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for  
PHASE V: STRESS ANALYSIS OF A  
DOUBLY-CURVED SKIN WITH A  
FLARED NOZZLE PORT

Contract NAS 8-2698  
(RAC 1452-6)

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION  
George C. Marshall Space Flight Center  
Huntsville, Alabama

REPUBLIC AVIATION CORPORATION  
Farmingdale, L. I., N. Y. 11735

## FOREWORD

This report was prepared by Drs. I. U. Ojalvo, M. Newman, M. Goldberg and Mr. N. Levine of Republic Aviation Corporation, Farmingdale, New York, under Contract NAS 8-2698, "Stress Analysis of a Doubly-Curved Skin with a Flared Nozzle Port."

The work was administered under the direction of Mr. David Hoppers of the Manufacturing Engineering Laboratory through Mr. Norman Schlemmer of the Propulsion and Vehicle Engineering Laboratory of the George C. Marshall Space Flight Center.

The Republic Program Manager is Dr. R. S. Levy.

## ACKNOWLEDGEMENTS

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## SUMMARY

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An analysis to determine the stresses and deflections caused by comparatively small flared nozzle openings in large pressure vessels and an associated digital computer program are described. The structural problem is idealized as a shell of revolution with axis normal to a thin, shallow, parent shell which is not axisymmetrical. The midsurfaces of the two shells are assumed to mate at a common intersecting circle and the entire configuration is subjected to internal pressurization and membrane edge forces.

*Author*

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## SECTION I INTRODUCTION

The evaluation of the magnitude of stresses in the neighborhood of nozzle ports is a technically important problem in the design of pressurized fuel tanks. Concurrent with stress concentration effects, resulting from geometric discontinuities, the nozzles serve as localized loading points which can transmit both thrusts and bending moments to the primary shell. Problems of this nature occur in the elliptical end closures of large boosters of the Saturn type, as well as with pressure vessels in the form of, spheres, cylinders, and tori (see Figure 1).

Recognizing a need for efficient nozzle fabrication techniques, Republic Aviation Corporation has recently developed a method of drawing flared nozzles from curved thin shells Reference [1]. Using the developed technique, the specific location of the nozzle drawing axis and the angle made with the local shell normal are quite arbitrary, thus enhancing the possibilities of the technique for general and, in particular, space vehicle applications. A significant advantage of the nozzle-forming procedure, is that it eliminates the need for complex weldments along awkward nozzle-shell junctions.

Although shell-nozzle interaction problems are of practical importance, comparatively little analytical work has been published in this area. Notable contributions have been made by Bijlaard [2, 3, 4] and Galletly [5] in the analysis of discontinuity stresses adjacent to central circular opening in spherical caps. In addition, Conway and Lissa [6] have considered the oblique intersection of a rigid cylinder with a shallow spherical shell. However, questions regarding the influence of unequal radii of curvature in the pressure vessel, at nozzle junctions, have remained unanswered.

This document describes a general approximate procedure for determining the stresses and deflections in flared nozzles with axes normal to comparatively large pressure vessels, as well as the stress distribution in the parent shell near the nozzle junction. Expressions for these discontinuity stress are derived

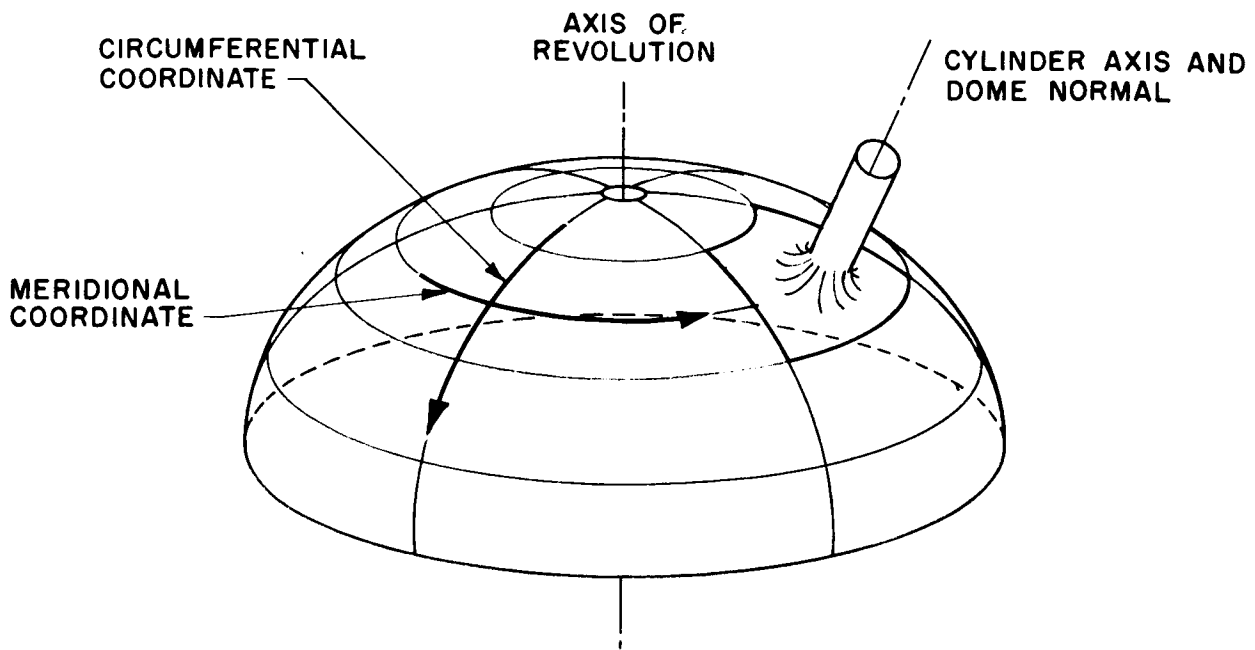


Figure 1a. Oblate Spheroid (Ellipsoid) Intersected by a Circular Cylinder With Axis Along a Surface Normal

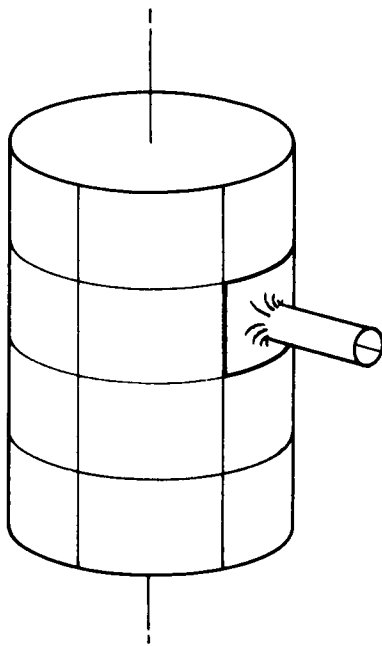


Figure 1b. Circular Cylinder Intersected Normally by a Second Circular Cylinder

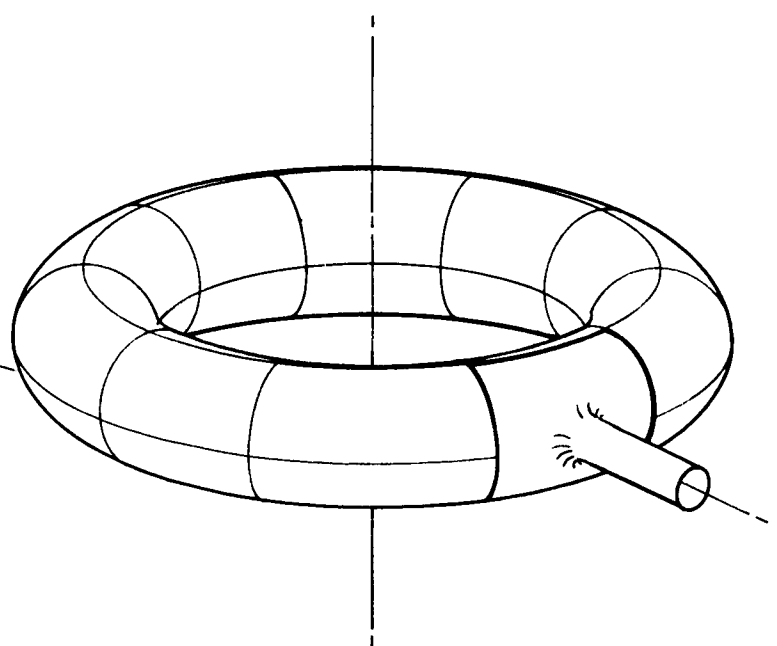


Figure 1c. Circular Torus Intersected by a Circular Cylinder with Axis along a Surface Normal



for an idealized model, which brings out the salient features of the problem.

It is assumed that the nozzle geometry, exclusive of the parent dome shell, can be approximated (see Figure 2) by a surface of revolution, consisting of a right circular cylinder joined to a bell-mouthed shell of variable thickness. The nozzle axis is taken normal to a thin, shallow, parent shell of double curvature which is idealized as a segment of an elliptic paraboloid. Because of the shallowness of the dome segment and the coincidence of nozzle axis and local shell normal, the curve describing the intersecting shells can be approximated as a circle. The entire system is subjected to internal pressure and it is assumed that membrane boundary conditions exist at the top of the cylinder and at the edges of the dome segment.

Sections II and III present the mathematical models for the cylinder-flare combination and dome, respectively. The analytical and numerical solution procedures associated with generating solution families for each are complete in their respective sections. Section IV deals with the joining of these two segments to form a composite shell by a point-matching [7, 8] scheme with least squares option.

The computer program, which numerically generates the appropriate solution families, joins the shell segments, and calculates the overall stresses and deformations, is described in Section V. A complete set of operating instructions is contained in the separate companion report entitled, "User's Manual for Stress Analysis of a Doubly-Curved Skin with a Flared Nozzle Port," Republic Aviation Corporation Report No. RAC 1452-7, dated 15 May 1965. Concurrent with the submission of this report, the computer program is being made available to the National Aeronautics and Space Administration, where it will be applied to the analysis of specific test problems.

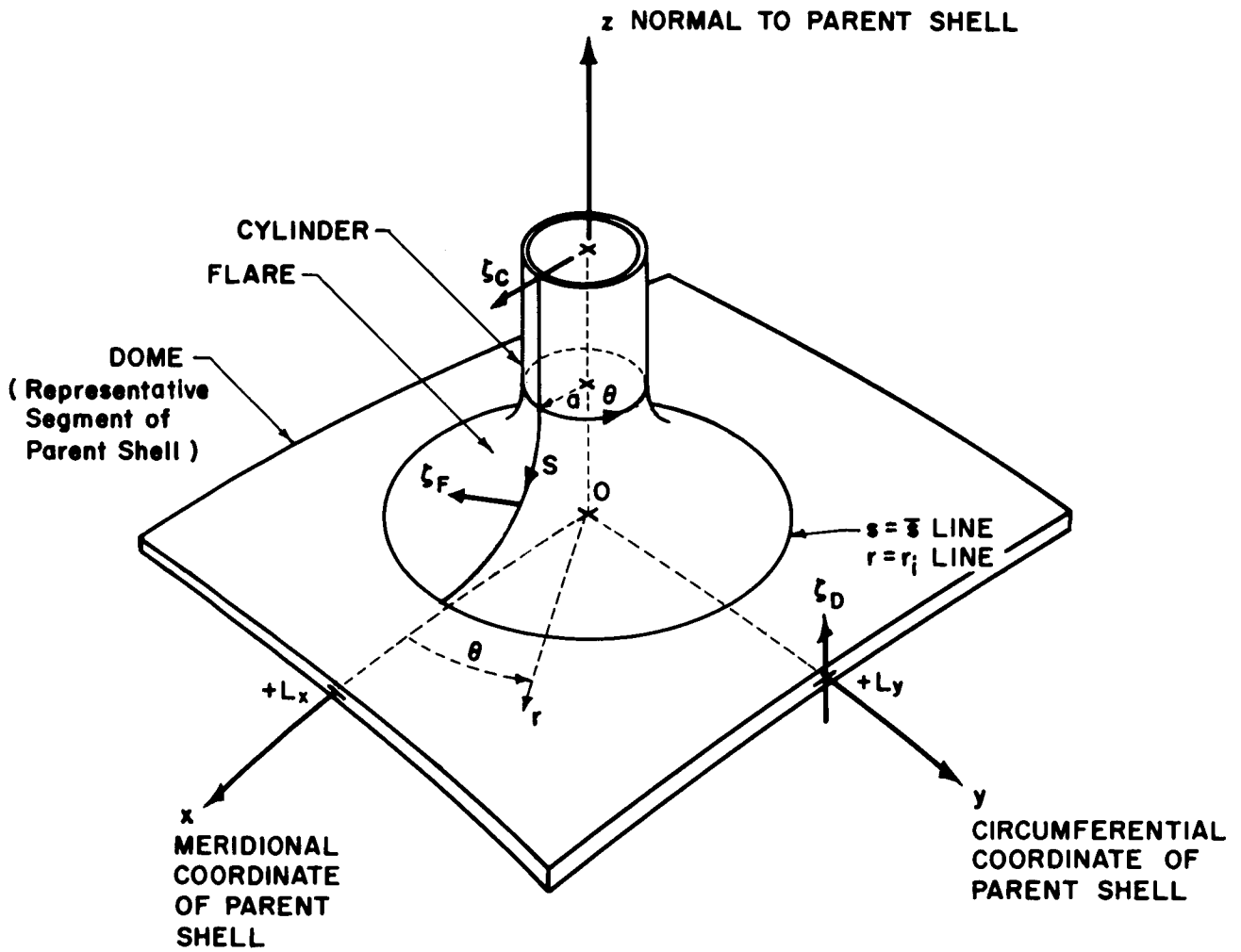


Figure 2. Idealization of a Flared-Nozzle as a Series of Shells

## SECTION II CYLINDER-FLARE ANALYSIS

### A. INTRODUCTION

The following development describes the general numerical procedures, based on Flugge's linear elastic shell equations [9], for determining the stresses and deflections in the cylinder-flare portion of the composite shell. It is assumed that the nozzle geometry, exclusive of the parent dome shell, can be approximated by a surface of revolution, consisting of a circular cylinder joined to a bell mouthed shell of variable thickness. Because of the axisymmetric shape of the flare model, it is possible to expand all dependent variables into Fourier series in the circumferential direction such that the Fourier coefficients are functions only of the meridional coordinate. The shell equations then yield an uncoupled set of ordinary differential equations for the coefficients of each Fourier component. Finite difference forms of these equations are then solved by a modification of the matrix methods employed in Reference [10].

### B. FORMULATION OF BASIC EQUATIONS

The flare consists of a shell of revolution (Figure 3) of variable wall thickness,  $t$ . The distance along the middle-surface meridional curve, measured from an edge of the shell is denoted by  $x$ ; the normal distance from the axis to the curve by  $r(x)$ . The functions  $r_1(x)$  and  $r_2(x)$  describe the meridional and hoop radii of curvature of the middle surface, respectively.

