COMMENTS ON SYNTHESIS TECHNIQUES EMPLOYING THE DIRECT METHOD

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Synthesis techniques based on Liapunov's direct method often require control signals which are sign functions.[1-5] The argument of the sign function in each case is a linear combination of system states. Because Flügge-Lotz[6] has shown that the desired solution may not exist for equations which have sign functions as forcing functions, authors customarily replace the discontinuous sign function by a continuous function, usually the saturation function defined as

$$\text{sat } k\gamma = \begin{cases} 1 \text{ for } \gamma > 1/k \\ k\gamma \text{ for } -1/k \leq \gamma \leq 1/k \\ -1 \text{ for } \gamma < -1/k \end{cases} \tag{1}$$

where $k > o$ is a constant.

This avoids the theoretical difficulty discussed by Flügge-Lotz, but, in some cases, leads to another concerning the validity of conclusions reached regarding stability in the region where the saturation function has a smaller magnitude than the theoretically specified sign function. This problem is investigated here in relation to the synthesis technique employing a model reference[2] for control of single input, single output, linear time-varying plants with parameters varying in an unknown manner within known, finite bounds.

The class of plants considered can be described by the equation

$$\dot{\underline{x}} = (A + \Delta A)\,\underline{x} + \underline{b}u \tag{2}$$

in which x is the plant output, u its input (the control signal), $\underline{x}^T = (x_1, x_2, \ldots x_n)$, $x_{i+1} = \dot{x}_i$ for $i = 1, 2, \ldots n-1$, A is an nxn constant stable matrix (all its eigenvalues have negative real parts) of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & 0 \\ 0 & & & \cdots & 1 \\ -a_1 & -a_2 & & \cdots & -a_n \end{bmatrix}$$

and $\underline{b}^T = [0, 0, 0, \ldots b_n]$.

It is assumed that $\infty > b_{n(max)} \geq b_n \geq b_{n(min)} > 0$, and that $b_{n(min)}$ is known. The elements of the matrix $\Delta A(t)$ are the variations of plant parameters from their desired values which are the elements of the A matrix. Because of the forms of A and $\underline{x}$, $\Delta A(t)$ is of the form

$$\Delta A(t) = \begin{bmatrix} & & \bigcirc & & \\ \Delta a_1(t) & & \Delta a_2(t) & \dots & \Delta a_n(t) \end{bmatrix}$$

where the Δa_i's may very in an unknown manner within known finite bounds.

The desired behavior for the plant is given by the equation of the model reference which is

$$\dot{\underline{x}}_d = A\,\underline{x}_d + \underline{c}\,r \tag{3}$$

where $\underline{x}_d^T = \left[x_{d1}, x_{d2}, \dots x_{dn} \right]$, $x_{d1} = x_d$ which is the model output, $x_{d(i+1)} = \dot{x}_{di}$ for $i = 1, 2, \dots n-1$, $\underline{c}^T = \left[0, 0, 0 \dots c_n\right]$, and r is the reference input to the system.

An error for the system is defined by

$$\underline{e} = \underline{x}_d - \underline{x} \tag{4}$$

Subtracting (2) from (3) gives

$$\dot{\underline{e}} = A\underline{e} + \underline{c}\,r - \underline{b}\,u - \Delta A(t)\,\underline{x} \tag{5}$$

A function of quadratic form

$$V(\underline{e}) = \underline{e}^T P\underline{e} \tag{6}$$

has time derivative

$$\dot{V}(\underline{e}, r, u, \underline{x}) = \underline{e}^T (A^T P + PA)\,\underline{e} + 2\underline{e}^T P\left[-\underline{b}\,u + \underline{c}\,r - \Delta A(t)\underline{x} \right] \tag{7}$$

If

$$A^T P + PA = -Q \tag{8}$$

and Q is a symmetric positive definite matrix, then P is a symmetric positive definite matrix because of the assumptions on A.[7] For convenience, let Q = I, the identity matrix.

