# Collisions and Nonlinear Effects in Plasmas

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by

Adrian Anatol Dolinsky

APPROVAL SHEET

Title of Thesis: Collisions and Nonlinear Effects in Plasmas

Name of Candidate: Adrian Anatol Dolinsky
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Thesis and Abstract Approved: ______________________________
Derek A. Tidman
Research Professor
Institute for Fluid Dynamics
and Applied Mathematics

Date Approved: May 7, 1965

# VITA

Name: Adrian Anatol Dolinsky.

Permanent Address: 9018 Piney Branch Road
Silver Spring, Maryland.

Degree and date to be conferred: Ph. D., 1965.

Date of birth: September 30, 1935.

Place of birth: Wojnilow (Voinilov), Ukraine.

Secondary education: Seton Hall Preparatory School
South Orange, New Jersey, 1954.

| Collegiate institutions attended | Dates | Degree | Date of Degree |
|---|---|---|---|
| Fordham College | 1954-58 | B. S. | 1958 |
| University of Maryland | 1958-65 | Ph.D. | 1965 |

Major: Plasma Physics.

Minors: Mathematics, Nuclear Physics.

Publications: "Numerical Integration of Kinetic Equations", Physics of Fluids 8, 436 (1965).

Positions held: Research Assistant in the Institute of Fluid Dynamics and Applied Mathematics of the University of Maryland, 1961-1965.

ABSTRACT

Title of Thesis: Collisions and Nonlinear Effects in Plasmas

Adrian Anatol Dolinsky, Doctor of Philosophy, 1965

Thesis directed by: Professor Derek A. Tidman



Nonlinear and collision effects in the behavior of plasmas are investigated for an electron gas embedded in a neutralizing, uniformly smeared out background of positive charge. Nonlinearity enters into the description of the behavior of a plasma through the collision term (arising from interparticle correlations) and the self-consistent electric field term (i.e. the ensemble average of the sum of Coulomb fields of all of the plasma particles) in an exact kinetic equation. It is impossible (at the present time) to treat both nonlinear terms simultaneously. For this reason the investigation is divided into two separate parts: In PART ONE the effect of the collision term on the behavior of a spatially homogeneous plasma is investigated; in PART TWO the self-consistent electric field term is treated under conditions which enable us to drop the collision term.



In PART ONE the problem of relaxation of the exact Balescu-Lenard kinetic equation is solved numerically as an initial value problem for isotropic velocity distribution functions. Several different forms of the initial distribution function are selected: a Gaussian, peaked at about 0.28 of the electron thermal velocity; a resonance function; and a Maxwellian coexisting with a sharply peaked Gaussian (the peak of the

Gaussian being located at 2.0 electron thermal velocities). The Fokker-Planck kinetic equation is also solved numerically under the same restrictions and with the same initial distribution functions. A comparison of the solutions of the two kinetic equations shows very small difference between them, and a probable reason for this is advanced. In addition, a relaxation time is defined, and the long time behavior of the distribution functions is studied.

In PART TWO the problem of light-by-light scattering in a plasma is investigated. Two coherent, monochromatic, plane-polarized, plane electromagnetic waves (produced by two lasers) pass through a large volume of a quiescent electron plasma and are scattered. When the frequencies of the impinging waves are tuned so that their difference is approximately equal to the frequency of the natural longitudinal plasma oscillations, these oscillations are excited. However, they are limited by the action of several physical mechanisms: the Landau damping, the collisional damping, and the nonlinear effects.

We are interested in the nature of the nonlinear effects. For this reason, the plasma is assumed to be describable by means of the collisionless plasma moment equations coupled with the Maxwell equations. The amount of nonlinearity is assumed to be small, and the equations are handled by the method of multiple time and spatial scales, a generalization due to Frieman and Sandri of a perturbation scheme developed for nonlinear mechanics by Krylov, Bogoliubov, and Mitropolsky.

The results show that there is a slow rotation and/or change in magnitude of the amplitudes of the two impinging electromagnetic waves (as they pass through the plasma). The rotation is both in space and in time. At the same time, a longitudinal electric field is built up slowly inside the plasma, and its amplitude changes slowly in space and in time. All of the above variations in space and in time proceed at rates which are proportional to the strength of the impinging radiation. Furthermore, the strength of the longitudinal field is at most of the order of magnitude of the strengths of the incident electromagnetic waves. This indicates the effectiveness of nonlinearity in limiting the longitudinal plasma oscillations.

# COLLISIONS AND NONLINEAR
# EFFECTS IN PLASMAS

by

Adrian Anatol Dolinsky

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PREFACE

We shall be concerned with the behavior of fully ionized plasmas, i.e. gaseous mixtures of several species of charged particles at sufficiently high temperatures and low densities to assure complete ionization for all times. Under such circumstances quantum effects can be neglected provided the De Broglie wavelengths of particles are much smaller than the average interparticle distances. At the same time, we shall assume that particle thermal velocities and macroscopic streaming velocities are small compared to the velocity of light. Consequently relativistic effects are also negligible. Such plasmas can therefore be described by the laws of classical physics for a many-body system of particles interacting through Coulomb forces.

A complete statistical description of a plasma would be by means of a probability distribution function in the phase space of all of the particles. This probability distribution function must obey Liouville's equation. However, a solution of Liouville's equation is generally impossible. Besides, a description by means of a probability distribution function in the phase space of all particles yields more information than is necessary for many purposes. Many physical properties of a plasma can, however, be determined from a knowledge of a one-particle distribution function for each species of particles. By a one-particle distribution function we mean the average (i.e. ensemble average) particle number density of a given species in the six-dimensional position-velocity space.

We would like to write down a differential equation from which a one-particle distribution function can be determined for all times if it is known at some initial time, say $t = 0$. Such an equation ought to contain no more detailed information than is contained in one-particle distribution functions; i.e., only terms containing one-particle distribution functions should be present. Such an equation (also called a kinetic equation) can be derived from the BBGKY (Born-Bogoliubov-Green-Kirkwood-Yvon) hierarchy of equations - which are derived from Liouville's equation - if some assumptions are made about the correlation functions for particles.

The first assumption is that the correlation functions are in some sense small compared to the order of magnitude of the one-particle distribution functions. This is usually true throughout most of the phase space of a many-body system of particles interacting through Coulomb forces. If it is also assumed that one is dealing with phenomena that vary slowly in space and time (compared to the plasma period $\omega_p^{-1}$ and Debye length), then the appropriate kinetic equation for the one-particle distribution function $f_\sigma(\underline{x},\underline{v},t)$ for the species $\sigma$ becomes

$$\frac{\partial f_\sigma}{\partial t} + \underline{v} \cdot \frac{\partial f_\sigma}{\partial \underline{x}} + \frac{e_\sigma}{m_\sigma} (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \frac{\partial f_\sigma}{\partial \underline{v}} = C(f_\sigma, f_\tau) ,$$

where $e_\sigma$ and $m_\sigma$ are, respectively, the charge and mass of a particle of species $\sigma$; $\underline{E}(\underline{x},t)$ is the electric field, which includes both an externally produced field and the self-consistent field of plasma particles (i.e. the sum of Coulomb fields of all particles, averaged over the ensemble); $\underline{B}(\underline{x},t)$ is an externally produced magnetic field; and $C(f_\sigma,f_\tau)$ is a collision term (of the order of magnitude of the pair correlation functions), arising from correlations between particles.

The derivation of an appropriate expression for $C(f_\sigma,f_\tau)$ is impossible without the introduction of additional assumptions. Some problems, however, can be treated, to a good approximation, by neglecting the collision term. The resulting kinetic equation is sometimes called the collisionless Boltzmann equation, or the Vlasov equation. It can be used, for example, to describe reasonably well the behavior of a plasma at very high temperatures and very low densities. In general, however, the collision term is important and should be retained. Two different forms of $C(f_\sigma,f_\tau)$ are widely used in plasma theory. One of them is called the Fokker-Planck collision term, or the Rosenbluth-MacDonald-Judd collision term; the other, a more exact collision term, is called the Balescu-Lenard collision term.

The Fokker-Planck collision term is derived in the same way and under the same assumptions as the collision term for a gas in which particles interact through strong, short-range forces. It can be obtained, for example, by making a Taylor expansion of the Boltzmann collision integral to treat distant collisions. Here, however, an additional assumption has to be made that only those two-particle collisions are to be counted for which the impact parameter for colliding particles is less than some characteristic length, which is chosen to be the Debye radius.

The assumptions under which the Fokker-Planck collision term is derived have many questionable features. First, it is assumed that a plasma particle collides with only one other plasma particle at any one time; i.e., only two-body collisions are assumed to exist. However, because of the long range nature of Coulomb forces, a particle will collide with many other particles simultaneously. Second, the time between two collisions is assumed to be much greater than the time duration of a collision. This is also incorrect for the same reason. Third, the screening of the charge of a particle by oppositely charged particles does not appear naturally, but has to be added in as an extra assumption. We may summarize by saying that the Fokker-Planck collision term treats collective effects improperly.

The more exact expression for the collision term, which is used in plasma theory, is the Balescu-Lenard collision term. It can be derived from the BBGKY hierarchy of equations by making the so-called Bogoliubov adiabatic hypothesis. This is that the higher interparticle correlation functions relax to their asymptotic long-time forms rapidly over the time scale in which the one-particle distribution functions are changing. (The Bogoliubov adiabatic hypothesis cannot be made for high frequency phenomena like electron plasma oscillations. In such phenomena the one-particle distribution functions change on a time scale comparable to the time scale of change of the interparticle correlation functions.) The resulting Balescu-Lenard collision term treats collective effects properly, taking into account automatically the screening of charged particles and the many-body collisions.

The kinetic equation is generally nonlinear. The nonlinear terms in the equation are the self-consistent electric field term and the collision term. Both nonlinear terms are important in the behavior of a plasma, and we shall be interested in both of them in this work. We shall be interested in the self-consistent field term, because its nonlinearity has not been studied sufficiently. However, this nonlinearity, even if small, is capable of limiting plasma oscillations effectively. We shall also be interested in the collision term, because it has not been investigated sufficiently: Only the Fokker-Planck kinetic equation has been studied so far to any great extent, whereas only the linearized version of the Balescu-Lenard kinetic equation has been integrated.

To simplify the mathematics, we shall limit ourselves to plasmas composed of only one species of particles, electrons, embedded in a uniformly smeared out background of positive charge to ensure charge neutrality on the average. It is not possible to treat the self-consistent field term and the collision term simultaneously. Further, the Balescu-Lenard collision term we use is valid only for a spatially homogeneous, field-free plasma; whereas the simultaneous presence of both the self-consistent field term and the

Fokker-Planck collision term makes the problem generally intractable (except when the kinetic equation is linearized). For this reason we divide our investigation into two separate parts and select two particular problems. In PART ONE the effect of collision terms on the behavior of a spatially homogeneous plasma is investigated. In PART TWO the self-consistent field term is treated under conditions which enable us to drop the collisional terms for the problem of light-light scattering in a plasma.

In PART ONE, to make the problem mathematically tractable, we limit ourselves to one-particle distribution functions which are isotropic in velocity space. The exact Balescu-Lenard equation is solved numerically as an initial value problem for such distribution functions. Several initial distribution functions are chosen: a Gaussian, peaked at 0.28 of the electron thermal velocity; a resonance function; and a very sharp Gaussian, peaked at 2.0 electron thermal velocities, coexisting with a Maxwellian. The exact Fokker-Planck equation is also solved numerically for the same initial distribution functions. The values of the plasma parameters are chosen such that differences between the solutions of the two kinetic equations - if there be any - will be noticeable.

Only small differences (a few percent) between the solutions of the two kinetic equations were observed for the initial distribution functions selected, and a possible explanation for this is advanced. The difference between the solutions of the two kinetic equations for the test particle problem is also analyzed, and a reason for this difference is given. In addition, a relaxation time is defined, and the long time behavior of the three initial distribution functions is investigated by means of a numerical integration of the Fokker-Planck equation.

In PART TWO we treat the problem of light-by-light scattering in a plasma: Two coherent, monochromatic, plane-polarized, plane electromagnetic waves impinge on a quiescent electron plasma and are scattered. When the frequencies of the two incident waves are tuned so that their difference is approximately equal to the frequency of the longitudinal plasma oscillations, those oscillations are excited. However, they do not grow linearly with time because of the limiting effect of several physical mechanisms.

We are interested in the nature of the mechanism of nonlinearity only. Therefore we assume the plasma to be describable by the collisionless moment and Maxwell equations. We also assume the nonlinear terms in these equations to be small compared to the linear terms. The equations can then be handled by the method of multiple time scales and spatial scales, a generalization due to Frieman and Sandri of a perturbation scheme developed by Krylov, Bogoliubov, and Mitropolsky for nonlinear mechanics.

The results show that there is a slow rotation and/or change in the magnitudes of the amplitudes of the two impinging electromagnetic waves as they pass through the plasma. The rate of rotation is proportional to the strength of the impinging radiation. At the same time, a longitudinal electrostatic oscillation is built up slowly inside the plasma. The rate of build-up of this oscillation is proportional to the rate of change of the amplitudes of the transverse fields; the strength of the amplitude of this oscillation is at most of the order of magnitude of the strengths of the transverse fields. All of these effects are due to a proper treatment of the small nonlinear terms, in the equations of motion, and cannot be obtained by simply carrying conventional perturbation theory to second order.

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# PART ONE

# NUMERICAL INTEGRATION OF KINETIC EQUATIONS

## I. INTRODUCTION

The problem of the relaxation to equilibrium of a fully ionized non-equilibrium plasma has been of interest for some time. In the absence of analytic solutions to the appropriate kinetic equations, which are non-linear, various authors have investigated problems that do not require the solution of a non-linear kinetic equation. As an example of such problems, one may mention the case of the relaxation to equilibrium of the distribution function of a test particle injected into a quiescent plasma. In these problems the appropriate kinetic equation can be linearized. Up to date the only investigations of the relaxation of a test-particle distribution function have been carried out by means of the Fokker-Planck kinetic equation. Thus Kranzer[1] studied the thermalization of a fast ion in a plasma by means of a numerical solution of the Fokker-Planck equation. Frisch[2] defined certain characteristic times which he called time lags in the thermalization of a fast ion injected into a plasma, and proceeded to calculate them without actually solving the Fokker-Planck equation. Ree and Kidder[3] obtained an analytic solution for the thermalization of a fast test ion injected into a plasma by approximating the friction and dispersion coefficients in the Fokker-Planck equation. Their solution is valid only when the speed of the test ion is less than the average speed of the plasma electrons, but large enough so that the plasma electrons interact more strongly with the test ion than do the plasma ions.

Attempts at an actual solution of a non-linear kinetic equation have up to now been confined only to the Fokker-Planck equation. Furthermore, they have been numerical solutions. In fact, the only investigation of the relaxation of a non-linear kinetic equation up to date is that of MacDonald, Rosenbluth and Chuck[4], who solved numerically, as an initial value problem, the non-linear Fokker-Planck equation for an electron-positron plasma which is spatially homogeneous and isotropic in velocity. It would be interesting to carry out a similar investigation for the non-linear Balescu-Lenard equation. This would be especially interesting in view of the fact that the Balescu-Lenard equation, by treating collective effects properly, gives a more general description of the behavior of a spatially homogeneous plasma than does the Fokker-Planck equation, which does not treat collective effects properly. The only attempt so far at a solution of the Balescu-Lenard (B/L) equation is the solution, as an initial value problem, of the linearized BL equation by Rosenberg and Wu[5]. These two authors took a multicomponent plasma and perturbed the distribution function of each species of particles slightly from the equilibrium Maxwellian distribution. Then they proceeded to investigate the decay of this small perturbation in the linear approximation.

This paper presents a numerical integration of the exact Balescu-Lenard (BL) kinetic equation for different initial distributions of an electron plasma embedded in a neutralizing, uniformly smeared out, positive-charge background. The Fokker-Planck equation with the Rosenbluth, MacDonald, Judd collision term (RMJ equation) is also integrated with the same initial distributions. By comparing the solutions of the two kinetic equations one hopes to arrive at an estimate of the importance of collective effects in the relaxation of these distribution functions.

The simplifying assumption made, in these calculations, is that the distribution functions are isotropic in velocity space. For a limited class of such distribution functions – for example, for distribution functions which are monotomically decreasing functions of $|\underset{\sim}{v}|$ – our results indicate that for most purposes there is a negligible difference (a few percent) between the predictions of the BL and RMJ kinetic equations. This is because these isotropic distributions are sufficiently stable that the $\underset{\sim}{v}$ and $\underset{\sim}{k}$ integrals in the BL equation ((A1) and (A2)) do not approach a zero of the Landau denominator, $D^{+}$, anywhere in the range of integration. Thus collective effects, which are treated properly in the BL equation, but not in the RMJ equation, are of little importance for such distributions.

We also define numerically a relaxation time in section (III), by considering how close all portions of a given initial distribution function will get to the final Maxwellian after a certain time, and whether or not they will stay close to the Maxwellian for all times after that time. Our conclusion is that a distribution function often oscillates about the final Maxwellian at certain points in velocity space. These points depend on the form of the initial distribution function. This behavior points out that the relaxation to the final Maxwellian cannot in general be taken to be an exponential decay (with the possible exception of the high-energy tail). This conclusion + agrees with the solution of the linearized Balescu-Lenard equation of Rosenberg and Wu[5], which is a superposition of exponential decays.

## II. KINETIC EQUATIONS

### A. BALESCU-LENARD (BL) EQUATION

Let $f(\underset{\sim}{v}_1,t)$ be the one-particle distribution function for a spatially uniform electron plasma embedded in a uniformly smeared out background of positive charge. $f(\underset{\sim}{v}_1,t)$ has two normalization conditions

$$\int f(\underset{\sim}{v}_1,t)\, d\underset{\sim}{v}_1 = 1 \tag{1}$$

and

$$\int v_1^2\, f(\underset{\sim}{v}_1,t)\, d\underset{\sim}{v}_1 = v_o^2 \tag{2}$$

where $v_o$ is the thermal speed of electrons.

For the purposes of numerical integration it is convenient to choose a set of dimensionless variables. Therefore we shall define three dimensionless variables $\underset{\sim}{V}_1$, $\tau$, and $g(\underset{\sim}{V}_1,\tau)$ by

$$\underset{\sim}{V}_1 \equiv \frac{\underset{\sim}{v}_1}{v_o} , \tag{3}$$

$$\tau \equiv \frac{t}{\tau_D(v_o)} , \tag{4}$$

and

$$g(\underset{\sim}{V}_1,\tau) \equiv v_o^3 \, f(\underset{\sim}{v}_1,t) , \tag{5}$$

where $\tau_D(v_o)$ is the Spitzer deflection time[6] for electrons moving with velocity $v_o$, given by the expression

$$\tau_D(v) = \frac{m^2 \, v^3}{8\pi \, n_o \, e^4 \, \ln \Lambda \left[\left(1 - \frac{1}{3}\frac{v_o^2}{v^2}\right) \mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{v}{v_o}\right) - \sqrt{\frac{2}{3\pi}}\,\frac{v_o}{v}\, e^{-\frac{3}{2}\frac{v^2}{v_o^2}}\right]} \tag{6}$$

where erf is the error function. From equations (1) and (2), the two dimensionless normalization conditions for $g(\underset{\sim}{V}_1,\tau)$ are

$$\int g(\underset{\sim}{V}_1,\tau) \, d\underset{\sim}{V}_1 = 1 \tag{7}$$

and

$$\int V_1^2 \, g(\underset{\sim}{V}_1, \ ) \, d\underset{\sim}{V}_1 = 1 \quad . \tag{8}$$

For isotropic velocity distributions, the BL equation can be written in dimensionless form as

$$\frac{\partial g}{\partial \tau} = \frac{\alpha}{V^2} \left\{ \frac{1}{V} \left( \frac{\partial^2 g}{\partial V^2} - \frac{1}{V} \frac{\partial g}{\partial V} \right) \int_o^V V_1^2 \, dV_1 \, G(V_1, \tau) \, \Phi(V_1, \tau) \right.$$

$$+ \frac{\partial g}{\partial V} \int_o^V V_1^2 \, dV_1 \, g(V_1, \tau) \, \Phi(V_1, \tau)$$

$$+ V \frac{\partial g}{\partial V} \, G(V, \tau) \, \Phi(V, \tau)$$

$$\left. + V^2 \, [g(V, \tau)]^2 \, \Phi(V, \tau) \right\} \quad , \tag{9}$$

where the functions $G(V,\tau)$ and $\Phi(V,\tau)$ occurring in (9) are defined by

$$G(V,\tau) \equiv \int_V^\infty V' dV' \, g(V', \tau) \tag{10}$$

and

$$\Phi(V,\tau) \equiv \frac{1}{4 \ln \frac{k_o}{k_D}} \left[ \ln H(V,\tau) - \frac{3}{\pi^2} \, \frac{\Gamma(V,\tau) \, L(V,\tau)}{V \, g(V,\tau)} \right] \quad , \tag{11}$$

where the functions $\Gamma(V,\tau)$, $H(V,\tau)$, and $L(V,\tau)$ are defined by

$$\Gamma(V,\tau) \equiv \frac{4\pi}{3} \; P \int_0^\infty \frac{V'^2 \, g(V',\tau) \, dV'}{V'^2 - V^2} \tag{12}$$

($P \int_0^\infty$ denoting the principal value integral),

$$H(V,\tau) \equiv \frac{\left[\left(\frac{k_o}{k_D}\right)^2 + \Gamma(V,\tau)\right]^2 + \left[\frac{2}{3}\pi^2 \, Vg(V,\tau)\right]^2}{[\Gamma(V,\tau)]^2 + \left[\frac{2}{3}\pi^2 \, Vg(V,\tau)\right]^2} , \tag{13}$$

and

$$L(V,\tau) \equiv \tan^{-1} \frac{2}{3}\pi^2 \; \frac{Vg(V,\tau)}{\Gamma(V,\tau)} - \tan^{-1} \frac{2\pi^2}{3} \frac{Vg(V,\tau)}{\left(\frac{k_o}{k_D}\right)^2 + \Gamma(V,\tau)} . \tag{14}$$

The quantity $k_o$ is the upper limit on the $k_1$ integration mentioned in Appendix A. Its value was taken as[7]

$$k_o = \frac{KT}{e^2} , \tag{15}$$

where $K$ is the Boltzmann constant, $T$ is the temperature, and $e$ is the electronic charge. $k_D$ is the Debye wave number, given by

$$k_D = \left(\frac{4\pi \; n_o \; e^2}{KT}\right)^{\frac{1}{2}} , \tag{16}$$

where $n_o$ is the electron particle density. $\alpha$ is defined by

$$\alpha \equiv \frac{16\pi^2 \; n_o \; e^4}{m^2 \; v_o^3} \quad \tau_D(v_o) \; \ln \frac{k_o}{k_D} , \tag{17}$$

where $m$ is the electron mass.

## B. THE FOKKER-PLANCK EQUATION WITH THE ROSENBLUTH-MACDONALD-JUDD COLLISION TERM (RMJ EQUATION)

The isotropic, spatially homogeneous RMJ equation for an electron plasma embedded in a uniform background of positive charge is[4,7]

$$\frac{\partial f}{\partial t} = \frac{16\pi^2 n_o e^4}{m^2}\left(\ell n \frac{k_o}{k_D}\right)\left\{\frac{1}{3}\frac{\partial^2 f}{\partial v^2}\left[\int_v^\infty v' f(v',t)\, dv'\right.\right.$$

$$\left. + \frac{1}{v^3}\int_o^v v'^4 f(v',t)\, dv'\right] + \frac{2}{3v}\frac{\partial f}{\partial v}\left[\int_o^\infty v' f(v',t)\, dv'\right.$$

$$\left. - \int_o^v v' dv' f(v',t) \quad (1 - \frac{v'}{v})^2 \, (1 + \frac{v'}{2v})\right]$$

$$\left. + [f(v,t)]^2 \right\} \quad . \qquad (18)$$

Transforming to the same dimensionless variables defined in (3)-(5), eq. (18) becomes

$$\frac{1}{\alpha}\frac{\partial g}{\partial \tau} = \frac{1}{3}\frac{\partial^2 g}{\partial V^2}\int_V^\infty V' g(V';\tau)\, dV' + \frac{1}{V^3}\int_0^V V'^4\, g(V';\tau)\, dV'$$

$$+ \frac{2}{3V}\frac{\partial g}{\partial V}\left[\int_0^\infty V' g(V';\tau)\, dV' - \int_0^V V'\, dV' g(V';\tau)\left(1 - \frac{V'}{V}\right)^2 \cdot\right.$$

$$\left. \cdot\left(1 + \frac{V'}{2V}\right)\right] + [g(V,\tau)]^2 \quad , \tag{19}$$

where all of the symbols have the same meaning they had in the dimensionless BL equation. The two normalizations given by (7) and (8) hold also in the case of the RMJ equation.

## C. RELATIONSHIP BETWEEN THE BL AND THE RMJ EQUATIONS

In the limit of $\frac{k_o}{k_D} \to \infty$ eq. (9) tends asymptotically to eq. (19). This can be seen from the following considerations:

$$\Gamma(V,\tau) \sim 1 \tag{20}$$

and

$$Vg(V,\tau) \sim 1 \quad . \tag{21}$$

For $\frac{k_D}{k_o} \ll 1$,

$$L(V,\tau) \cong \tan^{-1} \frac{2}{3}\pi^2 \frac{Vg(V,\tau)}{\Gamma(V,\tau)} \sim 1 \tag{22}$$

and

$$\ln H(V,\tau) \cong \ln\left(\frac{k_o}{k_D}\right)^4 \quad . \tag{23}$$

Therefore, by (20) - (23),

$$\Phi(V,\tau) \cong 1 \quad . \tag{24}$$

The double integral $\int_o^V v_1^2 \, dV_1 \, G(V_1,\tau)$ can be reduced to single integrals in the following way:

$$\int_o^V v_1^2 \, dV_1 \, G(V_1,\tau) = \int_o^V v_1^2 \, dV_1 \int_{V_1}^{\infty} V_2 \, dV_2 \, g(V_2,\tau)$$

$$= \int_o^V dV_1 \int_V^{\infty} dV_2 \, v_1^2 \, V_2 \, g(V_2,\tau)$$

$$+ \int_0^V dV_2 \int_0^{V_2} dV_1 \, V_1^2 \, V_2 \, g(V_2,\tau)$$

$$= \frac{1}{3} V^3 \int_V^\infty V' g(V',\tau) \, dV'$$

$$+ \frac{1}{3} \int_0^V V'^4 \, g(V',\tau) \, dV' \quad . \qquad (25)$$

Substituting first (24) and then (25) into (9) we obtain (19).

## D. DEPENDENCE OF THE KINETIC EQUATIONS ON $k_o$

It is shown in Appendix A that $k_o$ is the upper limit imposed on $|\underset{\sim}{k}_1|$ in (A2) to make the $\underset{\sim}{k}_1$ - integral convergent. Its value is more or less arbitrary, except that it must satisfy the condition

$$\frac{k_D}{k_o} << 1 \quad . \tag{26}$$

We have, somewhat arbitrarily, fixed its value by eq. (15). This choice indeed satisfies (26), because in this case

$$\frac{k_D}{k_o} = \frac{1}{4\pi} \frac{k_D^3}{n_o} \quad . \tag{27}$$

but the right-hand side of (27) is << 1 under the conditions under which the BL equation is assumed to hold.

Let us test the sensitivity of the BL equation to variations in $k_o$. Since $k_o$ enters only in the form $\left(\frac{k_o}{k_D}\right)$ into eq. (9), let us take (9) at $\tau = 0$ and differentiate it with respect to $\left(\frac{k_o}{k_D}\right)$. This boils down to evaluating the quantity $\frac{\partial}{\partial\left(\frac{k_o}{k_D}\right)}\ \alpha\ \Phi(V,0)$ in (9), whose value, by (11), (13), (14), and (17), is

$$\frac{\partial}{\partial\left(\frac{k_o}{k_D}\right)}\ \alpha\ \Phi(V,0) = \frac{\alpha\left(\frac{k_o}{k_D}\right)^3}{\ln \frac{k_o}{k_D}}\ \frac{1}{\left[\left(\frac{k_o}{k_D}\right)^2 + \Gamma(V,0)\right]^2 + \left[\frac{2\pi^2}{3}\ Vg(V,\tau)\right]^2}\ . \tag{28}$$

When $\frac{k_o}{k_D} \rightarrow \infty$, (28) reduces to

$$\frac{\partial}{\partial\left(\frac{k_o}{k_D}\right)}\ \alpha\ \Phi(V,0) = \frac{\alpha}{\frac{k_o}{k_D}\ \ln \frac{k_o}{k_D}}\ . \tag{29}$$

Applying $\frac{\partial}{\partial\left(\frac{k_o}{k_D}\right)}$ to (9) and substituting (29), we obtain

$$\frac{\partial^2}{\partial\tau\,\partial\left(\frac{k_o}{k_D}\right)}\ g(V,0) = \frac{1}{\frac{k_o}{k_D}\ \ln \frac{k_o}{k_D}}\ \left[\ \frac{\partial}{\partial\tau}\ g(V,0)\right]_{RMJ}\ , \tag{30}$$

where the expression $[\frac{\partial}{\partial\tau} g(V,0)]_{RMJ}$ is really the RMJ limit of (9) at $\tau = 0$, except that $k_o$ has not been restricted to the value $\frac{KT}{e^2}$.
By (26), we obtain the condition

$$\frac{\partial^2}{\partial\tau\,\partial\left(\frac{k_o}{k_D}\right)} g(V,0) << 1 \quad , \tag{31}$$

even if $k = \frac{KT}{e^2}$, (because of (26)).

$$\left(\frac{k_o}{k_D}\right)^2 >> \Gamma(V,\tau)$$

$$\sim \frac{2\pi^2}{3} Vg(V,\tau) \quad , \tag{32}$$

and (28) is well approximated by (29).

From the above considerations, we conclude that the relaxation of the one-particle distribution function is not very sensitive to changes in the value of $k_o$, at least for $\tau < \left(\frac{k_o}{k_D}\right)$. This is of course consistent with the nearly logarithmic dependence of the BL equation (similar to the RMJ equation) for large values of $\left(\frac{k_o}{k_D}\right)$. (The reasonable insensitivity to the cut-off value $k_o$ was also noted by Rosenberg and Wu[5] in the case of the linearized BL equation).

## III. RELAXATION TIME

One may try to define a relaxation time as a function of velocity for the one-particle distribution function. For this purpose, let us restrict ourselves to isotropic distributions and write all of the expressions in terms of dimensionless variables. We define a function $\varepsilon(V,\tau)$ by

$$\varepsilon(V,\tau) \equiv \frac{\int_{V-\delta}^{V+\delta} V^2 |g(V,\tau) - g_{max}(V)|\, dV}{\int_{V-\delta}^{V+\delta} V^2\, g_{max}(V)\, dV} , \tag{33}$$

where $g_{max}(V)$ is the final Maxwellian distribution, and $\delta$ is a small number. A relaxation time $\tau_R$ may then be defined to be that value of $\tau$ after which $\varepsilon(V,\tau)$ is less than some preassigned small positive number, $\Delta$.