The exact equilibrium equations of a differential shell element in terms of force and moment resultants are given by [9 ]

$$(r N_\phi)' + r_1 N'_{\theta\phi} - r_1 N_\theta \cos \phi - r Q_\phi = - r r_1 p_\phi \quad (1a)$$

$$(r N_{\phi\theta})' + r_1 N'_\theta + r_1 N_{\theta\phi} \cos \phi - r_1 Q_\theta \sin \phi = - r r_1 p_\theta \quad (1b)$$

$$r_1 N_\theta \sin \phi + r N_\phi + r_1 Q'_\theta + (r Q_\phi)' = r r_1 p \quad (1c)$$

$$(r M_\phi)' + r_1 M'_{\theta\phi} - r_1 M_\theta \cos \phi = r r_1 Q_\phi \quad (1d)$$

$$(r M_{\phi\theta})' + r_1 M'_\theta + r_1 M_{\theta\phi} \cos \phi = r r_1 Q_\theta \quad (1e)$$

$$\frac{M_{\phi\theta}}{r_1} - \frac{M_{\theta\phi}}{r_2} = N_{\phi\theta} - N_{\theta\phi} \quad (1f)$$

where  $( )' = \frac{\partial}{\partial \phi} ( )$

$( )' = \frac{\partial}{\partial \theta} ( )$  ,

$p$ ,  $p_\phi$  and  $p_\theta$  are the shell loads per unit of middle surface area in the  $\zeta$ ,  $\phi$  and  $\theta$  directions, respectively (Figure 3), and

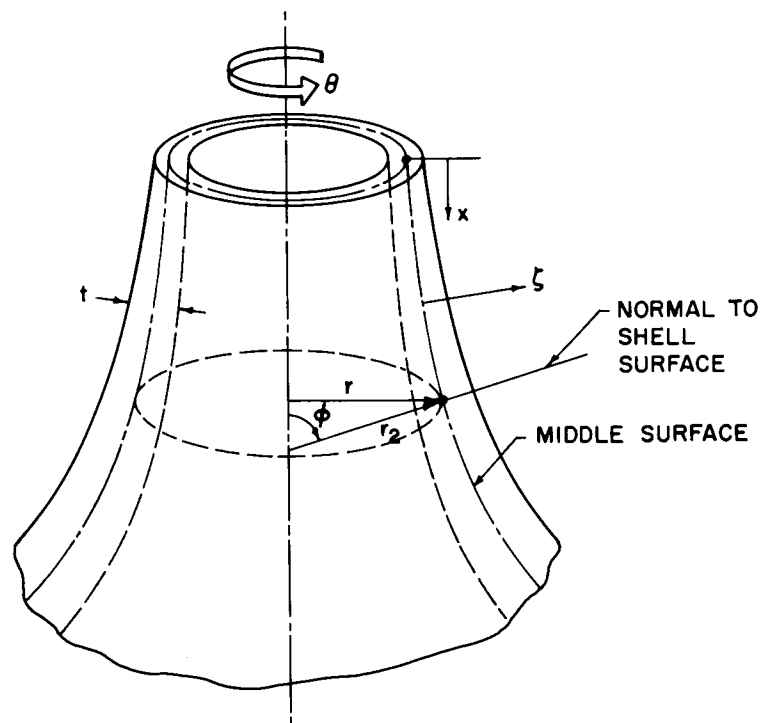


Figure 3. Flare Geometry

$$N_{\phi} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\phi\phi} \frac{r_2 + \zeta}{r_2} d\zeta \quad (2a)$$

$$N_{\theta} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\theta\theta} \frac{r_1 + \zeta}{r_1} d\zeta \quad (2b)$$

$$N_{\theta\phi} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\theta\phi} \frac{r_1 + \zeta}{r_1} d\zeta \quad (2c)$$

$$N_{\phi\theta} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\phi\theta} \frac{r_2 + \zeta}{r_2} d\zeta \quad (2d)$$

$$M_{\phi} = - \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\phi\phi} \frac{r_2 + \zeta}{r_2} \zeta d\zeta \quad (2e)$$

$$M_{\theta} = - \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\theta\theta} \frac{r_1 + \zeta}{r_1} \zeta d\zeta \quad (2f)$$

$$M_{\theta\phi} = - \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\theta\phi} \frac{r_1 + \zeta}{r_1} \zeta d\zeta \quad (2g)$$

$$M_{\phi\theta} = - \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\phi\theta} \frac{r_2 + \zeta}{r_2} \zeta d\zeta \quad (2h)$$

$$Q_{\phi} = - \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\phi\zeta} \frac{r_2 + \zeta}{r_2} d\zeta \quad (2i)$$

$$Q_{\theta} = - \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\theta\zeta} \frac{r_1 + \zeta}{r_1} d\zeta \quad (2j)$$

The stress components  $\sigma_{ij}$  ( $\begin{matrix} i = \phi, \theta \\ j = \phi, \theta, \zeta \end{matrix}$ ) are defined in accordance with conventional theory of elasticity notation. As a consequence, the force and moment resultants defined by equations (1) and (2) are positive when acting as shown in Figures 2c through 2e.

Equation (1f) is an identity which follows from the definitions of the force and moment resultants given by equations (2c), (2d), (2g), and (2h). Equations (1a) through (1e), therefore, constitute five independent equations involving ten resultants. The required connecting equations are provided by the constitutive relations [9]:

$$N_{\phi} = D \left[ \frac{v' + w}{r_1} + \nu \frac{u' + v \cos \phi + w \sin \phi}{r} \right] + \frac{K}{r_1^2} \frac{r_2 - r_1}{r_2} \left[ \frac{v - w'}{r_1} \frac{r_1}{r_1} + \frac{w'' + w}{r_1} \right], \quad (3a)$$

$$N_{\theta} = D \left[ \frac{u' + v \cos \phi + w \sin \phi}{r} + \nu \frac{v' + w}{r_1} \right] - \frac{K}{r r_1} \frac{r_2 - r_1}{r_2} \left[ -\frac{v}{r_1} \frac{r_2 - r_1}{r_2} \cos \phi + \frac{w \sin \phi}{r_2} + \frac{w''}{r} + \frac{w' \cos \phi}{r_1} \right], \quad (3b)$$

$$N_{\phi\phi} = D \frac{1-\nu}{2} \left[ \frac{u'}{r_1} + \frac{v' - u \cos \phi}{r} \right] + \frac{K}{r_1^2} \frac{1-\nu}{2} \frac{r_2 - r_1}{r_2} \left[ \frac{u'}{r_1} \frac{r_2 - r_1}{r_2} + \frac{u}{r_2} \frac{r_1 - r_2}{r_2} \cot \phi + \frac{w''}{r} - \frac{w'}{r} \frac{r_1}{r} \cos \phi \right], \quad (3c)$$

$$N_{\theta\theta} = D \frac{1-\nu}{2} \left[ \frac{u'}{r_1} + \frac{v' - u \cos \phi}{r} \right] + \frac{K}{r r_1} \frac{1-\nu}{2} \frac{r_2 - r_1}{r_2} \left[ \frac{v'}{r_1} \frac{r_2 - r_1}{r_2} - \frac{w''}{r_1} + \frac{w' \cos \phi}{r} \right], \quad (3d)$$

$$M_{\phi} = K \left[ \frac{1}{r_1^2} \left( w'' - w' \frac{r_1}{r_1} - w \frac{r_1 - r_2}{r_2} \right) - \frac{v'}{r_1 r_2} + \frac{v}{r_1^2} \frac{r_1}{r_1} + \nu \frac{w''}{r^2} + \nu \frac{w' \cos \phi}{r r_1} - \nu \frac{u'}{r r_2} - \nu \frac{v \cos \phi}{r r_1} \right], \quad (3e)$$

$$M_{\theta} = K \left[ \frac{w''}{r^2} + \frac{w' \cos \phi}{r r_1} - \frac{w}{r_2^2} \frac{r_2 - r_1}{r_1} - \frac{u'}{r r_1} - \frac{v \cos \phi}{r r_1} \frac{2r_2 - r_1}{r_2} \right. \\ \left. + \frac{\nu}{r_2^2} \left( w'' - w' \frac{r_1}{r_1} \right) - \frac{\nu}{r_1^2} \left( v' - v \frac{r_1}{r_1} \right) \right], \quad (3f)$$

$$M_{\phi\theta} = K \frac{1-\nu}{2} \left[ 2 \frac{w'}{r r_1} - 2 \frac{w'}{r^2} \cos \phi - \frac{u'}{r_1 r_2} \frac{2r_1 - r_2}{r_1} \right. \\ \left. + \frac{u}{r_2^2} \frac{2r_1 - r_2}{r_1} \cot \phi - \frac{v'}{r r_1} \right], \quad (3g)$$

$$M_{\theta\phi} = K \frac{1-\nu}{2} \left[ 2 \frac{w'}{r r_1} - 2 \frac{w'}{r^2} \cos \phi - \frac{u'}{r_1 r_2} + \frac{u}{r_2^2} \cot \phi - \frac{v'}{r r_1} \frac{2r_2 - r_1}{r_2} \right] \quad (3h)$$

where  $u, v, w$  are the middle surface displacements in the circumferential, meridional and normal directions, respectively (see Figure 4a), and

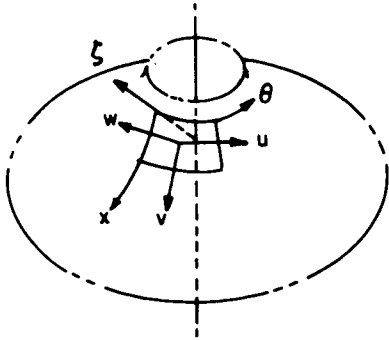
$$D = \frac{Et}{1-\nu^2} \quad (4a)$$

$$K = \frac{Et^3}{12(1-\nu^2)} \quad (4b)$$

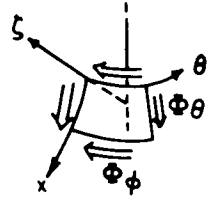
Equations (3) are based on the Kirchoff-Love hypothesis, which assumes the preservation of normals to the middle surface during deformation, and neglects direct stresses ( $\sigma_{\xi\xi}$ ). Equations (3), however, are based upon the retention of thickness-curvature terms up to the order of  $\left(\frac{t}{r}\right)^3$  and  $\left(\frac{t}{r}\right)^3$  in the strain-displacement relations and stress resultant expressions, whereas, in the usual thin shell theory, founded on Love's first approximation [11], first powers of these quantities are neglected in comparison with unity.

The set of field equations given by (1a) through (1e) and (3a) through (3h) constitute thirteen independent relations in the thirteen field quantities  $N_{\phi}, N_{\theta}, N_{\theta\phi}, N_{\phi\theta}, M_{\phi}, M_{\theta}, M_{\phi\theta}, M_{\theta\phi}, Q_{\phi}, Q_{\theta}, u, v$  and  $w$ . These quantities and the surface loads are now expanded into Fourier series as follows:

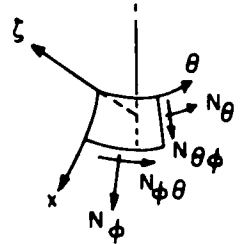
\* NOTE:  
 USE RIGHT-HAND RULE FOR  
 MOMENT AND ROTATION VECTORS



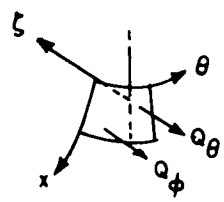
a. Displacements



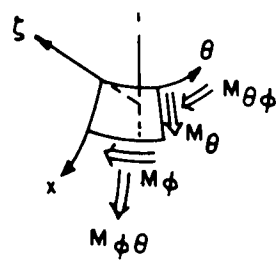
b. Rotations \*



c. Tangential Force Resultants



d. Transverse Force Resultants



e. Moments \*

Figure 4. Sign Conventions



$$\begin{pmatrix} N_\phi \\ N_\theta \end{pmatrix} = \sigma_c t_c \sum_{n=0}^{\infty} \begin{pmatrix} t_s^n \\ t_\theta^n \end{pmatrix} \cos n\theta$$

$$\begin{pmatrix} N_{\theta\phi} \\ N_{\phi\theta} \end{pmatrix} = \sigma_c t_c \sum_{n=1}^{\infty} \begin{pmatrix} t_{\theta s}^n \\ t_{s\theta}^n \end{pmatrix} \sin n\theta$$

$$\begin{pmatrix} M_\phi \\ M_\theta \end{pmatrix} = \frac{\sigma_c t_c^3}{r_c} \sum_{n=0}^{\infty} \begin{pmatrix} m_s^n \\ m_\theta^n \end{pmatrix} \cos n\theta$$

$$\begin{pmatrix} M_{\theta\phi} \\ M_{\phi\theta} \end{pmatrix} = \frac{\sigma_c t_c^3}{r_c} \sum_{n=1}^{\infty} \begin{pmatrix} m_{\theta s}^n \\ m_{s\theta}^n \end{pmatrix} \sin n\theta \quad (5)$$

$$Q_\phi = \sigma_c t_c \sum_{n=0}^{\infty} f_s^n \cos n\theta$$

$$Q_\theta = \sigma_c t_c \sum_{n=1}^{\infty} f_\theta^n \sin n\theta$$

$$u = \frac{r_c \sigma_c}{E} \sum_{n=1}^{\infty} u_\theta^n \sin n\theta$$

$$\begin{pmatrix} v \\ w \end{pmatrix} = \frac{r_c \sigma_c}{E} \sum_{n=0}^{\infty} \begin{pmatrix} u_s^n \\ w^n \end{pmatrix} \cos n\theta$$

$$\begin{pmatrix} p \\ p_\phi \end{pmatrix} = \frac{\sigma_c t_c}{r_c} \sum_{n=0}^{\infty} \begin{pmatrix} p^n \\ p_s^n \end{pmatrix} \cos n\theta \quad \left. \vphantom{\begin{pmatrix} p \\ p_\phi \end{pmatrix}} \right\}$$

$$p_\theta = \frac{\sigma_c t_c}{r_c} \sum_{n=1}^{\infty} p_\theta^n \sin n\theta \quad \left. \vphantom{p_\theta} \right\} \quad (6)$$

The quantities  $\sigma_c$ ,  $t_c$  and  $r_c$  represent a reference stress level, thickness and radial dimension of the shell, respectively, which tend to normalize the dimensionless Fourier coefficients  $t_s^n$ ,  $t_\theta^n$ ,  $\dots$ ,  $w^n$ . The expansions (5) and (6) and the development which follows are valid for loads and boundary conditions which produce symmetric deformations about the plane  $\theta = 0$ . Antisymmetric problems can be accommodated by interchanging the roles of  $\sin n\theta$  and  $\cos n\theta$  in Equations (5) and (6). Solutions for general asymmetric loading conditions can then be obtained by decomposing the complete problem into symmetric and antisymmetric components.

If Equations (5) and (6) are substituted into (1) and (3) the equations for the Fourier coefficients decouple into separate sets for each Fourier index  $n$ . Each set constitutes an eighth order system of ordinary differential equations with the meridional coordinate  $x$  as independent variable. To facilitate subsequent numerical computations, it is convenient to express the  $n^{\text{th}}$  general system as four second order differential equations in the Fourier coefficients  $u_s^n$ ,  $u_\theta^n$ ,  $w^n$  and  $m_s^n$ . This is accomplished in the following manner: Solve Equations (1d) through (1f) for  $Q_\phi$ ,  $Q_\theta$  and  $N_{\theta\phi}$  and substitute the resulting expressions into Equations (1a) through (1c). Next, eliminate terms (3a) and (3f) by making use of (3e). Substituting (5) and (6) into the resulting equations and considering only normal pressure loading then yields the following for the  $n^{\text{th}}$  set of Fourier coefficients\* :

$$\left. \begin{aligned} \rho \left[ \gamma t_s + t'_s - \gamma t_\theta \right] + n t_{s\theta} - \lambda^2 \left[ \rho \omega_s m'_s \right. \\ \left. + \rho \gamma \omega_s (m_s - m_\theta) + n (\omega_s - \omega_\theta) m_{\theta s} \right. \\ \left. + n \omega_s m_{s\theta} \right] = 0 \\ \rho \left[ \gamma t_{s\theta} + t'_{s\theta} \right] - n t_\theta - \lambda^2 \left[ \rho \gamma (\omega_s + \omega_\theta) m_{s\theta} \right. \\ \left. + \rho \omega_\theta m'_{s\theta} - n \omega_\theta m_\theta \right] = 0 \end{aligned} \right\} \quad (7)$$

---

\* For convenience the superscript  $n$  on the Fourier coefficients will be omitted in the following equations.

