} i\left(i\quad\frac{M-r-1}{2}\right)^4 - \frac{r^2}{2}\sum_{0}^{M-r-1} i\left(i-\frac{M-r-1}{2}\right)^2 +$$

$$\frac{r^4}{16}\,[M-r]$$

$$= \frac{(M-r-1)(M-r)(M-r+1)(3[M-r]^2-7)}{240} -$$

$$\frac{(M-r-1)(M-r)(M-r+1)r^2}{24} + \frac{r^4(M-r)}{16}$$

$$\sum_{0}^{M-r-1}\left(i-\frac{M-1}{2}\right)\left(i+r-\frac{M-1}{2}\right)=\sum_{0}^{M-r-1}\left(i-\frac{M-r-1}{2}-\frac{r}{2}\right)\left(i-\frac{M-r-1}{2}+\frac{r}{2}\right)$$

$$=\sum_{0}^{M-r-1}\left(i-\frac{M-r-1}{2}\right)^2-\frac{r^2}{4}$$

$$=\frac{(M-r-1)(M-r)(M-r+1)}{12}-\frac{r^2(M-r)}{4}$$

Substituting these relations into the expression for the three variances and combining terms where such combination results in some simplification gives:

$$\sigma^2_{\hat{X}}=\frac{3(3M^2-7)^2}{4M(M^2-4)}\left[\overline{N^2}-\overline{N}^2\right]+\frac{450}{M^2(M^2-4)^2}\sum_{1}^{M-1}r\ [\overline{N(t)N(t+rh)}-\overline{N}^2][M-r]\ \times$$

$$\left[\frac{(3M^2-7)^2}{400}-\frac{(78M^2-20Mr-5r^2-52)r^2}{240}-\frac{(M-r-1)(M-r+1)(3M^2-7+6Mr-3r^2)}{240}\right]$$

$$\sigma^2_{\dot{\hat{X}}}=\frac{12}{M(M^2-1)h^2}\left[\overline{N^2}-\overline{N}^2\right]+\frac{24}{M^2(M^2-1)^2h^2}\sum_{1}^{M-1}r\ [\overline{N(t)N(t+rh)}-\overline{N}^2][(M-r)^3-(1+3r^2)(M-1)]$$

$$\sigma^2_{\hat{\dot{X}}} = \frac{720}{M(M^2-1)(M^2-4)h^4}\left[\overline{N^2}-\overline{N}^2\right] + \frac{1800}{M^2(M^2-1)^2(M^2-4)^2h^4}\sum_{1}^{M-1} r\ [\overline{N(t)N(t+rh)}-\overline{N}^2] \times$$

$$[M-r][3/5(M-r-1)(M-r+1)(3[M-r]^2-7)-2(M-r-1)(M-r+1)(M^2+3r^2-1) -$$

$$(M^2-1)(M^2-6r^2-1) + 9r^4]$$