Diffraction of Random Waves in a

Homogeneous Anisotropic Medium

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Abstract

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This paper considers the diffraction of random waves in a homogeneous anisotropic medium. These random waves are produced, for example, by reflection from a rough surface such as the moon or by transmitting through the ionosphere containing irregularities. Dut to anisotropy both depolarization effect and the modification of spectral density functions of the wave by the medium may occur. Both effects may be important in certain ionospheric applications. In the forward scatter approximation it has been found that the sum of spectra of orthogonally polarized waves is uninfluenced by the medium.

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1. Introduction

The study of wave propagation in the magneto-ionic medium is concerned

with solving the wave equation of the form*

$$\nabla \times \nabla \times (\overleftrightarrow{\Gamma} \cdot \vec{D}) - \omega^2 \mu_0 \epsilon_0 \vec{D} = 0 \tag{1}$$

where $\overleftrightarrow{\Gamma}$ is the relative inverse dielectric tensor. In (1) the electric

displacement vector $\vec{D}$ is used rather than the usual electric field intensity

$\vec{E}$ since it is known that $\vec{D}$ is transverse while $\vec{E}$ is, in general, not [1].

The plane wave solution of (1) has been studied quite extensively, especially

its refractive indices known as the Appleton-Hartree formula.

The present investigation is concerned with the study of propagation

of random waves in such a medium. Specifically the statistical properties

of the wave are given at an initial plane. Through diffraction the fields

propagate in the half space. The statistical properties of the wave at

a plane parallel to the initial plane are desired. The method used is

similar to that used by others who assumed the medium to be isotropic [2].

Hence this study represents a slight generalization of the earlier work.

However, it is found that due to anisotropic nature of the medium the

additional effects such as depolarization of the wave and the modification

of spectral density of the field by the medium will appear.

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*exp jωt time dependence is assumed throughout. All quantities are in
 rationalized MKS units.

where $\overleftrightarrow{I}$ is the identity matrix. The immediate conclusion when (3) is substituted into (4) is the vanishing of D_z, hence the electric displacement is polarized entirely in the plane transverse to $\vec{k}$. With this simplification the remaining equations expressed in the matrix form are

$$\begin{bmatrix} \dfrac{1}{1-X} - \dfrac{\omega^2 \mu_0 \epsilon_0}{k^2} & -\dfrac{jXY_L}{(1-X)^2} \\[3ex] \dfrac{jXY_L}{(1-X)^2} & \dfrac{1}{1-X} - \dfrac{\omega^2 \mu_0 \epsilon_0}{k^2} \end{bmatrix} \begin{bmatrix} D_x \\[3ex] D_y \end{bmatrix} = 0 \qquad (5)$$

In order for the equation to have non-trivial solution the determinant of the coefficient matrix must vanish. If again Y^2 and higher order terms are ignored the following expressions for the refractive indices are obtained.

$$k^2/\omega^2 \mu_0 \epsilon_0 = 1 - X \mp XY \cos \Theta \qquad (6)$$

where Θ is the angle between the propagation vector and the steady magnetic field. The wave with the upper sign in (6) as the refractive index is designated as the ordinary wave and is circularly polarized in the left handed sense; the wave with the lower sign is designated as the extraordinary wave and is circularly polarized in the right handed sense. Now (6) is nothing more than the usual expression used in connection with the study of Faraday effect and it is valid under the quasi-longitudinal approximation. Fuller analysis shows that such approximation is valid for nearly all directions of propagation except the region at which the direction of propagation is but a few degrees from exactly perpendicular to the steady magnet field. Later development assumes (6) is valid for all regions and such an assumption

makes sense only if the quasi-transverse region makes negligible contribution to the total effect.

Let $k^{(0)}$ and $k^{(X)}$ denote the propagation constants of the ordinary and the extraordinary waves respectively. Their mean and difference can be found as

$$(k^{(0)} + k^{(X)})/2 = \omega[\mu_0\epsilon_0(1-X)]^{1/2} \equiv k_0 \qquad (7)$$

$$k^{(0)} - k^{(X)} = -\omega^2\mu_0\epsilon_0 \, XY \cos \theta/k_0 \qquad (8)$$