$$\begin{aligned}
& -\omega_s t_s - \omega_\theta t_\theta - \lambda^2 \left[ m_s'' + 2\gamma m_s' - \omega_s \omega_\theta m_s \right. \\
& \quad - \gamma m_\theta' + \left( \omega_s \omega_\theta - \frac{n^2}{\rho^2} \right) m_\theta + \frac{n}{\rho} (m_{s\theta}' + m_{\theta s}') \\
& \quad \left. + \frac{n\gamma}{\rho} (m_{s\theta} + m_{\theta s}) \right] = -p^n \\
& d \left\{ w'' + \nu \gamma w' + \left[ \omega_s (\omega_s - \omega_\theta) - \frac{\nu n^2}{\rho^2} \right] w - \omega_\theta u_s' \right. \\
& \quad \left. - (\omega_s' + \nu \gamma \omega_s) u_s - \frac{\nu n}{\rho} \omega_\theta u_\theta \right\} - m_s = 0
\end{aligned} \tag{7 cont'd}$$

$$\begin{aligned}
t_s &= b \left[ u_s' + (\omega_s + \nu \omega_\theta) w + \frac{\nu n}{\rho} u_\theta + \nu \gamma u_s \right] + \lambda^2 (\omega_s - \omega_\theta) m_s \\
& \quad + d \lambda^2 (\omega_s - \omega_\theta) \left[ \left( \omega_s \omega_\theta + \frac{\nu n^2}{\rho^2} \right) w + \omega_\theta u_s' - \nu \gamma w' \right. \\
& \quad \left. + \nu \omega_\theta \frac{n}{\rho} u_\theta + \nu \gamma \omega_s u_s \right] \\
t_\theta &= b \left[ \frac{n}{\rho} u_\theta + \gamma u_s + \omega_\theta w + \nu u_s' + \nu \omega_s w \right] \\
& \quad - d \lambda^2 (\omega_s - \omega_\theta) \left[ \gamma (\omega_\theta - \omega_s) u_s + \left( \omega_\theta^2 - \frac{n^2}{\rho^2} \right) w + \gamma w' \right] \\
t_{s\theta} &= b \left( \frac{1-\nu}{2} \right) \left[ u_\theta' - \frac{n}{\rho} u_s - \gamma u_\theta \right] \\
& \quad + d \lambda^2 \left( \frac{1-\nu}{2} \right) (\omega_s - \omega_\theta) \left[ (\omega_s - \omega_\theta) u_\theta' + \gamma (\omega_\theta - \omega_s) u_\theta \right. \\
& \quad \left. - \frac{n}{\rho} w' + \frac{\gamma n}{\rho} w \right] \\
m_\theta &= \nu m_s - d \left\{ \left[ \omega_\theta (\omega_s - \omega_\theta) - \nu \omega_s (\omega_\theta - \omega_s) + (1-\nu^2) \frac{n^2}{\rho^2} \right] w \right. \\
& \quad + \left[ \gamma (2\omega_s - \omega_\theta) - \nu^2 \gamma \omega_s \right] u_s + \frac{n}{\rho} (\omega_s - \nu^2 \omega_\theta) u_\theta \\
& \quad \left. - \gamma (1-\nu^2) w' + \nu (\omega_s - \omega_\theta) u_s' \right\}
\end{aligned} \tag{8}$$

$$\begin{aligned}
m_{\theta s} &= d \frac{(1-\nu)}{2} \left[ -\frac{2n}{\rho} w' + \frac{2n\gamma}{\rho} w - \omega_{\theta} u'_{\theta} + \gamma \omega_{\theta} u_{\theta} \right. \\
&\quad \left. + \frac{n}{\rho} (2\omega_s - \omega_{\theta}) u_s \right] \\
m_{s\theta} &= d \frac{(1-\nu)}{2} \left[ -\frac{2n}{\rho} w' + \frac{2n\gamma}{\rho} w - (2\omega_{\theta} - \omega_s) u'_{\theta} + \gamma (2\omega_{\theta} - \omega_s) u_{\theta} \right. \\
&\quad \left. + \frac{n}{\rho} \omega_s u_s \right]
\end{aligned}
\tag{8 cont'd}$$

The following dimensionless quantities and geometric identities have been employed in deriving Equations (7) and (8):

$$\begin{aligned}
s &= \frac{x}{r_c} \\
\lambda &= \frac{t}{r_c} \\
d &= \frac{(t/t_c)^3}{12(1-\nu^2)} \\
b &= \frac{t/t_c}{1-\nu^2}
\end{aligned}
\tag{9}$$

$$\begin{aligned}
\omega_s &= \frac{r_c}{r_1} \\
\omega_{\theta} &= \frac{r_c}{r_2} \\
\rho &= \frac{r}{r_c}
\end{aligned}
\tag{10}$$

$$\sin \phi = \rho \omega_{\theta}$$

$$\cos \phi = \rho \gamma$$

$$\frac{\partial}{\partial \phi} = \frac{1}{\omega_s} \frac{\partial}{\partial s} = \frac{1}{\omega_s} ( )'$$

$$\frac{\partial}{\partial s} = ( )' \tag{11}$$

$$\omega'_{\theta} = \gamma (\omega_s - \omega_{\theta})$$

$$\rho' = \rho \gamma$$

$$\gamma' = -(\gamma^2 + \omega_s \omega_{\theta})$$

The symbol ( )', which previously indicated differentiation with respect to  $\theta$  is now employed to denote differentiation with respect to  $s$  in the Fourier component Equations (7) and (8). No confusion should arise since  $\theta$  has been eliminated from the latter equations by Fourier decomposition.

When Equations (8) are substituted into (7), the resulting four second order differential equations are of the form:

$$\begin{aligned} & a_1 u''_s + a_2 u'_s + a_3 u_s + a_4 u'_{\theta} + a_5 u_{\theta} \\ & + a_5' w'' + a_6 w' + a_7 w + a_8 m'_s + a_9 m_s = 0 \end{aligned}$$

$$\begin{aligned} & a_{10} u'_s + a_{11} u_s + a_{12} u''_{\theta} + a_{13} u'_{\theta} + a_{14} u_{\theta} \\ & + a_{15} w'' + a_{16} w' + a_{17} w + a_{18} m_s = 0 \end{aligned}$$

(12)

$$\begin{aligned} & a_{19} u''_s + a_{19} u'_s + a_{20} u_s + a_{21} u''_{\theta} + a_{22} u'_{\theta} \\ & + a_{23} u_{\theta} + a_{24} w'' + a_{25} w' + a_{26} w + a_{27} m''_s \\ & + a_{28} m'_s + a_{29} m_s = -p \end{aligned}$$

$$\begin{aligned} & a_{30} u'_s + a_{31} u_s + a_{32} u_{\theta} + a_{33} w'' + a_{34} w' \\ & + a_{35} w + a_{36} m_s = 0 \end{aligned}$$

where the a's are listed in Appendix I.

Equations (12) can be conveniently written in the following matrix form:

$$E Z'' + F Z' + G Z = e \quad (13)$$

where  $Z$  is the solution vector

$$Z = \begin{pmatrix} u_s \\ u_\theta \\ w \\ m_s \end{pmatrix} \quad (14a)$$

and

$$E = \begin{pmatrix} a_1 & 0 & a_{5'} & 0 \\ 0 & a_{12} & a_{15} & 0 \\ a_{19'} & a_{21} & a_{24} & a_{27} \\ 0 & 0 & a_{33} & 0 \end{pmatrix} \quad (14b)$$

$$F = \begin{pmatrix} a_2 & a_4 & a_6 & a_8 \\ a_{10} & a_{13} & a_{16} & 0 \\ a_{19} & a_{22} & a_{25} & a_{28} \\ a_{30} & 0 & a_{34} & 0 \end{pmatrix} \quad (14c)$$

$$G = \begin{pmatrix} a_3 & a_5 & a_7 & a_9 \\ a_{11} & a_{14} & a_{17} & a_{18} \\ a_{20} & a_{23} & a_{26} & a_{29} \\ a_{31} & a_{32} & a_{35} & a_{36} \end{pmatrix} \quad (14d)$$

$$e = \begin{pmatrix} 0 \\ 0 \\ -p \\ 0 \end{pmatrix} \quad (14e)$$

### C. BOUNDARY CONDITIONS

The eighth order system of equations describing deformations of the shell must be solved subject to four appropriately specified boundary conditions at the edges  $S = 0$  and  $S = \bar{S}$ . In general, physically realizable edge conditions can be written in the following form:

$$K_{11}N_c + K_{12}v = L_1 \quad (15a)$$

$$K_{21}\widehat{N}_{\varphi\theta} + K_{22}u = L_2 \quad (15b)$$

$$K_{31}\widehat{Q}_\varphi + K_{32}w = L_3 \quad (15c)$$

$$K_{41}\Phi_\varphi + K_{42}M_\varphi = L_4 \quad (15d)$$

Where  $K_{11}, \dots, K_{42}$  and  $L_1, \dots, L_4$  are specified constants and the quantities not previously defined are: (a) an effective membrane shear stress, resultant

$$\widehat{N}_{\varphi\theta} = N_{\varphi\theta} - \frac{M_{\varphi\theta}}{r_2}, \quad (16a)$$

(b) an effective transverse shear stress resultant

$$\widehat{Q}_\varphi = \frac{1}{rr_1} \left[ \frac{\partial}{\partial\varphi} (rM_\varphi) + r_1 \frac{\partial}{\partial\theta} (M_{\theta\varphi}) - r_1 M_{\theta\varphi} \cos\varphi \right] + \frac{1}{r} \frac{\partial M_{\varphi\theta}}{\partial\theta}, \quad (16b)$$

and

(c) the meridional slope change

$$\Phi_\varphi = \frac{1}{r_1} \left[ \frac{\partial w}{\partial\varphi} - v \right], \quad (16c)$$

(see Figure 4b).

The Fourier coefficients corresponding to these additional quantities, i.e.,  $\widehat{t}_{s\theta}^n$ ,  $\widehat{f}_s^n$  and  $\widehat{\varphi}_s^n$  are defined by the expansions

$$\begin{Bmatrix} \widehat{N}_{\varphi\theta} \\ \widehat{Q}_\varphi \end{Bmatrix} = \sigma_c t_c \sum_{n=0}^{\infty} \begin{Bmatrix} \widehat{t}_{s\theta}^n \sin n\theta \\ \widehat{f}_s^n \cos n\theta \end{Bmatrix}$$

$$\Phi_{\phi} = \frac{\sigma_c}{E} \sum_{n=0}^{\infty} \bar{\phi}_s^n \cos n\theta, \quad (17)$$

so that from (5) and (15) the boundary conditions for the  $n^{\text{th}}$  Fourier component can be written as

$$\Omega Y + \Lambda Z = \ell \quad (18)$$

where

$$\Omega = \begin{pmatrix} \lambda K_{11} & 0 & 0 & 0 \\ 0 & \lambda K_{21} & 0 & 0 \\ 0 & 0 & \lambda K_{31} & 0 \\ 0 & 0 & 0 & \frac{K_{41}}{Er_c} \end{pmatrix} \quad (19a)$$

$$\Lambda = \begin{pmatrix} \frac{K_{12}}{E} & 0 & 0 & 0 \\ 0 & \frac{K_{22}}{E} & 0 & 0 \\ 0 & 0 & \frac{K_{32}}{E} & 0 \\ 0 & 0 & 0 & \lambda^3 r_c K_{42} \end{pmatrix} \quad (19b)$$

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{pmatrix} = \sigma_c r_c \sum_{n=0}^{\infty} \begin{pmatrix} \ell_1^n \cos n\theta \\ \ell_2^n \sin n\theta \\ \ell_3^n \cos n\theta \\ \ell_4^n \cos n\theta \end{pmatrix} \quad (19c)$$

and

$$Y = \begin{pmatrix} t_s \\ \hat{t}_{s\theta} \\ \hat{f}_s \\ \bar{c}_s \end{pmatrix} \quad (20)$$



In order to express the boundary conditions entirely in terms of the solution vector  $z$ , Equations (16) are first written in Fourier component form and the resulting Fourier coefficients are then expressed in terms of  $u_s$ ,  $u_\theta$ ,  $w$  and  $m_s$  by means of equations (8). The results of these calculations are given in matrix form by

$$Y = HZ' + JZ \quad (21)$$

where

$$H = \begin{pmatrix} b_1 & 0 & b_{4'} & 0 \\ 0 & b_6 & b_8 & 0 \\ b_{10'} & b_{11} & b_{13} & b_{15} \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (22a)$$

$$J = \begin{pmatrix} b_2 & b_3 & b_4 & b_{4''} \\ b_5 & b_7 & b_9 & 0 \\ b_{10} & b_{12} & b_{14} & b_{16} \\ -\omega_s & 0 & 0 & 0 \end{pmatrix} \quad (22b)$$

and the  $b$  coefficients are given in Appendix II.

Substituting (21) into (18) yields

$$\Omega HZ' + (\Lambda + \Omega J)Z = \ell \quad (23)$$

Thus, the field equations (13) are to be solved subject to boundary conditions of the type (23) at the edges  $S = 0$  and  $S = \bar{S}$ .

D. FINITE DIFFERENCE FORMULATION AND MATRIX SOLUTION OF THE DIFFERENCE EQUATIONS

The range of the independent variable  $0 \leq S \leq \bar{S}$  is divided into  $N$  intervals by the grid points  $i=0, 1, \dots, N$ . The  $i^{\text{th}}$  grid size is then defined by

$$\Delta_i = S_i - S_{i-1} \quad (i=1, 2, \dots, N) \quad (24)$$

The vector field  $Z$  is now replaced by the mesh vectors  $Z_i$ , defined only at the grid points. At all interior points,  $1 \leq i \leq N-1$ , the first two derivatives of  $Z$  are replaced by parabolic central difference expressions of the form

$$Z'_i = \frac{1}{(\Delta_{i+1} + \Delta_i)} \left[ \frac{\Delta_i}{\Delta_{i+1}} Z_{i+1} - \left( \frac{\Delta_i}{\Delta_{i+1}} - \frac{\Delta_{i+1}}{\Delta_i} \right) Z_i - \frac{\Delta_{i+1}}{\Delta_i} Z_{i-1} \right] \quad (25a)$$

$$Z''_i = \frac{2}{(\Delta_{i+1} + \Delta_i)} \left[ \frac{Z_{i+1}}{\Delta_{i+1}} - \left( \frac{1}{\Delta_{i+1}} + \frac{1}{\Delta_i} \right) Z_i + \frac{Z_{i-1}}{\Delta_i} \right] \quad (25b)$$

At the end points  $i=0, N$  first derivatives of  $Z$  are approximated by forward and backward difference formulas, respectively, which involve the same truncation orders as for the central difference formulas (25).

$$Z'_0 = -Z_2 \frac{\Delta_1}{\Delta_2 (\Delta_1 + \Delta_2)} + Z_1 \frac{\Delta_1 + \Delta_2}{\Delta_1 \Delta_2} - Z_0 \left( \frac{\Delta_2}{\Delta_1 (\Delta_1 + \Delta_2)} + \frac{2}{\Delta_1 + \Delta_2} \right) \quad (26a)$$

$$Z'_N = Z_N \frac{\Delta_{N-1}}{\Delta_N (\Delta_N + \Delta_{N-1})} + \frac{2}{\Delta_N + \Delta_{N-1}} - Z_{N-1} \frac{\Delta_N + \Delta_{N-1}}{\Delta_N \Delta_{N-1}} + Z_{N-2} \frac{\Delta_N}{\Delta_{N-1} (\Delta_N + \Delta_{N-1})} \quad (26b)$$

This, employing Equations (25) and (26) the boundary value problems given by Equations (13) and (23) are approximated by the matrix difference equations:

$$\bar{A}_0 Z_1 + \bar{B}_0 Z_0 + \bar{C}_0 Z_2 = g_0 \quad (27a)$$

$$A_i Z_{i+1} + B_i Z_i + C_i Z_{i-1} = g_i \quad (i=1, 2, \dots, N-1) \quad (27b)$$

$$\bar{A}_N Z_{N-2} + \bar{B}_N Z_N + \bar{C}_N Z_{N-1} = \bar{g}_N \quad (27c)$$

where

$$\begin{aligned}\bar{A}_0 &= \frac{\Delta_1 + \Delta_2}{\Delta_1 \Delta_2} \Omega_0 H_0 \\ \bar{B}_0 &= \Lambda_0 + \Omega_0 \left[ J_0 - \frac{H_0}{\Delta_1} \left( \frac{\Delta_2 + 2\Delta_1}{\Delta_1 + \Delta_2} \right) \right]\end{aligned}\quad (28a)$$

$$\begin{aligned}\bar{C}_0 &= -\frac{\Delta_1}{\Delta_2 (\Delta_1 + \Delta_2)} \Omega_0 H_0 \\ g_0 &= l_0 \\ A_i &= \frac{2E_i}{\Delta_{i+1}} + \frac{\Delta_i}{\Delta_{i+1}} F_i \\ B_i &= -2E_i \left( \frac{1}{\Delta_{i+1}} + \frac{1}{\Delta_i} \right) - \left( \frac{\Delta_i}{\Delta_{i+1}} - \frac{\Delta_{i+1}}{\Delta_i} \right) F_i + (\Delta_{i+1} + \Delta_i) G_i \\ C_i &= \frac{2E_i}{\Delta_i} - \frac{\Delta_{i+1}}{\Delta_i} F_i\end{aligned}\quad (28b)$$

$$\begin{aligned}g_i &= (\Delta_i + \Delta_{i+1}) e_i, \\ \bar{A}_N &= \Omega_N H_N \frac{\Delta_N}{\Delta_{N-1} (\Delta_N + \Delta_{N-1})} \\ \bar{B}_N &= \Lambda_N + \Omega_N \left[ J_N + \frac{H_N}{\Delta_N} \left( \frac{\Delta_{N-1} + 2\Delta_N}{\Delta_N \Delta_{N-1}} \right) \right] \\ \bar{C}_N &= -\Omega_N H_N \frac{\Delta_N + \Delta_{N-1}}{\Delta_N \Delta_{N-1}} \\ g_N &= l_N ,\end{aligned}\quad (28c)$$

and the subscripts zero and N refer to the grid values at  $i=0, N$ , respectively.

Equations (27) form a system of  $(N+1)$  linear algebraic equations in the  $N+1$  unknown mesh vectors  $(Z_i)_{i=0,1,\dots,N}$ . We assume that as  $N \rightarrow \infty$  the solution of this algebraic system converges to the solution of the differential system.

Thus for a sufficiently small mesh size, the mesh vectors  $Z_i$  are expected to accurately approximate the vector field  $Z(s)$ .