It is of course possible that $\varepsilon(V,\tau)$ as a function of $\tau$ decreases for a while to less than $\Delta$ and then increases again before finally approaching zero. These occurrences are easily recognized in the program, and the relaxation time is that value of $\tau$, say $\tau_R$, such that $\varepsilon(V,\tau) < \Delta$ for $\tau > \tau_R$.

# IV. NUMERICAL INTEGRATION

## A. BL EQUATION

The principal value integral in the expression for $\Gamma(V)$ was approximated by the first two non-vanishing terms of a series expansion about the singular point. Thus we obtained

$$\Gamma(V) = \frac{4\pi}{3} \left\{ \int_0^{V-h} \frac{V'^2\, g(V')\, dV'}{V'^2 - V^2} + \int_{V+h}^{\infty} \frac{V'^2\, g(V')\, dV'}{V'^2 - V^2} \right.$$

$$+ \left(V \frac{\partial g}{\partial V} + \frac{3}{2} g\right) h + \frac{1}{18} \left(V \frac{\partial^3 g}{\partial V^3} + \frac{9}{2} \frac{\partial^2 g}{\partial V^2}\right.$$

$$\left.\left. + \frac{3}{2V} \frac{\partial g}{\partial V} - \frac{3}{4V^2} g\right) h^3 \right\} , \qquad (34)$$

where $h$ is a small number.

The numerical integration of the BL equation was carried out by using the difference equation

$$g_i^{n+1} = g_i^n + \frac{\alpha \Delta\tau}{V_i^2} \left\{ \frac{1}{V_i} \left[ \left( \frac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{(\Delta V)^2} \right) \right. \right.$$

$$- \frac{1}{V_i} \left( \frac{g_{i+1}^n - g_{i-1}^n}{2 \Delta V} \right) \Bigg] \int_o^{V_i} V^2 \, dV \, G(V) \, \Phi(V)$$

$$+ \left( \frac{g_{i+1}^n - g_{i-1}^n}{2 \Delta V} \right) \int_o^{V_i} V^2 \, dV \, g(V) \, \Phi(V)$$

$$+ V_i \left( \frac{g_{i+1}^n - g_{i-1}^n}{2 \Delta V} \right) G_i^n \, \Phi_i^n + V_i^2 \left( g_i^n \right)^2 \Phi_i^n \Bigg\} \tag{35}$$

for all $V_i$'s, except $V_i = V_1 = 0$ and $V_i = V_M$, where $V_M$ is the maximum value of $V$ used. At $V_i = V_M$ the difference equation was

$$g_M^{n+1} = g_M^n + \frac{\alpha \Delta \tau}{V_M^2} \Bigg\{ \frac{1}{V_M} \Bigg[ \left( \frac{2g_M^n - 5g_{M-1}^n + 4g_{M-2}^n - g_{M-3}^n}{(\Delta V)^2} \right)$$

$$- \frac{1}{V_M} \left( \frac{3g_M^n - 4g_{M-1}^n + g_{M-2}^n}{2 \Delta V} \right) \Bigg] \int_o^{V_M} V^2 \, dV \, G(V) \, \Phi(V)$$

$$+ \left( \frac{3g_M^n - 4g_{M-1}^n + g_{M-2}^n}{2 \Delta V} \right) \int_o^{V_M} V^2 \, dV \, g(V) \, \Phi(V)$$

$$+ V_M \left( \frac{3g_M^n - 4g_{M-1}^n + g_{M-2}^n}{2 \Delta V} \right) G_M^n \, \Phi_M^n$$

$$+ V_M^2 \left( g_M^n \right)^2 \Phi_M^n \Bigg\} \tag{36}$$

In the above equations superscripts refer to time points, and subscripts, to space points. h was chosen to be equal to $\Delta V$ . The quantities $G_i^n$ , $\Phi_i^n$ , and $\Gamma_i^n$ are define by the equations

$$G_i^n \equiv G(V_i, \tau_n) \quad , \tag{37}$$

$$\Phi_i^n \equiv \Phi(V_i, \tau_n) \quad , \tag{38}$$

and

$$\Gamma_i^n \equiv \Gamma(V_i, \tau_n) \quad . \tag{39}$$

The values of integrals were approximated by finite sums. The size of subintervals in the range of integration was chosen to be $\Delta V$ in all cases. Whenever the number of subintervals was even, the integrals were evaluated by using Simpson's rule. Whenever the number of sub-intervals was odd, a combination of Simpson's rule and Newton-Cotes three-eighths quadrature formula was used. Whenever only one subinterval was available, the trapezoidal rule was used.

At $V_i = V_1 = 0$ , the value of $g_1^{n+1}$ was determined by the equation

$$g_1^{n+1} = g_2^{n+1} \quad . \tag{40}$$

This was based on the fact that

$$\frac{\partial g}{\partial V}(0, \tau) = 0 \quad , \tag{41}$$

if the BL equation is to hold at $V = 0$ for all times.

## B. RMJ EQUATION

The numerical integration was carried out by using the difference equation

$$g_i^{n+1} = g_i^n + \alpha\Delta\tau \left\{ \left( \frac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{3(\Delta V)^2} \right) (G_i^n + S_i^n) + \frac{1}{3V_i} \left( \frac{g_{i+1}^n - g_{i-1}^n}{\Delta V} \right) (G_i^n - \frac{1}{2} S_i^n + Q_i^n) + \left(g_i^n\right)^2 \right\} , \quad (42)$$

where the quantities $G_i^n$, $S_i^n$, and $Q_i^n$ are defined by the equations

$$G_i^n \equiv \int_{V_i}^{V_M} V\, g(V)\, dV , \quad (43)$$

$$S_i^n \equiv \frac{1}{V_i^3} \int_o^{V_i} V^4 \; g(V) \; dV \quad , \tag{44}$$

and

$$Q_i^n \equiv \frac{3}{2V_i} \int_o^{V_i} V^2 \; g(V) \; dV \quad . \tag{45}$$

Eq. (42) was used for all point, except $V_i = V_1 = 0$ and $V_i = V_M$ .

At $V_i = V_1 = 0$ eq. (40) was used, as in the case of the BL equation. At $V_i = V_M$ the difference equation was

$$g_M^{n+1} = g_M^n + \alpha\Delta\tau \left\{ \left( \frac{2g_M^n - 5g_{M-1}^n + 4g_{M-2}^n - g_{M-3}^n}{3(\Delta V)^2} \right) \right.$$

$$\cdot \; (G_M^n + S_M^n) + \frac{1}{3V_M} \left( \frac{3g_M^n - 4g_{M-1}^n + g_{M-2}^n}{\Delta V} \right)$$

$$\left. \cdot \; (G_M^n - \frac{1}{2} S_M^n + Q_M^n) + \left(g_M^n\right)^2 \right\} \quad . \tag{46}$$

## C. INITIAL DISTRIBUTION FUNCTIONS

The following different initial distribution functions and different values of $\left(\frac{k_o}{k_D}\right)$ were used:

### 1. Initial Gaussian Function

$$g(V,0) = 0.2289\ e^{-2.03(V-0.28)^2} , \tag{47}$$

with

$$V_M = 5.0 \tag{48}$$

and

$$\Delta V = 0.05 . \tag{49}$$

$g(V,\tau)$ was computed from $\tau = 0$ to $\tau = 2.4$ from the BL case, and from $\tau = 0$ to $\tau = 7.2$ for the RMJ case at intervals $\Delta\tau$, where

$$\Delta\tau = 0.004 , \tag{50}$$

for two different values of $\left(\frac{k_o}{k_D}\right)$:

a) $$\frac{k_o}{k_D} = 1.4178 \times 10^8 \tag{51}$$

and

b) $$\frac{k_o}{k_D} = 300 \tag{52}$$

2. Initial Resonance Function

$$g(V,0) = \frac{8}{\pi^2} \frac{1}{(V^2+1)^4} , \qquad (53)$$

with

$$V_M = 20.0 , \qquad (54)$$

$$\Delta V = 0.1 , \qquad (55)$$

and

$$\frac{k_o}{k_D} = 50 . \qquad (56)$$

$g(V,\tau)$ was computed from $\tau = 0$ to $\tau = 0.4$ for the BL case, and from $\tau = 0$ to $\tau = 3.4$ for the RMJ case at intervals $\Delta\tau$, where

$$\Delta\tau = 0.01 . \qquad (57)$$

3. Initial Maxwellian Function Coexisting with a High-Energy Gaussian Function

$$g(V,0) = \frac{0.8936}{0.512} \left(\frac{3}{2\pi}\right)^{3/2} e^{-\frac{3}{1.28} V^2}$$

$$+ 0.01192 \; e^{-100(V-2)^2} , \qquad (58)$$

with

$$V_M = 4.0 \quad , \tag{59}$$

$$\Delta V = 0.02 \quad , \tag{60}$$

and

$$\frac{k_o}{k_D} = 100 \quad . \tag{61}$$

$g(V,\tau)$ was computed from $\tau = 0$ to $\tau = 0.19$ for the BL case, and from $\tau = 0$ to $\tau = 1.3$ for the RMJ case at intervals $\Delta\tau$, where

$$\Delta\tau = 0.0005 \quad . \tag{62}$$

The calculations were performed on the IBM 7090 electronic computer.

## D. RELAXATION TIME, $\varepsilon(V,\tau)$

The number $\delta$ in eq. (33) was set equal to $\Delta V$. The integrals were performed using Simpson's rule.

$\varepsilon(V,\tau)$ was evaluated for the initial Gaussian with $\frac{k_o}{k_D} = 300$ (the Gaussian with $\frac{k_o}{k_D} = 1.4178 \times 10^8$ was not done, because it is equivalent to the Gaussian with $\frac{k_o}{k_D} = 300$ but with $\Delta\tau$ increased slightly), the initial resonance function, and the high-energy Gaussian coexisting with a Maxwellian. The quantities $\varepsilon(V,\tau)$ were computed from the solutions of the RMJ equation only, because earlier calculations showed the BL and the RMJ solutions to be almost identical for the above initial distribution functions.

$\varepsilon(V,\tau)$ was computed for the initial Gaussian for values of $\tau$ in the range $0 \leq \tau \leq 7.2$ ; for the initial resonance function, in the range $0 \leq \tau \leq 3.4$ ; and for the initial high-energy Gaussian coexisting with a Maxwellian, in the range $0 \leq \tau \leq 1.3$ .

## V. RESULTS OF NUMERICAL CALCULATIONS

### A. COLLECTIVE EFFECTS

The most important result of the numerical integrations was that no significant difference was found between the solutions of the BL equation and the solutions of the RMJ equation. For the initial Gaussian function with $\frac{k_o}{k_D} = 1.4178 \times 10^8$, the results were essentially identical for the two kinetic equations. The difference was at most 1%. This result was expected because $\ln\left(\frac{k_o}{k_D}\right)^4 \gg 1$. The same kind of behavior was found in the case of the initial Gaussian function with $\frac{k_o}{k_D} = 300$, and the initial resonance function, with $\frac{k_o}{k_D} = 50$.

This seems to be an

interesting result, because in these two cases quantities of the order of unity cannot be neglected relative to $\ln\left(\frac{k_o}{k_D}\right)^4$ . Table I shows the values of $g(V,\tau)$ at $\tau = 2.4$ , calculated from both the BL and the RMJ equations, for the initial Gaussian with $\frac{k_o}{k_D} = 300$ , for several values of V . Table II shows the values of $g(V,\tau)$ at $\tau = 0.4$ , calculated from both the BL and the RMJ equations, for the initial resonance function.

Table I

| V | $\tau = 0$<br>$g(V,0)$ | $\tau = \infty$<br>$g_{max}(V)$ | $\tau=2.4$<br>$g(V,\tau)$<br>RMJ | $\tau=2.4$<br>$g(V,\tau)$<br>BL |
|---|---|---|---|---|
| 0 | 0.2055 | 0.3299 | 0.3257 | 0.3243 |
| 0.25 | 0.2285 | 0.3004 | 0.2979 | 0.2973 |
| 0.5 | 0.2076 | 0.2268 | 0.2256 | 0.2253 |
| 0.75 | 0.1464 | 0.1419 | 0.1419 | 0.1418 |
| 1.0 | 0.08006 | 0.07362 | 0.07409 | 0.07410 |
| 1.25 | 0.03397 | 0.03166 | 0.03205 | 0.03206 |
| 1.5 | 0.01118 | 0.01129 | 0.01140 | 0.01141 |
| 1.75 | 0.002856 | 0.003337 | 0.003274 | 0.003273 |
| 2.0 | 0.0005658 | 0.0008178 | 0.0007334 | 0.0007328 |
| 2.25 | 0.000087 | 0.0001661 | 0.0001233 | 0.0001232 |
| 2.5 | 0.000010 | 0.000028 | 0.000015 | 0.000015 |

Table II

| V | $\tau = 0$ $g(V,0)$ | $\tau = \infty$ $g_{max}(V)$ | $\tau=0.4$ $g(V,\tau)$ RMJ | $\tau=0.4$ $g(V,\tau)$ BL |
|---|---|---|---|---|
| 0 | 0.7789 | 0.3299 | 0.5979 | 0.5934 |
| 0.4 | 0.4477 | 0.2595 | 0.4067 | 0.4066 |
| 0.8 | 0.1121 | 0.1263 | 0.1241 | 0.1234 |
| 1.2 | 0.02287 | 0.03805 | 0.02320 | 0.02320 |
| 1.6 | 0.005046 | 0.007091 | 0.004667 | 0.004692 |
| 2.0 | 0.001297 | 0.0008178 | 0.001192 | 0.001198 |
| 2.4 | 0.0003882 | 0.0000584 | 0.0003622 | 0.0003633 |
| 2.8 | 0.0001327 | 0.0000026 | 0.0001258 | 0.0001261 |
| 3.2 | 0.0000508 | 0.0 | 0.0000488 | 0.0000488 |
| 3.6 | 0.0000213 | 0.0 | 0.0000207 | 0.0000207 |
| 4.0 | 0.0000097 | 0.0 | 0.0000095 | 0.0000095 |

Perhaps the most interesting case was that of the initial Maxwellian coexisting with a sharp high-energy Gaussian peaked at $V = 2$. This case is similar to the test particle problem. But the behavior of this case was very similar to the behavior of the previous cases. The relaxation of the Maxwellian part of the initial distribution proceeded without really exhibiting collective effects. This is not surprising any more in view of the behavior of all of the previous cases. However, even the peak of the Gaussian failed to exhibit collective effects. The difference between the BL and the RMJ solutions for the peak was less than 2%. A difference of about 4% was observed to the right of the peak at velocities which were between 2.2 and 2.3 thermal speeds. However, these differences are too small to show unmistakably the importance of collective effects. Table III shows the values of $g(V,\tau)$ at $\tau = 0.19$ in the vicinity of the Gaussian peak for the solutions of the BL and the RMJ equations.

Table III

| V | $\tau = 0$<br>$g(V,0)$ | $\tau = \infty$<br>$g_{max}(V)$ | $\tau=0.19$<br>$g(V,\tau)$<br>RMJ | $\tau=0.19$<br>$g(V,\tau)$<br>BL |
|---|---|---|---|---|
| 1.70 | 0.00066010 | 0.00432260 | 0.00163051 | 0.00162568 |
| 1.74 | 0.00049089 | 0.00351647 | 0.00205418 | 0.00204975 |
| 1.78 | 0.00043724 | 0.00284698 | 0.00279051 | 0.00278595 |
| 1.82 | 0.00071164 | 0.00229391 | 0.00380954 | 0.00380243 |
| 1.86 | 0.00185265 | 0.00183944 | 0.00499052 | 0.00497691 |
| 1.90 | 0.00450775 | 0.00146794 | 0.00611075 | 0.00608780 |
| 1.94 | 0.00840283 | 0.00116586 | 0.00688563 | 0.00685531 |
| 1.98 | 0.01151361 | 0.00092151 | 0.00706345 | 0.00703384 |

TABLE III - continued

| V | $\tau = 0$<br>$g(V, 0)$ | $\tau = \infty$<br>$g_{max}(V)$ | $\tau = 0.19$<br>$g(V, \tau)$<br>RMJ | $\tau = 0.19$<br>$g(V, \tau)$<br>BL |
|---|---|---|---|---|
| 2.0 | 0.01197109 | 0.00081779 | 0.00688734 | 0.00686224 |
| 2.02 | 0.01149523 | 0.00072488 | 0.00653863 | 0.00652063 |
| 2.06 | 0.00834550 | 0.00056748 | 0.00541764 | 0.00541858 |
| 2.10 | 0.00440466 | 0.00044213 | 0.00398559 | 0.00400460 |
| 2.14 | 0.00169192 | 0.00034282 | 0.00258245 | 0.00261129 |
| 2.18 | 0.00047530 | 0.00026454 | 0.00146207 | 0.00149051 |
| 2.22 | 0.00009982 | 0.00020316 | 0.00071784 | 0.00073915 |
| 2.26 | 0.00001746 | 0.00015528 | 0.00030369 | 0.00031639 |
| 2.30 | 0.00000384 | 0.00011811 | 0.00011028 | 0.00011643 |

Graphs 1, 2, 3, and 4 are, respectively, the plots of the solutions of the BL equation (or the RMJ equation, since the two give almost identical results) for the four cases mentioned above.

## B. RELAXATION TO A MAXWELLIAN

Graphs 5, 6, and 7 are, respectively, the plots of the solutions of the RMJ equation for the initial Gaussian with $\frac{k_o}{k_D} = 300$, the initial resonance function, and the initial high-energy Gaussian coexisting with a Maxwellian. Graphs 5 and 6 agree with the earlier calculations of Rosenbluth et. al.[4], as well as with the findings of Ree et. al.[3], that the high-energy tail of an initial distribution function relaxes much slower than the low-energy portions. Graph 5 shows that the point $V = 0$, which at $\tau = 0$ is below the final Maxwellian, approaches the final Maxwellian and then overshoots it. However, in the time interval $0 \leq \tau \leq 7.2$ the point $V = 0$ was not found to start descending toward the final Maxwellian. The two normalization conditions, equations (7) and (8), remained good throughout the whole time interval. The error at $\tau = 7.2$ in the particle-number normalization was less than 0.013%, while the error in the energy normalization was less than 1.8%. We think that the distribution was not followed long enough in time to permit the point $V = 0$ to start descending toward the final Maxwellian. The fact that it seems to take a very long time for this to occur is not surprising, since the initial distribution function is very broad and its gradients in velocity space are small.

Graph 6 shows the distribution function for $\tau > 0$ dipping below the initial distribution and moving farther away from the final

Maxwellian in the higher energy portion of the graph. This tendency to dip seems to increase with time and to move down the high-energy tail. However, what happens to the dip after a very long time can only be guessed, because the distribution function was not followed long enough in time. One of the reasons for not following the distribution function longer in time was the large error creeping into the energy normalization. Particle normalization remained good (error was less than 0.11% at $\tau = 3.4$) but the energy normalization error was $\sim 4.8\%$ at $\tau = 3.4$. The value of the energy normalization showed a tendency to decrease monotonically with time. The change from $\tau$ to $(\tau + \Delta\tau)$ was steadily decreasing as $\tau$ got bigger and bigger, but this decrease was not fast enough. It was present in spite of the fact that the stability criterion on the magnitudes of $\Delta V$ and $\Delta\tau$ was satisfied. Extension of the range of $V$ from $0 \leq V \leq 20$ to $0 \leq V \leq 40$ to include a greater portion of the high-energy tail or readjustments in the values of $\Delta V$ and $\Delta\tau$, within the scope of the stability criterion, did not improve the situation much. Since the cause of the trouble could not be pinpointed, the decision was made to take the maximum value of $\tau$ to be that $\tau$ for which the error in energy normalization was less than 5%.

Graph 7 demonstrates the fact that the rate of relaxation of a portion of an initial distribution depends strongly on the gradients of that portion in velocity space. Thus a high-energy portion with large gradients may relax faster than a low-energy portion with small gradients.

## C. RELAXATION TIME

Graphs 8, 9, and 10 are the plots on semi-log paper of the quantity $\varepsilon(V,\tau)$ as a function of $\tau$, with the values of V serving as the curve parameters. Graph 8 is for the initial Gaussian with $\frac{k_o}{k_D} = 300$, Graph 9 is for the initial resonance function, and Graph 10 is for the high-energy Gaussian coexisting with a Maxwellian. (The case of the initial Gaussian with $\frac{k_o}{k_D} = 1.4178 \times 10^8$ was not treated separately, because, as it was stated already, increasing the value of $\frac{k_o}{k_D}$ in the RMJ equation is equivalent to keeping $\frac{k_o}{k_D}$ constant and increasing $\Delta\tau$ slightly in the finite difference analogue of the differential equation). Graphs 8, 9, and 10 show the impossibility of defining a relaxation time, $\tau_R$. For one thing, $\varepsilon(V,\tau)$ in Graph 8 is an increasing function of $\tau$ for $6 < \tau < 7.2$, for all V's but $V = 2.25$. In Graph 9, $\varepsilon(V,\tau)$ keeps increasing for $2.5 < \tau \lesssim 3.4$ for $V = 2$. It is not known to what value $\varepsilon(V,\tau)$ will increase before decreasing again. Besides, $\varepsilon(V,\tau)$ may keep on oscillating as $\tau$ increases, until $\varepsilon \to 0$ as $\tau \to \infty$; but we do not know the size of the amplitudes of these oscillations as a function of time. The curve with $V = 2.25$ in Graph 8; all of the curves in Graph 9, with the exception of the one with $V = 2.0$; and all of the curves in Graph 10 for $\tau > 0.7$ are monotonically decreasing with time. In fact, for large values of $\tau$ they approximate straight lines on the semi-log paper. But we should not conclude from this fact that for these curves the decay becomes exponential after a certain time. A look at Graphs 6 and 7 discloses that these curves may cease decreasing and start increasing after a while. The curves in Graphs 9 and 10 were not followed long enough in time to exhibit this behavior.

We conclude from the above discussion that it is impossible to define a relaxation time, $\tau_R$, as explained in (III), within the time limits used in the calculations. We also suspect that, in general, an initial distribution function does not decay to a final Maxwellian exponentially, even if the exponential decay is assumed to set in after some time, and not immediately. This suspicion applies to finite V. As for the high-energy tail of a distribution, it is still possible to visualize an exponential decay there. For example, in the case of the initial Gaussian, $\varepsilon(V,\tau=0) \to 1$ as $V \to \infty$. If we make use of the fact that the high-energy tail of a distribution function relaxes very slowly toward the final Maxwellian, $\varepsilon(V,\tau) \to 1$ as $V \to \infty$, even for large $\tau$'s. This would give us almost a straight line when plotted on the semi-log paper. Therefore it is possible for the relaxation to assume the form of an exponential decay in the high-energy tail. This argument would also be valid for other initial distribution functions which approach zero faster than the final Maxwellian as $V \to \infty$ (like the Gaussian above). In Appendix C we present a mathematical proof of the impossibility of an exponential decay of an initial distribution function to a final Maxwellian.

## VI. DISCUSSION

The lack of any significant difference between the solutions of the BL and the RMJ equations for the cases treated in this paper has to be taken as a matter of fact. It is somewhat surprising in cases in which $\ell n \frac{k_o}{k_D}$ is of the order of unity. The greatest puzzle is presented by the case of a Maxwellian coexisting with a sharp, high-energy Gaussian because of its similarity with the test particle problem.

The solution of the BL equation for the test particle problem indicates that collective effects may become important when the test-particle velocities are high and $\ell n \frac{k_o}{k_D} \sim O(1)$. By means of arguments analogous to those based on the solution of the RMJ equation, we obtain some characteristic times for the test particles, such as the "slowing down time", $\tau_s$, given by

$$\tau_s = \frac{M_t u^3}{e_t^2 \bar{\omega}_p^2 \left[ \left( 1 + \frac{\theta k_D^2}{M_t \bar{\omega}_p^2} \right) \ell n \frac{k_o}{k_D} + \ell n \frac{k_D u}{\bar{\omega}_p} \right]} \tag{63}$$

the "deflection time", $\tau_D$, given by

$$\tau_D = \frac{M_t^2 u^5}{2e_t^2 \theta \bar{\omega}_p^2 \left[ \frac{k_D^2 u^2}{\bar{\omega}_p^2} \ell n \frac{k_o}{k_D} - \ell n \frac{k_D u}{\bar{\omega}_p^2} \right]} \tag{64}$$

and the "energy exchange time", $\tau_W$, given by

$$\tau_W = \frac{M_t u^3}{2e_t^2 \bar{\omega}_p^2 \left[\ln \frac{k_o}{k_D} + \ln \frac{k_D u}{\bar{\omega}_p}\right]} \quad . \tag{65}$$

Here $e_t$ and $M_t$ are the test-particle charge and mass respectively; $u$ is the test particle velocity; $\theta$, defined by $\theta \equiv KT$, is the field particle temperature, $\bar{\omega}_p$ is the plasma frequency of the field particles; and $k_D$ is the Debye wave number of the field particles. In equations (64), (65), and (66) the term containing $\left(\ln \frac{k_o}{k_D}\right)$ is the same as the one obtained from the solution of the RMJ equation for the test particle problem. The term containing $\left(\ln \frac{k_D u}{\bar{\omega}_p}\right)$ derives from the collective effects.

On the other hand, if we assume an isotropic velocity distribution for test particles and, by analogy with the treatment of MacDonald, Rosenbluth, and Chuck[4], write the test-particle distribution function in the form

$$f_t(v,t) = g(v,t)\, e^{-\frac{M_t v^2}{2\theta}} \quad , \tag{66}$$

we can define a characteristic time it takes the inflection point of $g(v,t)$ to diffuse into the high-energy tail of the distribution by

$$\tau_o = \frac{M_t\, v_{inf}^3}{e_t^2\, \bar{\omega}_p^2 \left[\ln \frac{k_o}{k_D} + \ln \frac{k_D v_{inf}}{\bar{\omega}_p}\right]} \quad , \tag{67}$$

where $v_{inf}$ is the velocity at the inflection point of $g(v,t)$. (The derivations of characteristic times are given in Appendix B.)

The reason for the disparity between the test particle problem and the numerical solutions of problems discussed in this paper has to be sought in the behavior of the Landau denominator, $D^{+}(-\underset{\sim}{k}_1, i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)$, defined in (A6) of Appendix A, which appears on the right-hand side of the BL equation (eq. (A2)). In the RMJ equation $D^{+} = 1$, because the collective effects are neglected. In the BL equation, the value of the Landau denominator varies and may even assume the value zero. When this happens, the integrand of the integral, on the right-hand side of eq. (A2), may contain a singularity if the zero of $D^{+}$ is not canceled by a zero of the numerator of the integrand. We shall see that in the problems which were solved numerically in this paper the integrand has no singularities, while in the test particle problem the integrand does have singularities.

Let us confine ourselves to isotropic distributions. By (A9), (A13), and (A17) we see that $\text{Im}(D^{+}) = 0$ only when $u_1 = \frac{\underset{\sim}{k}_1}{k_1} \cdot \underset{\sim}{v}_1 = 0$ or $u_1 = \frac{\underset{\sim}{k}_1}{k_1} \cdot \underset{\sim}{v}_1 \to \infty$. When $u_1 = 0$, we see by (A16), (A13), and (A9) that $\text{Re}(D^{+}) \neq 0$. Therefore $D^{+} \neq 0$. When $u_1 \to \infty$, $\text{Im}(D^{+}) \to 0$, since $f(|u_1|) \to 0$, and $\text{Re}(D^{+}) \to 1 - \frac{\omega_p^2}{k_1^2 u_1^2}$. It is possible to find a $v_1$ and a $k_1$ in the range $0 < |k_1| \leq k_o$ such that

$$\frac{\omega_p^2}{k_1^2 u_1^2} = 1 \qquad (68)$$

This choice will yield $D^+ = 0$.

Suppose we look at the problems which were solved numerically. When $|v_1| \to \infty$, by (A11) and (A12), $f(v_1) \to 0$ faster than $F(u_1)$. Therefore the numerator in the integrand of the integral on the right side of eq. (A2), is of the order of $[f(v_1)]^2$, and the Landau denominator is, by (A17), also of the order $[f(v_1)]^2$. Hence, for $|v_1| \to \infty$, the zero of $D^+$ is canceled by the zero of the numerator of the integrand, and the integrand does not get too close to any of its singularities.

Let us now look at the test particle problem. Here, on account of the tenuity of the test particle distribution, only the field particle distribution enters into the evaluation of the Landau denominator. For $|v_1| \to \infty$, $D^+$ is of the order of the square of the field particle distribution function. In the numerator of the integrand of eq. (A2), $F(u_1)$ and $\frac{\partial F}{\partial u_1}$ refer to the field particles, while $f(v_1)$ and $\frac{\partial f}{\partial u_1}$ refer to the test particles. There exists a high velocity range in which the test particle distribution is still finite while the field particle distribution is already approaching zero. Therefore the Landau denominator will vanish faster than the numerator of the integrand, and the integrand will get very close to a singularity.

The preceding arguments confirm the fact that collective effects become significant in the solution of the BL equation only when the integrand in eq. (A2) gets very close to a singularity, at which $D^{+} = 0$ , in the range of integration. Such a situation may be realized, for example, in the anisotropic case of two contrastreaming electron plasmas. This problem is certainly worth a more thorough investigation.

## APPENDIX A

## DERIVATION OF THE ISOTROPIC BL EQUATION

The general anisotropic BL equation for a spatially uniform electron plasma embedded in a uniformly smeared out background of positive charge has the form[7]

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{v}_1} \cdot \underset{\sim}{J}(\underset{\sim}{v}_1, t) \quad , \tag{A1}$$

where $f(\underset{\sim}{v}_1, t)$ is the one-particle distribution function with the two normalization conditions given by equations (1) and (2). $\underset{\sim}{J}(\underset{\sim}{v}_1, t)$ is defined by the expression

$$\underset{\sim}{J}(\underset{\sim}{v}_1, t) \equiv \frac{2n_o e^4}{m^2} \int \frac{\underset{\sim}{k}_1 \, d\underset{\sim}{k}_1}{k_1^4} \frac{\left[f(\underset{\sim}{v}_1) \frac{\partial F}{\partial u_1} - F(u_1) \frac{\partial f}{\partial u_1}\right]}{|D^+(-\underset{\sim}{k}_1, i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1|^2} \quad , \tag{A2}$$

$F(u)$ is defined by

$$F(u) \equiv \int f(\underset{\sim}{v}_2, t) \, \delta\left(\frac{\underset{\sim}{k}_1}{k_1} \cdot \underset{\sim}{v}_2 - u\right) d\underset{\sim}{v}_2 \quad , \tag{A3}$$

$u_1$ and $\frac{\partial}{\partial u_1}$ are defined by

$$u_1 \equiv \frac{\underset{\sim}{k}_1}{k_1} \cdot \underset{\sim}{v}_1 \tag{A4}$$

and

$$\frac{\partial}{\partial u_1} \equiv \frac{\underset{\sim}{k}_1}{k_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} \quad . \tag{A5}$$

$D^+(-\underset{\sim}{k}_1, i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)$, the Landau denominator, is given by the expression

$$D^+(-\underset{\sim}{k}_1, i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1) = 1 - \frac{\omega_p^2}{k_1^2} \int_{-\infty}^{\infty} \frac{\partial F}{\partial u} \frac{du}{u - u_1 + i\varepsilon} \quad , \tag{A6}$$

with $\omega_p$, the plasma frequency, given by

$$\omega_p = \left(\frac{4\pi n_o e^2}{m}\right)^{\frac{1}{2}} , \tag{A7}$$

and $\varepsilon$ being a small positive number.