According to two theorems given in Reference 8, the following statement can be made concerning (5). If

$$u = \left| -\frac{c_n}{b_n} r + \sum_{i=1}^{n} \frac{\Delta a_i}{b_n} x_i \right|_{max} \operatorname{sgn} (\operatorname{grad}_e V)_n \tag{9}$$

then (5) is asymptotically stable, i.e., $e \rightarrow 0$ as $t \rightarrow \infty$. But there is the difficulty posed by Flügge-Lotz to contend with, so (9) must be replaced by

$$u = \left| -\frac{c_n}{b_n} r + \sum_{i=1}^{n} \frac{\Delta a_i}{b_n} x_i \right|_{max} \operatorname{sat} k \gamma \tag{10}$$

where $\gamma = (\operatorname{grad}_e V)_n$

Now it no longer can be argued that (5) is asymptotically stable in the region where

$$\left| \gamma \right| < 1/k \tag{11}$$

since u as given by (10) may not have sufficient magnitude to guarantee that $\dot{V} < 0$ in this region. What, then, can be said regarding stability? The following discussion considers this question. It is to be understood throughout that u as given by (10) is being considered, not that of (9).

The term $-2\, \underline{e}^T P\, \underline{b}\, u$ in (7) is always negative because of (10). Therefore, neglecting this term, $\dot{V}$ satisfies the inequality

$$\dot{V} \leq -\sum_{i=1}^{n} e_i^2 + \gamma \left[c_n r - \sum_{i=1}^{n} \Delta a_i (x_{di} - e_i) \right] \tag{12}$$

The variables x_{di} and e_i are introduced into the square bracket of (12) by adding and subtracting $\Delta A(t)\, \underline{x}_d$ within the square brackets of (7). This removes the components of $\underline{x}$ from the problem. The reason for doing this is explained below.

If r is bounded, then so are the components of $\underline{x}_d$ since a stable model is used. Thus

$$\left| c_n r - \sum_{i=1}^{n} \Delta a_i x_{di} \right| \leq M < \infty \tag{13}$$

An upper bound cannot be established for

$$\left| c_n r - \sum_{i=1}^{n} \Delta a_i x_i \right| \tag{14}$$

without making the unwarranted assumption that the system is stable. This explains the need for eliminating components of $\underline{x}$ from the problem. Another upper bound which can be established and which is useful is

$$\left| \sum_{i=1}^{n} \Delta a_i e_i \right| \leq \left| \Delta a \right|_m \sum_{i=1}^{n} \left| e_i \right| \leq \left| \Delta a \right|_m \left(n \sum_{i=1}^{n} e_i^2 \right)^{\frac{1}{2}} \tag{15}$$

where $\left| \Delta a \right|_m = \max_i \left\{ \left| \Delta a_i \right| \right\}$; i=1, 2, ... n

Use of (13) and (15) leads to a more conservative inequality for $\dot{V}$ which is

$$\dot{V} \leq -R^2 + \gamma (M + k_1 R) \tag{16}$$

where $R = \left(\sum_{i=1}^{n} e_i^2 \right)^{\frac{1}{2}}$

and $k_1 = \sqrt{n} \left| \Delta a \right|_{max}$

In the region where $\left| \gamma \right| < 1/k$

$$\dot{V} \leq -R^2 + \frac{k_1}{k} R + \frac{M}{k} \tag{17}$$

From (17) it can be seen that

$$\dot{V} < 0 \tag{18}$$

everywhere outside the spherical region

$$R(k) < \frac{k_1}{2k} + \frac{1}{2} \sqrt{\left(\frac{k_1}{k} \right)^2 + \frac{4M}{k}} \tag{19}$$

By choosing k large enough R (k) can be made arbitrarily small.

It can be concluded that as $t \to \infty$, u will cause $\underline{e}$ to be within the region common to R (k) and $\left| \gamma \right| < 1/k$. For n = 2, the situation is as shown in Figure 1. Asymptotic stability of (5) cannot be concluded. Limit cycles may exist, but can be made arbitrarily small be choosing k large enough.