Matrix solutions of the difference equations are obtained by a modified Gaussian elimination scheme slightly different from the one developed in Reference [10]. The procedure is as follows:

Solve (27b) for  $Z_2$  when  $i = 1$ ,

$$Z_2 = A_1^{-1} (g_1 - B_1 Z_1 - C_1 Z_0) \quad (29)$$

and substitute (29) into (27a) to obtain

$$A_0 Z_1 + B_0 Z_0 = g_0 \quad (27c)$$

where

$$A_0 = \bar{A}_0 - \bar{C}_0 A_1^{-1} B_1, \quad B_0 = \bar{B}_0 - \bar{C}_0 A_1^{-1} C_1, \quad g_0 = \bar{g}_0 - \bar{C}_0 A_1^{-1} g_1$$

Next solve (27b) for  $Z_{N-2}$  when  $i = N-1$ ,

$$Z_{N-2} = C_{N-1}^{-1} (g_{N-1} - A_{N-1} Z_N - B_{N-1} Z_{N-1})$$

and substitute into (27c) to obtain

$$B_N Z_N + A_N Z_{N-1} = g_N \quad (27e)$$

where

$$B_N = \bar{B}_N - \bar{A}_N C_{N-1}^{-1} A_{N-1}, \quad C_N = \bar{C}_N - \bar{A}_N C_{N-1}^{-1} B_{N-1},$$

$$g_N = \bar{g}_N - \bar{A}_{N-1} C_{N-1}^{-1} g_{N-1}$$

Next, Equations (27d) and (27b) (for  $i = 1$ ) are solved simultaneously for

$Z_0$  in terms of  $Z_1$  and  $Z_2$  and then for  $Z_1$  in terms of  $Z_2$ :

$$Z_0 = C_1^{-1} [g_1 - A_1 Z_2 - B_1 Z_1] \quad (30)$$

$$Z_1 = - [B_0 C_1^{-1} B_1 - A_0]^{-1} [B_0 C_1^{-1} A_1 Z_2 - B_0 C_1^{-1} g_1 + g_0] \quad (31)$$

(This inversion procedure avoids singularities which can occur in the  $B_0$  matrix for certain types of boundary conditions). It is next assumed that the value of the mesh vector at a given grid point can be expressed in terms of the value at the following grid point in the form

$$Z_{i-1} = - P_{i-1} Z_i + X_{i-1} \quad (i=2, 3, \dots, N-1). \quad (32)$$

Recurrence relations for the P's and X's are established by substituting (32) into (27b);

$$P_i = [B_i - C_i P_{i-1}]^{-1} A_i$$

$$X_i = [B_i - C_i P_{i-1}]^{-1} [g_i - C_i X_{i-1}], \quad (i = 2, 3, \dots, N-1). \quad (33)$$

However, from (31) the initial values are

$$P_1 = [B_0 C_1^{-1} B_1 - A_0]^{-1} [B_0 C_1^{-1} A_1] \quad (34a)$$

$$X_1 = [B_0 C_1^{-1} B_1 - A_0]^{-1} [B_0 C_1^{-1} g_1 - g_0], \quad (34b)$$

and therefore all the P's and X's up to  $P_{N-1}$  and  $X_{N-1}$  can be determined from (33) by recursions. The value of  $Z_N$  can now be determined by substituting

$$Z_{N-1} = -P_{N-1} Z_N + X_{N-1} \quad (35)$$

into (27e), yielding

$$Z_N = \left[ B_N - C_N P_{N-1} \right]^{-1} \left[ g_N - C_N X_{N-1} \right] \quad (36)$$

and therefore all the values of  $Z_i$  down to  $Z_2$  can be determined by recursions of (32). Finally  $Z_1$  and  $Z_0$  are found in turn from (31) and (30). Thus, the solution for all the  $Z_i \Big|_{i=0,1,\dots,N}$  involves only the inversion of 4 x 4 matrices at each grid point, which can be rapidly accomplished on a digital computer.

After the  $Z_i$  vectors have been obtained in the above manner, numerical approximations to the remaining field quantities can be calculated from obvious finite difference counterparts of equations (8).

The stresses  $\sigma_\phi$ ,  $\sigma_\theta$  and  $\sigma_{\phi\theta}$  are obtained by first finding the total grid point displacements from the last two equations of (5). These are then substituted into finite difference equivalents of the strain-displacement relations:<sup>[9]</sup>

$$\begin{aligned} \epsilon_\phi &= \frac{v'}{r_1} - \frac{v}{r_1} \frac{r_i'}{r_1} \frac{\zeta}{r_1 + \zeta} - \frac{w''}{r_1} \frac{\zeta}{r_1 + \phi} + \frac{w'}{r_1} \frac{r_i'}{r_1} \frac{\zeta}{r_1 + \zeta} + \frac{w}{r_1 + \zeta} \\ \epsilon_\theta &= \frac{u'}{r} + \frac{v}{r_1} \cot \phi \frac{r_1 + \zeta}{r_2 + \zeta} - \frac{w''}{r \sin \phi} \frac{\zeta}{r_2 + \zeta} \\ &\quad - \frac{w'}{r_1} \cot \phi \frac{\zeta}{r_2 + \zeta} + \frac{w}{r_2 + \zeta}, \\ \gamma_{\phi\theta} &= \frac{u'}{r_2} \frac{r_2 + \zeta}{r_1 + \zeta} - \frac{ur_1}{r_2^2} \frac{r_2 + \zeta}{r_1 + \zeta} \cot \phi + \frac{v'}{r_1 \sin \phi} \frac{r_1 + \zeta}{r_2 + \zeta} \\ &\quad - \frac{w'' \zeta}{r_1 \sin \zeta} \left( \frac{1}{r_2 + \zeta} + \frac{r_1}{r_2} \frac{1}{r_1 + \zeta} \right) \\ &\quad + \frac{w'}{r_2} \frac{\cot \phi}{\sin \phi} \left( \frac{\zeta}{r_2 + \zeta} + \frac{r_1}{r_2} \frac{\zeta}{r_1 + \zeta} \right). \end{aligned} \quad (37)$$

The stresses are finally calculated by substituting the resulting strains into Hooke's law,

$$\begin{aligned}
\sigma_{\varphi} &= \frac{E}{1-\nu^2} (\epsilon_{\varphi} + \nu \epsilon_{\theta}) \\
\sigma_{\theta} &= \frac{E}{1-\nu^2} (\epsilon_{\theta} + \nu \epsilon_{\varphi}) \\
\sigma_{\varphi\theta} &= \sigma_{\theta\varphi} = \frac{E}{2(1+\nu)} \gamma_{\varphi\theta}
\end{aligned} \tag{38}$$

### E. UNIT SOLUTIONS

Since the boundary conditions cannot be satisfied continuously at the flare-dome intersection, a solution procedure which matches conditions point-wise will be employed. Toward this end, we wish to develop unit solutions which satisfy the homogeneous flare equations and which, when multiplied by suitable constants, can be made to satisfy appropriate boundary conditions. To develop the unit solutions, we specialize the constants of Equations (15) so that the unit problem boundary conditions, satisfied for each  $n$ , are:

- (1)  $s = 0$ :  $t_s = t_{s\theta} = f_s = m_s = 0$   
 $s = \bar{s}$ :  $u_s = 1, u_{\theta} = w = m_s = 0$
- (2)  $s = 0$ : same as for (1)  
 $s = \bar{s}$ :  $u_s = 0, u_{\theta} = 1, w = m_s = 0$
- (3)  $s = 0$ : same as for (1)  
 $s = \bar{s}$ :  $u_s = u_{\theta} = 0, w = 1, m_s = 0$
- (4)  $s = 0$ : same as for (1)  
 $s = \bar{s}$ :  $u_s = u_{\theta} = w = 0, m_s = 1$

In addition, for  $n = 0$ , we require the particular solution corresponding to the internal pressure =  $p$  (a constant) and boundary conditions, which are:

$$\begin{aligned}
s = 0: t_s^0 &= \text{internal pressure membrane force} \\
t_{s\theta} &= f_s = m_s = 0 \\
s = \bar{s}: u_s &= u_{\theta} = w = m_s = 0
\end{aligned}$$

Multiplying the resulting unit solutions by constants and summing, we obtain, for eight of the more important derived quantities:

$$U_F(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \cos n \theta \sum_{k=1}^4 C_k^n u_{s1k}^n(s) + u^0(s) \quad (39a)$$

$$V_F(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \sin n \theta \sum_{k=1}^4 C_k^n u_{\theta k}^n(s) \quad (39b)$$

$$W_F(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \cos n \theta \sum_{k=1}^4 C_k^n w_k^n(s) + w^0(s) \quad (39c)$$

$$M_S(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \cos n \theta \sum_{k=1}^4 C_k^n m_k^n(s) + m^0(s) \quad (39d)$$

$$N_S(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \cos n \theta \sum_{k=1}^4 C_k^n t_{s_k}^n(s) + t_s^0(s) \quad (39e)$$

$$N_{S\theta}(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \sin n \theta \sum_{k=1}^4 C_k^n t_{s\theta_k}^n(s) \quad (39f)$$

$$\bar{Q}_S(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \cos n \theta \sum_{k=1}^4 C_k^n f_{s_k}^n(s) + f_s^0(s) \quad (39g)$$

$$\bar{\Phi}_S(S, \theta) = \sum_{n=0, 2, \dots}^{\bar{n}} \cos n \theta \sum_{k=1}^4 C_k^n \bar{\varphi}_k^n(s) + \bar{\varphi}^0(s) \quad (39h)$$

where the subscript F refers to "Flare," the subscript S relates to the meridional coordinate and  $U_F$  and  $V_F$  are the meridional and circumferential deflections, respectively.

The remaining derived quantities may be formed in a similar manner where necessary.

Only the even integer (n) Fourier terms have been used, since the nozzle-dome geometry is assumed to be symmetric about the xz and yz planes. Writing the solutions in this form yields  $4 \left(\frac{\bar{n}}{2} + 1\right)$  constants  $C_k^n$  to satisfy the nozzle-dome intersection compatibility and equilibrium conditions. However, because of the problem's symmetry it can be shown that  $C_2^0 = 0$ , thus reducing the number of unknowns to  $2\bar{n} + 3$ .



### SECTION III

#### DOMe ANALYSIS

##### A. INTRODUCTION

The governing equations for the bending and stretching of an arbitrarily shaped, thin, shallow shell were originally developed by Marguere [12]. For completeness, these equations are rederived (Appendix II) for the particular geometry of an elliptic-paraboloidal shell. The median surface of this shell is prescribed by (see Figure 5)

$$z = -\frac{1}{2} \left( \frac{x^2}{R_x} + \frac{y^2}{R_y} \right) \quad (1)$$

where, to the order of approximation in the analysis,  $R_x$  and  $R_y$  are the principal radii of curvature.

Marguerre's shallow shell theory is characterized by a linear system of eleven coupled equations for as many unknowns. In particular, this set consists of the equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (2a)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (2b)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \quad (2c)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y \quad (2d)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{N_x}{R_x} - \frac{N_y}{R_y} = -p(x, y) \quad (2e)$$

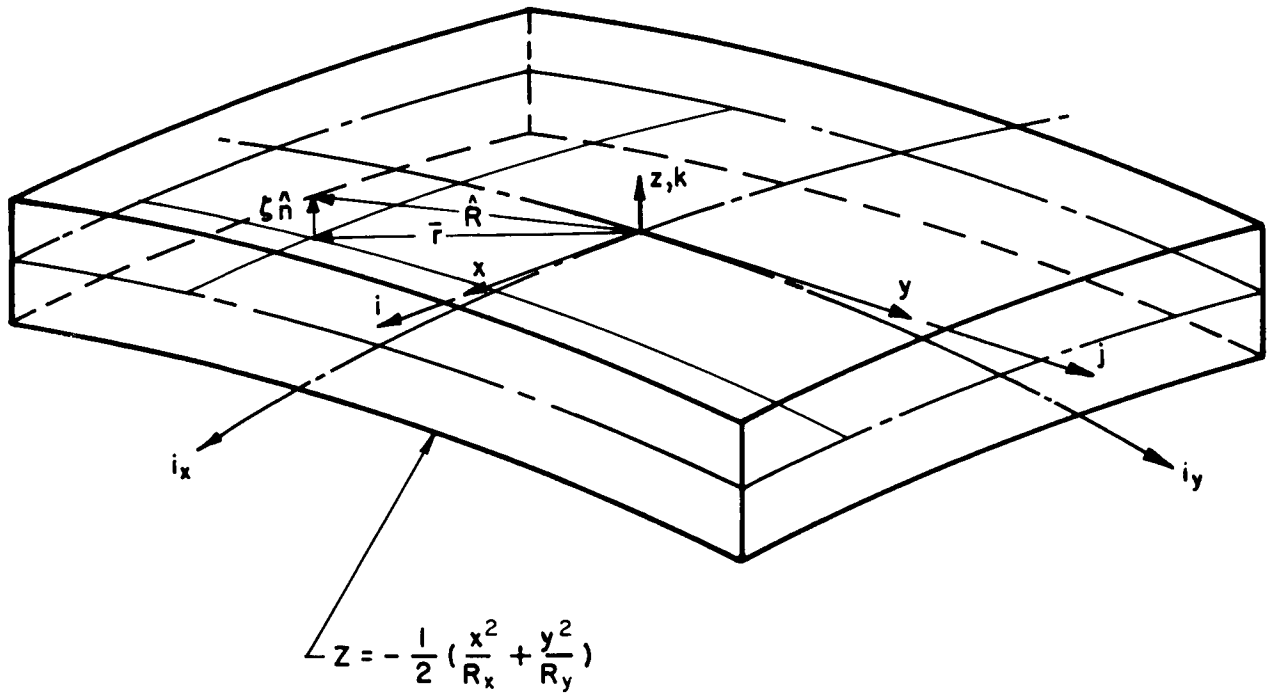


Figure 5. Coordinates and Notation for Shallow Elliptic-Paraboloidal Shell

and the stress resultant-displacement relations

$$N_x = A \left[ \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} + \left( \frac{1}{R_x} + \frac{\nu}{R_y} \right) w \right] \quad (3a)$$

$$N_y = A \left[ \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} + \left( \frac{1}{R_y} + \frac{\nu}{R_x} \right) w \right] \quad (3b)$$

$$N_{xy} = \frac{(1-\nu)A}{2} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] \quad (3c)$$

$$M_x = -D \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] \quad (3d)$$

$$M_y = -D \left[ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right] \quad (3e)$$

$$M_{xy} = -(1-\nu)D \frac{\partial^2 w}{\partial x \partial y} \quad (3f)$$

The notation and sign convention is shown in Figure 6. Associated with Equations 2 and 3 are the admissible boundary conditions which specify either

$$u \text{ or } \left[ N_x \cos(\ell, x) + N_{xy} \cos(\ell, y) \right] \quad (4a)$$

$$v \text{ or } \left[ N_{xy} \cos(\ell, x) + N_y \cos(\ell, y) \right] \quad (4b)$$

$$w \text{ or } \left[ \left( Q_x + \frac{\partial M_{xy}}{\partial y} \right) \cos(\ell, x) + \left( Q_y + \frac{\partial M_{xy}}{\partial x} \right) \cos(\ell, y) \right] \quad (4c)$$

$$\frac{\partial w}{\partial x} \text{ or } M_x \quad (4d)$$

$$\frac{\partial w}{\partial y} \text{ or } M_y \quad (4e)$$

Equations 2 and 3 can be contracted to the single fourth order partial differential equation (Appendix III)