Let us also define the quantity $\Psi$ by the expression

$$\Psi \equiv - \int_{-\infty}^{\infty} \frac{\partial F}{\partial u} \frac{du}{u - u_1 + i\varepsilon} . \tag{A8}$$

Thus (A6) can be rewritten as

$$D^+ = 1 + \frac{\omega_p^2}{k_1^2} \Psi . \tag{A9}$$

Let us now specialize all of the above formulas to the case of isotropic velocity distributions. We can write

$$f(\underset{\sim}{v}_1, t) = f(v_1, t) . \tag{A10}$$

In such cases (A3) shows that both $F(u)$ and $\frac{\partial F}{\partial u}$ do not depend on $\underset{\sim}{k}_1$.

To simplify integrations, choose the z-axis in the direction of $\underset{\sim}{v}_1$. Let $(k_1, \theta, \phi)$ be the polar-spherical coordinates of the vector $\underset{\sim}{k}_1$. We can now perform two integrations in (A3) as follows:

We can now perform two integrations in (A3) as follows

$$F(u) = \int_0^\infty v^2 \, dv \, f(v) \int_0^\pi \sin\theta d\theta \, \delta(v_1 \cos\theta - u) \int_0^{2\pi} d\phi$$

$$= 2\pi \int_0^\infty v^2 \, dv \, f(v) \int_{-1}^{1} d\mu \, \delta(v \mu - u)$$

$$= 2\pi \int_{|u|}^\infty v \, f(v) \, dv \quad . \tag{A11}$$

From (A11) we get

$$\frac{\partial F}{\partial u} = -2\pi u \, f(|u|) \quad . \tag{A12}$$

Let us now simplify the expression for $\Psi$. If $\psi_r$ and $\psi_i$ are defined to be, respectively, the real and the imaginary parts of $\Psi$, i.e.

$$\Psi = \psi_r + i\psi_i \quad , \tag{A13}$$

we obtain the following expressions for $\psi_r$ and $\psi_i$:

$$\psi_r = -P \int_{-\infty}^{\infty} \frac{\partial F}{\partial u} \frac{du}{u - v_1 \cos \theta} \tag{A14}$$

and

$$\psi_i = \pi \frac{\partial F}{\partial u} (v_1 \cos \theta) \quad . \tag{A15}$$

With the aid of (A12), equations (A14) and (A15) can be written in the form

$$\psi_r = 2\pi \quad P \int_{-\infty}^{\infty} \frac{u\, f(|u|)\, du}{u - u_1}$$

$$= 4\pi \quad P \int_{0}^{\infty} \frac{u^2\, f(u)\, du}{u^2 - u_1^2} \tag{A16}$$

and

$$\psi_i = -2\pi^2 \, u_1 \, f(|u_1|) \quad , \tag{A17}$$

where we have made use of eq. (A4). We can see from (A14) and (A15) that $\Psi$ does not depend on $k_1$ or $\phi$ . Consequently $D^+$ does not depend on $\phi$ .

Let us now try to simplify the expression for $\underset{\sim}{J}(\underset{\sim}{v}_1, t)$ . Eq. (A2) can be rewritten in the form

$$J(\underset{\sim}{v}_1,t) = \frac{2n_o e^4}{m^2} \left\{ f(v_1) \int \frac{\underset{\sim}{k}_1 \, d\underset{\sim}{k}_1}{k_1^4} \; \frac{\frac{\partial F}{\partial u_1}}{|D^+|^2} \right.$$

$$\left. - \frac{\partial f}{\partial \underset{\sim}{v}_1} \cdot \int \frac{\underset{\sim}{k}_1 \, \underset{\sim}{k}_1 \, d\underset{\sim}{k}_1}{k_1^5} \; \frac{F(u_1)}{|D^+|^2} \right\} . \tag{A18}$$

In eq. (A18) the integrations over $\phi$ can be performed immediately. If we also change the variable of integration over $\theta$ from $\theta$ to $u_1$, such that

$$du_1 = -v_1 \sin\theta \; d\theta \quad , \tag{A19}$$

and then interchange the order of integrations over $k_1$ and $u_1$, we obtain

$$J(\underset{\sim}{v}_1,t) = \frac{4\pi n_o e^4}{m^2} \frac{\underset{\sim}{v}_1}{v_1^3} \left\{ f(v_1) \int_{-v_1}^{v_1} u_1 \, du_1 \frac{\partial F}{\partial u_1} \int_o^{k_o} \frac{dk_1}{k_1} \frac{1}{|D^+|^2} \right.$$

$$\left. - \frac{1}{v_1} \frac{\partial f}{\partial v_1} \int_{-v_1}^{v_1} u_1^2 \, du_1 \, F(u_1) \int_o^{k_o} \frac{dk_1}{k_1} \frac{1}{|D^+|^2} \right\} . \tag{A20}$$

We have chosen an upper cut-off $k_o$ on the integral over $|\underset{\sim}{k}_1|$ in eq. (A20). Its meaning will become clearer when the integral over $|\underset{\sim}{k}_1|$ is evaluated. The integral over $|\underset{\sim}{k}_1|$ is an even function of $u_1$. Consequently the integrands in the two integrals over $u_1$ are even functions of $u_1$.

If we make use of (A12) and substitute (A20) into (A1), we obtain

$$\frac{\partial f}{\partial t} = \frac{8\pi n_o e^4}{m^2} \frac{1}{v_1^2} \frac{\partial}{\partial v_1} \left\{ 2\pi \, f(v_1) \int_0^{v_1} u_1^2 \, du_1 \, f(u_1) \int_0^{k_o} \frac{dk_1}{k_1} \frac{1}{|D^+|^2} \right.$$

$$\left. + \frac{1}{v_1} \frac{\partial f}{\partial v_1} \int_0^{v_1} u_1^2 \, du_1 \, F(u_1) \int_0^{k_o} \frac{dk_1}{k_1} \frac{1}{|D^+|^2} \right\} . \qquad \text{(A21)}$$

Let us now perform the integration over $k_1$.

$$\int_0^{k_o} \frac{dk_1}{k_1} \frac{1}{|D^+|^2} = \int_0^{k_o} \frac{dk_1}{k_1} \frac{1}{\left| 1 + \frac{\omega_p^2}{k_1^2} \Psi \right|^2}$$

$$= \frac{1}{2i \, \text{Im} \, \Psi} \int_0^{k_o} k_1 \, dk_1 \left[ \frac{\Psi}{k_1^2 + \omega_p^2 \Psi} - \frac{\Psi^*}{k_1^2 + \omega_p^2 \Psi^*} \right]$$

$$= \frac{\text{Im} \left[ \Psi \, \ln \frac{k_o^2 + \omega_p^2 \Psi}{\omega_p^2 \Psi} \right]}{2 \, \text{Im} \, \Psi}$$

$$= \frac{1}{4} \left[ \log \frac{|k_o^2 + \omega_p^2 \psi|^2}{|\omega_p^2 \psi|^2} \right.$$

$$\left. + 2 \frac{\psi_r}{\psi_i} \tan^{-1} \frac{\omega_p^2 \psi_i}{k_o^2 + \omega_p^2 \psi_r} - 2 \frac{\psi_r}{\psi_i} \tan^{-1} \frac{\psi_i}{\psi_r} \right] . \quad \text{(A22)}$$

Let us define the quantities $H(u_1)$, $L(u_1)$, and $\Phi(u_1)$ by

$$H(u_1) \equiv \ln \frac{|k_o^2 + \omega_p^2 \psi|^2}{|\omega_p^2 \psi|^2} , \quad \text{(A23)}$$

$$L(u_1) \equiv \tan^{-1} \frac{\omega_p^2 \psi_i}{k_o^2 + \omega_p^2 \psi_r} - \tan^{-1} \frac{\psi_i}{\psi_r} , \quad \text{(A24)}$$

and

$$\Phi(u_1) \equiv \frac{1}{\ln \frac{k_o}{k_D}} \int_o^{k_o} \frac{dk_1}{k_1} \frac{1}{|D^+|^2} , \quad \text{(A25)}$$

As one can see, $k_o$ has to be a finite number if (A22) - (A25) are to remain finite. We shall define its value somewhat arbitrarily by (15).

We shall now introduce the dimensionless variables defined by (3) - (5). By (A16) and (12), we obtain

$$\psi_r = \frac{3}{v_o^2}\,\Gamma(V,\tau) \quad . \tag{A26}$$

Substituting (A26) and (A17) into (A23) and (A24) and making use of (3) - (5), we obtain

$$H(u_1) = H(V) \tag{A27}$$

and

$$L(u_1) = L(V) \quad , \tag{A28}$$

where $H(V)$ and $L(V)$ are given, respectively, by (13) and (14). A look at (A25) shows that

$$\Phi(u_1) = \Phi(V) \quad , \tag{A29}$$

where $\Phi(V)$ is given by (11). Further, substituting (3) - (5) into (A11), we obtain

$$F(u) = \frac{2\pi}{v_o}\;G(V) \quad , \tag{A30}$$

where $G(V)$ is given by (10).

Substituting (3) - (5), (A25), (A29), and (A30) into (A21) we obtain eq. (9).

## APPENDIX B

## THE TEST PARTICLE PROBLEM

The BL equation for the test particle problem can be written in the form[7]

$$\frac{\partial f_t}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{v}} \cdot [\underset{\approx}{F}(\underset{\sim}{v})\ f_t(\underset{\sim}{v})] + \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{v}\ \partial \underset{\sim}{v}} : [\underset{\approx}{T}(\underset{\sim}{v})\ f_t(\underset{\sim}{v})] \tag{B1}$$

where $f_t(\underset{\sim}{v})$ is the test particle distribution function. $\underset{\approx}{F}(\underset{\sim}{v})$ is given by

$$\underset{\approx}{F}(\underset{\sim}{v}) = \underset{\approx}{F}_1(\underset{\sim}{v}) + \underset{\approx}{F}_2(\underset{\sim}{v}) \quad , \tag{B2}$$

with $\underset{\approx}{F}_1(\underset{\sim}{v})$ being defined by[8]

$$\underset{\approx}{F}_1(\underset{\sim}{v}) = -\frac{e_t^2}{M_t}\ (\bar{\omega}_p^2 + \frac{\theta}{M_t}\ k_D^2)\left(\ell n\ \frac{k_o}{k_D}\right)\frac{\underset{\sim}{v}}{v^3} \tag{B3}$$

and $\underset{\approx}{F}_2(v)$ being defined by[8]

$$\underset{\approx}{F}_2(\underset{\sim}{v}) = -\frac{e_t^2}{M_t}\ \ \bar{\omega}_p^2\left(\ell n\ \frac{k_D v}{\bar{\omega}_p}\right)\frac{v}{\underset{\sim}{v}^3} \quad . \tag{B4}$$

$\underset{\approx}{T}(\underset{\sim}{v})$ is given by

$$\underset{\approx}{T}(\underset{\sim}{v}) = \underset{\approx}{T}_1(\underset{\sim}{v}) + \underset{\approx}{T}_2(\underset{\sim}{v}) \quad , \tag{B5}$$

with $\underset{\approx}{T}_1(\underset{\sim}{v})$ being defined by[8]

$$\underset{\approx}{T}_1(\underset{\sim}{v}) = \frac{e_t^2\,\theta\,\bar{\omega}_p^2}{M_t^2}\left(\ell n\,\frac{k_o}{\bar{\omega}_p^2}\right)\left[\frac{k_D^2}{\bar{\omega}_p^2}\left(\frac{v^2\,\underset{\approx}{1} - \underset{\sim}{v}\,\underset{\sim}{v}}{v^3}\right) - \left(\frac{v^2\,\underset{\approx}{1} - 3\underset{\sim}{v}\,\underset{\sim}{v}}{v^5}\right)\right] \qquad \text{(B6)}$$

and $\underset{\approx}{T}_2(\underset{\sim}{v})$ being defined by[8]

$$\underset{\approx}{T}_2(\underset{\sim}{v}) = -\frac{e_t^2\,\theta\,\bar{\omega}_p^2}{M_t^2}\left(\ell n\,\frac{k_D v}{\bar{\omega}_p}\right)\left(\frac{v^2\,\underset{\approx}{1} - 3\,\underset{\sim}{v}\,\underset{\sim}{v}}{v^5}\right). \qquad \text{(B7)}$$

When we substitute (B2) - (B7) into (B1), perform all of the differentiations, and drop terms of the order $\frac{v_o}{v}$, where $v_o$ is the thermal velocity of field particles, we obtain the differential equation

$$\frac{\partial f_t}{\partial t} \cong \left\{\frac{e_t^2}{M_t}\left(\bar{\omega}_p^2 - \frac{\theta k_D^2}{M_t}\right)\left(\ell n\,\frac{k_o}{k_D}\right)\frac{\underset{\sim}{v}}{v^3}\cdot\frac{f_t}{\underset{\sim}{v}} + \frac{1}{2}\,\underset{\approx}{T}_1 : \frac{\partial^2 f_t}{\partial v\,\partial v}\right\} + \left\{\frac{e_t^2\,\bar{\omega}_p^2}{M_t}\left(\ell n\,\frac{k_D v}{\bar{\omega}_p}\right)\frac{\underset{\sim}{v}}{v^3}\cdot\frac{\partial f_t}{\partial \underset{\sim}{v}} + \frac{1}{2}\,\underset{\approx}{T}_2 : \frac{\partial^2 f_t}{\partial v\,\partial v}\right\}. \qquad \text{(B8)}$$

The terms inside the first pair of braces on the right side of (B8) are identical with the ones obtained from the solution of the RMJ equation.[6] The terms inside the second pair of braces are due to the collective effects.

If we now confine ourselves to isotropic distributions, the following relations hold:

$$\underset{\approx}{1} : \frac{\partial^2 f_t}{\partial \underset{\sim}{v}\, \partial \underset{\sim}{v}} = \frac{\partial^2 f}{\partial v^2} + \frac{2}{v} \frac{\partial f_t}{\partial v} \tag{B9}$$

and

$$\underset{\sim}{v}\, \underset{\sim}{v} : \frac{\partial^2 f_t}{\partial \underset{\sim}{v}\, \partial \underset{\sim}{v}} = v^2 \frac{\partial^2 f_t}{\partial v^2} \tag{B10}$$

Substituting (B9) and (B10) into (B8), and dropping terms of the form $\frac{f_t}{v^3}$ as well as terms of order one relative to terms of order $\left(\ell n \frac{k_D v}{\overline{\omega}_p}\right)$, we obtain the differential equation

$$\frac{M_t}{e_t^2\, \overline{\omega}_p^2} \frac{\partial f_t}{\partial t} \doteq \frac{1}{M_t v^2} \left[ \ell n \frac{k_o}{k_D} + \ell n \frac{k_D v}{\overline{\omega}_p} \right] \left[ \theta \frac{\partial}{\partial v} \left( \frac{1}{v} \frac{\partial f_t}{\partial v} \right) + M_t \frac{\partial f_t}{\partial v} \right] . \tag{B11}$$

Let us now define a function $g_t(v,t)$ by

$$f_t(v,t) = g_t(v,t)\, e^{-\frac{M_t v^2}{2\theta}} \quad . \tag{B12}$$

$g_t(v,t)$ satisfies the diffusion equation

$$\frac{M_t}{e_t^2\, \bar{\omega}_p^2} \frac{\partial g_t}{\partial t} = \frac{1}{M_t v^2} \left[ \ell n \frac{k_o}{k_D} + \ell n \frac{k_D v}{\bar{\omega}_p} \right] \left[ \frac{\theta}{v} \frac{\partial^2 g_t}{\partial v^2} \right.$$

$$\left. - M_t \left( 1 + \frac{\theta}{M_t v^2} \right) \frac{\partial g_t}{\partial v} \right] \quad . \tag{B13}$$

If $g_t(v,t)$ has an inflection point at $v_{inf}$, the speed with which this point diffuses into the high-speed region is given by

$$\left( \frac{\partial v}{\partial t} \right)_{g_t = \mathrm{const}} = \frac{e_t^2\, \bar{\omega}_p^2}{M_t v_{inf}^2} \left[ \ell n \frac{k_o}{k_D} + \ell n \frac{k_D v_{inf}}{\bar{\omega}_p} \right] \quad . \tag{B14}$$

We can define a characteristic time[4] $\tau_o$ by

$$\tau_o \equiv \frac{v_{inf}}{\left( \frac{\partial v}{\partial t} \right)_{g_t = \mathrm{const}}} = \frac{M_t\, v_{inf}^3}{e_t^2\, \bar{\omega}_p^2 \left[ \ell n \frac{k_o}{k_D} + \ell n \frac{k_D v_{inf}}{\bar{\omega}_p} \right]} \quad . \tag{B15}$$

Let us now go to the anisotropic case. Let us assume that

$$f_t(\underset{\sim}{v}, t=0) = \delta(\underset{\sim}{v} - \underset{\sim}{u}) \quad . \tag{B16}$$

We shall take velocity moments of eq. (B8) at $t = 0$. First, let us multiply (B8) through by $\underset{\sim}{v}$, and then integrate over $\underset{\sim}{v}$. Making use of (B16) and integrating by parts, we get

$$\frac{\partial \underset{\sim}{u}}{\partial t} \cong - \frac{e_t^2 \, \overline{\omega}_p^2}{M_t} \left[ \left(1 + \frac{\theta k_D^2}{M_t \, \overline{\omega}_p^2}\right) \ln \frac{k_o}{k_D} + \ln \frac{k_D u}{\overline{\omega}_p} \right] \frac{\underset{\sim}{u}}{u^3} \quad . \tag{B17}$$

We can define a "slowing down time"[6] by

$$\tau_s \equiv - \frac{u}{\left(\frac{\partial u}{\partial t}\right)} = \frac{M_t \, u^3}{e_t^2 \, \overline{\omega}_p^2 \left[ \left(1 + \frac{\theta k_D^2}{M_t \overline{\omega}_p^2}\right) \ln \frac{k_o}{k_D} + \ln \frac{k_D u}{\overline{\omega}_p} \right]} \quad . \tag{B18}$$

Let us now multiply (B8) through by $\underset{\sim}{v}\,\underset{\sim}{v}$, and then integrate over $\underset{\sim}{v}$. We obtain

$$\frac{\partial}{\partial t} (\underset{\sim}{u}\,\underset{\sim}{u}) = \frac{e_t^2 \, \overline{\omega}_p^2}{M_t} \left( \ln \frac{k_o}{k_D} \right) \left[ - \frac{\theta}{M_t} \left( \frac{u^2 \underset{\approx}{1} - 3 \, \underset{\sim}{u}\,\underset{\sim}{u}}{u^5} \right) \right.$$
$$\left. + \frac{\theta k_D^2}{M_t \overline{\omega}_p^2} \left( \frac{u^2 \underset{\approx}{1} - 3 \, \underset{\sim}{u}\,\underset{\sim}{u}}{u^3} \right) - 2 \, \frac{\underset{\sim}{u}\,\underset{\sim}{u}}{u^3} \right]$$

$$+ \frac{e_t^2 \bar{\omega}_p^2}{M_t} \left( \ln \frac{k_D u}{\bar{\omega}_p} \right) \left[ - \frac{\theta}{M_t} \left( \frac{u^2 \underset{\approx}{1} - 3 \underset{\sim}{u} \underset{\sim}{u}}{u^5} \right) - \frac{2 \underset{\sim}{u} \underset{\sim}{u}}{u^3} \right] . \tag{B19}$$

We can define a "deflection time"[6] by

$$\tau_D = \frac{u^2}{\frac{\partial}{\partial t} (u_\perp^2)} \cong \frac{M_t^2 u^5}{2 e_t^2 \theta \bar{\omega}_p^2 \left[ \frac{k_D^2 u^2}{\bar{\omega}_p^2} \ln \frac{k_o}{k_D} - \ln \frac{k_D u}{\bar{\omega}_p} \right]} , \tag{B20}$$

where $u_\perp$ is the component of velocity perpendicular to $\underset{\sim}{u}$ .

We can also define an "energy exchange time"[6] by

$$\tau_W \equiv - \frac{u^2}{\frac{\partial}{\partial t} (u^2)} = \frac{M_t u^3}{2 e_t^2 \bar{\omega}_p^2 \left[ \ln \frac{k_o}{k_D} + \ln \frac{k_D u}{\bar{\omega}_p} \right]} . \tag{B21}$$

## APPENDIX C

## THE EXPONENTIAL DECAY

Here we shall present a proof of the impossibility of an exponential decay of a distribution function. It is based on an adaptation and generalization of the method used by Rosenberg and Wu[5] to solve the linearized Balescu-Lenard equation.

Let us write the kinetic equation for a spatially homogeneous plasma in the symbolic form

$$\frac{\partial f}{\partial t} = C(f,f) \quad , \tag{C1}$$

where $C(f,f)$ is a collision operator which has not yet been specified. Thus we have not yet limited ourselved to any particular kinetic equation. Let us restrict ourselves to collision operators which are bilinear functions of $f(\underset{\sim}{v},t)$ . The collision operator of the Fokker-Planck equation satisfies this requirement, but the collision operator of the Balescu-Lenard equation does not. If $f_o(v)$ is the Maxwellian distribution to which $f(\underset{\sim}{v},t)$ will relax, we define a function $f_1(\underset{\sim}{v},t)$ by the expression

$$f_1(\underset{\sim}{v},t) \equiv f(\underset{\sim}{v},t) - f_o(v) \quad . \tag{C2}$$

Substituting (C2) into (C1), (C1) can be rewritten in the form

$$\frac{\partial f_1}{\partial t} = C(f_o,f_1) + C(f_1,f_o) + C(f_1,f_1) \quad . \tag{C3}$$

The term $C(f_o,f_o) = 0$ and therefore was not written explicitly in (C3). If $f_o(v)$ is incorporated into the definitions of the operators $C(f_o,f_1)$ and $C(f_1,f_o)$, the right side of (C3) can be said to consist of two linear functions of $f_1$ and one bilinear function of $f_1$. When $f_1$ is small compared to $f_o$ and eq. (C3) is linearized, the term $C(f_1,f_1)$ is simply droppped from the equation. If the Balescu-Lenard equation is linearized, it also satisfies the linearized eq. (C3).

Let us further restrict ourselves to $f_1$'s which are isotropic in velocity space. Suppose a complete orthonormal set of real functions of $|\underset{\sim}{v}|$, $\{\phi_n(v)\}$, has been selected, and $f_1$ is expanded in terms of the members of this set, so that

$$f_1(v,t) = \sum_{n=0}^{\infty} a_n(t) \phi_n(v) . \tag{C4}$$

Then eq. (C3) can be written in the form

$$\frac{\partial a_k}{\partial t} = \sum_n A_{kn} a_n + \sum_{m,n} B_{kmn} a_m a_n , \tag{C5}$$

where $A_{kn}$ and $B_{kmn}$ are defined by

$$A_{kn} \equiv (\phi_k, C(f_o,\phi_n)) + (\phi_k, C(\phi_n, f_o)) \tag{C6}$$

and

$$B_{kmn} \equiv (\phi_k, C(\phi_m,\phi_n)) . \tag{C7}$$

The symbol $(\phi_k, C(f_o,\phi_n))$ denotes the scalar product of $\phi_k(v)$ and $C(f_o,\phi_n)$, etc.. If we select a different orthonormal set of real functions of $|v|$, say $\{\bar{\phi}_n(v)\}$, and define the matrix element $\bar{A}_{kn}$ by

$$\bar{A}_{kn} \equiv [(\bar{\phi}_k, C(f_o,\bar{\phi}_n)) + (\bar{\phi}_k, C(\bar{\phi}_n,f_o))] \quad , \tag{C8}$$

then the matrix A can be obtained from the matrix $\bar{A}$ by an orthonormal transformation. Let us restrict ourselves now to the Fokker-Planck equation. Let us also assume that the set $\{\bar{\phi}_n\}$ is the same complete set used by Rosenberg and Wu[5]. Rosenberg and Wu[5] showed that $\bar{A}$ has real, non-positive eigenvalues in the case of the linearized Balescu-Lenard equation. This must also be true in the case of the linearized Fokker-Planck equation; and since the matrix A, or $\bar{A}$, is not changed when the Fokker-Planck equation is linearized, $\bar{A}$ must have real, non-positive eigenvalues in the case of the non-linear Fokker-Planck equation. Hence also the matrix A must have real, non-positive eigenvalues. Let us denote a particular eigenvalue by $(-\gamma^{(\nu)})$ and the corresponding eigenvector by $X^{(\nu)}$, so that the equation $AX^{(\nu)} = -\gamma^{(\nu)} X^{(\nu)}$ is satisfied.

We shall now expand the function $a_n(t)$ in terms of the eigenvectors of A. Thus we obtain

$$a_n(t) = \sum_\nu b^{(\nu)}(t)\, X_n^{(\nu)} \quad . \tag{C9}$$

Substituting into (C5), we obtain

$$\frac{\partial b^{(\lambda)}}{\partial t} = -\gamma^{(\lambda)} b^{(\lambda)} + \sum_{\mu,\nu} D_{\lambda\mu\nu} b^{(\mu)} b^{(\nu)} , \tag{C10}$$

where $D_{\lambda\mu\nu}$ is defined by

$$D_{\lambda\mu\nu} \equiv \sum_{k,m,n} B_{kmn} X_k^{(\lambda)} X_m^{(\mu)} X_n^{(\nu)} . \tag{C11}$$

Since $D_{\lambda\mu\nu} \neq 0$ , (C10) does not have any solutions of the form $b^{(\lambda)} = (\text{const}) e^{-|\text{const}|t}$ . Hence an initial distribution function cannot relax to a final Maxwellian via the Fokker-Planck equation by means of a simple exponential decay.

In the linear approximation eq. (C10) reduces to the equation

$$\frac{\partial b^{(\nu)}}{\partial t} = -\gamma^{(\nu)} b^{(\nu)} . \tag{C12}$$

Eq. (C12) has the solution

$$b^{(\nu)} = C^{(\nu)} e^{-\gamma^{(\nu)} t} , \tag{C13}$$

where $C^{(\nu)}$ is some constant, determined by initial conditions. Substituting (C13) into (C9), and subsequently into (C4), we obtain (in the linear approximation)

$$f_1(v,t) = \sum_n \sum_\nu C^{(\nu)} e^{-\gamma^{(\nu)} t} X_n^{(\nu)} \phi_n(v) . \tag{C14}$$

Since $f_o(v)$ satisfies the normalization conditions (1) and (2), we must have

$$\int_o^\infty v^2 \, f_1(v,t) \, dv = \int_o^\infty v^4 \, f_1(v,t) \, dv = 0 \quad . \tag{C15}$$

Let us define the numbers $\alpha^{(\nu)}$ and $\beta^{(\nu)}$ by the following equations:

$$\alpha^{(\nu)} \equiv \sum_n C^{(\nu)} \, X_n^{(\nu)} \int_o v^2 \, \phi_n(v) \, dv \tag{C16}$$

and

$$\beta^{(\nu)} \equiv \sum_n C^{(\nu)} \, X_n^{(\nu)} \int_o^\infty v^4 \, \phi_n(v) \, dv \quad . \tag{C17}$$

Then, eq. (C15) yields the following two equations:

$$\sum_\nu \alpha^{(\nu)} \, e^{-\gamma^{(\nu)} t} = 0$$

and

$$\sum_\nu \beta^{(\nu)} \, e^{-\gamma^{(\nu)} t} = 0 \tag{C18}$$

Eqs. (C18) must hold for all times, including $t = 0$. When $t = 0$, (C18) become

$$\sum_\nu \alpha^{(\nu)} = \sum_\nu \beta^{(\nu)} = 0 \quad . \tag{C19}$$

If we had only one exponential decay in $f_1(v,t)$, by (C14), $C^{(\nu)} \neq 0$ when $\nu = \mu$, and $C^{(\nu)} = 0$ when $\nu \neq \mu$. Consequently $\alpha^{(\nu)} \neq 0$ and $\beta^{(\nu)} \neq 0$ when $\nu = \mu$, while $\alpha^{(\nu)} = \beta^{(\nu)} = 0$ when $\nu \neq \mu$. It would then follow from eq. (C19) that $\alpha^{(\mu)} = \beta^{(\mu)} = 0$. Therefore $f_1(v,t)$ has to contain more than one exponential decay even in the linear approximation.