It is known that a wave of any polarization can be decomposed into two characteristic waves, each will propagate independently in the medium with its corresponding propagation constant and polarization. The resultant is the sum of these two waves. In the present case the resultant wave is particularly simple if it is assumed that the resultant is polarized linearly at some initial position. The resultant wave will propagate with a propagation constant equal to the mean of the propagation constants of the characteristic waves and its polarization is kept linear with the plane of polarization rotated continuously along the direction of propagation (Faraday effect). As computed in (7) the mean of the propagation constants is just the propagation constant of the corresponding isotropic medium (i.e., in the absence of the steady magnetic field). The rotation of the polarization is through an angle equal to one-half of the difference of the propagation constants multiplied by the distance of travel. For a plane wave propagating in z direction with the electric displacement polarized in y direction at z = 0 the resultant takes the form

$$D_x = -D_0 \sin \Omega \cdot \exp j(\omega t - k_0 z)$$

$$D_y = D_0 \cos \Omega \cdot \exp j(\omega t - k_0 z)$$

(9)

where the rotational angle is

$$\Omega = -\frac{\omega^2 \mu_0 \epsilon_0 XY \cos \Theta}{2k_0} \cdot z \qquad (10)$$

In general if the wave is assumed to be polarized in yz-plane at $z = 0$ and the direction of propagation has a polar angle α and azimuthal angle β the field is given by

$$D_x = -D_0(\sin^2 \beta + \cos^2 \beta \cos^2 \alpha)^{1/2} \sin \Omega \exp j(\omega t - \vec{k}_0 \cdot \vec{r})$$

$$D_y = D_0(\sin^2 \beta + \cos^2 \beta \cos^2 \alpha)^{-1/2}(\cos \alpha \cos \Omega + \sin^2 \alpha \sin \beta \cos \beta \sin \Omega) \exp j(\omega t - \vec{k}_0 \cdot \vec{r})$$

$$D_z = D_0(\sin^2 \beta + \cos^2 \beta \cos^2 \alpha)^{-1/2}(\cos \beta \cos \alpha \overbrace{\sin \alpha}^{} \sin \Omega - \sin \beta \sin \alpha \cos \Omega) \exp j(\omega t - \vec{k}_0 \cdot \vec{r})$$

(11)

For cases of small polar angles (11) can be approximated by

$$D_x = -D_0 \sin \Omega \exp j(\omega t - \vec{k}_0 \cdot \vec{r})$$

$$D_y = D_0 \cos \Omega \exp j(\omega t - \vec{k}_0 \cdot \vec{r}) \qquad (12)$$

$$D_z = 0$$

The Faraday rotational angle Ω appeared in (11) and (12) is the same as (10) except z is to be replaced by r. Since Ω plays an important role in the diffraction its discussion will be postponed to a later section.

3.   Diffraction of Random Waves

The problem assumes that the statistical properties of the fields are given at some initial plane, say at z = 0. Our interest is in learning the statistical properties of the displacement vector at some plane parallel to the initial plane after propagating in the kind of medium discussed in section 2. This suggests the use of the notation $\vec{D}(\vec{r}_\perp ;z)$ which stresses the fact that statistical properties are desired for various values of z in the two dimensional space $\vec{r}_\perp = (x,y)$, when their properties at z = 0 are known. The problem is explicitly posed when the following (boundary) conditions are assumed.

1) $D(\vec{r}_\perp ;0)$ is polarized in yz-plane. However we are most interested in the case of small polar angles, it will be assumed the approximate expression (12) is valid for all plane waves. This means that at z = 0, the wave can be assumed to be polarized entirely along y-axis.

2) $D(\vec{r}_\perp ;0)$ is a complex homogeneous random fields [3] with known correlation functions that are absolutely integrable over all $\vec{r}_\perp$. This assures the existence of the spectral density functions.

4)   As z → ∞, the retarded solution is taken.

The method used here follows that used in the theory of time series analysis. Like the theory of time series analysis, the Fourier transform of $D(\vec{r}_\perp ;0)$ does not in general exist because of the condition 3. This difficulty can be overcome as done by Wiener [4] by truncating $D(\vec{r}_\perp ;0)$. Therefore, let

$$D_R(\vec{r}_\perp ;0) = \begin{cases} D(\vec{r}_\perp ;0) & \vec{r} \ \epsilon \ R \\ \\ 0 & \vec{r} \ \epsilon \ R \end{cases} \qquad (13)$$

where $R = \left\{ x,y \mid X/2 < x < -X/2, \; Y/2 < y < -Y/2 \right\}$.* Define in the mean

square sense the amplitude spectral density function**

$$D_R(\vec{\kappa};0) = \iint\limits_{-\infty}^{\infty} D_R(\vec{r}_\perp;0) \, e^{-j\vec{\kappa}\cdot\vec{r}_\perp} \, d^2 r_\perp$$

$$= \iint\limits_{R} D(\vec{r}_\perp;0) \, e^{-j\vec{\kappa}\cdot\vec{r}_\perp} \, d^2 r_\perp \tag{14}$$

Then through Fourier inversion

$$D_R(\vec{r}_\perp;0) = \frac{1}{(2\pi)^2} \iint\limits_{-\infty}^{\infty} D_R(\vec{\kappa};0) \, e^{j\vec{\kappa}\cdot\vec{r}_\perp} \, d^2\kappa \tag{15}$$