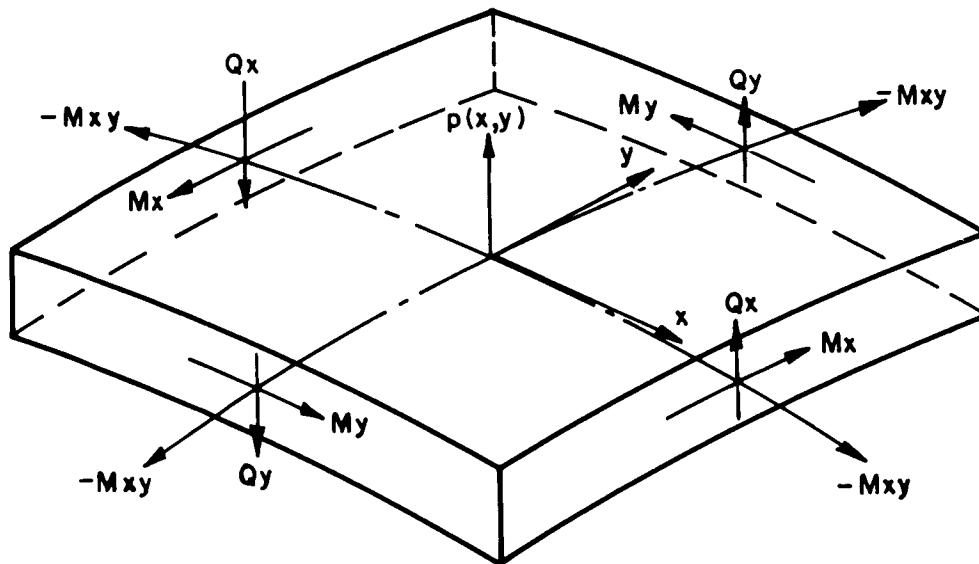
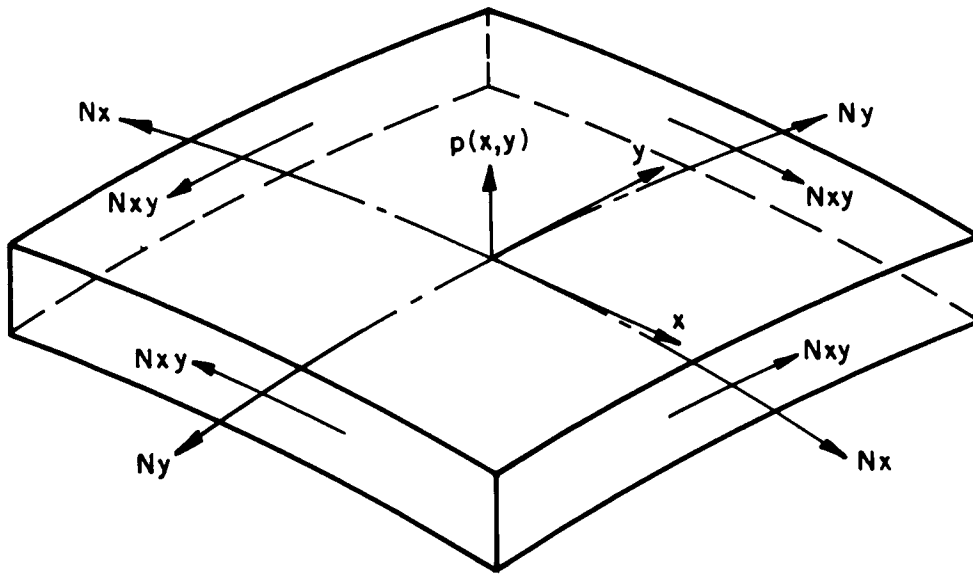


Figure 6. Sign Convention for the Stress Resultants

$$\nabla^4 \varphi - i \delta^2 \nabla_k^2 \varphi = \frac{P(x,y)}{D} \quad (5a)$$

where  $\varphi$ , the complex deflection-stress function, is defined by

$$\varphi = w + \frac{i F}{[(1-\nu^2) AD]^{1/2}} \quad (5b)$$

with

$$\begin{aligned} i &= \sqrt{-1} \\ \delta^4 &= \frac{(1-\nu^2)A}{D} \end{aligned} \quad (5c)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla_k^2 = \frac{1}{R_x} \frac{\partial^2}{\partial y^2} + \frac{1}{R_y} \frac{\partial^2}{\partial x^2}$$

and  $F$  is related to the stress resultants by

$$\begin{aligned} N_x &= \frac{\partial^2 F}{\partial y^2} \\ N_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} \\ N_y &= \frac{\partial^2 F}{\partial x^2} \end{aligned} \quad (5d)$$

Thus, Marguerre's shallow shell theory reduces to the solution of Equation 5a subject to the requisite boundary conditions selected from Equations 4.

## B. BOUNDARY CONDITIONS

The controlling equation for the deformation of the elliptic-paraboloidal shell (Equation 5a) requires the stipulation of four boundary conditions at each

edge to establish a unique solution. Four of these will be specified some distance away from the junction of the flare and dome.

In accordance with the local character of the discontinuity stress, the bending effects attenuate rapidly, and the stress field approaches the membrane field. Thus, away from the intersection, the following conditions are imposed,

at  $x = L_x$  (Figure 7)

$$\begin{aligned}
 N_x &= \text{membrane force} \\
 N_{xy} &= 0 \\
 Q_x + \frac{\partial M_{xy}}{\partial y} &= 0 \\
 M_x &= 0
 \end{aligned}
 \tag{6a}$$

at  $y = L_y$

$$\begin{aligned}
 N_y &= \text{membrane force} \\
 N_{xy} &= 0 \\
 Q_y + \frac{\partial M_{xy}}{\partial x} &= 0 \\
 M_y &= 0
 \end{aligned}
 \tag{6b}$$

The symmetry of the shell leads to the additional restrictions along  $x = 0$

$$\begin{aligned}
 u &= 0 \\
 N_{xy} &= 0 \\
 \frac{\partial w}{\partial x} &= 0 \\
 Q_x + \frac{\partial M_{xy}}{\partial y} &= 0
 \end{aligned}
 \tag{7a}$$

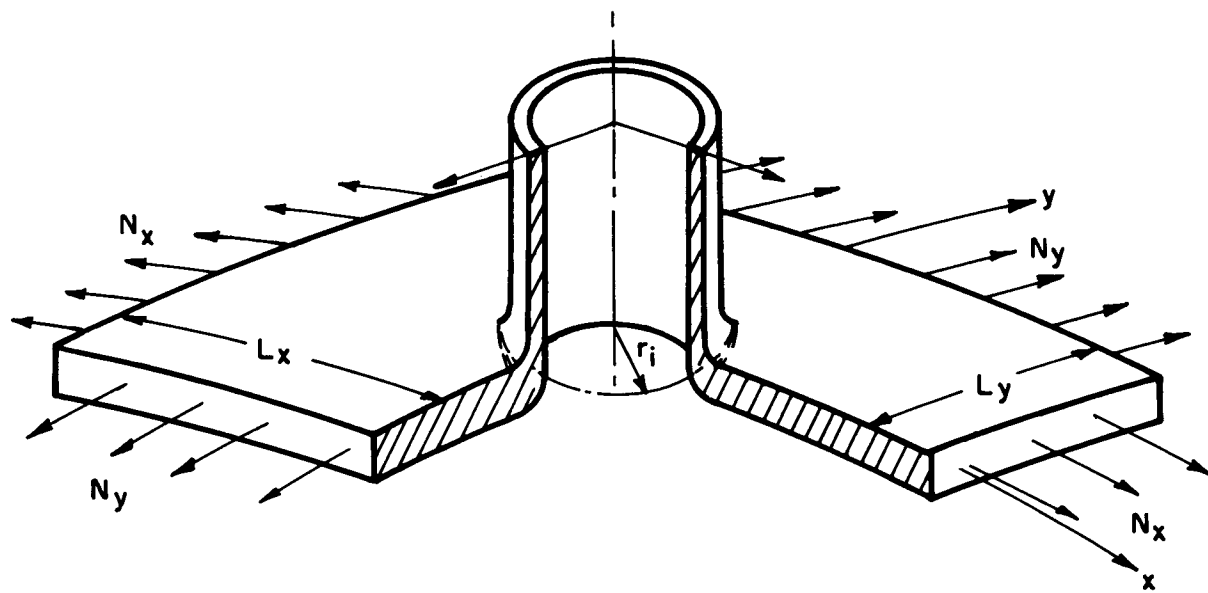


Figure 7. Intersection of the Flared Nozzle with the Elliptic-Paraboloidal Shell

and at  $y = 0$

$$\begin{aligned}
 v &= 0 \\
 N_{xy} &= 0 \\
 \frac{\partial w}{\partial y} &= 0 \\
 Q_y &= \frac{\partial M_{xy}}{\partial x} = 0
 \end{aligned}
 \tag{7b}$$

Furthermore, the continuity conditions at the junction of the flare and the dome require that the displacements, normal slope, and stress resultants of each shell be equal. These compatibility relations will be described in detail in the section dealing with the "point-matching" of the flare to the dome. However, for subsequent use, the Cartesian components of the displacements and forces in the elliptic-paraboloidal shell are transformed to curvilinear components. To this end, the displacements are (Figure 8a)

$$\begin{aligned}
 u_r &= u \cos \theta + v \sin \theta \\
 v_\theta &= -u \sin \theta + v \cos \theta
 \end{aligned}
 \tag{8a}$$

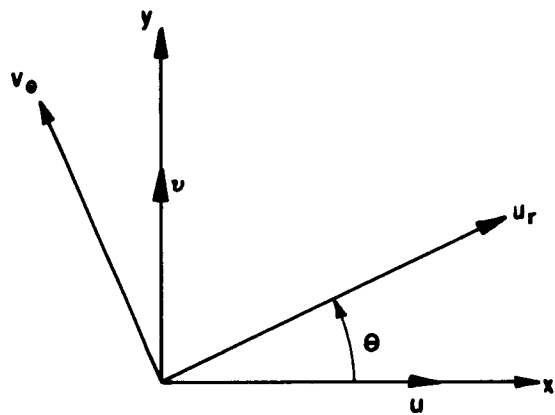
and the in-plane force resultants are (Figure 8b)

$$\begin{aligned}
 N_r &= N_x \cos^2 \theta + N_y \sin^2 \theta + 2 N_{xy} \sin \theta \cos \theta \\
 N_{r\theta} &= (N_y - N_x) \sin \theta \cos \theta + N_{xy} (\cos^2 \theta - \sin^2 \theta) \\
 N_\theta &= N_x \sin^2 \theta + N_y \cos^2 \theta - 2 N_{xy} \sin \theta \cos \theta
 \end{aligned}
 \tag{8b}$$

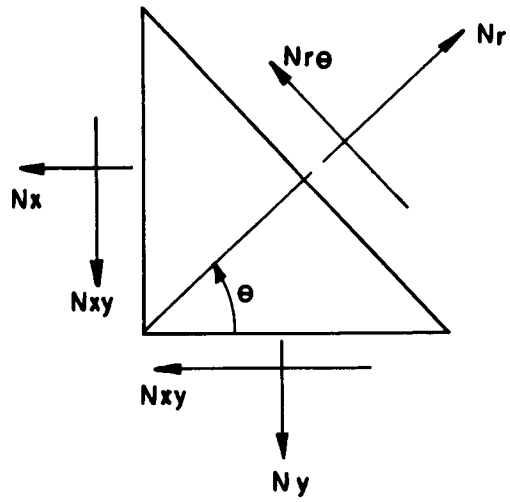
and the moment resultants are (Figure 8c)

$$\begin{aligned}
 M_r &= M_x \cos^2 \theta + M_y \sin^2 \theta + 2 M_{xy} \sin \theta \cos \theta \\
 M_{r\theta} &= (M_y - M_x) \sin \theta \cos \theta + M_{xy} (\cos^2 \theta - \sin^2 \theta) \\
 M_\theta &= M_x \sin^2 \theta + M_y \cos^2 \theta - 2 M_{xy} \sin \theta \cos \theta
 \end{aligned}
 \tag{8c}$$

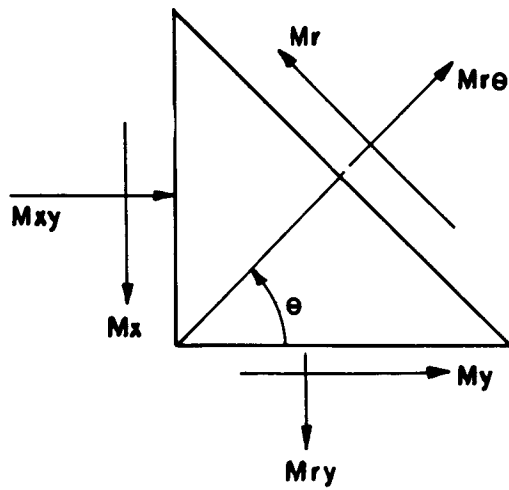




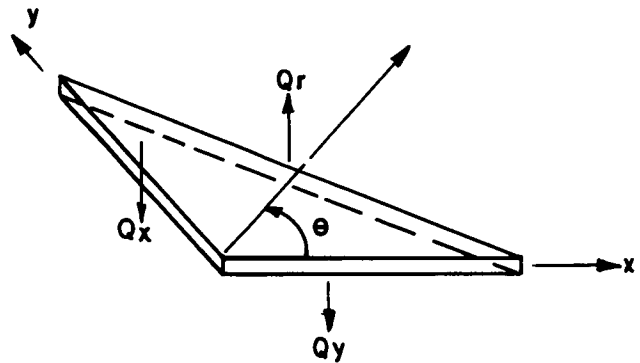
a. displacements



b. force resultants



c. moment resultants



d. transverse shear resultants

Figure 8. Cartesian and Polar Coordinate Representations of the Displacements and Stress Resultants

and the transverse shear resultants are (Figure 8d)

$$\begin{aligned} Q_r &= Q_x \cos \theta + Q_y \sin \theta \\ Q_\theta &= -Q_x \sin \theta + Q_y \cos \theta \end{aligned} \tag{8d}$$

## C. SOLUTION OF THE PROBLEM: ELLIPTIC-PARABOLOIDAL SHELL

### 1. Technical Approach

The mathematical difficulties associated with the simultaneous satisfaction of the field equation (Equation 5a) and the associated boundary conditions (Equations 6, 7, and the interaction conditions) are sizable. Consequently, an approximate solution technique known as "Least-Squares Point-Matching", will be employed, to effect a solution to the composite (interaction) problem.

To prepare the way for the application of the point-matching procedure, two families of functions which satisfy the homogeneous field Equation, 5a, are constructed. Each family is in the form of an infinite series with coefficients which must be determined from the boundary conditions. The solution (applicable to the elliptic-paraboloidal shell) is composed of three distinct parts. These are (1) a particular solution of Equation 5a (suitable for the description of the membrane field), (2) a solution to the homogeneous equation which decays exponentially in the y direction and has a trigonometric variation in the x direction, and (3) a companion solution to the homogeneous equation which decays exponentially in the x direction and has a trigonometric variation in the y direction.

The required solutions will be developed in subsection 2, and the boundary conditions satisfied by each of the solutions will be discussed in subsection 3.

### 2. Component Solutions

#### a. Particular Solution

A particular solution of Equation 5a, which satisfies the far field boundary conditions (Equation 6) as well as the symmetry requirements

(Equations 7), can be constructed from the membrane theory of shells. After neglecting the quantities multiplied by the flexural rigidity,  $D$ , Equation 5a reduces to

$$\nabla_k^2 F = p \quad (9)$$

A suitable choice of  $F(x, y)$  is

$$F(x, y) = p \frac{R_y}{4} \left[ y^2 + \left( 2 - \frac{R_y}{R_x} \right) x^2 \right] \quad (10)$$

and from 5d, the stress resultants are

$$\begin{aligned} N_x &= p \frac{R_y}{2} \\ N_{xy} &= 0 \\ N_y &= p \frac{R_y}{2} \left( 2 - \frac{R_y}{R_x} \right) \end{aligned} \quad (11a)$$

Substituting Equation 11a into 3a, 3b, and 3c, arbitrarily setting the normal deflection,  $w$ , equal to zero, and then integrating, leads to the membrane displacements

$$\begin{aligned} u &= p \frac{R_y}{2Et} \left[ 1 - 2\nu + \nu \frac{R_y}{R_x} \right] x \\ v &= p \frac{R_y}{2Et} \left[ 2 - \nu - \frac{R_y}{R_x} \right] y \end{aligned} \quad (11b)$$

$$w = 0$$

#### b. Complementary Solution

A complementary solution to Equation 5a, which gives rise to stress resultants that decay exponentially as  $y$  increases, can be presented in the form

$$\varphi_n = C e^{-\beta y} \cos \alpha_n x \quad (12a)$$

where  $\alpha_n$  is taken as

$$\alpha_n = \frac{n\pi}{2L_x} ; \quad n = \text{integer} \quad (12b)$$

and  $\beta$  is a characteristic number to be determined and  $C$  is an arbitrary constant. The constant  $L_x$  is the projected semi-span in the  $x$  direction (Figure 7).