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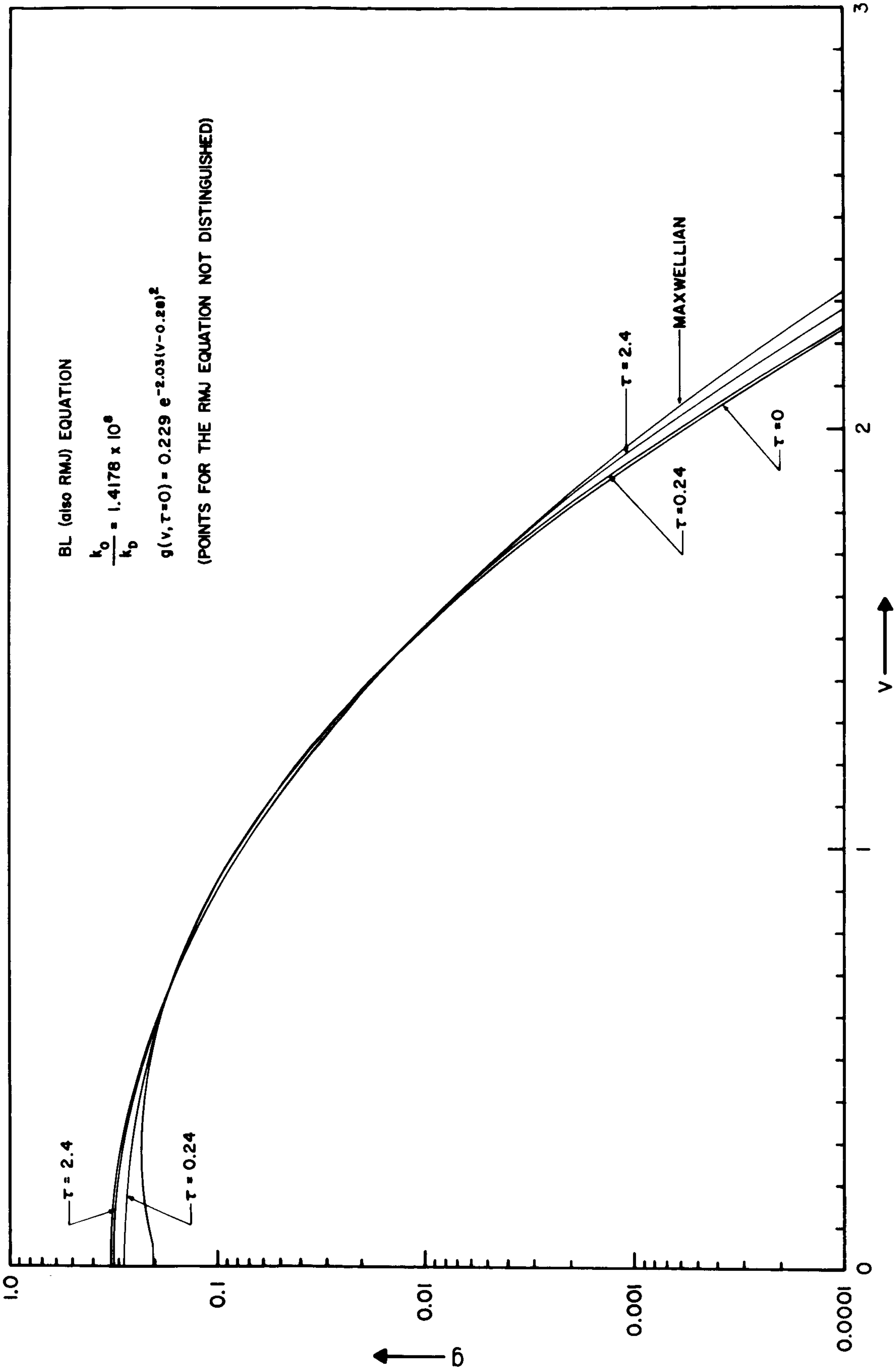
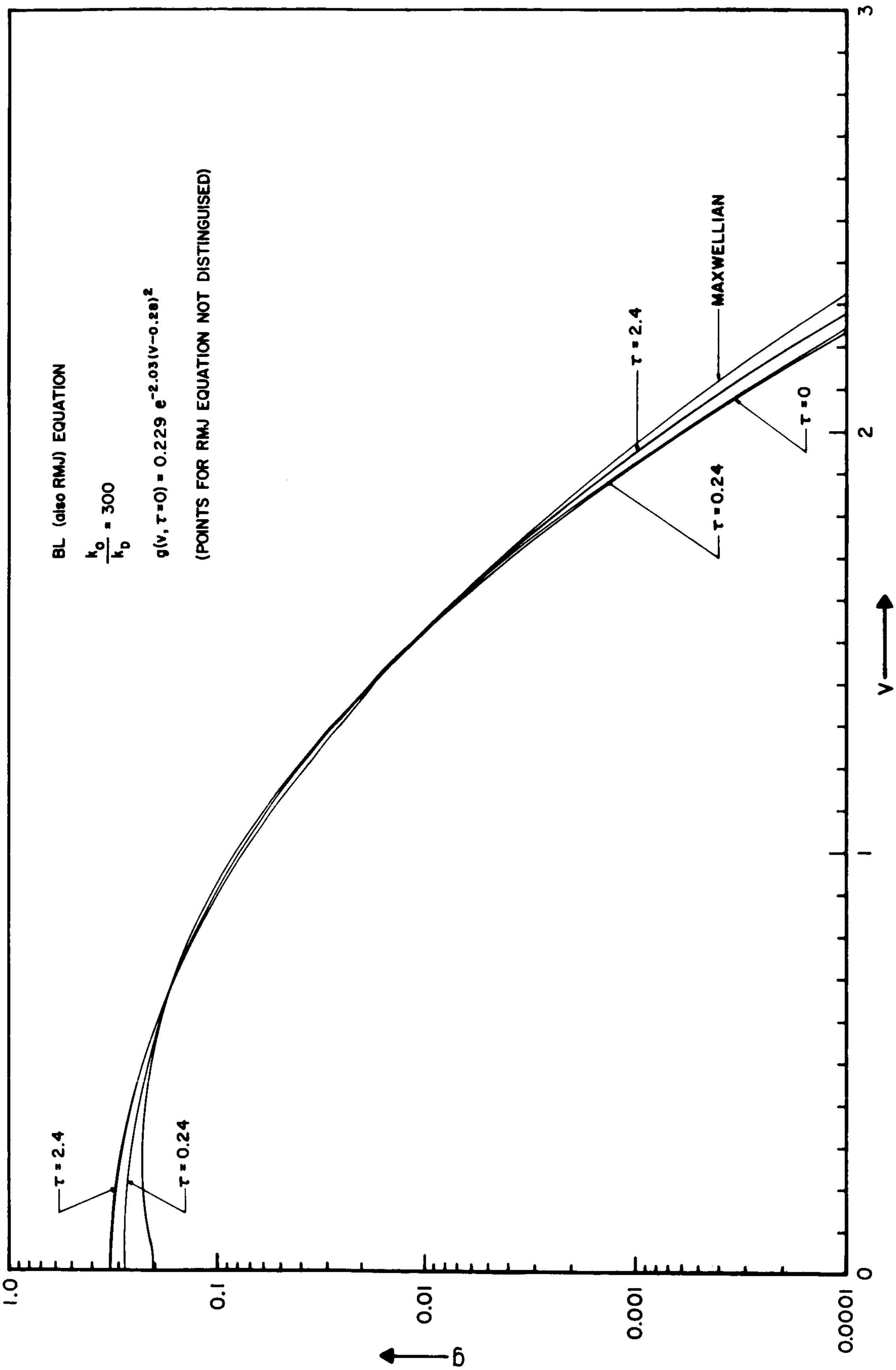
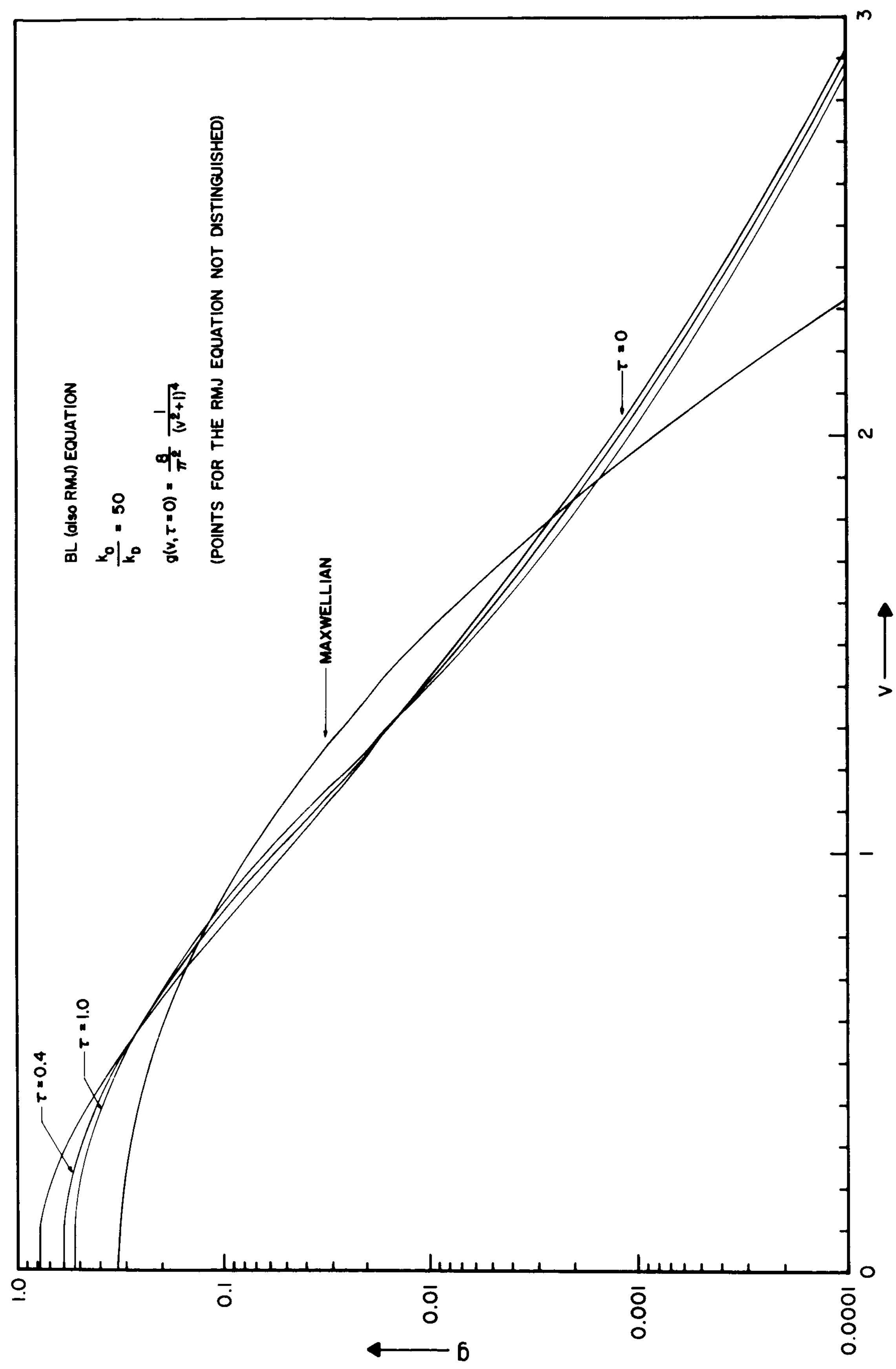

BL (also RMJ) EQUATION
$\frac{k_o}{k_D} = 1.4178 \times 10^8$
$g(v, \tau=0) = 0.229\ e^{-2.03(v-0.28)^2}$
(POINTS FOR THE RMJ EQUATION NOT DISTINGUISHED)
τ = 2.4
MAXWELLIAN
τ = 0.24
τ = 0
τ = 2.4
τ = 0.24
1.0
0.1
0.01
0.001
0.0001
0
1
2
3
g
V

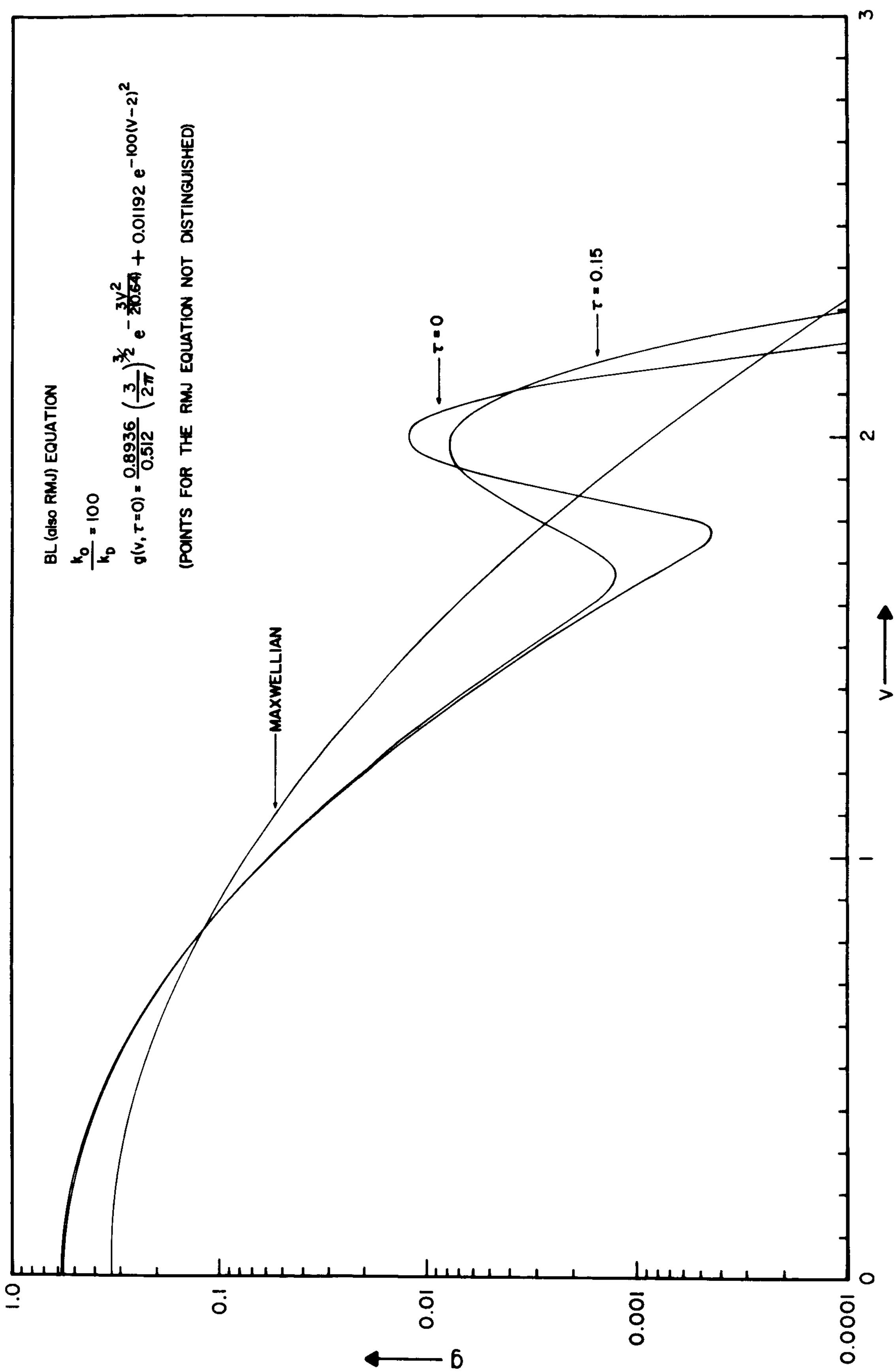
Graph 1


BL (also RMJ) EQUATION
$\frac{k_o}{k_D} = 300$
$g(v, \tau = 0) = 0.229\ e^{-2.03(v-0.28)^2}$
(POINTS FOR RMJ EQUATION NOT DISTINGUISED)
τ = 2.4
τ = 0.24
τ = 2.4
MAXWELLIAN
τ = 0
τ = 0.24
1.0
0.1
0.01
0.001
0.0001
0
1
2
3
g
v

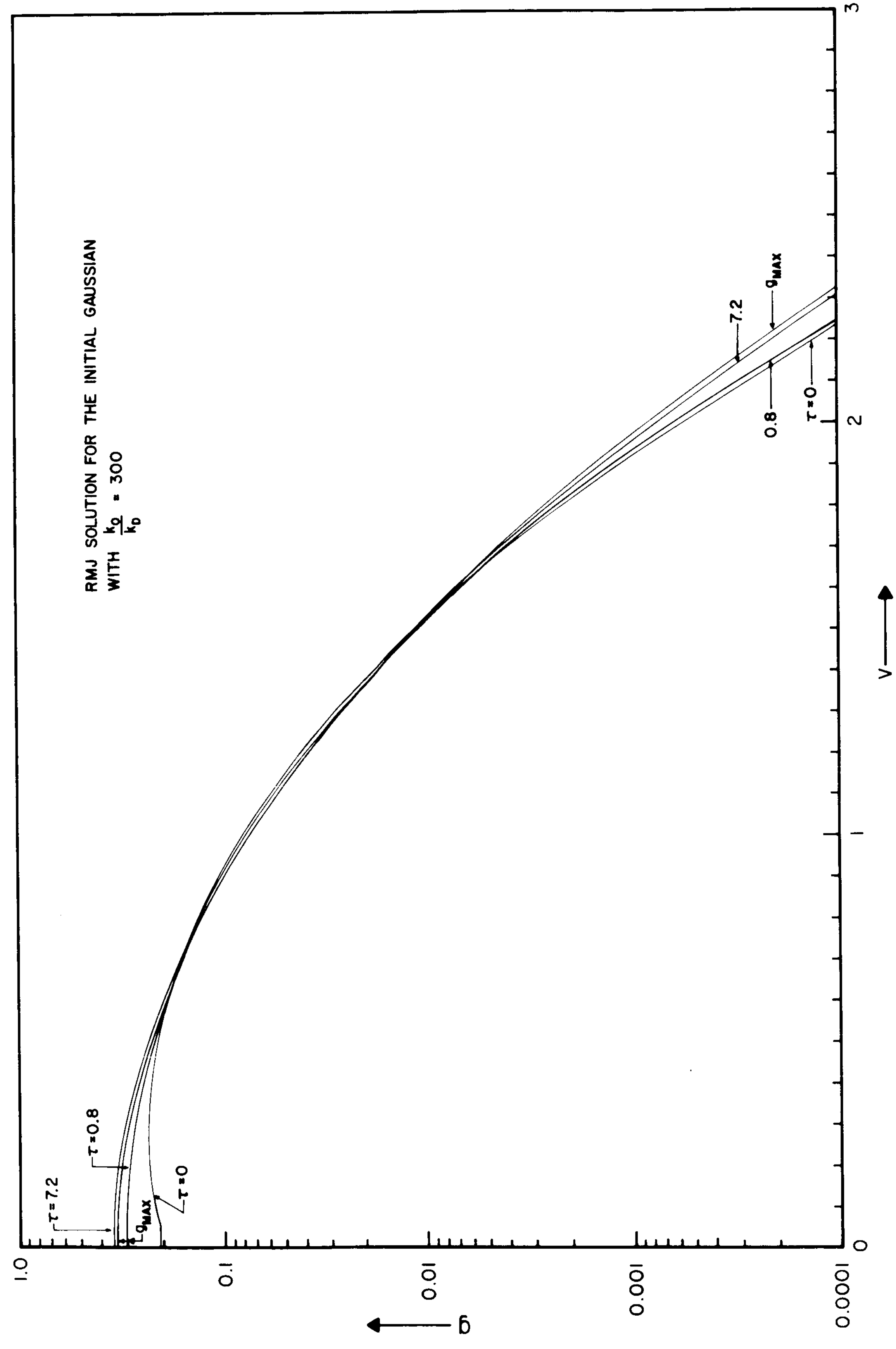
Graph 2


BL (also RMJ) EQUATION
$\frac{k_o}{k_D} = 50$
$g(v, \tau = 0) = \frac{8}{\pi^2} \frac{1}{(v^2+1)^4}$
(POINTS FOR THE RMJ EQUATION NOT DISTINGUISHED)
MAXWELLIAN
τ = 0
τ = 0.4
τ = 1.0
1.0
0.1
0.01
0.001
0.0001
g
0
1
2
3
v

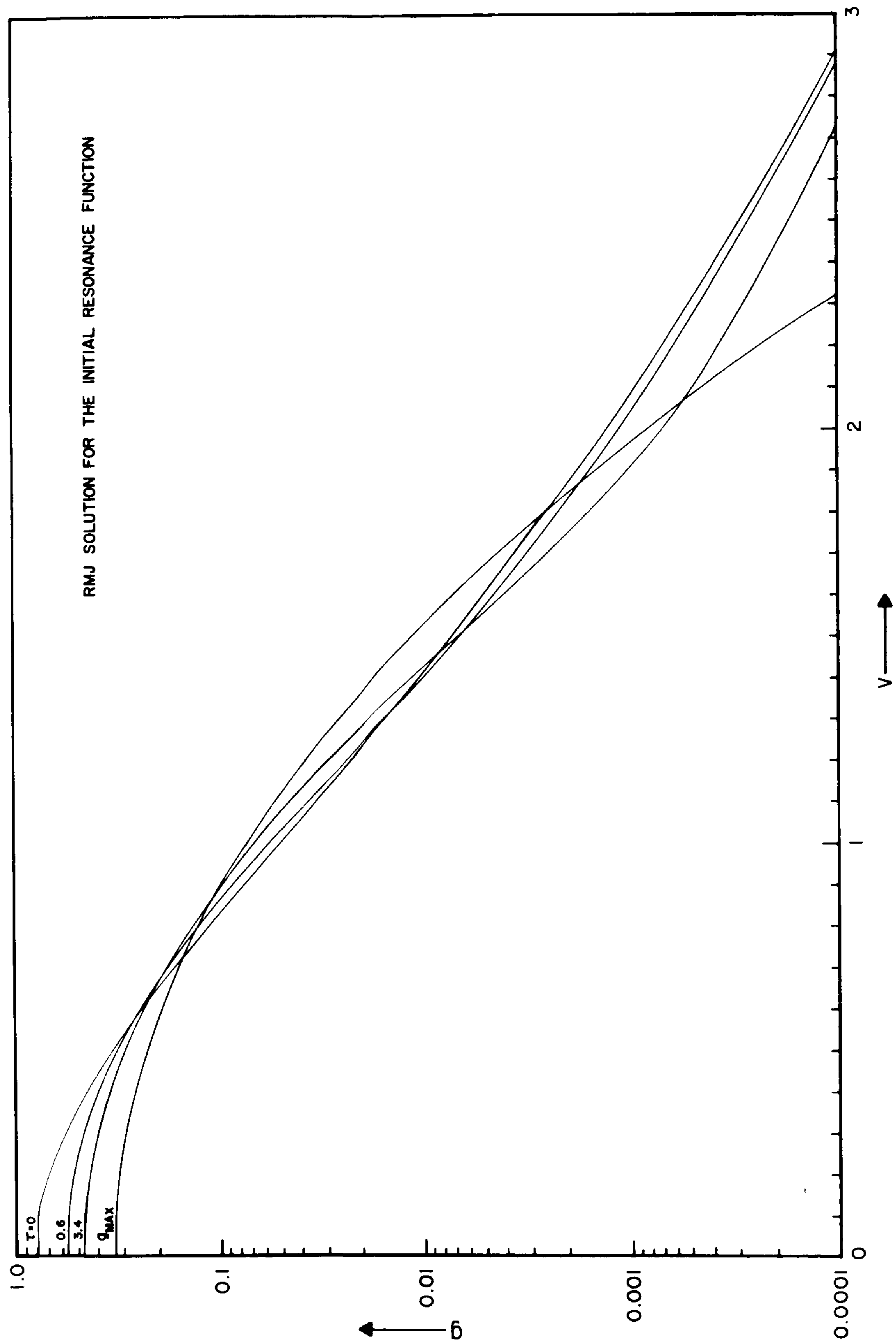
Graph 3


BL (also RMJ) EQUATION
$\frac{k_0}{k_D} = 100$
$g(v, \tau = 0) = \frac{0.8936}{0.512} \left(\frac{3}{2\pi}\right)^{3/2} e^{-\frac{3v^2}{2(0.64)}} + 0.01192\, e^{-100(v-2)^2}$
(POINTS FOR THE RMJ EQUATION NOT DISTINGUISHED)
MAXWELLIAN
τ = 0
τ = 0.15
1.0
0.1
0.01
0.001
0.0001
g
0
1
2
3
v

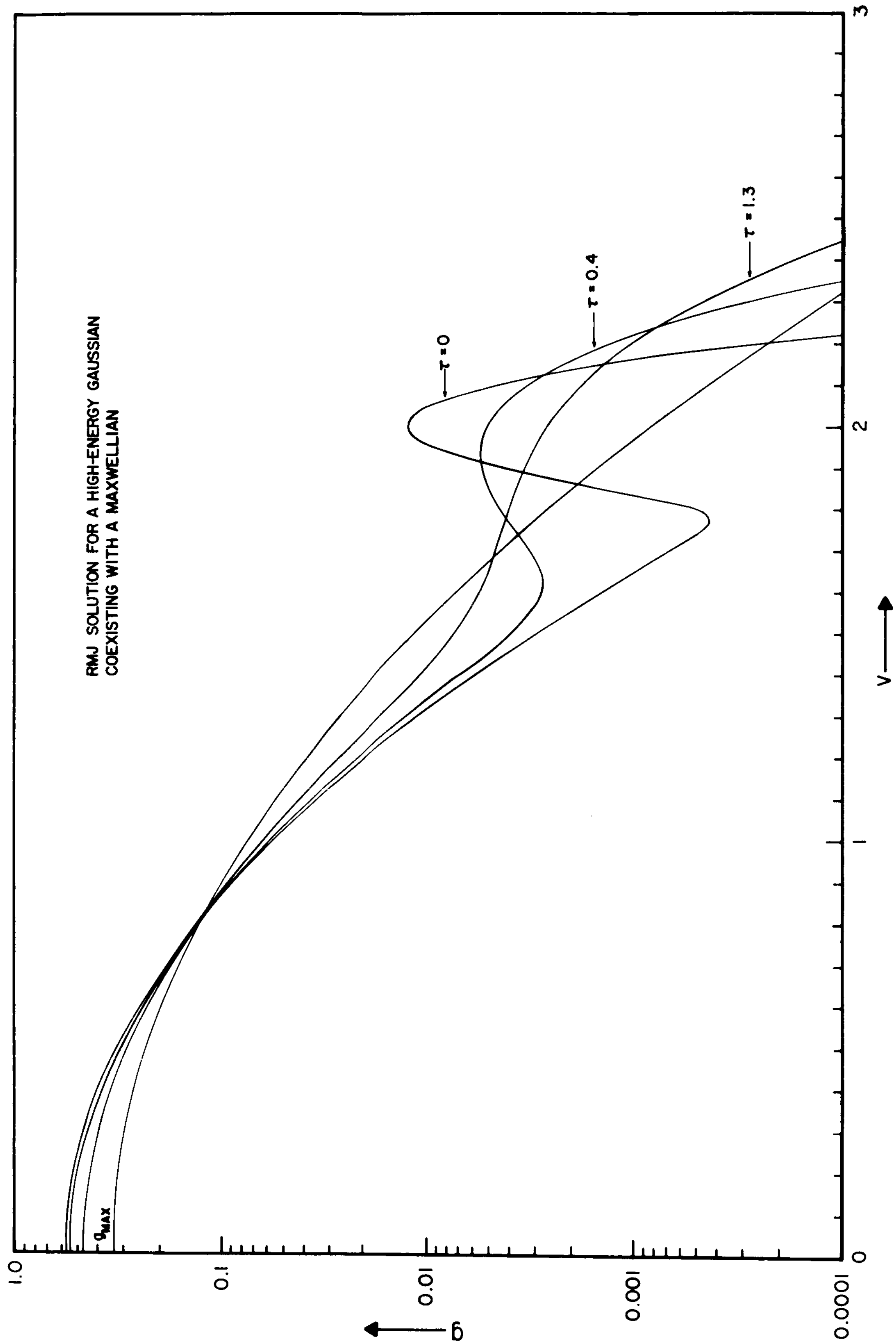
Graph 4


RMJ SOLUTION FOR THE INITIAL GAUSSIAN
WITH $\frac{k_o}{k_D}$ = 300
τ = 7.2
τ = 0.8
$g_{MAX}$
τ = 0
7.2
$g_{MAX}$
0.8
τ = 0
1.0
0.1
0.01
0.001
0.0001
g
0
1
2
3
V

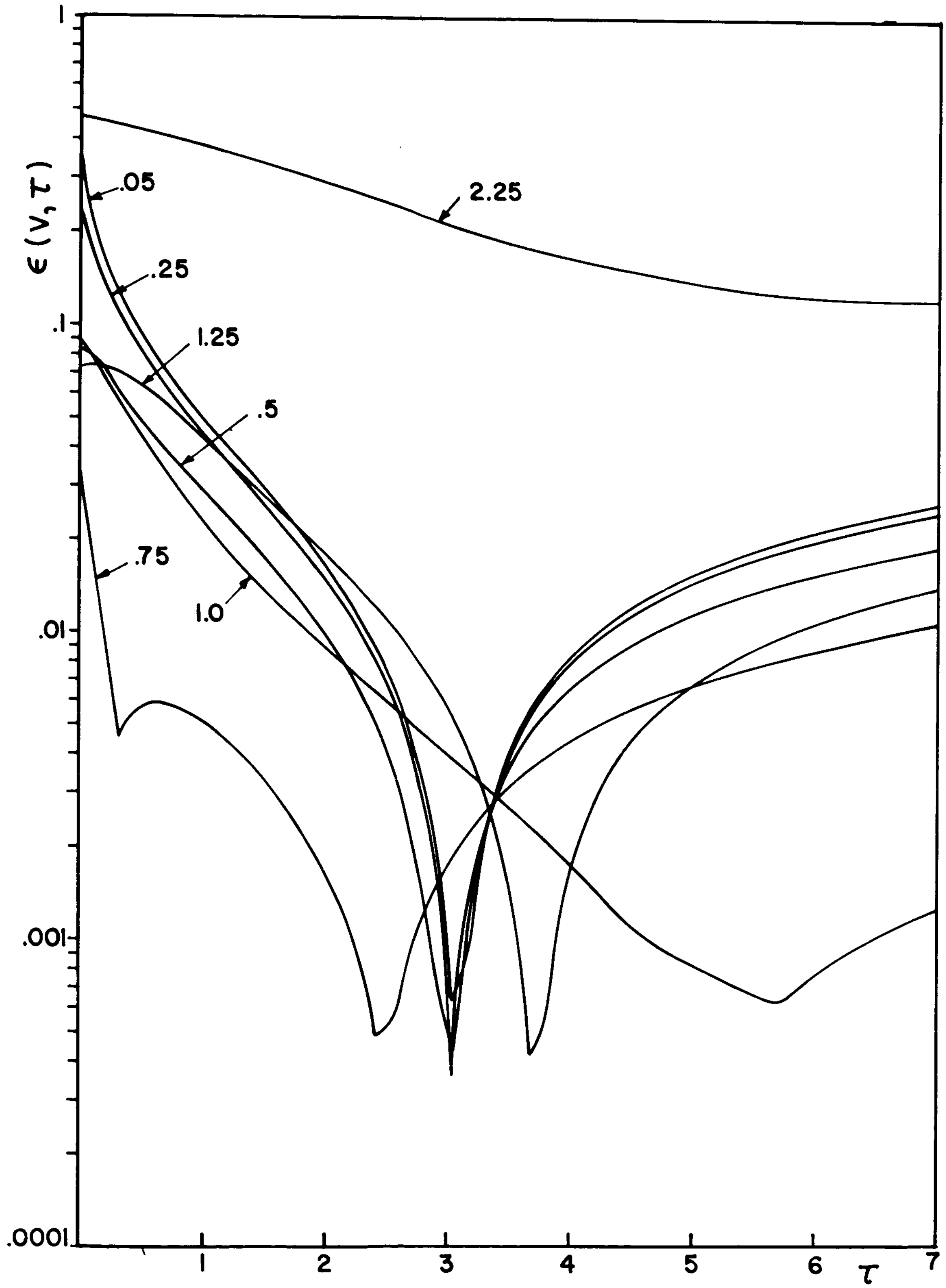
Graph 5


RMJ SOLUTION FOR THE INITIAL RESONANCE FUNCTION
r=0
0.6
3.4
$g_{MAX}$
1.0
0.1
0.01
0.001
0.0001
g
0
1
2
3
V

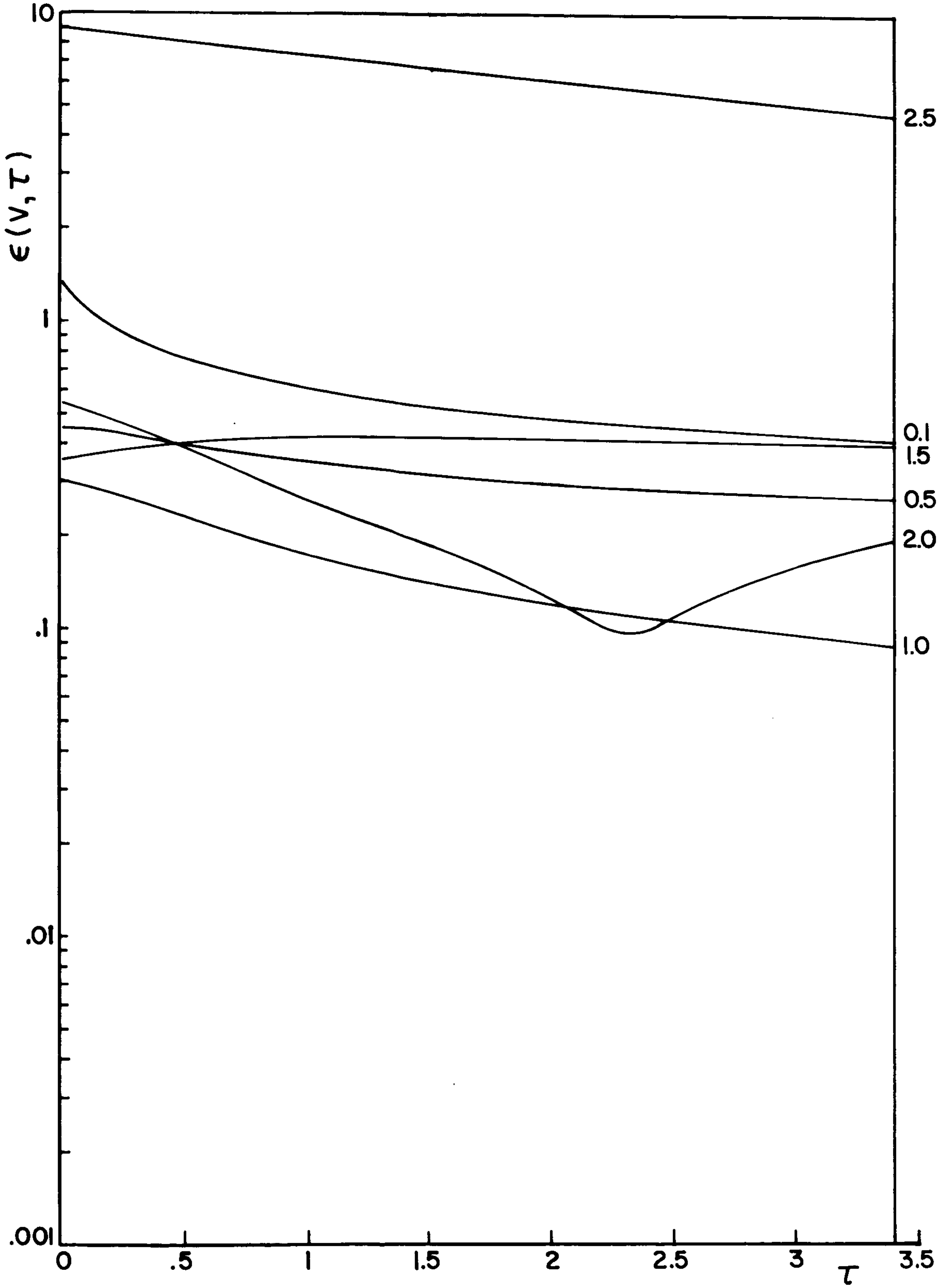
Graph 6


RMJ SOLUTION FOR A HIGH-ENERGY GAUSSIAN
COEXISTING WITH A MAXWELLIAN
τ = 0
τ = 0.4
τ = 1.3
$g_{MAX}$
1.0
0.1
0.01
0.001
0.0001
g
0
1
2
3
V