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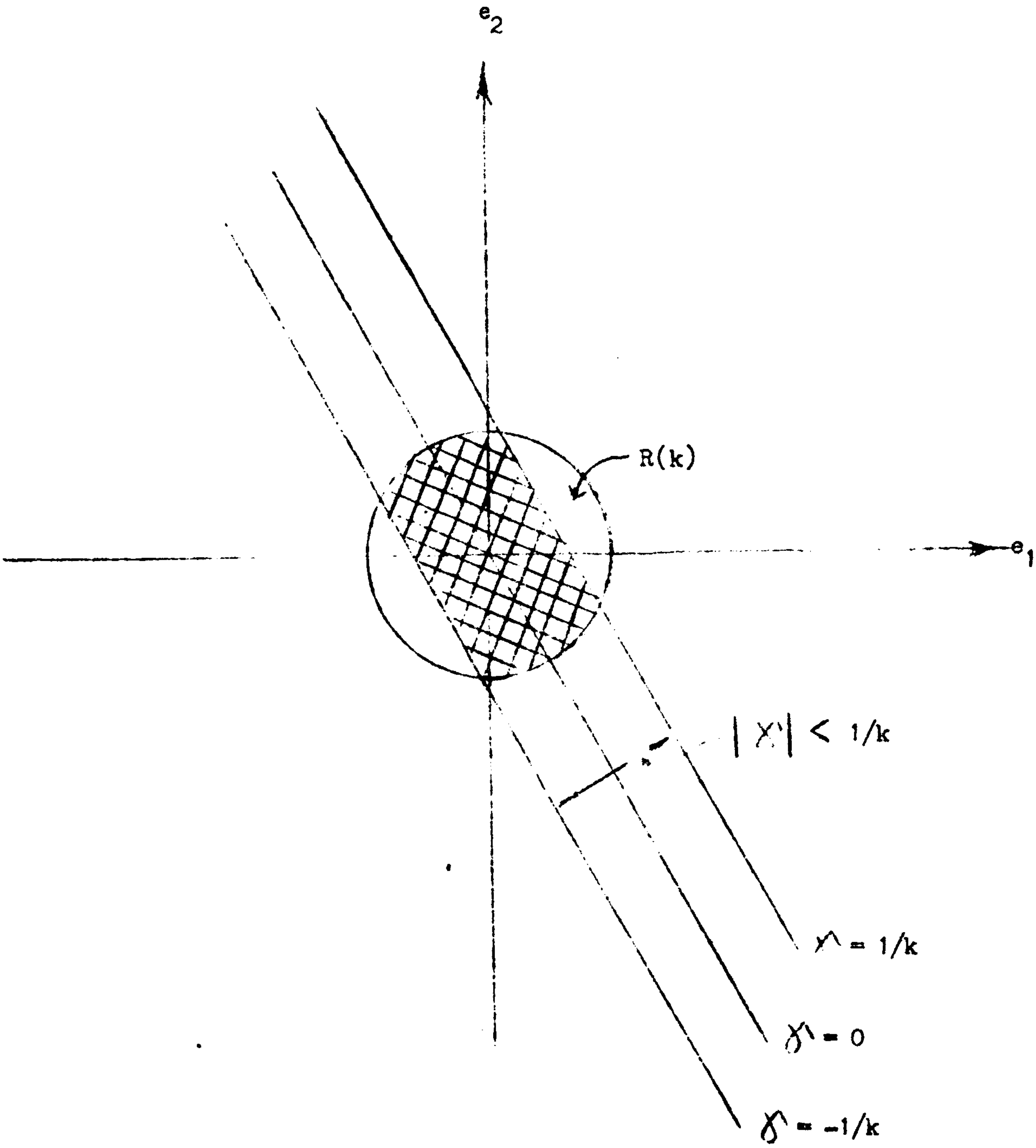

Figure 1. Cross Hatched Area Indicating Where $\dot{V}$ May Be Positive

[illegible]ENCES

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