Substituting from Equation 12a into 5a leads to the characteristic equation

$$\left(\beta^2 - \alpha_n^2\right)^2 - \frac{i\delta^2}{R_x} \left(\beta^2 - \alpha_n^2\right) - i\delta^2 \left(\frac{1}{R_x} - \frac{1}{R_y}\right) \alpha_n^2 = 0 \quad (13)$$

for the acceptable values of  $\beta$ . They are,

$$\begin{aligned} \beta_{1,2} &= \pm \left(p_1 + i q_1\right) \\ \beta_{3,4} &= \pm \left(p_2 + i q_2\right) \end{aligned} \quad (14)$$

with the definitions

$$p_1^2 = \frac{1}{2} \left(\alpha_n^2 - \frac{\delta^2 b}{2R_x}\right) \left(1 \pm \sqrt{1 + \frac{(1+a)\delta^2}{2R_x \alpha_n^2 - \delta^2 b}}\right)^2 \quad (15a)$$

$$p_2^2 = \frac{1}{2} \left(\alpha_n^2 + \frac{\delta^2 b}{2R_x}\right) \left(1 \pm \sqrt{1 + \frac{(1-a)\delta^2}{2R_x \alpha_n^2 + \delta^2 b}}\right)^2 \quad (15b)$$

$$q_1^2 = \frac{1}{2} \left(\alpha_n^2 - \frac{\delta^2 b}{2R_x}\right) \left(-1 \pm \sqrt{1 + \frac{(1+a)\delta^2}{2R_x \alpha_n^2 - \delta^2 b}}\right)^2 \quad (15c)$$

$$q_2^2 = \frac{1}{2} \left(\alpha_n^2 + \frac{\delta^2 b}{2R_x}\right) \left(-1 \pm \sqrt{1 + \frac{(1-a)\delta^2}{2R_x \alpha_n^2 + \delta^2 b}}\right)^2 \quad (15d)$$

$$a^2 = \frac{1}{2} \left(1 + \sqrt{1 + \frac{16R_x^4 \alpha_n^4}{\delta^4} \left(\frac{1}{R_x} - \frac{1}{R_y}\right)^2}\right) \quad (15e)$$

$$b = - \frac{2 R_x^2 \alpha_n^2}{a \delta^2} \left( \frac{1}{R_x} - \frac{1}{R_y} \right) \quad (15f)$$

Since  $p_1$ ,  $p_2$ ,  $q_1$ , and  $q_2$  are real numbers, the choice of signs preceding the radical is dependent upon the signs of the terms  $\left( \alpha_n^2 - \frac{\delta^2 b}{2R_x} \right)$  and  $\left( \alpha_n^2 + \frac{\delta^2 b}{2R_x} \right)$ .

For example, in computing  $p_1$ , if

$$\alpha_n^2 - \frac{\delta^2 b}{2R_x} > 0 \quad (16a)$$

the positive sign is required, and if

$$\alpha_n^2 - \frac{\delta^2 b}{2R_x} < 0 \quad (16b)$$

the negative sign is required. Furthermore, if

$$\alpha_n^2 - \frac{\delta^2 b}{2R_x} = 0 \quad (16c)$$

equation 15a reduces to

$$p_1^2 = \frac{(1+a) \delta^2}{4 R_x} \quad (16d)$$

The remaining quantities,  $q_1$ ,  $p_2$ , and  $q_2$  are obtained in the same manner.

Employing Equation 14, the expanded form of Equation 12a becomes,

$$\varphi_n = \cos \alpha_n \times \left[ C_1 e^{(p_1+iq_1)y} + C_2 e^{-(p_1+iq_1)y} + C_3 e^{(p_2+iq_2)y} + C_4 e^{-(p_2+iq_2)y} \right] \quad (17)$$

where  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  are complex constants. Since Equation 17 is valid for any  $n$ , the complete solution will be composed of the sum over all values on  $n$ . With this understanding, further discussions will be concerned only with a generic term of the series.

Retaining the terms which give rise to negative exponents in 17, since the stress resultants must decay as  $y$  grows large, and letting

$$\begin{aligned} C_2 &= A_1 + i B_1 \\ C_4 &= A_2 + i B_2 \end{aligned}$$

Equations 17 becomes

$$\begin{aligned} \varphi_n = \cos \alpha_n \times & \left[ \left\{ A_1 \cos q_1 y + B_1 \sin q_1 y \right\} e^{-p_1 y} \right. \\ & + \left\{ A_2 \cos q_2 y + B_2 \sin q_2 y \right\} e^{-p_2 y} \\ & + i \left\{ B_1 \cos q_1 y - A_1 \sin q_1 y \right\} e^{-p_1 y} \\ & \left. + i \left\{ B_2 \cos q_2 y - A_2 \sin q_2 y \right\} e^{-p_2 y} \right] \quad (18) \end{aligned}$$

Equating real and imaginary parts, as prescribed by Equation 5b, leads to the deflection

$$\begin{aligned} w_n = \cos \alpha_n \times & \left[ \left\{ A_1 \cos q_1 y + B_1 \sin q_1 y \right\} e^{-p_1 y} \right. \\ & \left. + \left\{ A_2 \cos q_2 y + B_2 \sin q_2 y \right\} e^{-p_2 y} \right] \quad (19a) \end{aligned}$$

and the stress function

$$\begin{aligned} \frac{F_n}{[(1-\nu^2)AD]^{1/2}} = \cos \alpha_n \times & \left[ \left\{ -A_1 \sin q_1 y + B_1 \cos q_1 y \right\} e^{-p_1 y} \right. \\ & \left. + \left\{ -A_2 \sin q_2 y + B_2 \cos q_2 y \right\} e^{-p_2 y} \right] \quad (19b) \end{aligned}$$

The stress resultants associated with Equations 19 are obtained from the interconnecting relations (Equations 3d, 3e, 3f, and 5d) and are

$$\begin{aligned}
N_x = & [(1-\nu^2)AD]^{1/2} \cos \alpha_n x \left[ e^{-p_1 y} \left\{ \left[ (q_1^2 - p_1^2) A_1 + 2q_1 p_1 B_1 \right] \sin q_1 y \right. \right. \\
& + \left. \left. \left[ 2q_1 p_1 A_1 - (q_1^2 - p_1^2) B_1 \right] \cos q_1 y \right\} \right. \\
& + e^{-p_2 y} \left\{ \left[ (q_2^2 - p_2^2) A_2 + 2q_2 p_2 B_2 \right] \sin q_2 y \right. \\
& + \left. \left. \left[ 2q_2 p_2 A_2 - (q_2^2 - p_2^2) B_2 \right] \cos q_2 y \right\} \right] \quad (20a)
\end{aligned}$$

$$\begin{aligned}
N_{xy} = & \alpha_n [(1-\nu^2)AD]^{1/2} \sin \alpha_n x \left[ e^{-p_1 y} \left\{ (p_1 A_1 - q_1 B_1) \sin q_1 y \right. \right. \\
& - \left. \left. (q_1 A_1 + p_1 B_1) \cos q_1 y \right\} \right. \\
& + e^{-p_2 y} \left\{ (p_2 A_2 - q_2 B_2) \sin q_2 y - (q_2 A_2 + p_2 B_2) \cos q_2 y \right\} \left. \right] \quad (20b)
\end{aligned}$$

$$\begin{aligned}
N_y = & -\alpha_n^2 [(1-\nu^2)AD]^{1/2} \cos \alpha_n x \left[ e^{-p_1 y} \left\{ -A_1 \sin q_1 y + B_1 \cos q_1 y \right\} \right. \\
& + e^{-p_2 y} \left\{ -A_2 \sin q_2 y + B_2 \cos q_2 y \right\} \left. \right] \quad (20c)
\end{aligned}$$

$$\begin{aligned}
M_x = & D \cos \alpha_n x \left[ e^{-p_1 y} \left\{ \left[ (\alpha_n^2 - \nu [p_1^2 - q_1^2]) A_1 + 2\nu p_1 q_1 B_1 \right] \cos q_1 y + \right. \right. \\
& + \left. \left. \left[ -2\nu p_1 q_1 A_1 + (\alpha_n^2 - \nu [p_1^2 - q_1^2]) B_1 \right] \sin q_1 y \right\} \right. \\
& + e^{-p_2 y} \left\{ \left[ (\alpha_n^2 - \nu [p_2^2 - q_2^2]) A_2 + 2\nu p_2 q_2 B_2 \right] \cos q_2 y \right. \\
& + \left. \left. \left[ -2\nu p_2 q_2 A_2 + (\alpha_n^2 - \nu [p_2^2 - q_2^2]) B_2 \right] \sin q_2 y \right\} \right] \quad (20d)
\end{aligned}$$

$$\begin{aligned}
M_y = & -D \cos \alpha_n x \left[ e^{-p_1 y} \left\{ \left[ (p_1^2 - q_1^2 - \nu \alpha_n^2) A_1 - 2p_1 q_1 B_1 \right] \cos q_1 y + \right. \right. \\
& \left. \left[ 2p_1 q_1 A_1 + (p_1^2 - q_1^2 - \nu \alpha_n^2) B_1 \right] \sin q_1 y \right\} + \\
& e^{-p_2 y} \left\{ \left[ (p_2^2 - q_2^2 - \nu \alpha_n^2) A_2 - 2p_2 q_2 B_2 \right] \cos q_2 y + \right. \\
& \left. \left[ 2p_2 q_2 A_2 + (p_2^2 - q_2^2 - \nu \alpha_n^2) B_2 \right] \sin q_2 y \right\} \right] \quad (20e)
\end{aligned}$$

$$\begin{aligned}
M_{xy} = & (1-\nu) D \alpha_n \sin \alpha_n x \left[ e^{-p_1 y} \left\{ (-p_1 A_1 + q_1 B_1) \cos q_1 y - \right. \right. \\
& \left. (q_1 A_1 + p_1 B_1) \sin q_1 y \right\} + \\
& e^{-p_2 y} \left\{ (-p_2 A_2 + q_2 B_2) \cos q_2 y - (q_2 A_2 + p_2 B_2) \sin q_2 y \right\} \right] \quad (20f)
\end{aligned}$$

The transverse shear resultants are, from Equations 2c and 2d,

$$\begin{aligned}
Q_x = & \alpha_n D \sin \alpha_n x \left[ e^{-p_1 y} \left\{ \left( \left[ p_1^2 - q_1^2 - \alpha_n^2 \right] A_1 - 2p_1 q_1 B_1 \right) \cos q_1 y \right. \right. \\
& \left. \left. + \left( 2p_1 q_1 A_1 + \left[ p_1^2 - q_1^2 - \alpha_n^2 \right] B_1 \right) \sin q_1 y \right\} + \right. \\
& e^{-p_2 y} \left\{ \left( \left[ p_2^2 - q_2^2 - \alpha_n^2 \right] A_2 - 2p_2 q_2 B_2 \right) \cos q_2 y \right. \\
& \left. \left. + \left( 2p_2 q_2 A_2 + \left[ p_2^2 - q_2^2 - \alpha_n^2 \right] B_2 \right) \sin q_2 y \right\} \right] \quad (21a)
\end{aligned}$$



$$\begin{aligned}
Q_y = & D \cos \alpha_n x \left[ e^{-p_1 y} \left\{ \left( \left[ p_1^3 - 3p_1 q_1^2 - p_1 \alpha_n^2 \right] A_1 + \left[ q_1^3 - 3p_1^2 q_1 + q_1 \alpha_n^2 \right] B_1 \right) \cos q_1 y \right. \right. \\
& + \left. \left( - \left[ q_1^3 - 3p_1^2 q_1 + q_1 \alpha_n^2 \right] A_1 + \left[ p_1^3 - 3p_1 q_1^2 - p_1 \alpha_n^2 \right] B_1 \right) \sin q_1 y \right\} \\
& + e^{-p_2 y} \left\{ \left( \left[ p_2^3 - 3p_2 q_2^2 - p_2 \alpha_n^2 \right] A_2 + \left[ q_2^3 - 3p_2^2 q_2 + q_2 \alpha_n^2 \right] B_2 \right) \cos q_2 y \right. \\
& + \left. \left( - \left[ q_2^3 - 3p_2^2 q_2 + q_2 \alpha_n^2 \right] A_2 + \left[ p_2^3 - 3p_2 q_2^2 - p_2 \alpha_n^2 \right] B_2 \right) \sin q_2 y \right\} \right] \quad (21b)
\end{aligned}$$

The displacements  $u$  and  $v$  are evaluated from Equations 3a, 3b, and 3c where  $N_x$ ,  $N_{xy}$ ,  $N_y$ , and  $w$  are now known functions. Solving the first order partial differential equations results in

$$\begin{aligned}
u = & \left[ \frac{D}{(1-\nu^2)A} \right]^{1/2} \frac{\sin \alpha_n x}{\alpha_n} \left[ e^{-p_1 y} \left\{ \left[ \left( q_1^2 - p_1^2 - \alpha_n^2 \nu \right) A_1 + \left( 2p_1 q_1 - \frac{\{(1-\nu^2)A\}}{D R_x} \right)^{1/2} B_1 \right] \sin q_1 y \right. \right. \\
& + \left. \left[ \left( 2q_1 p_1 - \frac{\{(1-\nu^2)A\}}{D R_x} \right)^{1/2} A_1 - \left( q_1^2 - p_1^2 - \alpha_n^2 \nu \right) B_1 \right] \cos q_1 y \right\} + \\
& e^{-p_2 y} \left\{ \left[ \left( q_2^2 - p_2^2 - \alpha_n^2 \nu \right) A_2 + \left( 2p_2 q_2 - \frac{\{(1-\nu^2)A\}}{D R_x} \right)^{1/2} B_2 \right] \sin q_2 y \right. \\
& + \left. \left[ \left( 2p_2 q_2 - \frac{\{(1-\nu^2)A\}}{D R_x} \right)^{1/2} A_2 - \left( q_2^2 - p_2^2 - \alpha_n^2 \nu \right) B_2 \right] \cos q_2 y \right\} \right] \quad (22a)
\end{aligned}$$

$$\begin{aligned}
v = & \left[ \frac{D}{(1-\nu^2)A} \right]^{1/2} \frac{\cos \alpha_n x}{\alpha_n} \left[ e^{-p_1 y} \left\{ \left[ (p_1^3 - 3p_1 q_1^2 + \alpha_n^2 p_1 \nu + q_1 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} - 2(1+\nu) \alpha_n^2 p_1) A_1 \right. \right. \right. \\
& + (q_1^3 - 3p_1^2 q_1 - q_1 \alpha_n^2 \nu + p_1 \left\{ \frac{(1-\nu^2)A}{R_x^2 D} \right\}^{1/2} + 2(1+\nu) \alpha_n^2 q_1) B_1 \left. \left. \left. \right] \sin q_1 y \right. \right. \\
& + \left[ (q_1^3 - 3q_1 p_1^2 - q_1 \alpha_n^2 \nu + p_1 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} + 2(1+\nu) \alpha_n^2 q_1) A_1 \right. \\
& - (p_1^3 - 3q_1^2 p_1 + p_1 \alpha_n^2 \nu + q_1 \left\{ \frac{(1-\nu^2)A}{R_x^2 D} \right\}^{1/2} - 2(1+\nu) \alpha_n^2 p_1) B_1 \left. \left. \left. \right] \cos q_1 y \right\} \right. \\
& + e^{-p_2 y} \left\{ \left[ (p_2^3 - 3p_2 q_2^2 + \alpha_n^2 p_2 \nu + q_2 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} - 2(1+\nu) \alpha_n^2 p_2) A_2 \right. \right. \right. \\
& + (q_2^3 - 3p_2^2 q_2 - q_2 \alpha_n^2 \nu + p_2 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} + 2(1+\nu) \alpha_n^2 q_2) B_2 \left. \left. \left. \right] \sin q_2 y \right. \right. \\
& + \left[ (q_2^3 - 3q_2 p_2^2 - q_2 \alpha_n^2 \nu + p_2 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} + 2(1+\nu) \alpha_n^2 q_2) A_2 \right. \\
& - (p_2^3 - 3q_2^2 p_2 + p_2 \alpha_n^2 \nu + q_2 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} - 2(1+\nu) \alpha_n^2 p_2) B_2 \left. \left. \left. \right] \cos q_2 y \right\} \right. \\
& \left. \left. \left. \right] \right]
\end{aligned}$$

(22b)

Thus, all the field quantities derivable from the assumed solution (Equation 12a) are now known for integral values of  $n$ . The special case of  $n = 0$  must be examined independently. This case corresponds to a one dimensional problem in the coordinate  $y$ . Therefore, Equation (5a) reduces to (with  $p = 0$ )

$$\frac{d^4 \varphi_0}{dy^4} - i \frac{\delta^2}{R_x} \frac{d^2 \varphi_0}{dy^2} = 0 \quad (23a)$$

and  $\varphi_0$  is

$$\varphi_0 = C_0' + C_1' y + C_2' e^{-\beta_0(1+i)y} + C_3' \beta_0(1+i)y \quad (23b)$$

where

$$\beta_0 = \left( \frac{\delta^2}{2R_x} \right)^{1/2} \quad (23c)$$

and  $C_0'$ ,  $C_1'$ ,  $C_2'$  and  $C_3'$  are complex constants.