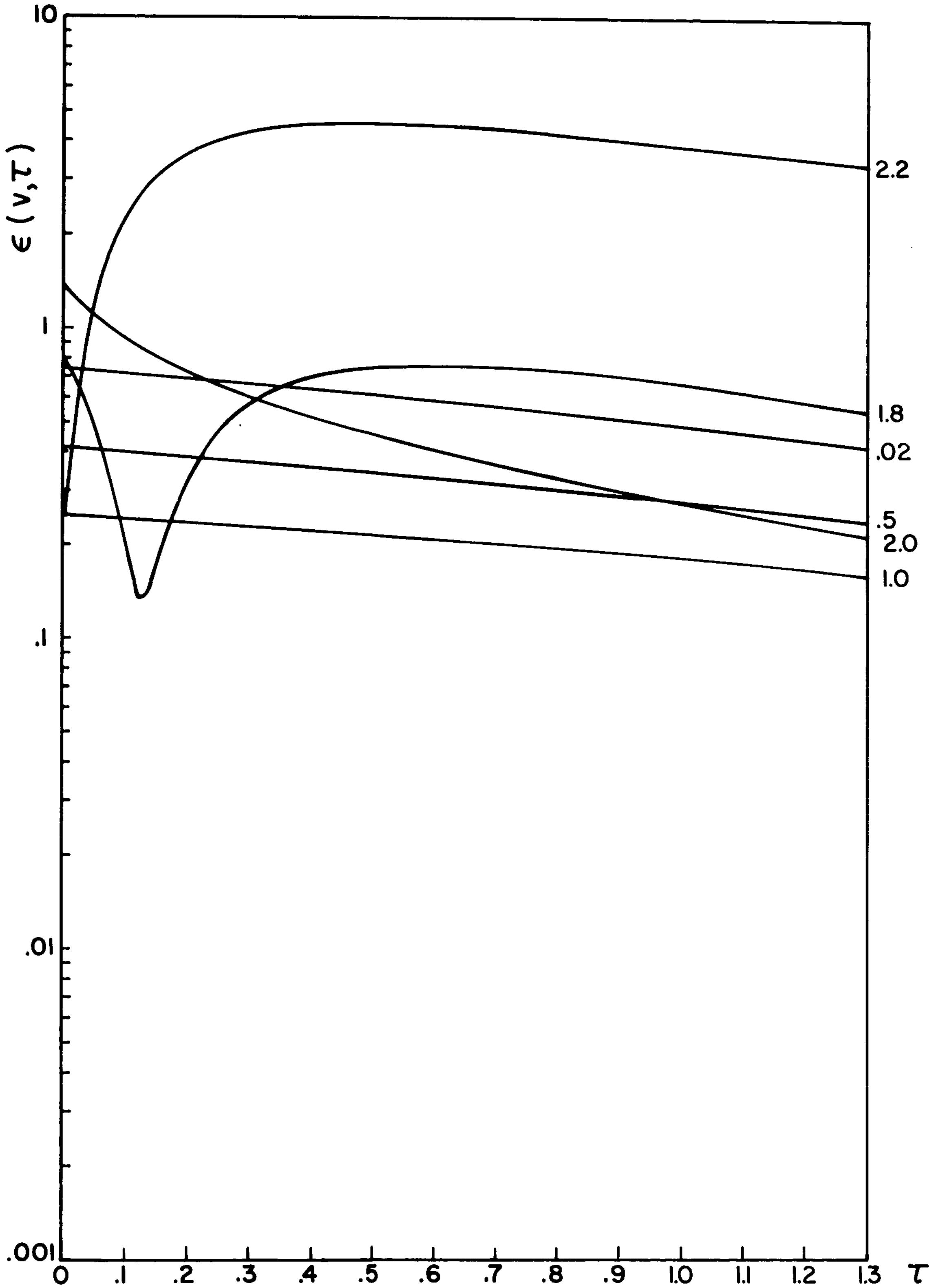
Graph 7




Graph 8




Graph 9




Graph 10

# PART TWO

# NONLINEAR EFFECTS IN THE LIGHT-BY-LIGHT SCATTERING IN A PLASMA

## I. INTRODUCTION

Recently there has been considerable interest in the scattering of light by light inside a plasma. Platzman, Buchsbaum, and Tzoar[1] calculated, using quantum mechanics, the incoherent cross section for the scattering of light by light in the presence of a plasma to lowest order in the plasma parameter, i.e. to lowest order in the reciprocal of the number of particles in the Debye sphere. Kroll, Ron, and Rostoker[2] calculated, by solving the Vlasov equation, the scattering cross section for two plane electromagnetic waves each one of which is monochromatic and coherent. With the present-day state of laser technology such a light-by-light scattering experiment is feasible.

This scattering process is of practical interest, because it can be used, among other things, as a density probe for plasmas. It has advantages over the process of incoherent scattering of a single light beam incident on a plasma, because the scattered energy flux per unit incident energy flux is much larger for the light-light scattering process than it is for the scattering of a single incident light beam (as was pointed out by Platzman et al.[1] and by Kroll et. al.[2]).

The reason for this fact is that a single light beam passing through a quiescent plasma is only scattered by the thermal density fluctuations, which are small. On the other hand, the presence of two incident light beams enables us to tune their frequencies so that their difference is equal to the natural frequency of longitudinal oscillations.

The two light beams are then able to excite coherent plasma density oscillations, and are in turn scattered by these oscillations. These density oscillations are much larger than the thermal density fluctuations, and therefore enhance the scattering process.

We shall make the following model for the scattering process. Two infinite plane waves, with wave vectors $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$, and frequencies $\omega_1$ and $\omega_2$ respectively, impinge on a quiescent plasma, confined in a large volume $V$. A detector is placed very far from the plasma and measures the scattered energy flux over a long period of time $T$.

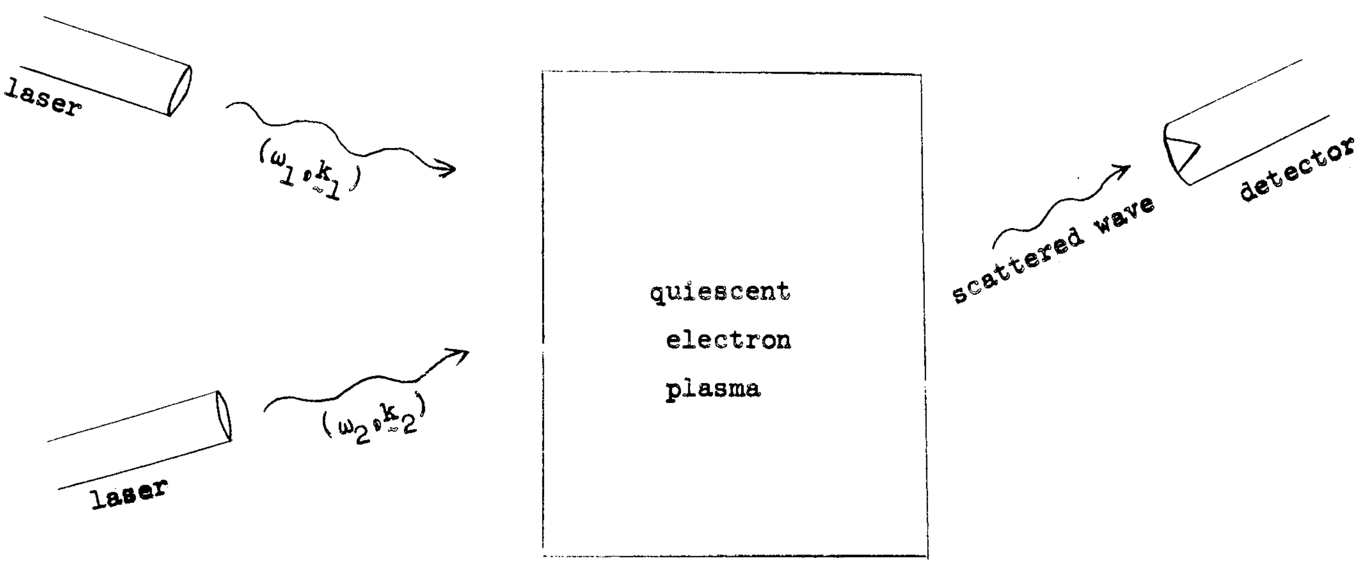



Figure XI

For the sake of simplicity we shall assume the volume $V$ to be a rectangular box and the plasma to consist of one species of particles, electrons, with average particle density $N_o$. To ensure charge neutrality, the electron plasma is embedded in a uniformly smeared out background of positive charge of charge density $N_o e$.

The differential cross section per unit frequency interval for the scattering of light by a plasma is[2,3]

$$\frac{d\sigma}{d\omega} = \frac{r_o^2}{2\pi} S(\underset{\sim}{k} - \frac{\omega'}{c}\hat{n},\ \omega'-\omega)\ (1 - \frac{1}{2}\sin^2\theta) \tag{1}$$

where $r_o = \frac{e^2}{mc^2}$, the classical electron radius; $\theta$ is the angle of scattering, i.e. the angle between the incident energy flux and the scattered energy flux; $\underset{\sim}{k}$ is the wave vector of incident light; $\omega$ is the frequency of incident light; $\omega'$ is the frequency of scattered light; $\hat{n}$ is a unit vector pointing in the direction of the scattered flux; and $S(\underset{\sim}{k},\omega)$ is the spectral density, defined by

$$S(\underset{\sim}{k},\omega) = \lim_{\substack{V\to\infty \\ T\to\infty}} \frac{2|n(\underset{\sim}{k},\omega)|^2}{N_o VT}, \tag{2}$$

where $n(\underset{\sim}{k},\omega)$ is the Fourier transform of $n(\underset{\sim}{x},t)$ which is defined to be the fluctuation of the electron density about the equilibrium density $N_o$.

The calculation of $n(x,t)$ is difficult, because the equations describing the behavior of the plasma are non-linear. To make the problem tractable, one usually restricts oneself to incident light beams whose amplitudes are small in the sense that the changes they produce in the plasma variables are small compared with the values of these variables for the quiescent plasma (i.e. $|n(x,t)|/N_o \ll 1$). This enables us to introduce formally a small parameter $\varepsilon$, which is a measure of the strength of the amplitudes of the incident light waves, and to use some kind of perturbation theory.

If one chooses to describe the behavior of the plasma by means of the collisionless moment equations and the Maxwell equations, and applies the conventional linearization process to these equations, one obtains an $n(x,t)$ which grows linearly with time. (This will be pointed out more explicitly in Sec. IVC.) Since the density must remain finite, there have to exist physical mechanisms which limit the density oscillations but were left out of the above mathematical scheme. The neglected mechanisms are the Landau damping, the collisional damping, and the non-linear effects.

All of these mechanisms operate simultaneously. But for a particular choice of numerical values of plasma parameters and incident electric fields, one mechanism usually dominates. Which one is dominant in a particular situation is determined by the numerical values selected.

The dominant mechanism yields smaller density oscillations than all other mechanisms, because it limits these oscillations more effectively.

Since it is very difficult to calculate the action of all of the limiting mechanisms simultaneously, the effect of each mechanism is calculated separately, with the exclusion of all other mechanisms. Kroll, Ron, and Rostoker[2] were interested in the Landau damping mechanism. There-for they linearized the Vlasov and the Maxwell equations, and calculated the Landau damping effect from these linearized equations. The density oscillations, as calculated by Kroll et. al.[2], turned out to be inversely proportional to the Landau damping decrement.

Since in the linearized theory the collisional damping, as calculated from the Fokker-Planck equation, introduces an additional damping decrement, which plays a role analogous to the Landau damping decrement, Kroll et. al.[2] managed to incorporate the collisional damping mechanism into their results by adding the collisional damping decrement to the Landau damping decrement. We can see from their results the reason why the linearized, collisionless moment equations yield density oscillations which increase linearly with time. Linearized, collisionless moment equations neglect both the collisions and Landau damping. Therefore from the viewpoint of those equations the collisional and the Landau damping decrements are both zero. Hence the density oscillations will grow with time.

We have neglected the effects of collisions and Landau damping, and have calculated the contributions of non-linear effects. For this purpose we have limited ourselves to collisionless moment equations and Maxwell equations. To make the problem mathematically tractable, we have assumed the nonlinearity in the equations to be small. This enabled us to treat the nonlinearity by the generalization due to Frieman and Sandri[4] of an expansion technique for nonlinear mechanics due to Bogoliubov, Krylov, and Mitropolsky[5].

The generalization due to Frieman and Sandri is known as the multiple time-scale method. It introduces into the problem many time scales, each scale being of a different order in $\varepsilon$ . The purpose of these "slow" length and time scale variables is to introduce enough freedom in the equations to cancel secular (i.e. $t$ or $x$ proportional) terms in the perturbation expansion. We have adapted the Frieman-Sandri method to our problem by also introducing many spatial scales, defined in an analogous way.

We have derived an expression for the differential cross section for the scattering of light by light. We have also derived an expression by which one can determine quantitatively which mechanism limits plasma oscillations more effectively for a particular set of numerical values of plasma parameters and impinging wave parameters. Our results indicate that nonlinear effects are sometimes much more important than damping effects. This is particularly true when the impinging waves are fairly strong. On the other hand, when the impinging waves are very weak, the damping effects dominate.

## II. ELECTRON-PLASMA EQUATIONS

### A. MOMENT AND MAXWELL EQUATIONS

Let $P_o$ be the pressure of the quiescent plasma; $\underset{\approx}{p}(\underset{\sim}{x},t)$, the fluctuation of the pressure tensor about $P_o \underset{\approx}{1}$ ($\underset{\approx}{1}$ is the unit dyadic); $\underset{\sim}{v}(\underset{\sim}{x},t)$, the velocity; $\underset{\sim}{E}$, the electric field; and $\underset{\sim}{B}$, the magnetic field. We assume (as was pointed out in (I)) the plasma to be described by the following low temperature, collisionless moment equations:

$$\frac{\partial n}{\partial t} + N_o \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v} = - \frac{\partial}{\partial \underset{\sim}{x}} \cdot n\underset{\sim}{v} \quad , \tag{3}$$

$$(N_o + n) \left[ \frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}) \underset{\sim}{v} \right] = - \frac{1}{m} \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{p}$$

$$- \frac{e}{m} (N_o + n) (\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B}) \quad , \tag{4}$$

and

$$\frac{\partial \underset{\approx}{p}}{\partial t} + (\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}) \underset{\approx}{p} + (P_o \underset{\approx}{1} + \underset{\approx}{p}) (\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v})$$

$$+ [(P_o \underset{\approx}{1} + \underset{\approx}{p}) \cdot \frac{\partial}{\partial \underset{\sim}{x}}] \underset{\sim}{v} + \widetilde{[(P_o \underset{\approx}{1} + \underset{\approx}{p}) \cdot \frac{\partial}{\partial \underset{\sim}{x}}] \underset{\sim}{v}}$$

$$= - \frac{e}{mc} [(P_o \underset{\approx}{1} + \underset{\approx}{p}) \times \underset{\sim}{B} + \widetilde{(P_o \underset{\approx}{1} + \underset{\approx}{p}) \times \underset{\sim}{B}}] \quad , \tag{5}$$

where the notation $\tilde{\underset{\approx}{A}}$ means the transpose of the dyadic $\underset{\approx}{A}$.

In (5) the heat conduction term has been left out because we are dealing with a low temperature plasma. The term $(P_o \underset{\approx}{1} \times \underset{\sim}{B} + \widetilde{P_o \underset{\approx}{1} \times \underset{\sim}{B}})$ in (5) vanishes. This can be seen by writing it in component form:

$$(P_o \underset{\approx}{1} \times \underset{\sim}{B} + \widetilde{P_o \underset{\approx}{1} \times B})_{ij} = P_o(\delta_{ik\ell}\, \delta_{jk} + \delta_{jk\ell}\, \delta_{ik})\, B_\ell$$

$$= P_o(\delta_{ij\ell} + \delta_{ji\ell})\, B_\ell = 0$$

where $\delta_{jk}$ is the Kronecker delta, and $\delta_{ik\ell}$ is the Levi-Civita density, the antisymmetric unit tensor of the third rank (with the value zero whenever any two indices are equal, with the value (+1) whenever $(ik\ell)$ form an even permutation of (123), and with the value (-1) whenever $(ik\ell)$ form an odd permutation of (123)). Summation over repeated indices is assumed.

To the three moment equations we add the four Maxwell equations:

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{E} = -4\pi en \quad , \tag{6}$$

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{B} = 0 \quad , \tag{7}$$

$$\frac{\partial}{\partial \underset{\sim}{x}} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} \quad , \tag{8}$$

and

$$\frac{\partial}{\partial \underset{\sim}{x}} \times \underset{\sim}{B} = \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} - \frac{4\pi N_o e}{c} \underset{\sim}{v} - \frac{4\pi e}{c} n\underset{\sim}{v} \quad . \tag{9}$$

Equations (3) - (9) are assumed to constitute the complete set of equations describing the behavior of the plasma.

## B. WAVE EQUATIONS

For many purposes it is more convenient to work with nonlinear wave equations. By a non-linear wave equation we mean a nonlinear partial differential equation having a linear and a nonlinear term. The non-linear term may contain several plasma field variables. The linear term, however, contains only one plasma variable, and has the form of the linear wave equation for that variable: That is, the non-linear wave equation is essentially the linear wave equation with a nonlinear driving term.

We shall be concerned with the $\underset{\sim}{E}$ field only, but shall want to examine the longitudinal and transverse components of $\underset{\sim}{E}$ separately. (By the longitudinal component of a vector we mean that component which has no curl.) For this reason we write down the wave equation for $\underset{\sim}{E}$, and then by taking the divergence and then the curl of that equation, we obtain wave equations for $n(\underset{\sim}{x},t)$ and for $\underset{\sim}{B}(\underset{\sim}{x},t)$, respectively. The wave equation for $\underset{\sim}{E}(\underset{\sim}{x},t)$ is

$$\Bigg[ \frac{1}{c^2} \frac{\partial^4}{\partial t^4} + \frac{\omega_p^2}{c^2} \frac{\partial^2}{\partial t^2} - \left(1 + \frac{1}{3} \frac{v_o^2}{c^2}\right) \nabla^2 \frac{\partial^2}{\partial t^2}$$

$$+ \frac{1}{3} v_o^2 \nabla^2 \nabla^2 + \left(1 - \frac{2}{3} \frac{v_o^2}{c^2}\right) \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial \underset{\sim}{x}} \left( \frac{\partial}{\partial \underset{\sim}{x}} \cdot \right)$$

$$- \frac{1}{3} v_o^2 \nabla^2 \nabla^2 \frac{\partial}{\partial \underset{\sim}{x}} \left( \frac{\partial}{\partial \underset{\sim}{x}} \cdot \right) \Bigg] \underset{\sim}{E} = - \frac{4\pi e}{c^2} \frac{\partial^2}{\partial t^2} \Bigg[ N_o \left( \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} \right.$$

$$\left. + \frac{e}{mc} \underset{\sim}{v} \times \underset{\sim}{B} \right) - \frac{\partial}{\partial t} (n\underset{\sim}{v}) \Bigg] - \frac{4\pi e}{c^2} \frac{\partial^2}{\partial t^2} \left[ n \left\{ \frac{\partial \underset{\sim}{v}}{\partial t} \right. \right.$$

$$+ \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} + \frac{e}{m} (\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B}) \Big\} \Big]$$

$$- \frac{4\pi e}{3} \frac{v_o^2}{c^2} \nabla^2 \frac{\partial}{\partial t} (n\underset{\sim}{v}) - \frac{8\pi e}{3} \frac{v_o^2}{c^2} \frac{\partial}{\partial t} \frac{\partial}{\partial \underset{\sim}{x}} \left( \frac{\partial}{\partial \underset{\sim}{x}} \cdot n\underset{\sim}{v} \right)$$

$$+ \frac{4\pi e}{mc^2} \frac{\partial}{\partial t} \frac{\partial}{\partial \underset{\sim}{x}} \cdot \Big[ \left( \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\approx}{p} + \underset{\approx}{p} \left( \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v} \right)$$

$$+ \left( \underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\sim}{v} + \widetilde{\left( \underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\sim}{v}} + \frac{e}{mc} (\underset{\approx}{p} \times \underset{\sim}{B} + \widetilde{\underset{\approx}{p} \times \underset{\sim}{B}}) \Big] , \qquad (10)$$

where $v_o \equiv \left( \frac{3P_o}{N_o m} \right)^{1/2}$, the thermal velocity, and $\omega_p = \left( \frac{4\pi N_o e^2}{m} \right)^{1/2}$, the electron plasma frequency. (Eq. (10) is derived in the Appendix.)

Taking $\left( - \frac{1}{4\pi e} \frac{\partial}{\partial \underset{\sim}{x}} \cdot \right)$ of eq. (10) and substituting eq. (6), we obtain

$$\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 - v_o^2 \nabla^2 \right) \frac{\partial^2 n}{\partial t^2} = \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial \underset{\sim}{x}} \cdot \Big[ N_o \Big( \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v}$$

$$+ \frac{e}{mc} \underset{\sim}{v} \times \underset{\sim}{B} \Big) - \frac{\partial}{\partial t} (n\underset{\sim}{v}) \Big] + \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial \underset{\sim}{x}} \cdot \Big[ n \Big\{ \frac{\partial \underset{\sim}{v}}{\partial t}$$

$$+ \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} + \frac{e}{m} \left( \underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B} \right) \Big\} \Big]$$

$$+ v_o^2 \frac{\partial}{\partial t} \nabla^2 \frac{\partial}{\partial \underset{\sim}{x}} \cdot (n\underset{\sim}{v}) - \frac{1}{m} \frac{\partial}{\partial t} \frac{\partial^2}{\partial \underset{\sim}{x} \partial \underset{\sim}{x}} : \Big[ \left( \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\approx}{p}$$

$$+ \underset{\approx}{p} \left( \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v} \right) + \left( \underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\sim}{v} + \widetilde{\left( \underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\sim}{v}}$$

$$+ \frac{e}{mc} \left( \underset{\approx}{p} \times \underset{\sim}{B} + \widetilde{\underset{\approx}{p} \times \underset{\sim}{B}} \right) \Big] . \qquad (11)$$

Taking $\left(-c\, \frac{\partial}{\partial \underset{\sim}{x}} \times\right)$ of eq. (10) and substituting eq. (8), we obtain

$$
\begin{aligned}
&\left[\frac{1}{c^2}\frac{\partial^4}{\partial t^4} + \frac{\omega_p^2}{c^2}\frac{\partial^2}{\partial t^2} - \left(1 + \frac{1}{3}\frac{v_o^2}{c^2}\right)\nabla^2 \frac{\partial^2}{\partial t^2} \right. \\
&\left. + \frac{1}{3} v_o^2 \nabla^2 \nabla^2\right] \frac{\partial \underset{\sim}{B}}{\partial t} = \frac{4\pi e}{c}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial \underset{\sim}{x}} \times \left[N_o\left(\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\, \underset{\sim}{v} \right.\right. \\
&\left.\left. + \frac{e}{mc}\, \underset{\sim}{v} \times \underset{\sim}{B}\right) - \frac{\partial}{\partial t}(n\underset{\sim}{v})\right] + \frac{4\pi e}{c}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial \underset{\sim}{x}} \times \left[n\left\{\frac{\partial \underset{\sim}{v}}{\partial t}\right.\right. \\
&\left.\left. + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\, \underset{\sim}{v} + \frac{e}{m}\left(\underset{\sim}{E} + \frac{1}{c}\, \underset{\sim}{v} \times \underset{\sim}{B}\right)\right\}\right] \\
&+ \frac{4\pi e c}{3}\frac{v_o^2}{c^2}\nabla^2 \frac{\partial}{\partial \underset{\sim}{x}} \times \frac{\partial}{\partial t}(n\underset{\sim}{v}) - \frac{4\pi e}{mc}\frac{\partial}{\partial t}\frac{\partial}{\partial \underset{\sim}{x}} \times \\
&\frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[\left(\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\approx}{p} + \underset{\approx}{p}\left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v}\right) + \left(\underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\sim}{v} \right. \\
&\left. + \widetilde{\left(\underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\sim}{v}} + \frac{e}{mc}\left(\underset{\approx}{p} \times \underset{\sim}{B} + \widetilde{\underset{\approx}{p} \times \underset{\sim}{B}}\right)\right] .
\end{aligned}
\tag{12}
$$

We can see immediately that the linear part of eq. (11) will satisfy the dispersion relation

$$
\omega^2 = \omega_p^2 + v_o^2\, k^2 \tag{13}
$$

when the non-linear terms are set equal to zero. Eq. (13) is the dispersion relation for longitudinal plasma oscillations. Similarly, the linear part of (12) will satisfy the dispersion relation

$$\omega^4 - (\omega_p^2 + c^2k^2 + \frac{1}{3} v_o^2k^2)\omega^2 + \frac{1}{3} c^2 v_o^2k^4 = 0 \tag{14}$$

when the non-linear terms are set equal to zero. Eq. (14) is the dispersion relation for transverse plasma oscillations. The nonlinear terms in (11) and (12) describe the coupling of these two modes.

## III. PERTURBATION EXPANSION

### A. THE NATURE OF THE PERTURBATION SCHEME

According to the discussion in (I), the amplitudes of the impinging electromagnetic waves are $O(\varepsilon)$ quantities. These waves produce small disturbances in the quiescent plasma which are of $O(\varepsilon)$ also. As a result of this $n(x,t)$, $\underset{\sim}{v}(\underset{\sim}{x},t)$, $\underset{\approx}{p}(\underset{\sim}{x},t)$, $\underset{\sim}{E}(\underset{\sim}{x},t)$, and $\underset{\sim}{B}(\underset{\sim}{x},t)$ are all $O(\varepsilon)$ quantitites. On the other hand, the nonlinear terms in equations (3) - (12), being quadratic in the above $O(\varepsilon)$ quantities, are of $O(\varepsilon^2)$.

Neglecting for the moment the nonlinear terms in equations (3) - (12), we obtain a set of linear equations, with all of the terms in them of $O(\varepsilon)$. The transverse components of the solutions of these linearized equations have temporal variations on the scale of $O\left(\frac{1}{\omega_1}\right)$ and spatial variations on the scale of $O\left(\frac{1}{k_1}\right)$. On the other hand, the longitudinal components have temporal variations on the scale of $O\left(\frac{1}{\omega_p}\right)$. We consider these temporal and spatial scales to be of $O(1)$. The amplitudes of the solutions, however, are of $O(\varepsilon)$.

However, the presence of nonlinear terms in eqs. (3) - (12), which are of a higher order in $\varepsilon$, introduces not only small changes on the fast scales in the amplitudes of the solutions, but also small changes in the frequencies and wavelengths of those solutions. These small shifts in frequencies and wavelengths imply the presence of additional time and spatial scales which are of $O(\varepsilon)$.