Note that since $D_R(\vec{r}_\perp;0)$ is assumed to have zero mean, so is $D_R(\vec{\kappa};0)$. The autocorrelation is obtained by

$$K(\xi,\eta;0) = \; < D^*(x,y;0) \, D(x+\xi,\, y+\eta;0) > \tag{16}$$

where the angular brackets are used to denote the ensemble average. Since

K is assumed to be absolutely integrable over the infinite range there

exists a non-negative spectral density function defined by

$$S(\vec{\kappa};0) = \lim_{\substack{X\to\infty \\ Y\to\infty}} \frac{1}{XY} \; < D_R(\vec{\kappa};0) \, D_R^*(\vec{\kappa};0) > \tag{17}$$

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*Note that here X and Y are specific values of x and y respectively, not those defined by (2).

**Fourier transform pairs are distinguished by their arguments, same symbols are used here.

By Wiener-Khintchine theorem $K(\xi, \eta; 0)$ and $S(\vec{\kappa}; 0)$ form a Fourier transform pair [5]. In case $D(\vec{r}_\perp; 0)$ is additionally ergodic the autocorrelation function defined by (16) can also be obtained through spatial average

$$K(\xi, \eta; 0) = \lim_{\substack{X \to \infty \\ Y \to \infty}} \frac{1}{XY} \iint_{-\infty}^{\infty} D_R^*(x, y; 0) \, D_R(x + \xi, \, y + \eta; 0) \, dx \, dy \tag{18}$$

with probability 1.

The problem assumes that $S(\vec{\kappa}; 0)$ (or equivalently $K(\xi, \eta; 0)$ is given and the expressions of the spectral densities and the autocorrelations for some finite values of z are desired. Now in the half space $z > 0$, the fields are given by the superposition of many plane waves of the form (12). With $\exp j\omega t$ factor supressed, these are given by

$$D_{xR}(\vec{r}_\perp; z) = -\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} D_R(\vec{k}_\perp; 0) \sin \Omega (\vec{k}_\perp; z) \, e^{-j\sqrt{k_0^2 - k^2}\, z} \, e^{-j\vec{k}_\perp \cdot \vec{r}_\perp} \, d^2 k_\perp \tag{19}$$

$$D_{yR}(r; z) = \frac{1}{(2\pi)^2} \iint D_R(k; 0) \cos \Omega (k; z) \, e^{-j\sqrt{k_0^2 - k^2}\, z} \, e^{-j\vec{k}_\perp \cdot \vec{r}_\perp} \, d^2 k_\perp$$

where $\vec{k}_\perp = (k_x, k_y)$. Above expressions are certainly valid when $z = 0$ at which, since $\Omega = 0$, (19) reduces to (15). This sets $\vec{\kappa} = \vec{k}_\perp$. The first property of $D_{xR}(\vec{r}_\perp; z)$ and $D_{yR}(\vec{r}_\perp; z)$ can be obtained by averaging (19) over the ensemble, it concludes that both components have zero mean for any value of z. As indicated earlier the case of interest is when all component waves propagate in the forward direction like the consideration of Fresnel diffraction problems. In other words, $D_R(\vec{k}_\perp; 0)$ has appreciable value only

over the region $k_0^2 < k_\perp^2$ (corresponding to evanescent waves) can be ignored. Therefore, the elements of the spectral matrix are obtained as

$$S_{xx}(\vec{k}_\perp;z) = S(\vec{k}_\perp;0) \sin^2 \Omega$$

$$S_{xy}(\vec{k}_\perp;z) = S_{yx}(\vec{k}_\perp;z) = S(\vec{k}_\perp;0) \cos \Omega \sin \Omega \qquad (20)$$

$$S_{yy}(\vec{k}_\perp;z) = S(\vec{k}_\perp;0) \cos^2 \Omega$$

For the special case of isotropic medium (20) shows that there is no depolarization effect; and the spectral density and hence also the autocorrelation function are independence of z as found by others [2]. When the medium is anisotropic these are no longer true. However, it is interesting to note that the sum

$$S_{xx}(\vec{k}_\perp;z) + S_{yy}(\vec{k}_\perp;z) = S(\vec{k}_\perp;0) \qquad (21)$$

which is independent of z. This has some practical importance since it means that the sum of spectral densities of orthogonally polarized waves is uninfluenced by the medium.

Since the Faraday rotation angle plays a dominant role in determining the spectral density functions it will be discussed in the next section.