Again, since the field quantities are to decay exponentially, only the negative exponential is retained. Furthermore, the rigid body constant  $C_0'$  is also omitted. Therefore, analogous to the case  $n \neq 0$ , the displacements and stress resultants for  $n = 0$  are (with  $C_2' = A_0 + i B_0$ )

$$w_0 = (A_0 \cos \beta_0 y + B_0 \sin \beta_0 y) e^{-\beta_0 y} \quad (24a)$$

$$\frac{F_0}{[(1-\nu^2)AD]^{1/2}} = (-A_0 \sin \beta_0 y + B_0 \cos \beta_0 y) e^{-\beta_0 y} \quad (24b)$$

$$N_x = [(1-\nu^2)AD]^{1/2} 2\beta_0^2 e^{-\beta_0 y} (A_0 \cos \beta_0 y + B_0 \sin \beta_0 y) \quad (24c)$$

$$N_{xy} = 0 \quad (24d)$$

$$N_y = 0 \quad (24e)$$

$$M_x = -2\nu D \beta_o^2 e^{-\beta_o y} (A_o \sin \beta_o y - B_o \cos \beta_o y) \quad (24f)$$

$$M_{xy} = 0 \quad (24g)$$

$$M_y = -2D \beta_o^2 e^{-\beta_o y} (A_o \sin \beta_o y - B_o \cos \beta_o y) \quad (24h)$$

$$Q_x = 0 \quad (24i)$$

$$Q_y = -2D \beta_o^3 e^{-\beta_o y} \left( -[A_o - B_o] \sin \beta_o y + [A_o + B_o] \cos \beta_o y \right) \quad (24j)$$

$$u_o = 0 \quad (24k)$$

$$v_o = \frac{e^{-\beta_o y}}{2\beta_o} \left\{ (\sin \beta_o y - \cos \beta_o y) A_o - (\cos \beta_o y + \sin \beta_o y) B_o + \text{constant} \right\} \quad (24l)$$

All quantities related to the product type solution (Equation 12a) displaying exponentially decaying stress fields in the y direction have now been evaluated. The companion solution, i.e., the solution which gives rise to an exponentially decaying stress field in the x direction, can be developed in a similar manner. Here the stress function is taken as

$$\Phi = \sum_n e^{\bar{\beta} x} \cos \bar{\alpha}_m y \quad (25a)$$

where

$$\alpha_m = \frac{m\pi}{2L_y} \quad (25b)$$

Proceeding as before, Equations 12a to 24 would yield the similar relations (Equations 13, 15, and 17 through 24), with the following changes:

replace:

$$\begin{array}{ccc} R_x & \text{by} & R_y \\ R_y & \downarrow & R_x \\ L_x & & L_y \\ L_y & & L_x \\ x & & y \\ y & & x \\ u_n & & v_m \\ v_n & & u_m \\ w_n & & w_m \\ M_x & & M_y \\ M_y & & M_x \\ M_{xy} & & M_{xy} \\ N_x & & N_y \\ N_y & & N_x \\ N_{xy} & & N_{xy} \\ p_1 & & p_3 \\ p_2 & & p_4 \\ q_1 & & q_3 \\ q_2 & & q_4 \\ \alpha_n & & \bar{\alpha}_m \\ A_1 & & A_3 \\ A_2 & & A_4 \\ B_1 & & B_3 \\ B_2 & & B_4 \end{array}$$

(26)

The equations resulting from this cyclic interchange (Equation 26) will not be listed since they are employed symbolically in subsequent analysis. That is, the first solution (Equations 12a to 24) is tabulated and the companion solution can always be obtained by the cyclic interchange listed in Equation 26.

### 3. Satisfaction of the Boundary Conditions

It is not possible to satisfy all the boundary conditions continuously with the type solutions employed herein. Therefore, a solution procedure will be used, which satisfies the boundary conditions at discrete points. An efficient function to be employed in this procedure can be constructed from each of the component complementary solutions. To this end, each family (with unknown constants such as  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$ ) is made to satisfy as many boundary conditions as possible while leaving one free constant. Thus, three of the four unknowns are evaluated from some of the boundary conditions, and the fourth unknown will be evaluated in accordance with the remaining conditions by the "Least-Squares Point-Matching" procedure.

The symmetry conditions with respect to the x axis, Equation 7a, are identically satisfied for all values of y and n (including  $n = 0$ ) by virtue of the trigonometric variation. Similar requirements with respect to the y direction Equation 7b, lead to the relations, for all values of x and  $n \geq 1$ ,

$$\begin{aligned}
 & \left[ q_1^3 - 3q_1 p_1^2 + (2+\nu) \alpha_n^2 q_1 + p_1 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} \right] A_1 \\
 & - \left[ p_1^3 - 3q_1^2 p_1 - (2+\nu) \alpha_n^2 p_1 + q_1 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} \right] B_1 \\
 & + \left[ q_2^3 - 3q_2 p_2^2 + (2+\nu) \alpha_n^2 q_2 + p_2 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} \right] A_2 \\
 & - \left[ p_2^3 - 3q_2^2 p_2 - (2+\nu) \alpha_n^2 p_2 + q_2 \left\{ \frac{(1-\nu^2)A}{DR_x} \right\}^{1/2} \right] B_2 = 0 \quad (27a)
 \end{aligned}$$

$$q_1 A_1 + p_1 B_1 + q_2 A_2 + p_2 B_2 = 0 \quad (27b)$$

$$-p_1 A_1 + q_1 B_1 - p_2 A_2 + q_2 B_2 = 0 \quad (27c)$$

$$\begin{aligned} & \left[ p_1^3 - 3p_1 q_1^2 - (2-\nu) \alpha_n^2 p_1 \right] A_1 + \left[ q_1^3 - 3p_1^2 q_1 + (2-\nu) \alpha_n^2 q_1 \right] B_1 \\ & + \left[ p_2^3 - 3p_2 q_2^2 - (2-\nu) \alpha_n^2 p_2 \right] A_2 + \left[ q_2^3 - 3p_2^2 q_2 + (2-\nu) \alpha_n^2 q_2 \right] B_2 = 0 \quad (27d) \end{aligned}$$

Making use of Equations 15, 27b and 27c in Equations 27a and 27d and simplifying, results in the following compact set of equations

$$(q_1 b - p_1 a) A_1 + (p_1 b + q_1 a) B_1 - (q_2 b - p_2 a) A_2 - (p_2 b + q_2 a) B_2 = 0 \quad (28a)$$

$$q_1 A_1 + p_1 B_1 + q_2 A_2 + p_2 B_2 = 0 \quad (28b)$$

$$-p_1 A_1 + q_1 B_1 - p_2 A_2 + q_2 B_2 = 0 \quad (28c)$$

$$-(p_1 b + q_1 a) A_1 + (q_1 b - p_1 a) B_1 + (p_2 b + q_2 a) A_2 - (q_2 b - p_2 a) B_2 = 0 \quad (28d)$$

The above system of homogeneous equations is linearly independent and therefore has the trivial solution,  $A_1 = A_2 = B_1 = B_2 = 0$ . Thus, the form of the solution does not admit to termwise satisfaction of the symmetry conditions. However, in the solution procedure to be employed, all the boundary conditions which are not satisfied continuously will be satisfied at specified points. With this in mind, Equation 28d is arbitrarily omitted and the remaining three equations are solved in the terms of  $A_1$ . Consequently, all the field quantities will be given in terms of the undetermined constant  $A_1$ . This constant,  $A_1$ , will be evaluated by "Least-Squares Point-Matching". One of the conditions to be matched is the vanishing of the transverse shear force at  $y = 0$ , which is required to satisfy Equation 28d.

A similar argument for the companion solution (Equation 26) results in an additional constant  $A_3$ , which must be evaluated in the same manner as was employed for  $A_1$ .

For the special case  $n = 0$ , the symmetry conditions lead to the non-vanishing field equations

$$w = A_0 (\cos \beta_0 y + \sin \beta_0 y) e^{-\beta_0 y} \quad (29a)$$

$$N_x = [(1-\nu^2)AD]^{1/2} 2\beta_0^2 e^{-\beta_0 y} A_0 (\cos \beta_0 y + \sin \beta_0 y) \quad (29b)$$

$$M_x = -2\nu D\beta_0^2 e^{-\beta_0 y} A_0 (\sin \beta_0 y - \cos \beta_0 y) \quad (29c)$$

$$M_y = -2D\beta_0^2 e^{-\beta_0 y} A_0 (\sin \beta_0 y - \cos \beta_0 y) \quad (29d)$$

$$Q_y = -4D\beta_0^3 e^{-\beta_0 y} A_0 \cos \beta_0 y \quad (29e)$$

$$v = \frac{e^{-\beta_0 y}}{\beta_0} A_0 [1 - \cos \beta_0 y] \quad (29f)$$

where again,  $A_0$  is the undetermined coefficient and is to be evaluated by point-matching methods.

Thus, the dome solutions may be combined with undetermined parameters  $A_i$  and  $B_j$  as follows:

$$u_D = \sum_{i=0}^I A_i (\bar{u}_{1i}(x, y) \cos \theta + \bar{v}_{1i}(x, y) \sin \theta) + \sum_{j=0}^J B_j (\bar{u}_{2j} \cos \theta + \bar{v}_{2j} \sin \theta) + u_p \quad (30a)$$

$$v_D = \sum_{i=0}^I A_i (\bar{u}_{1i}(x, y) \sin \theta + \bar{v}_{1i} \cos \theta) + \sum_{j=0}^J B_j (-\bar{u}_{2j} \sin \theta + \bar{v}_{2j} \cos \theta) + v_p \quad (30b)$$

$$w_D = \sum_{i=0}^I A_i w_{1i}(x, y) + \sum_{j=0}^J B_j w_{2j}(x, y)$$



$$M_r = \sum_{i=0}^I A_i \left( m_{1ix} \frac{1+\cos 2\theta}{2} + m_{1iy} \frac{1-\cos 2\theta}{2} + m_{1ixy} \sin 2\theta \right) \\ + \sum_{j=0}^J B_j \left( m_{2jx} \frac{1+\cos 2\theta}{2} + m_{2jy} \frac{1-\cos 2\theta}{2} + m_{2jxy} \sin 2\theta \right)$$

$$N_r = \sum_{i=0}^I A_i \left( n_{1ix} \frac{1+\cos 2\theta}{2} + n_{1iy} \frac{1-\cos 2\theta}{2} + n_{1ixy} \sin 2\theta \right) \\ + \sum_{j=0}^J B_j \left( n_{2jx} \frac{1+\cos 2\theta}{2} + n_{2jy} \frac{1-\cos 2\theta}{2} + n_{2jxy} \sin 2\theta \right) + N_p(x, y)$$

$$\bar{N}_{r\theta} = \sum_{i=0}^I A_i \left( n_{1iy} - n_{1ix} \frac{\sin 2\theta}{2} + n_{1ixy} \cos 2\theta \right) \\ + \sum_{j=0}^J B_j \left( n_{2jy} - n_{2jx} \frac{\sin 2\theta}{2} + n_{2jxy} \cos 2\theta \right) + \bar{N}_p \quad (30f)$$

$$\bar{Q}_r = \sum_{i=0}^I A_i \left( q_{1ix} \cos \theta + q_{1iy} \sin \theta - 2m_{1ixy} \sin 2\theta + (m_{1ix} - m_{1iy}) \cos 2\theta \right) \\ + \sum_{j=0}^J B_j \left( q_{2jx} \cos \theta + q_{2jy} \sin \theta - 2m_{2jxy} \sin 2\theta + (m_{2jx} - m_{2jy}) \cos 2\theta \right) \quad (30g)$$

$$\bar{\Phi}_r = \sum_{i=0}^I A_i \left( \varphi_{1ix} \cos \theta + \varphi_{1iy} \sin \theta \right) + \sum_{j=0}^J B_j \left( \varphi_{2jx} \cos \theta + \varphi_{2jy} \sin \theta \right) \quad (30h)$$

where the  $\bar{u}_{1i}$ ,  $\bar{u}_{2j}$ ,  $\bar{v}_{1i}$ , ...,  $\varphi_{2jy}$  are the unit solutions, functions of  $x$  and  $y$  and  $y(\theta = \tan^{-1} \frac{y}{x})$ , as developed in subsection C-2. For example,  $\bar{u}_{1i}$  used in

Equation (30a) is the  $u$  of (22a), whereas the  $\bar{u}_{2j}$  of (30a) is the  $u$  of (22a) after the replacements indicated by Equation (26) have been made.

Although the solution, characterized by negative exponentials in the  $y$  direction, gives rise to a decaying stress field, the companion solution will have a trigonometric variation in the same direction. Therefore, the far field boundary conditions, Equations 6, will be satisfied in a least-squares sense. For example, consider the first condition of 6.2. At  $x = L_x$ , the particular solution satisfies the boundary condition and the remaining two solutions must vanish. For  $n$  equal to an odd integer,  $N_x$  given by Equation 20a vanishes identically. However, this will not be the case for  $n$  equal to even integers. The companion solution will decay exponentially for all values of the index. The vanishing of the complementary solutions at  $x = L_x$  must be accomplished point-wise.

The general description presented here is characteristic of all the far field boundary conditions (Equations 6). Satisfaction of these conditions as well as the shell joining will be described in Section IV.

SECTION IV  
FLARE-DOME SHELL POINT-MATCHING

A. INTRODUCTION

As has been shown, solutions for the dome and flare shell equations of equilibrium and compatibility consist of truncated series [Section II Equations (39) and Section III Equations (30)], each term of which satisfies the homogeneous differential equations and is multiplied by an undetermined coefficient, and particular solutions which satisfy the nonhomogeneous shell equations.

The coefficients of the solutions to the homogeneous equations must now be determined to obtain expressions which satisfy the boundary conditions at the top edge of the cylinder-flare, the flare-dome junction, and the far field of the dome. In addition, the resulting solutions must satisfy conditions of symmetry about the x and y axes. The symmetry conditions follow from the assumption of constant (but different from one another) curvatures in the principal dome directions.

B. POINT-MATCHING

To determine the  $A_1$  and  $B_1$  of Equations (30) (Section II), boundary and symmetry conditions will be satisfied pointwise. For example, at the flare-dome intersection,

compatibility requires:

$$\begin{aligned}
 U_F &= U_D \\
 V_F &= V_D \\
 W_F &= W_D \\
 \phi_S &= \phi_r
 \end{aligned}
 \tag{1}$$

and equilibrium requires:

$$\begin{aligned}
 N_S &= N_r \\
 N_{S\theta} &= N_{r\theta} \\
 \bar{Q}_S &= \bar{Q}_r \\
 M_S &= -M_r
 \end{aligned}
 \tag{2}$$

Therefore, select one point,  $r_i$ ,  $\theta_t$  (see Figure 9) at the intersection curve of the two shells, and generate eight linear equations in the A, B, C for each point.

In a similar manner, point-matching the dome membrane conditions at  $x = L_x$  (Figure 9) and various  $y$ 's:

$$\begin{aligned} N_x &= \text{membrane force} \\ N_{xy} &= 0 \\ Q_x &= 0 \\ M_x &= 0 \end{aligned} \tag{3}$$

at  $y = L_y$  (Figure 9) and various  $x$ 's:

$$\begin{aligned} N_y &= \text{membrane force} \\ N_{xy} &= 0 \\ \bar{Q}_y &= 0 \\ M_y &= 0 \end{aligned} \tag{4}$$

leads to four simultaneous equations in the  $A_i$  and  $B_j$  for each point matched.

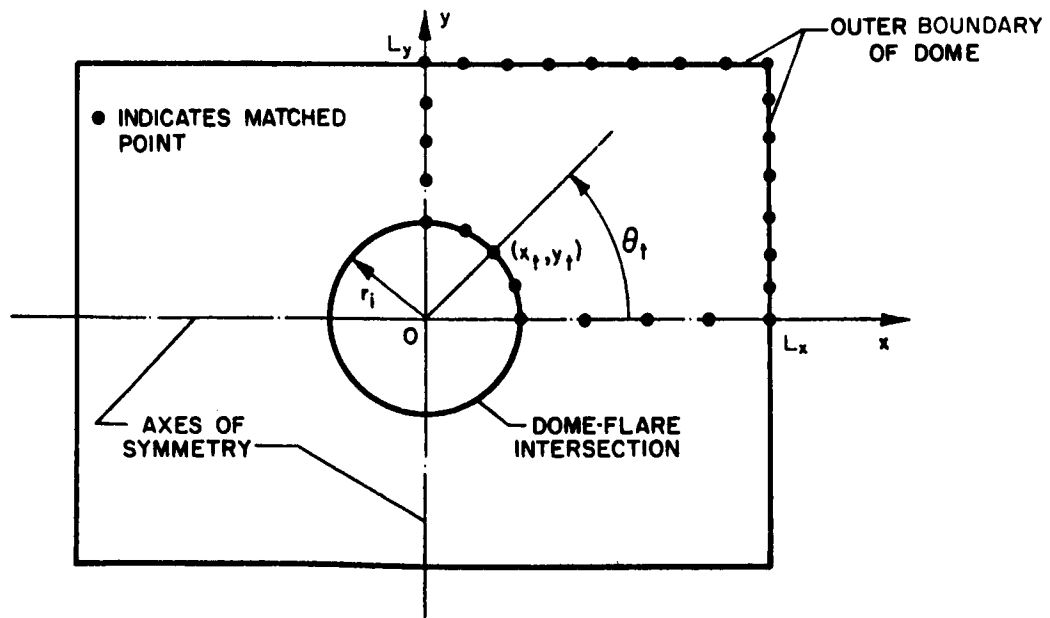


Figure 9. Typical Array of Matched Points

Symmetry conditions about the x and y axes, not automatically satisfied by the trial functions are point-matched:

$$\text{at } x = 0 \text{ and various } y\text{'s, } \bar{Q}_x = 0 \quad (5)$$

while

$$\text{at } y = 0 \text{ and various } x\text{'s, } \bar{Q}_y = 0 \quad (6)$$

Conditions at the cylinder edge need not be point-matched, since the  $M_S, N_S, N_{S\theta}, \bar{Q}_S$  automatically satisfy membrane boundary conditions created by internal pressurization.