We shall take these additional slow spatial and time scales into account by explicitly introducing many time variables, denoted by $t_o$, $\varepsilon t_1$, $\varepsilon^2 t_2, \ldots,$ and many spatial scales, denoted by $\underset{\sim}{x}_o$, $\varepsilon \underset{\sim}{x}_1$, $\varepsilon^2 \underset{\sim}{x}_2, \ldots,$ with $\frac{\partial t_o}{\partial t} = 1$, $\frac{\partial(\varepsilon t_1)}{\partial t} = \varepsilon$, $\frac{\partial(\varepsilon^2 t_2)}{\partial t} = \varepsilon^2, \ldots,$ and $\frac{\partial \underset{\sim}{x}_o}{\partial \underset{\sim}{x}} = \underset{\approx}{1}$, $\frac{\partial(\varepsilon \underset{\sim}{x}_1)}{\partial \underset{\sim}{x}} = \varepsilon \underset{\approx}{1}$, $\frac{\partial(\varepsilon^2 \underset{\sim}{x}_2)}{\partial x} = \varepsilon^2 \underset{\approx}{1}, \ldots$. We may write the actual spatial and time dependence of any function as the dependence on many time variables and many spatial variables, i.e. $f(\underset{\sim}{x}, t) = f(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, \varepsilon^2 \underset{\sim}{x}_2, \ldots, t_o, \varepsilon t_1, \varepsilon^2 t_2, \ldots)$.

We shall approximate the corrections to the amplitudes of the plasma variables due to non-linear terms by writing the solutions to eqs. (3) - (12) in the form of power series in $\varepsilon$. We write, accordingly,

$$\begin{pmatrix} n(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\sim}{v}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\approx}{p}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\sim}{E}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\sim}{B}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \end{pmatrix} = \sum_{k=1}^{N} \varepsilon^k \begin{pmatrix} n^{(k)}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\sim}{v}^{(k)}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\approx}{p}^{(k)}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\sim}{E}^{(k)}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \\ \underset{\sim}{B}^{(k)}(\underset{\sim}{x}_o, \varepsilon \underset{\sim}{x}_1, t_o, \varepsilon t_1) \end{pmatrix} \qquad (15)$$

## B. THE $O(\varepsilon)$ AND $O(\varepsilon^2)$ PLASMA EQUATIONS

To $O(\varepsilon)$, equations (3) - (12) are, respectively,

$$\frac{\partial n^{(1)}}{\partial t_o} + N_o \frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\sim}{v}^{(1)} = 0 \quad , \tag{16}$$

$$\frac{\partial \underset{\sim}{v}^{(1)}}{\partial t_o} = -\frac{1}{N_o m} \frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\approx}{p}^{(1)} - \frac{e}{m} \underset{\sim}{E}^{(1)} \quad , \tag{17}$$

$$\frac{\partial \underset{\approx}{p}^{(1)}}{\partial t_o} + P_o \underset{\approx}{1} \frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\sim}{v}^{(1)} + P_o \left( \frac{\partial}{\partial \underset{\sim}{x}_o} \underset{\sim}{v}^{(1)} + \widetilde{\frac{\partial}{\partial \underset{\sim}{x}_o} \underset{\sim}{v}^{(1)}} \right) = 0 \quad , \tag{18}$$

$$\frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\sim}{E}^{(1)} = -4\pi e n^{(1)} \quad , \tag{19}$$

$$\frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\sim}{B}^{(1)} = 0 \quad , \tag{20}$$

$$\frac{\partial}{\partial \underset{\sim}{x}_o} \times \underset{\sim}{E}^{(1)} = -\frac{1}{c} \frac{\partial \underset{\sim}{B}^{(1)}}{\partial t_o} \quad , \tag{21}$$

$$\frac{\partial}{\partial \underset{\sim}{x}_o} \times \underset{\sim}{B}^{(1)} = \frac{1}{c} \frac{\partial \underset{\sim}{E}^{(1)}}{\partial t_o} - \frac{4\pi N_o e}{c} \underset{\sim}{v}^{(1)} \quad , \tag{22}$$

$$\left[\frac{1}{c^2}\frac{\partial^4}{\partial t_o^4} + \frac{\omega_p^2}{c^2}\frac{\partial^2}{\partial t_o^2} - \left(1 + \frac{1}{3}\frac{v_o^2}{c^2}\right)\nabla_o^2\frac{\partial^2}{\partial t_o^2}\right.$$

$$+ \frac{1}{3}v_o^2\nabla_o^2\nabla_o^2 + \left(1 - \frac{2}{3}\frac{v_o^2}{c^2}\right)\frac{\partial^2}{\partial t_o^2}\frac{\partial}{\partial \underset{\sim}{x}_o}\left(\frac{\partial}{\partial \underset{\sim}{x}_o}\cdot\right)$$

$$\left.- \frac{1}{3}v_o^2\nabla_o^2\frac{\partial}{\partial \underset{\sim}{x}_o}\left(\frac{\partial}{\partial \underset{\sim}{x}_o}\cdot\right)\right]\underset{\sim}{E}^{(1)} = 0 , \tag{23}$$

$$\left(\frac{\partial^2}{\partial t_o^2} + \omega_p^2 - v_o^2\nabla_o^2\right)\frac{\partial n^{(1)}}{\partial t_o} = 0 , \tag{24}$$

and

$$\left[\frac{1}{c^2}\frac{\partial^4}{\partial t_o^4} + \frac{\omega_p^2}{c^2}\frac{\partial^2}{\partial t_o^2} - \left(1 + \frac{1}{3}\frac{v_o^2}{c^2}\right)\nabla_o^2\frac{\partial^2}{\partial t_o^2}\right.$$

$$\left.+ \frac{1}{3}v_o^2\nabla_o^2\nabla_o^2\right]\underset{\sim}{B}^{(1)} = 0 , \tag{25}$$

where $\nabla_o^2$ is the Laplacian with respect to the $\underset{\sim}{x}_o$ variable. As we can see, plane wave solutions of eq. (23) - (25) satisfy the dispersion relations (13) - (14), as could have been expected.

To $O(\varepsilon^2)$, the wave equations (10) - (12) are, respectively,

$$
\begin{aligned}
&\left[\frac{1}{c^2}\,\frac{\partial^4}{\partial t_o^4} + \frac{\omega_p^2}{c^2}\,\frac{\partial^2}{\partial t_o^2} - \left(1 + \frac{1}{3}\,\frac{v_o^2}{c^2}\right)\nabla_o^2\,\frac{\partial^2}{\partial t_o^2}\right.\\
&+ \frac{1}{3}\,v_o^2\,\nabla_o^2\,\nabla_o^2 + \left(1 - \frac{2}{3}\,\frac{v_o^2}{c^2}\right)\frac{\partial^2}{\partial t_o^2}\,\frac{\partial}{\partial \mathbf{x}_o}\left(\frac{\partial}{\mathbf{x}_o}\,\cdot\right)\\
&\left.- \frac{1}{3}\,v_o^2\,\nabla_o^2\,\frac{\partial}{\partial \mathbf{x}_o}\left(\frac{\partial}{\partial \mathbf{x}_o}\,\cdot\right)\right]\mathbf{E}^{(2)} = -\frac{4\pi e}{c^2}\,\frac{\partial^2}{\partial t_o^2}\left[N_o\left(\mathbf{v}^{(1)}\cdot\frac{\partial}{\partial \mathbf{x}_o}\,\mathbf{v}^{(1)}\right.\right.\\
&\left.\left.+ \frac{e}{mc}\,\mathbf{v}^{(1)}\times\mathbf{B}^{(1)}\right) - \frac{\partial}{\partial t_o}\left(n^{(1)}\mathbf{v}^{(1)}\right)\right] - \frac{4\pi e}{c^2}\,\frac{\partial^2}{\partial t_o^2}\left[n^{(1)}\left(\frac{\partial \mathbf{v}^{(1)}}{\partial t_o}\right.\right.\\
&\left.\left.+ \frac{e}{m}\,\mathbf{E}^{(1)}\right)\right] - \frac{4\pi e}{3}\,\frac{v_o^2}{c^2}\,\nabla_o^2\left(\frac{\partial}{\partial t_o}\,n^{(1)}\,\mathbf{v}^{(1)}\right)\\
&- \frac{8\pi e}{3}\,\frac{v_o^2}{c^2}\,\frac{\partial}{\partial t_o}\,\frac{\partial}{\partial \mathbf{x}_o}\left(\frac{\partial}{\partial \mathbf{x}_o}\cdot n^{(1)}\,\mathbf{v}^{(1)}\right)\\
&+ \frac{4\pi e}{mc^2}\,\frac{\partial}{\partial t_o}\,\frac{\partial}{\partial \mathbf{x}_o}\cdot\left[\mathbf{v}^{(1)}\cdot\frac{\partial}{\partial \mathbf{x}_o}\,\mathsf{P}^{(1)} + \mathsf{P}^{(1)}\left(\frac{\partial}{\partial \mathbf{x}_o}\cdot\mathbf{v}^{(1)}\right)\right.\\
&\widetilde{\left(\mathsf{P}^{(1)}\cdot\frac{\partial}{\partial \mathbf{x}_o}\right)\mathbf{v}^{(1)} + \left(\mathsf{P}^{(1)}\cdot\frac{\partial}{\partial \mathbf{x}_o}\right)}\,\mathbf{v}^{(1)}\\
&\left.+ \frac{e}{mc}\left(\mathsf{P}^{(1)}\times\mathbf{B} + \widetilde{\mathsf{P}^{(1)}\times\mathbf{B}^{(1)}}\right)\right]\\
&- \left[\frac{4}{c^2}\,\frac{\partial^4}{\partial(\epsilon t_1)\partial t_o^3} + \frac{2\omega_p^2}{c^2}\,\frac{\partial^2}{\partial(\epsilon t_1)\partial t_o}\right.
\end{aligned}
$$

$$- 2\left(1 + \frac{1}{3}\frac{v_o^2}{c^2}\right) \frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)} \circ \frac{\partial}{\partial \underset{\sim}{x}_o} \frac{\partial^2}{\partial t_o^2}$$

$$- 2\left(1 + \frac{1}{3}\frac{v_o^2}{c^2}\right) \nabla_o^2 \frac{\partial^2}{\partial(\epsilon t_1)\partial t_o} + \frac{4}{3} v_o^2 \frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)} \circ \frac{\partial}{\partial \underset{\sim}{x}_o} \nabla_o^2$$

$$+ 2\left(1 - \frac{2}{3}\frac{v_o^2}{c^2}\right) \frac{\partial^2}{\partial(\epsilon t_1)\partial t_o} \frac{\partial}{\partial \underset{\sim}{x}_o}\left(\frac{\partial}{\partial \underset{\sim}{x}_o} \circ\right)$$

$$+ \left(1 - \frac{2}{3}\frac{v_o^2}{c^2}\right) \frac{\partial^2}{\partial t_o^2} \left\{\frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)}\left(\frac{\partial}{\partial \underset{\sim}{x}_o} \circ\right) + \frac{\partial}{\partial \underset{\sim}{x}_o}\left(\frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)} \circ\right)\right\}$$

$$- \frac{1}{3} v_o^2 \nabla_o^2 \left\{\frac{\partial}{\partial \underset{\sim}{x}_o}\left(\frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)} \circ\right) + \frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)}\left(\frac{\partial}{\partial \underset{\sim}{x}_o} \circ\right)\right\}$$

$$- \frac{2}{3} v_o^2 \left(\frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)} \circ \frac{\partial}{\partial \underset{\sim}{x}_o}\right)\frac{\partial}{\partial \underset{\sim}{x}_o}\left(\frac{\partial}{\partial \underset{\sim}{x}_o} \circ\right)\Bigg] \underset{\sim}{E}^{(1)} \qquad (26)$$

and

$$\left(\frac{\partial^2}{\partial t_o^2} + \omega_p^2 - v_o^2 \nabla_o^2\right)\frac{\partial^2 n^{(2)}}{\partial t_o^2} = \frac{\partial^2}{\partial t_o^2}\frac{\partial}{\partial \underset{\sim}{x}_o} \circ \Bigg[N_o\Bigg(\underset{\sim}{v}^{(1)} \circ \frac{\partial}{\partial \underset{\sim}{x}_o}\underset{\sim}{v}^{(1)}$$

$$+ \frac{e}{mc}\underset{\sim}{v}^{(1)} \times \underset{\sim}{B}'^{(1)}\Bigg) - \frac{\partial}{\partial t_o}\left(n^{(1)}\underset{\sim}{v}^{(1)}\right)\Bigg] + \frac{\partial^2}{\partial t_o^2}\frac{\partial}{\partial \underset{\sim}{x}_o} \circ \Bigg[n^{(1)}\Bigg(\frac{\partial \underset{\sim}{v}^{(1)}}{\partial t_o}$$

$$+ \frac{e}{mc}\underset{\sim}{E}^{(1)}\Bigg)\Bigg] + v_o^2 \frac{\partial}{\partial t_o}\nabla_o^2 \frac{\partial}{\partial \underset{\sim}{x}_o} \circ \left(n^{(1)}\underset{\sim}{v}^{(1)}\right)$$

$$- \frac{1}{m} \frac{\partial}{\partial t_o} \frac{\partial^2}{\partial \underset{\sim}{x}_o \partial \underset{\sim}{x}_o} : \Bigg[ \left( \underset{\sim}{v}^{(1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} \right) \underset{\approx}{p}^{(1)} + \underset{\approx}{p}^{(1)} \left( \frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\sim}{v}^{(1)} \right)$$

$$+ \left( \underset{\approx}{p}^{(1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} \right) \underset{\sim}{v}^{(1)} + \widetilde{\left( \underset{\approx}{p}^{(1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} \right) \underset{\sim}{v}^{(1)}}$$

$$+ \frac{e}{mc} \left( \underset{\approx}{p}^{(1)} \times \underset{\sim}{B}^{(1)} + \widetilde{\underset{\approx}{p}^{(1)} \times \underset{\sim}{B}^{(1)}} \right) \Bigg]$$

$$+ 2 \left[ v_o^2 \frac{\partial}{\partial (\epsilon \underset{\sim}{x}_1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} - \frac{\partial^2}{\partial (\epsilon t_1) \partial t_o} \right] \frac{\partial^2 n^{(1)}}{\partial t_o^2} \quad . \tag{27}$$

## IV. THE $O(\varepsilon)$ SOLUTIONS AND SECULAR TERMS IN $O(\varepsilon^2)$

### A. INITIAL AND BOUNDARY CONDITIONS

We shall assume that the electric and magnetic fields of the two plane waves impinging on the plasma contain terms only of $O(\varepsilon)$, there being no terms of higher order in $\varepsilon$. Consequently we require that $\underset{\sim}{B}^{(1)}$ and the transverse component of $\underset{\sim}{E}^{(1)}$, denoted by $\underset{\sim}{E}_T^{(1)}$, be the electric and magnetic fields of the impinging waves. We define the scattered electromagnetic fields to be the transverse electric and magnetic fields which are of higher order in $\varepsilon$ than $O(\varepsilon)$.

From the physical standpoint, we are primarily interested in the scattering problem which was posed in the Introduction (I): Two lasers, located in vacuum, are turned on at time $t = 0$; the two electromagnetic waves emitted by the lasers enter the plasma, are scattered, leave the plasma, and are intercepted by detectors. We take the plasma to be in a quiescent state at $t = 0$. Therefore there will be no longitudinal electric field at $t = 0$. We shall consequently require that the longitudinal component of $\underset{\sim}{E}^{(1)}$, denoted by $\underset{\sim}{E}_L^{(1)}$, be zero everywhere inside the plasma at $t = 0$.

Let us take the volume of the plasma large enough so that quantities which are periodic functions of $\varepsilon \underset{\sim}{x}_1$ can go through the variation of at least one wavelength inside the plasma. On the other hand, the volume is assumed to be small enough so that the characteristic time for the transverse electromagnetic waves to pass through the volume be small compared to the characteristic time for the build-up of plasma oscillations. This implies that when the two lasers are turned on, the waves which they emit will penetrate the plasma completely before the longitudinal plasma oscillations become large enough to produce significant scattering of the waves.

We can therefore assume that, at $t = 0$, $\underset{\sim}{E}_T^{(1)}$ and $\underset{\sim}{B}^{(1)}$ are given everywhere inside the plasma, including the boundary, by

$$\left.\begin{aligned} \underset{\sim}{E}_T^{(1)} &= \underset{\sim}{A}_1 \sin \psi_1 + \underset{\sim}{A}_2 \sin \psi_2 \\ \underset{\sim}{B}^{(1)} &= \frac{c\, \underset{\sim}{k}_1 \times \underset{\sim}{A}_1}{\omega_1} \sin \psi_1 + \frac{c\, \underset{\sim}{k}_2 \times \underset{\sim}{A}_2}{\omega_2} \sin \psi_2 , \end{aligned}\right\} \qquad (28)$$

with

$$\underset{\sim}{k}_1 \cdot \underset{\sim}{A}_1 = \underset{\sim}{k}_2 \cdot \underset{\sim}{A}_2 = 0 , \qquad (29)$$

and $\psi_1$ and $\psi_2$ defined by

$$\left.\begin{aligned} \psi_1 &\equiv \underset{\sim}{k}_1 \cdot \underset{\sim}{x} - \omega_1 t + \phi_1 \\ \psi_2 &\equiv \underset{\sim}{k}_2 \cdot \underset{\sim}{x} - \omega_2 t + \phi_2 \end{aligned}\right\} . \qquad (30)$$

$\omega_1$ and $\underset{\sim}{k}_1$, and $\omega_2$ and $\underset{\sim}{k}_2$ satisfy the dispersion relation (14), respectively. $\underset{\sim}{A}_1$, $\underset{\sim}{A}_2$, $\phi_1$, and $\phi_2$ are independent of position inside the plasma at $t = 0$. They are determined by the output of the lasers.

We shall further assume that $\underset{\sim}{E}_T^{(1)}$ and $\underset{\sim}{B}^{(1)}$ are given by (28) on the interface between the plasma and vacuum, facing the two lasers, for all times, with $\underset{\sim}{A}_1$, $\underset{\sim}{A}_2$, $\phi_1$, and $\phi_2$ being constant on the interface and equal to their values at $t = 0$. We shall also assume that $\underset{\sim}{E}_L^{(1)} = 0$, on the same interface, for all times.

The scattering problem, which we have just described, with the initial and boundary conditions, is only one of the many problems we can pose. Another problem, that we can state, is the pure initial-value problem. In this problem, we assume the boundaries of the plasma to have been removed to infinity, so that the plasma covers all space. We then have to state only initial conditions for the problem. One may suppose, for example, that it is somehow possible to set up initial conditions which are identical with the initial conditions set up above for the actual scattering problem.

Again, another problem, that one can pose, is the pure boundary-value problem. In this problem, we are interested only in the steady state solutions of the equations describing the behavior of the plasma. We may simplify this problem by neglecting the initial conditions, and take into account only the boundary conditions. We may, for instance, take the same boundary conditions as were set up for the actual scattering problem above. There are other problems we can pose. We shall, however, discuss only the pure initial-value and the pure boundary-value problems in addition to the actual scattering problem.

## B. TRANSVERSE COMPONENTS OF $O(\varepsilon)$ SOLUTIONS

An inspection of eqs. (16) to (22) discloses immediately that their solutions have the following transverse components:

$$E_T^{(1)} = A_1(\varepsilon x_1, \varepsilon t_1) \sin \psi_1 + A_2(\varepsilon x_1, \varepsilon t_1) \sin \psi_2 , \tag{31}$$

where $\psi_1$ and $\psi_2$ are given by

$$\left.\begin{aligned} \psi_1 &= k_1 \circ x - \omega_1 t + \phi_1(\varepsilon x_1, \varepsilon t_1) \\ \psi_2 &= k_2 \circ x - \omega_2 t + \phi_2(\varepsilon x_1, \varepsilon t_1) \quad . \end{aligned}\right\} \tag{32}$$

$A_1$, $A_2$, $\phi_1$, and $\phi_2$ are some functions of $\varepsilon x_1$ and $\varepsilon t_1$ which have not yet been determined.

$$B^{(1)} = \frac{c(k_1 \times A_1)}{\omega_1} \sin \psi_1 + \frac{c(k_2 \times A_2)}{\omega_2} \sin \psi_2 \tag{33}$$

$$v_T^{(1)} = -\frac{e}{m\Omega_1} A_1 \cos \psi_1 - \frac{e}{m\Omega_2} A_2 \cos \psi_2 \tag{34}$$

where $\Omega_1$ and $\Omega_2$ are defined by $\Omega_1 \equiv \omega_1 \left( 1 - \frac{v_o^2 k_1^2}{3\omega_1^2} \right)$ and $\Omega_2 \equiv \omega_2 \left( 1 - \frac{v_o^2 k_2^2}{3\omega_2^2} \right)$

$$\begin{aligned} P_T^{(1)} = & -\frac{e P_o}{m\Omega_1 \omega_1} (k_1 A_1 + A_1 k_1) \cos \psi_1 \\ & -\frac{e P_o}{m\Omega_2 \omega_2} (k_2 A_2 + A_2 k_2) \cos \psi_2 \quad . \end{aligned} \tag{35}$$

## C. SECULAR BEHAVIOR IN $O(\varepsilon^2)$ EQUATIONS

Looking at the nonlinear terms on the right-hand side of eq. (27), we notice that the transverse components of the $O(\varepsilon)$ solutions (expressions (31) to (35)) contribute terms proportional to $\sin 2\psi_1$, $\sin 2\psi_2$, $\sin(\psi_1 + \psi_2)$, and $\sin(\psi_1 - \psi_2)$. If $\omega_1$, $\omega_2$, $\underset{\sim}{k}_1$, and $\underset{\sim}{k}_2$ are chosen so that $(\omega_1 - \omega_2)^2 = \omega_p^2 + v_o^2(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2$, i.e., $(\omega_1 - \omega_2)$ and $(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)$ satisfy the dispersion relation (13) for longitudinal plasma oscillations, and $|\underset{\sim}{k}_1 - \underset{\sim}{k}_2| \lesssim k_D$ (where $k_D = \left(\frac{4\pi N_o e}{KT}\right)^{1/2}$ is the Debye wave number), the nonlinear term containing the factor $\sin(\psi_1 - \psi_2)$ will be in resonance with the homogeneous solution of the left-hand side of eq. (27). This will produce an $n^{(2)}$ which is growing linearly with time.

The physical reason for this behavior is the fact that the nonlinear term containing $\sin(\psi_1 - \psi_2)$ is the divergence of a longitudinal driving force of frequency $(\omega_1 - \omega_2)$ which will keep on feeding energy into the plasma oscillations and thus cause them to increase with time.

However, we know that plasma oscillations must remain finite. Therefore the phase difference between the driving force and the longitudinal plasma oscillations must change slowly with space and/or time so that the driving force and the plasma oscillations will gradually get out of phase and the growth of the oscillations will be checked. But this requires that naturally oscillating longitudinal plasma field variables be non-vanishing. Otherwise it would be meaningless to talk about a slow phase drift of a plasma variable which is zero at all spatial points for all times. This can be seen from eq. (27), which requires the existence of $\underset{\sim}{E}_L^{(1)}$ (changing on the slow spatial and/or time scale) to eliminate the secular terms in the nonlinear driving force.

By eqs. (23) or (24), $\underset{\sim}{E}_L^{(1)} = a(\epsilon \underset{\sim}{x}_1, \epsilon t_1)(\underset{\sim}{k}_1 - \underset{\sim}{k}_2) \cos(\psi_1 - \psi_2) + b(\epsilon \underset{\sim}{x}_1, \epsilon t_1)(\underset{\sim}{k}_1 - \underset{\sim}{k}_2) \sin(\psi_1 - \psi_2)$. However, we can show that $b(\epsilon \underset{\sim}{x}_1, \epsilon t_1) = 0$ identically. The argument goes as follows. If $\underset{\sim}{E}_L^{(1)} = b \sin(\psi_1 - \psi_2)$, then, by eq. (19), $n^{(1)} \sim \cos(\psi_1 - \psi_2)$. The presence of $n^{(1)}$ introduces an $O(\epsilon^2)$ transverse current of the form $n^{(1)} \underset{\sim}{v}_T^{(1)}$ into eq. (26). Taking into account the form of $\underset{\sim}{v}_T^{(1)}$, as given by eq. (34), we see that $n^{(1)} \underset{\sim}{v}_T^{(1)}$ contains terms proportional to $(\underset{\sim}{A}_2 \cos \psi_1)$ and $(\underset{\sim}{A}_1 \cos \psi_2)$. The first of these is polarized parallel to $\underset{\sim}{A}_2$ but oscillates with phase $\psi_1$, the second one is polarized parallel to $\underset{\sim}{A}_1$, but oscillates with phase $\psi_2$. Both terms will consequently be in phase with the natural transverse plasma oscillations, and will drive these oscillations.

A slow spatial and/or temporal drift of the phase angles $\phi_1$ and $\phi_2$ may not be fast enough to get the natural transverse oscillations and the transverse current $(n^{(1)} \underset{\sim}{v}_T^{(1)})$ sufficiently quickly out of phase with each other to limit the oscillations. We therefore require, in general, an additional relative rotation of the directions of polarization of the current $(n^{(1)} \underset{\sim}{v}_T^{(1)})$ and the transverse plasma oscillations.

A glance at eqs. (26) and (31) shows that the nonlinear terms oscilating with $\psi_1$ or $\psi_2$ are proportional to $\sin\psi_1$ or $\sin\psi_2$. On the other hand, the slow variation of $\underset{\sim}{E}_T^{(1)}$ in eq. (26), containing $\sin\psi_1$ or $\sin\psi_2$, will be proportional to $\frac{\partial\phi_1}{\partial(\epsilon t_1)}$ and/or $\frac{\partial\phi_1}{\partial(\epsilon \underset{\sim}{x}_1)}$, or $\frac{\partial\phi_2}{\partial(\epsilon t_1)}$ and/or $\frac{\partial\phi_2}{\partial(\epsilon \underset{\sim}{x}_1)}$, not to the derivatives of $\underset{\sim}{A}_1$ or $\underset{\sim}{A}_2$ with respect to $\epsilon t_1$ and/or $\epsilon \underset{\sim}{x}_1$. Therefore there is no provision for the rotation of the directions of polarization to remove secular terms from eq. (26). The presence of $\underset{\sim}{E}_L^{(1)} = b(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)\sin(\psi_1-\psi_2)$, as we can see, creates secular terms in eq. (26) which cannot be removed. We shall therefore set $b(\epsilon\underset{\sim}{x}_1,\epsilon t_1) = 0$.

The presence of $\underset{\sim}{E}_L^{(1)} = a(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)\cos(\psi_1-\psi_2)$, on the other hand, creates no such problems. By eq. (19), $n^{(1)} \sim \sin(\psi_1-\psi_2)$. Hence the current $(n^{(1)}\,\underset{\sim}{v}_T^{(1)})$ will contain terms proportional to $(\underset{\sim}{A}_2\sin\psi_1)$ and $(\underset{\sim}{A}_1\sin\psi_2)$. The nonlinear terms in eq. (26) oscillating with phases $\psi_1$ or $\psi_2$ will be proportional to $\cos\psi_1$ or $\cos\psi_2$. But the slow variation of $\underset{\sim}{E}_T^{(1)}$ in eq. (26), containing $\cos\psi_1$ or $\cos\psi_2$, will be proportional to the derivatives of $\underset{\sim}{A}_1$ or $\underset{\sim}{A}_2$, with respect to $\epsilon\underset{\sim}{x}_1$ and/or $\epsilon t_1$. Therefore slow rotation of the amplitudes will be possible.

## D. LONGITUDINAL COMPONENTS OF $O(\varepsilon)$ SOLUTIONS

Let us define the quantities $\underset{\sim}{k}_3$, $\underset{\sim}{k}_4$, $\omega_3$, $\omega_4$, $\phi_3$, $\phi_4$, $\psi_3$ and $\psi_4$ by the following expressions:

$$\left.\begin{aligned}
\underset{\sim}{k}_3 &\equiv \underset{\sim}{k}_1 + \underset{\sim}{k}_2 \\
\underset{\sim}{k}_4 &\equiv \underset{\sim}{k}_1 - \underset{\sim}{k}_2 \\
\omega_3 &\equiv \omega_1 + \omega_2 \\
\omega_4 &\equiv \omega_1 - \omega_2 \\
\phi_3 &\equiv \phi_1 + \phi_2 \\
\phi_4 &\equiv \phi_1 - \phi_2 \\
\psi_3 &\equiv \psi_1 + \psi_2 \\
\psi_4 &\equiv \psi_1 - \psi_2 \quad .
\end{aligned}\right\} \tag{36}$$

We shall take $\underset{\sim}{E}_L^{(1)}$ to be of the form

$$\underset{\sim}{E}_L^{(1)} = a(\varepsilon \underset{\sim}{x}_1, \varepsilon t_1)\, \underset{\sim}{k}_4 \cos \psi_4 \quad , \tag{37}$$

where $a(\varepsilon \underset{\sim}{x}_1, \varepsilon t_1)$ is an unknown function to be determined by the solution of $O(\varepsilon^2)$ equations. The initial condition that $\underset{\sim}{E}_L^{(1)}$ be zero at $t = 0$ everywhere, yields the initial condition on $a(\varepsilon \underset{\sim}{x}_1, \varepsilon t_1)$: $a(\varepsilon \underset{\sim}{x}_1, \varepsilon t_1) = 0$ at $t = 0$, for all $\underset{\sim}{x}$, inside the plasma and on the boundary. The condition that $\underset{\sim}{E}_L^{(1)}$ be zero on the boundary for all times, yields the boundary condition on $a(\varepsilon \underset{\sim}{x}_1, \varepsilon t_1)$: $a(\varepsilon \underset{\sim}{x}_1, \varepsilon t_1) = 0$ on the boundary for all $t$.

An inspection of eq. (16) to (22) discloses immediately that their solutions have the following longitudinal components:

$$n^{(1)} = \frac{a\, k_4^2}{4\pi e} \sin \psi_4 \quad , \tag{38}$$

$$\underset{\sim}{v}_L^{(1)} = \frac{ea}{m\Omega_4} \underset{\sim}{k}_4 \sin \psi_4 \quad , \tag{39}$$

where $\Omega_4$ is defined by $\Omega_4 \equiv \omega_4 \left( 1 - \frac{v_o^2\, k_4^2}{\omega_4^2} \right)$, and

$$\underset{\approx}{p}_L^{(1)} = \frac{e\, P_o\, a}{m\Omega_4\, \omega_4} \left(k_4^2\, \underset{\approx}{1} + 2\, \underset{\sim}{k}_4\, \underset{\sim}{k}_4\right) \sin \psi_4 \quad . \tag{40}$$

## V. REMOVAL OF SECULAR BEHAVIOR FROM $O(\varepsilon^2)$ EQUATIONS

We shall now proceed to evaluate the nonlinear terms in $O(\varepsilon^2)$ equations and to determine the conditions which will remove secular behavior.