4. Dependence of Rotation Angle

Let Ω_z be the Faraday rotation angle suffered by a wave propagating in z direction. From (10)

$$\Omega_z = - \frac{\omega^2 \mu_0 \epsilon_0 \, X \, Y_z \, z}{2k_0} \qquad (22)$$

For a wave propagating in a general direction with propagation constant $k_0 = (\vec{k}_\perp, \sqrt{k_0^2 - k_\perp^2})$, the polarization of the wave will rotate in the right-handed sense from y-axis through an angle

$$\Omega(\vec{k}_\perp; z) = \Omega_z - \frac{\omega^2 \mu_0 \epsilon_0 \; X \; z}{2k_0} \; \frac{\vec{k}_\perp \cdot \vec{Y}}{\sqrt{k_0^2 - k_\perp^2}} \qquad (23)$$

Note that for a fixed z, Ω_z is a constant. Hence the dependence on $\vec{k}_\perp$ of the rotational angle is through the second term in (23). Because of the appearance of the dot product it suggests that it is advantageous to rotate the coordinates about z axis so that y'z plane is parallel to the plane of magnetic meridian and $\hat{x}$'z plane perpendicular to it. For this coordinate system $\vec{k}_\perp \cdot \vec{Y} = k_{y'} Y_{y'}$. Further, consistent with the forward propagation assumption $k_\perp$ can be ignored when compared with k_0. This results in an expression for the rotational angle which is only a function of $k_{y'}$. Let ϕ be the angle between the displacement vector at z = 0 and the magnetic meridian plane. Then

$$\Omega(k_{y'}; z) = \phi + \Omega_z - \frac{\omega^2 \mu_0 \epsilon_0 \; X \; Y_{y'} z}{2k_0^2} \cdot k_{y'} \qquad (24)$$

Let

$$z_c = \frac{2k_0^2}{\omega^2 \mu_0 \epsilon_0 \; X \left| Y_{y'} \right| k_b} = \frac{2n_0^2}{X \left| Y_{y'} \right| k_b} \qquad (25)$$

where k_b is the spectral width of S(k ;0) in the plane of magnetic meridian plane (i.e. y' component). Then (24) can be written as

$$\Omega(k_{y'}; z) = \phi + \Omega_z + \frac{z}{z_c} \frac{k_{y'}}{k_b} \qquad (26)$$

(26) indicates that in the region $z \ll z_c$, $\Omega(k_{y'};z)$ is nearly constant over the spectral width of interest and the spectral density of the field is not affected appreciably by the medium. When z has the same order of magnitude as z_c or even larger than z_c, the spectral densities of the fields are expected to be strongly affected by the medium.

5. Correlation Functions

The elements of the correlation matrix can be calculated by taking the Fourier inversion of the respective elements of the spectral matrix. As seen from (20), since the spectral densities are given in the product form the convolution theorem can be used to compute the correlation functions. Assume that the approximate expression (26) is valid the convolution integrals can be carried out easily, obtaining

$$K_{x'x'}(\xi',\eta';z) = \frac{1}{2} K(\xi',\eta';0) - \frac{1}{4} K(\xi',\eta' + 2z/z_c k_b;0) e^{j2(\phi + \Omega_z)}$$

$$- \frac{1}{4} K(\xi',\eta' - 2z/z_c k_b;0) e^{-j2(\phi + \Omega_z)}$$

$$K_{x'y'}(\xi',\eta';z) = K_{y'x'}(\xi',\eta';z) = \frac{1}{4} K(\xi',\eta' + 2z/z_c k_b;0) e^{j2(\phi + \Omega_z)}$$

$$- \frac{1}{4} K(\xi',\eta' - 2z/z_c k_b;0) e^{-j2(\phi + \Omega_z)} \qquad (27)$$

$$K_{y'y'}(\xi',\eta';z) = \frac{1}{2} K(\xi',\eta';0) + \frac{1}{4} K(\xi',\eta' + 2z/z_c k_b;0) e^{j2(\phi + \Omega_z)}$$

$$+ \frac{1}{4} K(\xi',\eta' - 2z/z_c k_b;0) e^{-j2(\phi + \Omega_z)}$$

By addition, it is seen that $K_{x'x'}(\xi',\eta';z) + K_{y'y'}(\xi',\eta';z) = K(\xi',\eta';0)$.
Since the Fourier transform is unique this is just the special case of (21).

6. Discussions

As an example take $f = 20$ mc, $f_p = 10$ mc, $f_{Hz} = 1$ mc, and $k_b = 0.1/m$,
the critical distance comes out to be approximately 1200 km. Hence it seems
that the effect of anisotropic nature of the medium may be important in
ionospheric studies of irregularities and in studies of moon reflections.
As suggested in section 3, one possible way of eliminating the influence
of anisotropic nature of the medium is through addition of spectra obtained
on orthogonal antennas as shown by (21).

The method described here can be extended to studying the scattering
from irregularities and from rough surfaces and other diffraction problems.

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