First, let us simplify the $O(\varepsilon^2)$ wave equations somewhat. We shall assume that the temperature (and hence the pressure $P_o$) of the quiescent plasma is low. Having made this assumption, we shall expand all quantities which are functions of $P_o$ in power series in $P_o$. The first term in the expansion of any quantity will be the value of that quantity at zero temperature. We shall be primarily interested in zero temperature values of quantities. Consequently, eq. (27) can be written in the form

$$\left(\frac{\partial^2}{\partial t_o^2} + \omega_p^2 - v_o^2 \nabla_o^2\right)\frac{\partial n^{(2)}}{\partial t_o} = \frac{\partial}{\partial t_o}\,\frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \left[ N_o \left( \underset{\sim}{v}^{(1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o}\, \underset{\sim}{v}^{(1)} + \frac{e}{mc}\, \underset{\sim}{v}^{(1)} \times \underset{\sim}{B}^{(1)} \right) - \frac{\partial}{\partial t_o}\left(n^{(1)}\, \underset{\sim}{v}^{(1)}\right)\right] + (\text{non-linear terms of } O(P_o)) + 2\left[v_o^2\, \frac{\partial}{\partial(\varepsilon \underset{\sim}{x}_1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} - \frac{\partial^2}{\partial(\varepsilon t_1)\partial t_o}\right]\frac{\partial n^{(1)}}{\partial t_o} \quad . \tag{41}$$

The $\underset{\sim}{v}^{(1)}$ quantity, appearing in eq. (41), will be approximated by using only the first term of the expansion in $P_o$. Thus, by eq. (34),

$$\underset{\sim}{v}_T^{(1)} \approx -\frac{e}{m\omega_1}\, \underset{\sim}{A}_1 \cos\psi_1 - \frac{e}{m\omega_2}\, \underset{\sim}{A}_2 \cos\psi_2 \, , \tag{42}$$

and, by eq. (39),

$$\underset{\sim}{v}_L^{(1)} = \frac{ea}{m\omega_4} \underset{\sim}{k}_4 \sin \psi_4 \quad . \tag{43}$$

We have written the linear terms of $O(P_o)$ out explicitly in eq. (41). By droppping these terms we would leave ourselves no linear terms containing spatial derivatives. We would then be unable to do any boundary value problems or mixed inital-value-boundary-value problems. This can be seen from the fact that spatial variations can limit the longitudinal mode only if the longitudinal dispersion relation depends on them. The longitudinal dispersion relation is given by eq. (13). When $P_o = 0$, eq. (13) reduces to $\omega^2 = \omega_p^2$. Hence there is no dependence on k at $P_o = 0$. Therefore we need a non-zero $P_o$. Consequently we shall carry the $O(P_o)$ linear terms along in eq. (41).

Eq. (26) can be simplified in the following way. Making use of eq. (6), we obtain

$$\frac{\partial}{\partial \underset{\sim}{x}} \left( \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{E} \right) = - 4\pi e \frac{\partial n}{\partial \underset{\sim}{x}} \quad .$$

Or using multiple spatial and time scales and expansion (15), we obtain

$$\epsilon^2 \frac{\partial}{\partial \underset{\sim}{x}_o} \left( \frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\sim}{E}^{(2)} \right) = - \epsilon^2 \; 4\pi e \; \frac{\partial n^{(2)}}{\partial \underset{\sim}{x}_o}$$

$$- \epsilon^2 \; 4\pi e \frac{\partial n^{(1)}}{\partial (\epsilon \underset{\sim}{x}_1)} - \epsilon^2 \frac{\partial}{\partial (\epsilon \underset{\sim}{x}_1)} \left( \frac{\partial}{\partial \underset{\sim}{x}_o} \cdot \underset{\sim}{E}^{(1)} \right)$$

$$- \epsilon^2 \; \frac{\partial}{\partial \underset{\sim}{x}_o} \left( \frac{\partial}{\partial (\epsilon \underset{\sim}{x}_1)} \cdot \underset{\sim}{E}^{(1)} \right) . \qquad (44)$$

Eq. (26) can now be written in the form

$$\left( \nabla_o^2 - \frac{1}{c^2} \; \frac{\partial^2}{\partial t_o^2} - \frac{\omega_p^2}{c^2} \right) \underset{\sim}{E}^{(2)} = -4\pi e \; \frac{\partial n^{(2)}}{\partial \underset{\sim}{x}_o}$$

$$+ \frac{4\pi e}{c^2} \left[ N_o \left( \underset{\sim}{v}^{(1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} \; \underset{\sim}{v}^{(1)} + \frac{e}{mc} \; \underset{\sim}{v}^{(1)} \times \underset{\sim}{B}^{(1)} \right) \right.$$

$$\left. - \frac{\partial}{\partial t_o} \left( n^{(1)} \; \underset{\sim}{v}^{(1)} \right) \right] + \text{(linear and non-}$$

$$\text{linear terms of } \; O(P_o)) - 2 \left[ \frac{\partial}{\partial (\epsilon \underset{\sim}{x}_1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} \right.$$

$$\left. - \frac{1}{c^2} \; \frac{\partial^2}{\partial (\epsilon t_1) \epsilon t_o} \right] \; \underset{\sim}{E}^{(1)} - 4\pi e \; \frac{\partial n^{(1)}}{\partial (\epsilon \underset{\sim}{x}_1)} \quad . \qquad (45)$$

## A. EVALUATION OF NON-LINEAR TERMS

Substituting the $O(\varepsilon)$ solutions into the nonlinear expression $N_o\left(\underset{\sim}{v}^{(1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} \underset{\sim}{v}^{(1)} + \frac{e}{mc} \underset{\sim}{v}^{(1)} \times \underset{\sim}{B}^{(1)}\right)$, we obtain

$$N_o\left(\underset{\sim}{v}^{(1)} \cdot \frac{\partial}{\partial \underset{\sim}{x}_o} \underset{\sim}{v}^{(1)} + \frac{e}{mc} \underset{\sim}{v}^{(1)} \times \underset{\sim}{B}^{(1)}\right)$$

$$= -\frac{\omega_p^2}{8\pi m}\left[\frac{A_1^2}{\omega_1^2} \underset{\sim}{k}_1 \sin 2\psi_1 + \frac{A_2^2}{\omega_2^2} \underset{\sim}{k}_2 \sin 2\psi_2\right.$$

$$\left. + \frac{(\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2)}{\omega_1 \omega_2}\left(\underset{\sim}{k}_3 \sin \psi_3 + \underset{\sim}{k}_4 \sin \psi_4\right)\right]$$

$$+ \frac{\omega_p a}{8\pi m}\left[-\frac{(\underset{\sim}{k}_1 \cdot \underset{\sim}{A}_2)}{\omega_2} \underset{\sim}{k}_1 \cos \psi_1\right.$$

$$- \frac{(\underset{\sim}{k}_2 \cdot \underset{\sim}{A}_1)}{\omega_1} \underset{\sim}{k}_2 \cos \psi_2 + \frac{(\underset{\sim}{k}_1 \cdot \underset{\sim}{A}_2)}{\omega_2} (2\underset{\sim}{k}_2 - \underset{\sim}{k}_1) \cos(2\psi_2 - \psi_1)$$

$$\left. + \frac{(\underset{\sim}{k}_2 \cdot \underset{\sim}{A}_1)}{\omega_1} (2\underset{\sim}{k}_1 - \underset{\sim}{k}_2) \cos(2\psi_1 - \psi_2)\right]$$

$$+ \frac{a^2 k_4^2}{8\pi m} \underset{\sim}{k}_4 \sin 2\psi_4 \; . \tag{46}$$

Similarly, we obtain for $\frac{\partial}{\partial t_o} (n^{(1)} \underset{\sim}{v}^{(1)})$ the expression

$$\frac{\partial}{\partial t_o} (n^{(1)} \underset{\sim}{v}^{(1)}) = \frac{a\, k_4^2}{8\pi m} \Bigg[ \frac{\omega_1}{\omega_2} \underset{\sim}{A}_2 \cos \psi_1$$

$$- \frac{\omega_2}{\omega_1} \underset{\sim}{A}_1 \cos \psi_2 - \frac{(2\omega_2 - \omega_1)}{\omega_2} \underset{\sim}{A}_2 \cos(2\psi_2 - \psi_1)$$

$$+ \frac{(2\omega_1 - \omega_2)}{\omega_1} \underset{\sim}{A}_1 \cos(2\psi_1 - \psi_2) \Bigg]$$

$$- \frac{a^2\, k_4^2}{4\pi m} \underset{\sim}{k}_4 \sin 2\psi_4 \; . \tag{47}$$

Since (46) and (47) appear as driving terms in eq. (45), we have a scattered transverse wave at frequency $(2\omega_2 - \omega_1)$ and a scattered wave at frequency $(2\omega_1 - \omega_2)$ .

## B. ELIMINATION OF SECULAR TERMS FROM LONGITUDINAL FIELDS

Substituting expressions (46) and (47) into eq. (41), we can write eq. (41) in the form

$$\left(\frac{\partial^2}{\partial t_o^2} + \omega_p^2 - v_o^2 \nabla_o^2\right) n^{(2)} = -\frac{\omega_p^2}{8\pi m} \frac{(\underset{\sim}{A}_1 \circ \underset{\sim}{A}_2)}{\omega_1 \omega_2} k_4^2 \cos \psi_4$$

$$+ \frac{\omega_p a}{8\pi m} \frac{(\underset{\sim}{k}_1 \circ \underset{\sim}{A}_2)}{\omega_2} k_1^2 \sin \psi_1 + \frac{\omega_p a}{8\pi m} \frac{(\underset{\sim}{k}_2 \circ \underset{\sim}{A}_1)}{\omega_1} k_2^2 \sin \psi_2$$

$$+ \frac{a k_4^2}{8\pi m} \frac{\omega_1}{\omega_2} (\underset{\sim}{k}_1 \circ \underset{\sim}{A}_2) \sin \psi_1 - \frac{a k_4^2}{8\pi m} \frac{\omega_2}{\omega_1} (\underset{\sim}{k}_2 \circ \underset{\sim}{A}_1) \sin \psi_2$$

$$+ (\text{other terms}) - \frac{\omega_4 a k_4^2}{2\pi e} \frac{\partial \phi_4}{\partial(\epsilon t_1)} \sin \psi_4$$

$$+ \frac{\omega_4 k_4^2}{2\pi e} \frac{\partial a}{\partial(\epsilon t_1)} \cos \psi_4 + \frac{v_o^2 k_4^2}{2\pi e} \left(\underset{\sim}{k}_4 \circ \frac{\partial a}{\partial(\epsilon \underset{\sim}{x}_1)}\right) \cos \psi_4$$

$$- \frac{v_o^2 a k_4^2}{2\pi e} \left(\underset{\sim}{k}_4 \circ \frac{\partial \phi_4}{\partial(\epsilon \underset{\sim}{x}_1)}\right) \sin \psi_4 \quad . \tag{48}$$

The term (<u>other terms</u>) in eq. (48) includes all finite temperature nonlinear terms and all of the zero temperature nonlinear terms which do not oscillate with phases $\psi_1$, $\psi_2$, or $\psi_4$.

The nonlinear secular term in eq. (48) is the term $\left(-\frac{\omega_p^2}{8\pi m} \frac{\underset{\sim}{A}_1 \circ \underset{\sim}{A}_2}{\omega_1 \omega_2} k_4^2 \cos \psi_4\right)$. The secular behavior will be eliminated from eq. (48) if the following relations are satisfied

$$\omega_4 \frac{\partial \phi_4}{\partial(\varepsilon t_1)} + v_o^2 \, \underset{\sim}{k}_4 \cdot \frac{\partial \phi_4}{\partial(\varepsilon \underset{\sim}{x}_1)} = 0 \tag{49}$$

and

$$\frac{k_4^2}{2\pi e}\left(\omega_4 \frac{\partial a}{\partial(\varepsilon t_1)} + v_o^2 \, \underset{\sim}{k}_4 \cdot \frac{\partial a}{\partial(\varepsilon \underset{\sim}{x}_1)}\right) = \frac{\omega_p^2 \, k_4^2}{8\pi m} \, \frac{(\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2)}{\omega_1 \, \omega_2}$$

$$+ \; (O(P_o) \;\; \text{terms}) \;. \tag{50}$$

The left-hand sides of eqs. (49) and (50) have the form of convective derivative. Eq. (49) states that $\phi_4$ remains constant to an observer traveling in the direction of $\underset{\sim}{k}_4$ with the velocity $\left(\frac{v_o^2}{\omega_4/k_4}\right)$. Since $\phi_4$ is the same for all spatial points at time $t = 0$, and retains its initial value on the boundary for all times, we take $\phi_4$ to be a constant (i.e. to be independent of $\varepsilon \underset{\sim}{x}_1$ and $\varepsilon t_1$). Eq. (50) states that the change in $\underline{a}$, which an observer traveling in the direction $\underset{\sim}{k}_4$ with the velocity $\left(\frac{v_o^2}{\omega_4/k_4}\right)$ sees, is proportional to the scalar product of the amplitudes $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$. When $(\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2) = 0$, the observer notices no change in $\underline{a}$.

## C. ELIMINATION OF SECULAR TERMS FROM TRANSVERSE FIELDS

Since $n^{(2)}$ enters into eq. (45), those terms in $n^{(2)}$ which oscillate with phases $\psi_1$ and $\psi_2$ will produce secular behavior in eq. (45). Let us evaluate these terms. For the sake of simplicity, we shall evaluate these terms in a zero-temperature plasma. Let us make the ansatz that $n^{(2)} = C_1 \sin \psi_1 + C_2 \sin \psi_2 +$ other terms , where $C_1$ and $C_2$ are unspecified constants. Substituting into eq. (48), we obtain

$$n^{(2)} = \frac{a(\omega_p k_1^2 + \omega_1 k_4^2)}{8\pi m\omega_2 (\omega_p^2 - \omega_1^2)} (\underset{\sim}{k}_1 \cdot \underset{\sim}{A}_2) \sin \psi_1$$

$$+ \frac{a(\omega_p k_2^2 - \omega_2 k_4^2)}{8\pi m\omega_1 (\omega_p^2 - \omega_2^2)} (\underset{\sim}{k}_2 \cdot \underset{\sim}{A}_1) \sin \psi_2$$

$$+ \text{(other terms not oscillating with phases } \psi_1 \text{ and } \psi_2)$$

$$+ O(P_o) \text{ terms} \quad . \tag{51}$$

Substituting (51), (46), and (47) into eq. (45), we can write eq. (45) in the form

$$\left( \nabla_o^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t_o^2} - \frac{\omega_p^2}{c^2} \right) \underset{\sim}{E}^{(2)} = \frac{ea\, k_4^2}{2m\, c^2} \frac{\omega_1}{\omega_2} \left[ \frac{(\underset{\sim}{k}_1 \circ \underset{\sim}{A}_2)}{k_1^2} \underset{\sim}{k}_1 \right.$$

$$\left. - \underset{\sim}{A}_2 \right] \cos \psi_1 - \frac{ea\, k_4^2}{2m\, c^2} \frac{\omega_2}{\omega_1} \left[ \frac{(\underset{\sim}{k}_2 \circ \underset{\sim}{A}_1)}{k_2^2} \underset{\sim}{k}_2 - \underset{\sim}{A}_1 \right] \cos \psi_2$$

$$+ \text{(other terms)} - 2\left[ \frac{\partial}{\partial(\varepsilon \underset{\sim}{x}_1)} \circ \frac{\partial}{\partial \underset{\sim}{x}_o} - \frac{1}{c^2} \frac{\partial}{\partial(\varepsilon t_1)} \frac{\partial}{\partial t_o} \right] \underset{\sim}{E}^{(1)}$$

$$- 4\pi e \frac{\partial n^{(1)}}{\partial(\varepsilon \underset{\sim}{x}_1)} \quad . \tag{52}$$

The nonlinear secular terms in eq. (52) are the two terms containing $\cos \psi_1$ and $\cos \psi_2$, respectively. They can be canceled only by slow spatial and/or time variations in $\underset{\sim}{E}_T^{(1)}$, since $n^{(1)}$ and $\underset{\sim}{E}_L^{(1)}$ do not contain any terms oscillating with phases $\psi_1$ and $\psi_2$. Therefore the secular behavior will be eliminated if the following relations are satisfied

$$\omega_1 \frac{\partial \phi_1}{\partial(\varepsilon t_1)} + c^2 \underset{\sim}{k}_1 \circ \frac{\partial \phi_1}{\partial(\varepsilon \underset{\sim}{x}_1)} = 0 \quad , \tag{53}$$

$$\omega_2 \frac{\partial \phi_2}{\partial(\varepsilon t_1)} + c^2 \underset{\sim}{k}_2 \circ \frac{\partial \phi_2}{\partial(\varepsilon \underset{\sim}{x}_1)} = 0 \quad , \tag{54}$$

$$\frac{\omega_1}{c^2} \frac{\partial \underset{\sim}{A}_1}{\partial(\varepsilon t_1)} + \left( \underset{\sim}{k}_1 \circ \frac{\partial}{\partial(\varepsilon \underset{\sim}{x}_1)} \right) \underset{\sim}{A}_1 = \frac{ea\, k_4^2}{4m\, c^2} \frac{\omega_1}{\omega_2} \left[ \frac{(\underset{\sim}{k}_1 \circ \underset{\sim}{A}_2)}{k_1^2} k_1 \right.$$

$$\left. - A_2 \right] + 0(P_o) \text{ terms} \quad , \tag{55}$$

and

$$\frac{\omega_2}{c^2} \frac{\partial \underset{\sim}{A}_2}{\partial(\epsilon t_1)} + \left( \underset{\sim}{k}_2 \circ \frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)} \right) \underset{\sim}{A}_2 = - \frac{ea\, k_4^2}{4m\, c^2} \frac{\omega_2}{\omega_1} \left[ \frac{(\underset{\sim}{k}_2 \circ \underset{\sim}{A}_1)}{k_2^2} \underset{\sim}{k}_2 - \underset{\sim}{A}_1 \right] + O(P_o) \text{ terms} \quad . \tag{56}$$

Eq. (53) states that $\phi_1$ remains constant for an observer traveling in the direction of $\underset{\sim}{k}_1$ with the velocity $\left( \frac{c^2}{\omega_1/k_1} \right)$. Eq. (54) states that $\phi_2$ remains constant for an observer traveling in the direction $\underset{\sim}{k}_2$ with the velocity $\left( \frac{c^2}{\omega_2/k_2} \right)$. Since both $\phi_1$ and $\phi_2$ are constant at $t = 0$ and retain their initial values on the boundary for all times, we can take $\phi_1$ and $\phi_2$ to be independent of $\epsilon \underset{\sim}{x}_1$ and $\epsilon t_1$ .

Eq. (55) states that an observer traveling in the direction of $\underset{\sim}{k}_1$ with the velocity $\left( \frac{c^2}{\omega_1/k_1} \right)$ sees a change in $\underset{\sim}{A}_1$ which is proportional to $\underline{a}$ and to the component of $\underset{\sim}{A}_2$ perpendicular to $\underset{\sim}{k}_1$ . When the component of $\underset{\sim}{A}_2$ perpendicular to $\underset{\sim}{k}_1$ is parallel or antiparallel to $\underset{\sim}{A}_1$ , only the magnitude of $\underset{\sim}{A}_1$ will change. However, when $\underset{\sim}{A}_2$ has a component perpendicular to both $\underset{\sim}{k}_1$ and $\underset{\sim}{A}_1$ , $\underset{\sim}{A}_1$ will rotate (and change its magnitude simultaneously). An analogous argument holds for the rate of change of $\underset{\sim}{A}_2$ .

## VI. PROPERTIES OF THE RELATIONS WHICH REMOVE SECULAR BEHAVIOR

We obtained in (V) the conditions which $a$, $\underset{\sim}{A}_1$, and $\underset{\sim}{A}_2$ must satisfy to remove secular behavior from the $O(\varepsilon^2)$ wave equations. Here we shall study some of the consequences of those conditions.

### A. THE PURE INITIAL VALUE PROBLEM

We shall neglect for the time being the presence of boundaries; i.e., we shall assume that the plasma covers all space and that the same initial conditions have been set up for this problem as for the actual physical problem with boundaries. Then we can study the case of no spatial dependence of $a$, $\underset{\sim}{A}_1$, and $\underset{\sim}{A}_2$. Eqs. (50), (55), and (56) will then reduce to

$$\frac{\partial a}{\partial(\varepsilon t_1)} = \frac{e\omega_p}{4m\,\omega_1\omega_2}\,(\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2) + O(P_o)\text{ terms}\,, \tag{57}$$

$$\frac{\partial \underset{\sim}{A}_1}{\partial(\varepsilon t_1)} = \frac{ea\,k_4^2}{4m\,\omega_2}\left[\frac{(\underset{\sim}{k}_1 \cdot \underset{\sim}{A}_2)}{k_1^2}\,\underset{\sim}{k}_1 - \underset{\sim}{A}_2\right] + O(P_o)\text{ terms}\,, \tag{58}$$

and

$$\frac{\partial \underset{\sim}{A}_2}{\partial(\varepsilon t_1)} = -\frac{ea\,k_4^2}{4m\,\omega_1}\left[\frac{(\underset{\sim}{k}_2 \cdot \underset{\sim}{A}_1)}{k_2^2}\,\underset{\sim}{k}_2 - \underset{\sim}{A}_1\right] + O(P_o)\text{ terms}\,. \tag{59}$$

Since no physics is lost by taking the temperature of the plasma to be zero, we shall do so, and shall therefore drop the $O(P_o)$ terms from eqs. (57), (58), and (59).

The following conservation laws can be obtained immediately from eqs. (57) - (59) :

$$\left.\begin{aligned} \frac{A_1^2}{\omega_1} + \frac{a^2\, k_4^2}{\omega_p} &= \frac{A_1^2(\varepsilon t_1 = 0)}{\omega_1} \\ \frac{A_2^2}{\omega_2} - \frac{a^2\, k_4^2}{\omega_p} &= \frac{A_2^2(\varepsilon t_1 = 0)}{\omega_2} \quad . \end{aligned}\right\} \qquad (60)$$

Eqs. (60) show that $\underline{a}$ is bounded. This means that the longitudinal field $\underset{\sim}{E}_L^{(1)}$ and hence the density $n^{(1)}$ are bounded.

To study eqs. (57) - (59) in more detail, we shall write them in component form. Let us choose a coordinate system such that $\underset{\sim}{k}_1 = (k_1, 0, 0)$ and $\underset{\sim}{k}_2 = (k_2 \cos\alpha, k_2 \sin\alpha, 0)$ . In this coordinate system eqs. (57) to (59) become

$$\left.\begin{aligned} \frac{\partial A_{1y}}{\partial(\varepsilon t_1)} &= -\frac{e\, k_4^2}{4m\,\omega_2}\, a\, A_{2y} \\ \frac{\partial A_{1z}}{\partial(\varepsilon t_1)} &= -\frac{e\, k_4^2}{4m\,\omega_2}\, a\, A_{2z} \\ \frac{\partial A_{2y}}{\partial(\varepsilon t_1)} &= \frac{e\, k_4^2}{4m\,\omega_1} \cos^2\alpha\, a\, A_{1y} \end{aligned}\right\} \qquad (61)$$

$$\left.\begin{aligned}
\frac{\partial A_{2z}}{\partial(\varepsilon t_1)} &= \frac{e\, k_4^2}{4m\,\omega_1}\, a\ A_{1z} \\
A_{1x} &= 0 \\
A_{2x} &= -\tan\alpha\ A_{2y} \\
\frac{\partial a}{\partial(\varepsilon t_1)} &= \frac{e\,\omega_p}{4m\,\omega_1\omega_2}\,(A_{1y}\,A_{2y} + A_{1z}\,A_{2z})\ .
\end{aligned}\right\}$$

The following conservation laws can be obtained from eqs. (61):

$$\left.\begin{aligned}
\frac{A_{1z}^2}{\omega_1} + \frac{A_{2z}^2}{\omega_2} &= \frac{A_{1z}^2(\varepsilon t_1 = 0)}{\omega_1} + \frac{A_{2z}^2(\varepsilon t_1 = 0)}{\omega_2} \\
\cos^2\alpha\ \frac{A_{1y}^2}{\omega_1} + \frac{A_{2y}^2}{\omega_2} &= \cos^2\alpha\ \frac{A_{1y}^2(\varepsilon t_1 = 0)}{\omega_1} + \frac{A_{2y}^2(\varepsilon t_1 = 0)}{\omega_2}\ .
\end{aligned}\right\} \quad (62)$$

Eqs. (62) show that if at $t = 0$ $A_{1z} = A_{2z} = 0$, then $A_{1z} = A_{2z} = 0$ for $t > 0$. Similarly, if at $t = 0$, $A_{1y} = A_{2y} = 0$, then $A_{1y} = A_{2y} = 0$ for $t > 0$ (and consequently $A_{2x} = 0$). Therefore, if $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are at $t=0$ in the plane of $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$, they will remain in that plane for $t > 0$. On the other hand, if $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are perpendicular to the plane of $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ at $t = 0$, they will remain so for $t > 0$. These two results are not surprising, because in both cases the component of $\underset{\sim}{A}_2$ which is perpendicular to $\underset{\sim}{k}_1$ is either parallel or antiparallel to $\underset{\sim}{A}_1$; and likewise, the component of $\underset{\sim}{A}_1$ which is perpendicular to $\underset{\sim}{k}_2$ is parallel or antiparallel to $\underset{\sim}{A}_2$. Therefore only magnitudes of $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ can change.

We shall now show that solutions to eqs. (57) - (59) can be obtained for some specialized cases of physical interest and that those solutions are periodic. For this purpose let us introduce new variables, defined by:

$$\left.\begin{aligned}
\tau &\equiv \varepsilon t_1 \\
y_1 &\equiv \frac{A_{1y}}{\sqrt{\omega_1}} \\
y_2 &\equiv \frac{A_{2y}}{\sqrt{\omega_2}} \\
z_1 &\equiv \frac{A_{1z}}{\sqrt{\omega_1}} \\
z_2 &\equiv \frac{A_{2z}}{\sqrt{\omega_2}} \\
v &\equiv \frac{e\, k_4^2}{4m\sqrt{\omega_1 \omega_2}}\, a \\
\beta &\equiv \cos^2 \alpha \\
\gamma &\equiv \frac{e^2\, \omega_p\, k_4^2}{16 m^2\, \omega_1 \omega_2}
\end{aligned}\right\} \qquad (63)$$

Then eqs. (61), when expressed in terms of the new variables, become

$$\left.\begin{aligned} \frac{dy_1}{d\tau} &= -y_2\, v \\ \frac{dy_2}{d\tau} &= \beta y_1\, v \\ \frac{dz_1}{d\tau} &= -z_2\, v \\ \frac{dz_2}{d\tau} &= z_1\, v \\ \frac{dv}{d\tau} &= \gamma(y_1\, y_2 + z_1\, z_2)\ . \end{aligned}\right\} \qquad (64)$$

The initial conditions can also be written in terms of the new varialbes. They are

$$\left.\begin{aligned} y_{10} &= y_1(\tau=0) \\ y_{20} &= y_2(\tau=0) \\ z_{10} &= z_1(\tau=0) \\ z_{20} &= z_2(\tau=0) \\ v &= 0 \quad \text{when} \quad \tau = 0 \end{aligned}\right\} \qquad (65)$$

Let us confine ourselves to the special case when $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ both lie in the plane of $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$. Eqs. (64) then reduce to the three equations:

$$\left.\begin{aligned} \frac{dy_1}{d\tau} &= -y_2\, v \;, \\ \frac{dy_2}{d\tau} &= \beta y_1\, v \;, \\ \text{and} \quad \frac{dv}{d\tau} &= \gamma\, y_1\, y_2 \;. \end{aligned}\right\} \tag{66}$$

Eqs. (66) have the properties of the derivatives of elliptic functions, defined as follows: If $u = \int_0^{\phi} \frac{d\phi'}{\sqrt{1-k^2 \sin^2 \phi'}}$ is an elliptic integral of the first kind, then $\mathrm{sn}(u,k) \equiv \sin \phi$, $\mathrm{cn}(u,k) \equiv \cos \phi$, and $\mathrm{dn}(u,k) \equiv \sqrt{1-k^2 \sin^2 \phi}$. From these definitions we obtain $\frac{d}{du}\,\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\,\mathrm{dn}(u,k)$, $\frac{d}{du}\,\mathrm{cn}(u,k) = -\,\mathrm{sn}(u,k)\,\mathrm{dn}(u,k)$, and $\frac{d}{du}\,\mathrm{dn}(u,k) = -k^2\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)$.

Therefore we make the ansatz that $y_1 = y_{10}\,\mathrm{cn}(\lambda\tau,\kappa)$, $y_2 = y_{20}\,\mathrm{dn}(\lambda\tau,\kappa)$, and $v = c\,\mathrm{sn}(\lambda\tau,\kappa)$, where $\lambda$, $\kappa$, and $c$ are unkown constants to be determined. Substituting the ansatz into eqs. (66), we obtain

$$\left.\begin{aligned} y_1 &= y_{10}\ \mathrm{cn}\left[\sqrt{\gamma}\ y_{20}\ \tau,\ i\sqrt{\beta}\ \frac{y_{10}}{y_{20}}\right] \\ y_2 &= y_{20}\ \mathrm{dn}\left[\sqrt{\gamma}\ y_{20}\ \tau,\ i\sqrt{\beta}\ \frac{y_{10}}{y_{20}}\right] \\ v &= \sqrt{\gamma}\ y_{10}\ \mathrm{sn}\left[\sqrt{\gamma}\ y_{20}\ \tau,\ i\sqrt{\beta}\ \frac{y_{10}}{y_{20}}\right]. \end{aligned}\right\} \quad (67)$$

The elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$ are periodic in $u$ with a period equal to $4\int_0^{1/2\pi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$. Therefore the solutions (67) are periodic functions of $\tau$.

The other special case, when $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are both perpendicular to the plane of $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$, can be solved in an identical way. In this case eqs. (64) reduce to

$$\left.\begin{aligned} \frac{dz_1}{d\tau} &= -z_2\ v \\ \frac{dz_2}{d\tau} &= z_1\ v \\ \frac{dv}{d\tau} &= \gamma\ z_1\ z_2\ . \end{aligned}\right\} \quad (68)$$

The solutions of eqs. (68) are identical with the solutions (67) of eqs. (66) when $\beta = 1$, and $z_1$, $z_2$, $z_{10}$, and $z_{20}$ replace $y_1$, $y_2$, $y_{10}$, and $y_{20}$, respectively.

We will now cite an example where the solutions of eqs. (64) are periodic elliptic functions although the component of $\underset{\sim}{A}_2$ which is perpendicular to $\underset{\sim}{k}_1$ is not parallel or antiparallel to $\underset{\sim}{A}_1$. Let us, first of all, derive some conservation laws applying to the components of $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$. Multiplying the first of eqs. (64) by $y_1$, the second by $y_2$, then adding the two equations and integrating, we obtain

$$y_1^2 + \frac{y_2^2}{\beta} = y_{10}^2 + \frac{y_{20}^2}{\beta} \quad . \tag{69}$$

Performing identical manipulations with the third and fourth equations of the set (64), we obtain

$$z_1^2 + z_2^2 = z_{10}^2 + z_{20}^2 \quad . \tag{70}$$

Let us now divide the first equation of the set (64) by the third. We obtain

$$\frac{dy_1}{dz_1} = \frac{y_2}{z_2} \quad .$$

Making use of eqs. (69) and (70), we can write

$$\frac{dy_1}{dz_1} = \beta^{1/2} \, \frac{\left(y_{10}^2 + \frac{y_{20}^2}{\beta} - y_1^2\right)^{1/2}}{\left(z_{10}^2 + z_{20}^2 - z_1^2\right)^{1/2}} \quad . \tag{71}$$

Let us define two new variables, $\theta_1$ and $\theta_2$, by means of the expressions

$$\left.\begin{aligned} y_1 &= (y_{10}^2 + y_{20}^2/\beta)^{1/2} \sin \theta_1 \\ z_1 &= (z_{10}^2 + z_{20}^2)^{1/2} \sin \theta_2 \ . \end{aligned}\right\} \quad (72)$$

From eqs. (72) and (71) we obtain

$$\left.\begin{aligned} \frac{d\theta_1}{\beta^{1/2}} &= d\theta_2 \\ \text{or} \qquad \frac{\theta_1 - c}{\beta^{1/2}} &= \theta_2 \end{aligned}\right\} \quad (73)$$

with $c$ defined by

$$c \equiv \sin^{-1}\left[\frac{y_{10}}{\left(y_{10}^2 + \frac{y_{20}^2}{\beta}\right)^{1/2}}\right] - \beta^{1/2} \sin^{-1}\left[\frac{z_{10}}{(z_{10}^2 + z_{20}^2)^{1/2}}\right] .$$

Then on substituting for $\frac{dy_1}{d\tau}$ in the first of eqs. (64), we obtain

$$\frac{\gamma^{-1/2}\, d\theta_1}{\left[y_{10}^2 + z_{10}^2 - \left(y_{10}^2 + \frac{y_{20}^2}{\beta}\right) \sin^2\theta_1 - (z_{10}^2 + z_{20}^2)\sin^2\left(\frac{\theta_1 - c}{\beta^{1/2}}\right)\right]^{1/2}} = -\beta^{1/2}\, dt \ . \quad (74)$$

If we now select the special case in which $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ are parallel, $\beta^{1/2} = 1$, and eq. (74) is the differential of an elliptic integral of the first kind. Hence $\theta_1$ is an elliptic function of $t$.

## B. THE PURE BOUNDARY VALUE PROBLEM

We shall now neglect the initial conditions but retain the boundary conditions. Thus we can study the case of no time dependence of $a$ , $\underset{\sim}{A}_1$ , and $\underset{\sim}{A}_2$ . Eqs. (50), (55), and (56) now reduce to

$$v_o^2\, \underset{\sim}{k}_4 \cdot \frac{\partial a}{\partial(\epsilon \underset{\sim}{x}_1)} = \frac{e\omega_p^2}{4m\, \omega_1\omega_2} (\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2) + O(P_o) \text{ terms} , \tag{75}$$

$$\left(\underset{\sim}{k}_1 \cdot \frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)}\right) \underset{\sim}{A}_1 = \frac{e\, k_4^2}{4m\, c^2} \frac{\omega_1}{\omega_2}\, a \left[\frac{(\underset{\sim}{k}_1 \cdot \underset{\sim}{A}_2)}{k_1^2}\, \underset{\sim}{k}_1 - \underset{\sim}{A}_2\right] + O(P_o) \text{ terms}, \tag{76}$$

and

$$\left(\underset{\sim}{k}_2 \cdot \frac{\partial}{\partial(\epsilon \underset{\sim}{x}_1)}\right) \underset{\sim}{A}_2 = -\frac{e\, k_4^2}{4m\, c^2} \frac{\omega_2}{\omega_1}\, a \left[\frac{(\underset{\sim}{k}_2 \cdot \underset{\sim}{A}_1)}{k_2^2}\, \underset{\sim}{k}_2 - \underset{\sim}{A}_1\right] + O(P_o) \text{ terms} . \tag{77}$$

As can be seen from eq. (75), a pure boundary value problem is an impossibility when $P_o = 0$ . Therefore we must assume a finite temperature for the quiescent plasma.

Let us take, for the sake of simplicity, the $y - z$ plane to be the boundary between the plasma and the vacuum, with the plasma on the positive side of the $y - z$ plane. Let us also assume $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ to be parallel to one another, for the time being, and to point in the direction of the positive x-axis. We can obtain some conservation laws from eqs. (75)-(77). For example, multiplying eq. (75) by $\underline{a}$ and dotting eq. (76) with

$A_1$, then adding and integrating, we obtain

$$\frac{A_1^2}{\omega_1} + \frac{v_o^2}{c^2}\,\frac{\omega_1 k_4}{\omega_p k_1}\,\frac{k_4^2 a^2}{\omega_p} = \frac{A_1^2(\epsilon x_1 = 0)}{\omega_1} \quad . \tag{78}$$

The conservation equation (78) shows that $a(\epsilon x_1)$ is bounded. This means that the longitudinal field $E_L^{(1)}$ and hence the density $n^{(1)}$ are bounded in space.

We would like to make a comparison between the values of $E_L^{(1)}(\epsilon t_1)$ and $E_L^{(1)}(\epsilon x_1)$. Since $A_{1,2}(\epsilon x_1) \sim A_{1,2}(\epsilon t_1)$, we obtain from eq. (78) and the first one of eqs. (62) that

$$\frac{a^2(\epsilon x_1)}{a^2(\epsilon t_1)} \sim \frac{c^2}{v_o^2}\,\frac{\omega_p k_1}{\omega_1 k_4} \quad .$$

But $\omega_1 \sim c\,k_1$, and $\omega_4 = \omega_1 - \omega_2 \sim \omega_p$. Furthermore $\omega_4 \sim c(k_1 - k_2) = c\,k_4$. Therefore

$$\frac{a^2(\epsilon x_1)}{a^2(\epsilon t_1)} \sim \frac{c^2}{v_o^2} \quad ,$$

or

$$\frac{|a(\epsilon x_1)|}{|a(\epsilon t_1)|} \sim \frac{c}{v_o} \quad . \tag{79}$$

since $c > v_o$ always, $\left\langle E_L^{(1)}(\epsilon t_1)\right\rangle_{\text{average}} < \left\langle E_L^{(1)}(\epsilon x_1)\right\rangle_{\text{average}}$ always.

We may note here that because of the close similarity between eqs. (57) - (59) on the one hand and eqs. (75) to (77) on the other, the behavior of the quantities $A_1$, $A_2$, and $a$ in space is very similar to the behavior of these quantities in time.

## C. MIXED, INITIAL-VALUE-BOUNDARY-VALUE PROBLEM

For the discussion of this problem we have to retain eqs. (50), (55), and (56) in their original form. If we took the temperature of the plasma to be identically zero, the term containing $\frac{\partial a}{\partial(\varepsilon x_1)}$ in eq. (50) would drop out. Since $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ must remain constant on the boundary for all times, $\underline{a}$ would grow linearly with time on the boundary. To prevent this occurrence, we must require that the temperature of the plasma be non-zero.

Let us now look at the physical content of eqs. (50), (55), and (56). At $t = 0$, $a = 0$, and $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ do not change in space or in time. If $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are perpendicular at $t = 0$, the convective derivative of $\underline{a}$ is zero. Because of the initial and boundary conditions on $\underline{a}$, $a = 0$, identically, for all points in space, for all times. Then, by eqs. (55) and (56), and by the initial and boundary conditions on $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$, $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ will retain their initial values at all spatial points for all times.

On the other hand, when $\underset{\sim}{A}_1 \circ \underset{\sim}{A}_2 \neq 0$ at $t = 0$, $\underline{a}$ begins to grow. The existence of a non-vanishing $\underline{a}$ then induces rotations in $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$. We may say that the changes in $\underline{a}$, $\underset{\sim}{A}_1$, and $\underset{\sim}{A}_2$ are propagated like convective currents with current velocities $(v_o^2/(\omega_4/k_4))$, $(c^2/(\omega_1/k_1))$, and $(c^2/(\omega_2/k_2))$, respectively.

Let us restrict ourselves now, for the sake of simplicity, to $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ which are parallel to each other and normally incident on the boundary between the plasma and vacuum, and the boundary coinciding with the y - z plane. Initially a , $\underset{\sim}{A}_1$ , and $\underset{\sim}{A}_2$ have the same values everywhere, including the boundary. At a time equal to t , $\underset{\sim}{A}_1$ will differ from its initial value. But for $x > (c^2/(\omega_1/k_1))t$ the instantaneous value of $\underset{\sim}{A}_1$ will be independent of x . For $x < (c^2/(\omega_1/k_1))t$, on the other hand, $\underset{\sim}{A}_1$ will generally differ from one spatial point to another. Thus an observer located at a point x , with $x > (c^2/(\omega_1/k_1))t$ , with $x > (c^2/(\omega_2/k_2))t$ , or with $x > (v_o^2/(\omega_4/k_4))t$ , depending on whichever convective velocity is the fastest, will not have yet experienced the effects of the boundary for the first t seconds. As far as he is concerned, he sees only an initial-value problem. On the other hand, an observer located at a point with the coordinate x less than the product of the fastest convective velocity and the time, will have already experienced the influence of the boundary. The reason for this behavior is the finite velocities of propagation of the changes in a , $\underset{\sim}{A}_1$ , and $\underset{\sim}{A}_2$ , respectively.

## VII. THE SCATTERING CROSS SECTION

We shall now estimate the scattering cross section for the light-by-light scattering process. The differential cross section per unit frequency interval is given by eq. (1). We have to calculate the spectral density $S(\underset{\sim}{k},\omega)$, which is defined by eq. (2). To lowest order in $\varepsilon$

$$S(\underset{\sim}{k},\omega) = \lim_{\substack{V\to\infty \\ T\to\infty}} \frac{2|\varepsilon n^{(1)}(\underset{\sim}{k},\omega)|^2}{N_o VT} \sim O(\varepsilon^2) \quad , \tag{80}$$

where $n^{(1)}(\underset{\sim}{k},\omega)$ is defined by

$$n^{(1)}(\underset{\sim}{x},t) = \frac{1}{VT} \sum_{\underset{\sim}{k},\omega} n^{(1)}(\underset{\sim}{k},\omega) \quad e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t)} \quad . \tag{81}$$

Since we are considering the resonance process, $n^{(1)}(\underset{\sim}{x},t) = \frac{a\, k_4^2}{4\pi e} \sin \psi_4$. Therefore

$$n^{(1)}(\underset{\sim}{k},\omega) = \int_V d\underset{\sim}{x} \int_{-T/2}^{T/2} dt \quad n^{(1)}(\underset{\sim}{x},t) \quad e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t)}$$

$$= \frac{k_4^2}{4\pi e} \int_V d\underset{\sim}{x} \int_{-T/2}^{T/2} dt \quad e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t)} \, a(\varepsilon \underset{\sim}{x},\varepsilon t) \sin \psi_4$$

$$= \frac{k_4^2}{8\pi e i} \left\{ e^{i\phi_4} \int_V d\underset{\sim}{x} \quad e^{-i(\underset{\sim}{k}-\underset{\sim}{k}_4)\cdot\underset{\sim}{x}} \int_{-T/2}^{T/2} dt \quad e^{-i(\omega+\omega_4)t} \, a(\varepsilon \underset{\sim}{x},\varepsilon t) \right.$$

$$- e^{-i\phi_4} \int_V d\underset{\sim}{x} \; e^{-i(\underset{\sim}{k}+\underset{\sim}{k}_4)\cdot\underset{\sim}{x}} \int_{-T/2}^{T/2} dt \; e^{-i(\omega-\omega_4)t} \; a(\epsilon\underset{\sim}{x},\epsilon t) \Big\}$$

$$= \frac{k_4^2}{8\pi e i} \left[ e^{i\phi_4} \, a(\underset{\sim}{k}-\underset{\sim}{k}_4,\omega+\omega_4) - e^{-i\phi_4} \, a(\underset{\sim}{k}+\underset{\sim}{k}_4,\omega-\omega_4) \right] \; .$$

The spectral density $S(\underset{\sim}{k},\omega)$ can now be written in the form

$$S(\underset{\sim}{k},\omega) = \epsilon^2 \lim_{\substack{V\to\infty \\ T\to\infty}} \frac{2|n^{(1)}(\underset{\sim}{k},\omega)|^2}{N_o VT}$$

$$= \epsilon^2 \frac{k_4^4}{32\pi^2 N_o e^2} \lim_{\substack{V\to\infty \\ T\to\infty}} \frac{1}{VT} \Big[ |a(\underset{\sim}{k}-\underset{\sim}{k}_4, \omega+\omega_4)|^2$$

$$+ |a(\underset{\sim}{k}+\underset{\sim}{k}_4,\omega-\omega_4)|^2 - e^{2i\phi_4} \, a^*(\underset{\sim}{k}+\underset{\sim}{k}_4,\omega-\omega_4) \, a(\underset{\sim}{k}-\underset{\sim}{k}_4,\omega+\omega_4)$$

$$- e^{-2i\phi_4} \, a^*(\underset{\sim}{k}-\underset{\sim}{k}_4,\omega+\omega_4) \, a(\underset{\sim}{k}+\underset{\sim}{k}_4,\omega-\omega_4) \Big] \; . \tag{82}$$

Let us take a closer look at $a(\underset{\sim}{k},\omega)$ . Since $a(\epsilon\underset{\sim}{x},\epsilon t)$ is a very slowly varying function of $\underset{\sim}{x}$ and $t$ , its Fourier transform $a(\underset{\sim}{k},\omega)$ is sharply peaked at $\underset{\sim}{k} = 0$ and $\omega = 0$ and has a small spread in $\underset{\sim}{k}$ and $\omega$ about this peak. Consequently the cross terms in eq. (82) are very small compared to the other terms, and we shall neglect them. Let us also neglect the spread in $\underset{\sim}{k}$ and $\omega$ . Thus $a(\underset{\sim}{k}-\underset{\sim}{k}_4,\omega+\omega_4)$ will be approximated by the quantity $(VT \, \bar{a} \, \delta_{\underset{\sim}{k},\underset{\sim}{k}_4} \, \delta_{\omega,-\omega_4})$ , where $\bar{a}$ is the value of $a(\underset{\sim}{x},t)$ at the peak. The value of $S(\underset{\sim}{k},\omega)$ will then be approximately

$$S(\underset{\sim}{k},\omega) \approx \varepsilon^2 \frac{\bar{a}^2\, k_4^4}{32\pi^2\, N_o e^2} \, [\delta(\underset{\sim}{k}-\underset{\sim}{k}_4)\, \delta(\omega+\omega_4)$$

$$+ \delta(\underset{\sim}{k}+\underset{\sim}{k}_4)\, \delta(\omega-\omega_4)\ ] \tag{83}$$

where $\delta(\underset{\sim}{k})$ and $\delta(\omega)$ are the Dirac delta functions. We shall not write $\varepsilon^2$ in the expression for $S(\underset{\sim}{k},\omega)$ from now on, because the presence of $\bar{a}^2$, which is of $O(\varepsilon^2)$, is sufficient to indicate that $S(\underset{\sim}{k},\omega)$ is of $O(\varepsilon^2)$.

We may note that if we were not dealing with a resonance process, $n^{(1)}(\underset{\sim}{x},t) = 0$, and the first non-vanishing term in the expansion of $n(\underset{\sim}{x},t)$ would be $n^{(2)}(\underset{\sim}{x},t)$ (which is of $O(\varepsilon^2)$). Consequently $S(\underset{\sim}{k},\omega)$ would be of $O(\varepsilon^4)$. Therefore the resonance process enhances the scattering cross section significantly.

We would like to compare our cross section with that obtained by Kroll, Ron, and Rostoker[2]. The cross sections will differ only because of the differences in the spectral densities. The spectral density $S_{\kappa RR}$ of Kroll, Ron, and Rostoker[2] is, when expressed in our notation,

$$S_{kRR}(\underset{\sim}{k},\omega) = \frac{1}{128\pi^2\, N_o m^2} \; \frac{k^4(\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2)^2}{\omega_1^2\, \omega_2^2\, |\varepsilon(\underset{\sim}{k},\omega)|^2} \; [\delta(\underset{\sim}{k}-\underset{\sim}{k}_4)\, \delta(\omega+\omega_4)$$

$$+ \delta(\underset{\sim}{k}+\underset{\sim}{k}_4)\ \ \delta(\omega-\omega_4)] \quad , \tag{84}$$

where $\varepsilon(\underset{\sim}{k},\omega)$ , the longitudinal dielectric function, is approximated by

$$|\varepsilon(\underset{\sim}{k},\omega)|^2 = \left[ 1 - \left(\frac{\omega_p}{\omega}\right)^2 \right]^2 + \Gamma^2 \rightarrow \Gamma^2 .$$

Here $\Gamma$ is the Landau damping decrement or the collisional damping decrement, whichever is larger.

Let us define $R$ to be the ratio of $S_{kRR}(\underset{\sim}{k},\omega)$ to our $S(\underset{\sim}{k},\omega)$ . Then

$$R = \frac{\omega_p^2}{16\pi N_o m} \; \frac{k_4^2}{\omega_1^2 \, \omega_2^2} \; \frac{(\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2)^2}{(\bar{a}^2 \, k_4^2) \, \Gamma^2} . \tag{85}$$

When $R < 1$ , $S_{kRR}(\underset{\sim}{k},\omega) < S(\underset{\sim}{k},\omega)$ , and the density $n(\underset{\sim}{x},t)$ of Kroll et al.[2] is smaller than our density. This means that Landau damping and/or collisional damping is more effective in limiting the density oscillations than are the nonlinear effects. On the other hand, when $R > 1$ , the nonlinear effects are more effective than the damping mechanisms. Thus, given the numerical values of the plasma parameters and the electric fields produced by the two lasers, we can determine by means of the expression (85) which physical mechanism is the dominant one in limiting the longitudinal plasma oscillations.

Let us estimate the ratio $R$ by using a set of typical numerical values of the plasma and the incident electric field parameters. We shall use the set selected by Kroll, Ron, and Rostoker[2] in their calculations of the damping effects. Therefore we choose $N_o = 10^{14}$ cm$^{-3}$, $K_B T = 10$ eV, $\omega_p = 5.64 \times 10^{11}$ sec$^{-1}$, $\lambda$ (of incident electric field) $\sim 0.7 \times 10^{-4}$ cm, E (amplitude of the incident electric field) $\sim 10^8$ V/cm, and $\Gamma_c$ (collisional damping decrement) $\cong 1.1 \times 10^{-3}$. With this choice of parameters, according to Kroll et al.[2], the collisional damping dominates over the Landau damping. Since $|\underset{\sim}{A}_1|$, $|\underset{\sim}{A}_2|$, and $|a\underset{\sim}{k}_4|$ are of $O(\varepsilon)$, $|\underset{\sim}{A}_2|$ and $|a\underset{\sim}{k}_4|$ are both of $O(A_1)$, which in turn is of O(E incident). Substituting the above numerical values of the plasma and the incident electric field parameters into eq. (85), we obtain $10^{-8} < R < 10^{-7}$. (This estimate was made under the assumption that $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ are parallel.) Therefore for this choice of parameters the damping effects limit the longitudinal plasma oscillations more effectively than do the nonlinear effects.

## VIII. DISCUSSION

We have shown that the presence of even a small amount of nonlinearity, in the equations used to describe the behavior of a plasma, can effectively limit the amplitude of plasma oscillations driven by two light beams. In fact, under some circumstances, the nonlinear effects limit these oscillations more strongly than the Landau damping and the collisional damping mechanisms.

The nonlinear effects are always accompanied by a non-vanishing longitudinal electric field of $O(\varepsilon)$, $\underset{\sim}{E}_L^{(1)}$, whenever they limit plasma oscillations. This is a very interesting fact, because $\underset{\sim}{E}_L^{(1)}$, as well as the transverse field $\underset{\sim}{E}_T^{(1)}$, satisfy the $O(\varepsilon)$ plasma equations, which are linear, and therefore keep the $O(\varepsilon)$ transverse and the $O(\varepsilon)$ longitudinal components of fields completely separated from one another.

The transverse fields are determined by the output of the two lasers. But there is no experimental device which sets up a longitudinal field, $\underset{\sim}{E}_L^{(1)}$. All that is done is to make sure the plasma is in a quiescent state at the beginning of the experiment. The experimental set-ups for the case when the difference in frequencies of the two laser beams approximately equals to the natural frequency of longitudinal plasma oscillations, and for the case when it is not, are identical.

When the difference in frequencies of the impinging beams is not equal to the frequency of plasma oscillations, no secular terms arise in the equations of motion, and $\underset{\sim}{E}_L^{(1)}$ remains identically zero - there is only a second-order field $E_L^{(2)}$. However, when the frequency of one of the impinging waves is varied until it differs from the frequency of the other impinging wave by the frequency of plasma oscillations, a resonance process results: Longitudinal plasma oscillations are excited, are simultaneously limited by nonlinear effects, and $\underset{\sim}{E}_L^{(1)}$ appears spontaneously. This longitudinal oscillation in turn scatters the light beams.

It is also interesting to note what happens when the amplitudes of the two impinging waves are varied while keeping everything else constant. Let us assume for the moment that the electric fields $\underset{\sim}{E}_T^{(1)}$ and $\underset{\sim}{E}_L^{(1)}$ and the damping decrement $\Gamma$ have been made dimensionless.* The density fluctuation $n(\underset{\sim}{x},t)$ which is limited by nonlinear effects is $O(E_L^{(1)})$. Since $E_L^{(1)}$ is of $O(E_T^{(1)})$, $n(\underset{\sim}{x},t)$ is also of $O(E_T^{(1)})$. As $E_T^{(1)}$ increases or decreases, $n(\underset{\sim}{x},t)$ will also increase or decrease, respectively. On the other hand, the density fluctuations, $n(\underset{\sim}{x},t)$, which are limited by Landau and/or collisional damping are of $O\left(\frac{E_T^{(1)2}}{\Gamma}\right)$. They will also increase or decrease as $E_T^{(1)}$ increases or decreases, respectively. The damping decrement $\Gamma$, however, does not depend on $E_T^{(1)}$, and will not change when $E_T^{(1)}$ is varied.

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*$\Gamma$ is made dimensionless by dividing it by the plasma frequency $\omega_p$. $E_T^{(1)}$ and $E_L^{(1)}$ may be made dimensionless by dividing them by $(1/2\ N_o\ m\ v_o^2)^{1/2}$, the square root of the thermal energy density.

When $E_T^{(1)} << \Gamma$, $\frac{E_T^{(1)2}}{\Gamma} << E_T^{(1)}$, and the density $n(x,t)$ which is limited by a damping mechanism is smaller than the density $n(x,t)$ which is limited by nonlinear effects. On the other hand, when $E_T^{(1)} >> \Gamma$, $\frac{E_T^{(1)2}}{\Gamma} >> E_T^{(1)}$, and the situation is reversed. We conclude from this that damping effects dominate when $E_T^{(1)}$ is very weak, and the nonlinear effects dominate when $E_T^{(1)}$ is strong. The conclusion is borne out by the numerical calculations in Section VII.

This is not unreasonable, because $E_T^{(1)}$ is a measure of nonlinearity in the equations of motion, but $E_T^{(1)}$ does not affect $\Gamma$, the damping decrement. Keeping $E_T^{(1)}$ very small results in very small nonlinear terms, without affecting the damping. An increase in $E_T^{(1)}$, on the other hand, increases the magnitude of nonlinear terms, while still keeping the damping decrement unchanged. Therefore an increase in $E_T^{(1)}$ results in the increasing importance of nonlinearity as compared with the damping effects.

## APPENDIX

## DERIVATION OF WAVE EQUATIONS

We shall derive the wave equation for $\underset{\sim}{E}$ (eq. (10) in the text).

Taking the curl of eq. (8) and substituting into eq. (9), we obtain

$$\frac{\partial}{\partial \underset{\sim}{x}} \times \left( \frac{\partial}{\partial \underset{\sim}{x}} \times \underset{\sim}{E} \right) + \frac{1}{c^2} \frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = \frac{4\pi N_o e}{c^2} \frac{\partial \underset{\sim}{v}}{\partial t} + \frac{4\pi e}{c^2} \frac{\partial}{\partial t} (n\underset{\sim}{v}) \quad . \qquad \text{(A1)}$$

If we now substitute for $\frac{\partial \underset{\sim}{v}}{\partial t}$ from eq. (4) into eq. (A1), there results

$$\left[ \frac{\partial}{\partial \underset{\sim}{x}} \times \left( \frac{\partial}{\partial \underset{\sim}{x}} \times \right) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{c^2} \right] \underset{\sim}{E}$$

$$+ \frac{4\pi e}{mc^2} \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{p} = - \frac{4\pi N_o e}{c^2} \left( \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} + \frac{e}{mc} \underset{\sim}{v} \times \underset{\sim}{B} \right)$$

$$+ \frac{4\pi e}{c^2} \frac{\partial}{\partial t} (n\underset{\sim}{v}) - \frac{4\pi e}{c^2} n \left( \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} \right)$$

$$- \frac{4\pi e^2}{mc^2} n \left( \underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B} \right) \quad . \qquad \text{(A2)}$$

The term $\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{p}$ can be obtained from eq. (5) in the following way:

From eq. (3) we have

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v} = -\frac{1}{N_o}\frac{\partial n}{\partial t} - \frac{1}{N_o}\frac{\partial}{\partial \underset{\sim}{x}} \cdot (n\underset{\sim}{v}) \qquad \text{(A3)}$$

Substituting eq. (A3) into eq. (5), we obtain

$$\frac{\partial \underset{\approx}{p}}{\partial t} - \frac{P_o}{N_o}\underset{\approx}{1}\frac{\partial n}{\partial t} + P_o\left(\frac{\partial}{\partial \underset{\sim}{x}}\underset{\sim}{v} + \widetilde{\frac{\partial}{\partial \underset{\sim}{x}}\underset{\sim}{v}}\right)$$

$$= \frac{P_o}{N_o}\underset{\approx}{1}\frac{\partial}{\partial \underset{\sim}{x}} \cdot (n\underset{\sim}{v}) - \underset{\approx}{p}\left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v}\right) - \left(\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\approx}{p}$$

$$-\left(\underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\sim}{v} - \widetilde{\left(\underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\sim}{v}} - \frac{e}{mc}\left(\underset{\approx}{p} \times \underset{\sim}{B} + \widetilde{\underset{\approx}{p} \times \underset{\sim}{B}}\right) . \qquad \text{(A4)}$$

Applying the operator $\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot\right)$ to eq. (A4) and then substituting (A3) into it, we obtain

$$\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{p} - \frac{2P_o}{N_o}\frac{\partial^2}{\partial t^2}\frac{\partial n}{\partial \underset{\sim}{x}} + P_o \nabla^2 \frac{\partial \underset{\sim}{v}}{\partial t}$$

$$= \frac{2P_o}{N_o}\frac{\partial}{\partial t}\frac{\partial}{\partial \underset{\sim}{x}}\left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot n\underset{\sim}{v}\right) - \frac{\partial}{\partial t}\frac{\partial}{\partial \underset{\sim}{x}} \cdot \Bigg[\left(\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\approx}{p}$$

$$+ \underset{\approx}{p}\left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v}\right) + \left(\underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\sim}{v} + \widetilde{\left(\underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\sim}{v}}$$

$$+ \frac{e}{mc}\left(\underset{\approx}{p} \times \underset{\sim}{B} + \widetilde{\underset{\approx}{p} \times \underset{\sim}{B}}\right)\Bigg] . \qquad \text{(A5)}$$

We can obtain the term $P_o \nabla^2 \frac{\partial \underset{\sim}{v}}{\partial t}$ from eq. (4), which yields

$$P_o \nabla^2 \frac{\partial \underset{\sim}{v}}{\partial t} = - \frac{P_o}{N_o m} \nabla^2 \frac{\partial}{\partial \underset{\sim}{x}} \circ \underset{\approx}{p} - \frac{P_o e}{m} \nabla^2 \underset{\sim}{E}$$

$$- P_o \nabla^2 \left( \underset{\sim}{v} \circ \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} + \frac{e}{mc} \underset{\sim}{v} \times \underset{\sim}{B} \right) - \frac{P_o}{N_o} \nabla^2 \left[ n \left( \frac{\partial \underset{\sim}{v}}{\partial t} \right.\right.$$

$$\left.\left. + \underset{\sim}{v} \circ \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} \right) \right] - \frac{P_o e}{N_o m} \nabla^2 \left[ n(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B}) \right] . \qquad \text{(A6)}$$

Also, from eq. (6), we obtain

$$n = - \frac{1}{4\pi e} \frac{\partial}{\partial \underset{\sim}{x}} \circ \underset{\sim}{E} \quad . \qquad \text{(A7)}$$

Substituting (A6) and (A7) into (A5), we obtain

$$\left( \frac{\partial^2}{\partial t^2} - \frac{1}{3} v_o^2 \nabla^2 \right) \frac{\partial}{\partial \underset{\sim}{x}} \circ \underset{\approx}{p} + \frac{P_o e}{m} \left[ \frac{2}{\omega_p^2} \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial \underset{\sim}{x}} \left( \frac{\partial}{\partial \underset{\sim}{x}} \circ \underset{\sim}{E} \right) \right.$$

$$\left. - \nabla^2 \underset{\sim}{E} \right] = P_o \nabla^2 \left( \underset{\sim}{v} \circ \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} + \frac{e}{mc} \underset{\sim}{v} \times \underset{\sim}{B} \right)$$

$$+ \frac{P_o}{N_o} \nabla^2 \left[ n \left( \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \circ \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{v} \right) \right] + \frac{P_o e}{N_o m} \nabla^2 \left[ n \, (\underset{\sim}{E} \right.$$

$$\left. + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B}) \right] + \frac{2P_o}{N_o} \frac{\partial}{\partial t} \frac{\partial}{\partial \underset{\sim}{x}} \left( \frac{\partial}{\partial \underset{\sim}{x}} \circ n\underset{\sim}{v} \right)$$

$$
\begin{aligned}
& - \frac{\partial}{\partial t} \frac{\partial}{\partial \underset{\sim}{x}} : \left[ \left( \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\approx}{p} + \underset{\approx}{p} \left( \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v} \right) \right. \\
& + \left( \underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\sim}{v} + \widetilde{\left( \underset{\approx}{p} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\sim}{v}} \\
& \left. + \frac{e}{mc} \left( \underset{\approx}{p} \times \underset{\sim}{B} + \widetilde{\underset{\approx}{p} \times \underset{\sim}{B}} \right) \right] \qquad \text{(A8)}
\end{aligned}
$$

where we have defined the thermal velocity $v_o^2 \equiv \frac{3P_o}{N_o m}$ .

If we now apply the operator $\left( \frac{\partial^2}{\partial t^2} - \frac{1}{3} v_o^2 \nabla^2 \right)$ to eq. (A2) and then substitute (A8) into it, we obtain eq. (10) in the text.

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