By judicious selection of the points matched, an equal number of unknowns ( $A_i, B_j$  and  $C_k^n$ ) and linear point-matched algebraic equations may be obtained. This can be achieved by the appropriate choice of the maximum Fourier index  $\bar{n}$  and the maximum dome summation indices I and J. These are then solved simultaneously for the constants and reinserted in the series solutions of Sections II and III. However, if the number of unknowns is less than the number of equations, a solution may be obtained in the least-square sense. This is explained in the following subsection.

### C. LEAST-SQUARES

Let the system of linear equations, generated as described above, be denoted as

$$\begin{aligned} C_{11} x_1 + C_{12} x_2 + C_{13} x_3 + \dots + C_{1n} x_n - T_1 &= 0 \\ C_{21} x_1 + C_{22} x_2 + C_{23} x_3 + \dots + C_{2n} x_n - T_2 &= 0 \\ C_{31} x_1 + C_{32} x_2 + C_{33} x_3 + \dots + C_{3n} x_n - T_3 &= 0 \\ &\dots \\ C_{m1} x_1 + C_{m2} x_2 + C_{m3} x_3 + \dots + C_{mn} x_n - T_m &= 0 \end{aligned} \quad (7)$$

where  $m > n$ , and  $x_1, x_2 \dots x_n$  are the unknowns.

The equations (7) may be expressed in matrix form as

$$[C] \{x\} - \{T\} = 0 \quad (8)$$

where  $[C]$  is the  $m \times n$  matrix of the coefficients  $C_{11}, C_{12}, \dots, C_{mn}$ ;  $\{x\}$  is the column matrix  $x_1, x_2, \dots, x_n$ , and  $\{T\}$  is the column matrix  $T_1, T_2, \dots, T_m$ . The well-known least squares procedure is to pre-multiply the equation (8) by the transpose of  $[C]$ , say  $[C]'$ , that is, form the equation

$$[C]' [C] \{x\} - [C]' \{T\} = 0 \quad (9)$$

The resulting system of equations (9) is  $n$  equations in  $n$  unknowns. Furthermore, if  $[C]$  is of rank  $n$  and  $[C]' [C]$  is of rank  $n$ , then the system of equations has a unique solution. It shall now be established that the above procedure minimizes the sum of the squares of the residuals,  $\sum_{i=1}^m e_i^2$  where

$$e_i = C_{i1} x_1 + C_{i2} x_2 + \dots + C_{in} x_n - T_i, \quad (10)$$

thereby justifying the name "least-squares."

For the purposes of economy of notation, we rewrite (10) as

$$e_i = C_{ik} x_k - T_i \quad \begin{array}{l} (k = 1, 2, \dots, n) \\ (i = 1, 2, \dots, m; m > n) \end{array} \quad (11)$$

where the repeated subscript indicates summation and the rank of the  $[C]$  matrix is  $n$ .

Let  $y$  be defined by

$$y = e_i e_i \quad (12)$$

For stationary values,

$$\frac{\partial y}{\partial x_j} = 0 = 2 e_i \frac{\partial e_i}{\partial x_j} = 2 e_i C_{ij} \quad (j = 1, 2, \dots, n) \quad (13)$$

From (11) and (13),

$$C_{ik} C_{ij} x_k - T_i C_{ij} = 0 \quad (14)$$

Now  $C_{ik} C_{ij}$  is the  $kj$  element of  $[C]'$   $[C]$  which is taken to be of rank  $n$ , that is, it is nonsingular. Let  $x_k^0$  be the unique solution of (14).

Now consider  $y = y(x_k)$ . Then by Taylor's expansion about  $x_k^0$ ,

$$y(x_k) = y(x_k^0) + \left( \frac{\partial y}{\partial x_k} \right)_0 (x_k - x_k^0) + \left( \frac{\partial^2 y}{\partial x_j \partial x_k} \right)_0 (x_j - x_j^0) (x_k - x_k^0) \quad (15)$$

The series terminates because all higher derivatives are zero.

Therefore, since the first partial derivatives are zero,

$$y(x_k) - y(x_k^0) = \left( \frac{\partial^2 y}{\partial x_j \partial x_k} \right)_0 (x_j - x_j^0) (x_k - x_k^0) \\ y(x_k) - y(x_k^0) = 2 C_{ik} C_{ij} \delta_j \delta_k \quad (16)$$

where  $\delta_j = x_j - x_j^0$  (17)

$$y(x_k) - y(x_k^0) = 2 (C_{ij} \delta_j) (C_{ik} \delta_k) \\ = \text{sum of squares} \geq 0$$

Thus,  $y(x_k) - y(x_k^0) \geq 0$

But  $C_{ik} \delta_k = 0$  implies that  $\delta_k = 0$ , since the rank of  $C$  is  $n$ . Therefore

$$y(x_k) - y(x_k^0) > 0 \quad \text{if } x_k - x_k^0 \neq 0$$

and consequently  $y(x_k^0)$  is an absolute minimum.

SECTION V  
COMPUTER PROGRAM

A. INTRODUCTION

A computer program which employs the analyses presented in Sections II through IV has been developed. It consists of six separate parts as described below:

- Part 1: Computer unit solutions of the flare equations. Output from this portion of the program includes five tapes to be input in subsequent parts of the program.
- Part 2: Obtains the point-matching matrix  $[a_{ij}]$ , and right-hand side  $\{y_i\}$  for the system  $[a_{ij}]\{x_i\} = \{y_i\}$ , where the  $x_i$  are the unknowns  $A_i$ ,  $B_j$ , and  $C_k^n$ . Output from this part is used as input to Part 3.
- Part 3: Computes the least-square matrix if  $[a_{ij}]$  is not a square matrix (if it is square, this step is omitted), normalizes the point-matching equations, orders them to facilitate more accurate solutions, and solves for the  $A_i$ ,  $B_j$ , and  $C_k^n$ , using double precision. These are reproduced on tape which is used in Parts 4, 5, and 6.
- Part 4: Determines the dome stresses and deflections using the  $A_i$  and  $B_j$  obtained in Part 3.
- Part 5: Computes the flare stresses and deflections employing the  $C_k^n$  of Part 3, together with tapes 1 - 5 developed in Part 1.
- Part 6: Checks the point-matching equations of Part 2, to determine how well they are satisfied, using the  $A_i$ ,  $B_j$  and  $C_k^n$  of Part 3 and the  $[a_{ij}]$  of Part 2.

It should be pointed out that although the program appears segmented, it is possible to run each part consecutively in a single machine pass; if the input cards for each section are properly interspersed between the program decks of successive parts.



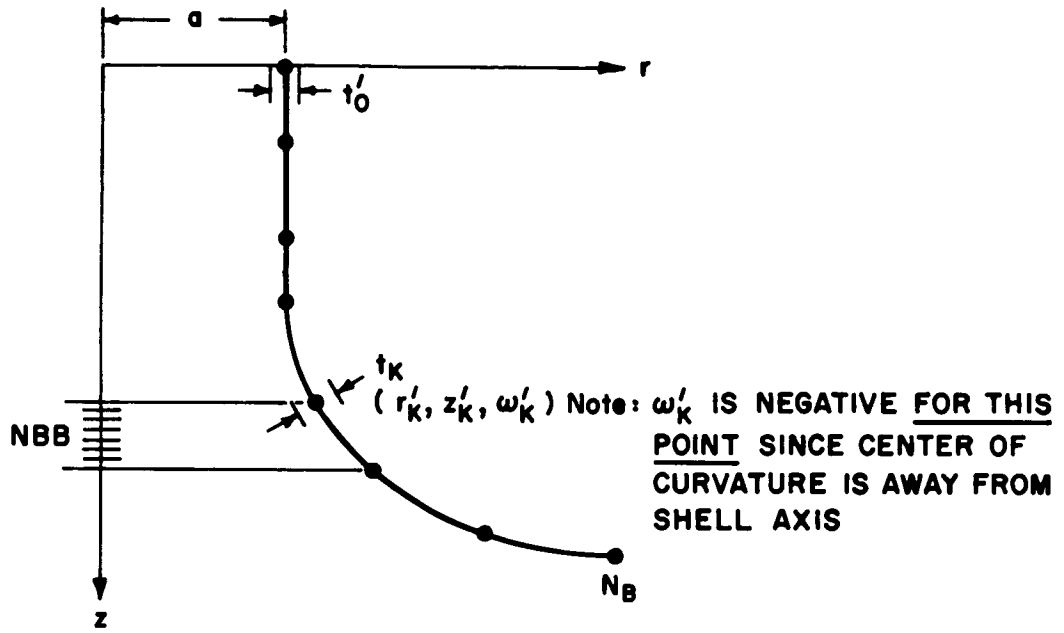
## B. GEOMETRY OF FLARE

Application of the matrix solution of Section II to typical flared nozzle problems requires the specification of certain geometric quantities, from which the system-properties matrices  $A_i$ ,  $B_i$ ,  $C_i$ , and  $g_i$ ,  $i = 0, 1, 2, \dots, N$ , may be computed. Two input options, explained below, are available for this purpose.

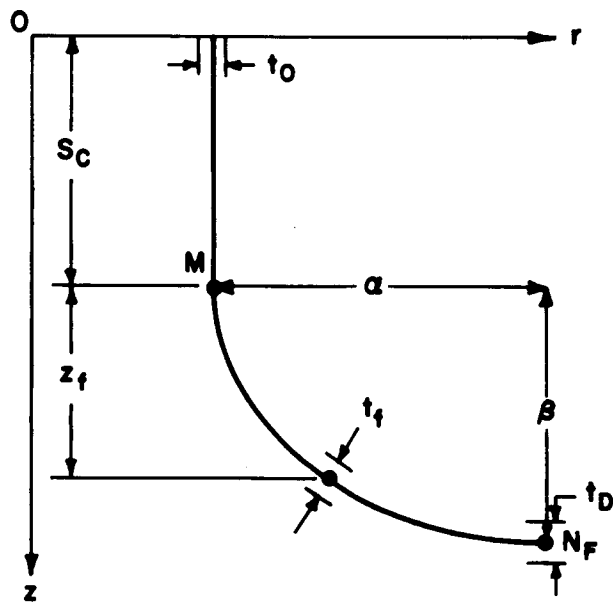
Option 1 inputs an array of axial and radial coordinates on the shell's meridional middle-surface, as well as the thickness and meridional curvature at these locations (see Figure 10a). A cubic curve fit is then made through the first three points with the additional condition that slopes of shell thickness, meridional curvature, and middle-surface, are zero with respect to the shell axis. This simplification is permissible, since the nozzle starts out as a circular cylinder of uniform wall thickness. The length of curve between the first two points is then divided into  $\bar{N}$  finite difference intervals, equally spaced with respect to the nozzle's axial coordinate. A cubic fit is then made, using the second, third, and fourth points, as well as the slope of the previous cubic, at the second input station. This back-stepping of the cubic fit is continued to insure a smooth flare contour. The interval between the second and third points is again subdivided into  $\bar{N}$  finite difference intervals, equally spaced with respect to the nozzle axis. This process is continued until finally, the last three input points  $\bar{N}-2$ ,  $\bar{N}-1$ , and  $\bar{N}$ , as well as the slope of the previous cubic slope at the  $\bar{N}-2$  station, are used to generate the last cubic, which is then subdivided into  $2\bar{N}$  equal intervals, between  $\bar{N}-2$  and  $N$ , along the cylinder axis. Thus, the nozzle geometry is subdivided into  $N = \bar{N} \times \bar{N}$  finite difference intervals with  $N + 1$  end points denoted by  $i = 0, 1, 2, \dots, N$ . Next, dimensionless arc-lengths,  $\Delta_i$ , are computed using the Pythagorean Theorem and coordinates of adjacent points. Then, the following quantities, in the order indicated, are computed employing parabolic central finite-difference differentiations where appropriate:

$$\rho_i, \rho'_i, \gamma_i, \omega_{\theta i} = \frac{\sqrt{1 - (\rho'_i)^2}}{\rho_i}, \omega'_s,$$

$$b_i = \frac{t_i/t_c}{1-\nu^2}, b'_i, d_i = \frac{(t_i/t_c)^3}{12(1-\nu^2)}, d'_i, \lambda = \frac{t_c}{r_c}$$



a. Input Option 1



b. Input Option 2

Figure 10. Cylinder-Flare Input Geometry

It is now possible to compute the a's and b's of Appendices 1 and 2, from which the matrices of Section II, Equations (14b, c, d, etc.) may be obtained.

Option 2 inputs the radius,  $a$ , of a circular cylinder and the semi-horizontal,  $\alpha$ , and vertical,  $\beta$ , axes of a quarter of an elliptical torus which mates smoothly with the cylinder (see Figure 10b). The variables  $z_f$ ,  $t_f$ , and  $t_D$  permit a quadratic thickness variation in the flare. The cylinder, of length  $S_c$ , is divided into  $M$  equally spaced difference intervals, while the total number of intervals for the combined shell is  $N_F$ . The elliptical dimensionless arc lengths, are given by

$$\Delta_i = \frac{\sqrt{(r_i - r_{i-1})^2 + (\bar{z}_i - \bar{z}_{i-1})^2}}{r_c}$$

where:  $r_i = a + \alpha (1 - \cos \phi_i)$

$\bar{z}_i = \beta \sin \phi_i + S_c$

and:  $\phi_i = \frac{(i-M)\pi}{2(N-M)}$ ,  $i = M + 1, M + 2, \dots, N$

The  $\omega_{si}$  and  $\omega'_{si}$  are computed from

$$\omega_{si} = -r_c \beta \alpha [\alpha^2 \sin^2 \phi + \beta^2 \cos^2 \phi_i]^{-3/2}$$

$$\omega'_{si} = 3 \left( \frac{\alpha}{\beta} - \frac{\beta}{\alpha} \right) \omega_s^2 \cos \phi_i \sin \phi_i$$

and the remaining quantities,  $\rho_i$ ,  $\rho'_i$ ,  $\gamma_i$ ,  $b_i$ ,  $b'_i$ ,  $d_i$ ,  $d'_i$ , and  $\omega_{\theta i}$  are computed as for Option 1.

### C. DOME GEOMETRY

For a complete description, the dome geometry simply requires the principal curvatures, the dome boundaries  $L_x$ ,  $L_y$ , a constant thickness  $t_D$ , and boundary radius,  $r_i$ , at which the dome joins with the flare.

The outer boundaries of the dome, defined by  $L_x$  and  $L_y$ , remain somewhat arbitrary in that they are usually not uniquely defined, but must be chosen within certain bounds. These bounds are determined as follows:

- (1) The minimum distance away from the flare intersection curve at which localized nozzle attachment effects must be essentially negligible is a lower bound for  $L_x$  and  $L_y$  and,
- (2)  $L_x$  and  $L_y$  must be sufficiently small so as not to violate the dome assumptions such as constant principal radii of curvature and shallowness.

Based on experimental data, the minimum  $L_x$  and  $L_y$  should be approximately four times the flare-dome intersection radius. However, the only true method of establishing whether the magnitudes of  $L_x$  and  $L_y$  are adequate, is to rerun any given problem with different  $L_x$ 's and  $L_y$ 's. This should be repeated until the stress and deflection results stabilize to the same value near the nozzle junction, while still decaying to the membrane stress field at  $L_x$  and  $L_y$ . These same comments are applicable for the cylinder length coordinate at  $z = 0$ , where it is assumed that membrane conditions prevail.

#### D. POINT-MATCHING

The points at which boundary conditions will be matched are indicated by dots in Figure 11.

NDY is the number of points along the y axis at which  $\bar{Q}_x$  is set equal to zero. The YD are the ordinates of these points. In a similar manner, NDX and NDXP establish the points at which  $\bar{Q}_y$  is set equal to zero.

NDXP is the number of points along  $y = L_y$  at which membrane boundary conditions, of which there are four, are satisfied, and XPD are the specific points. Similarly, NDYP and YPD relate to membrane conditions along  $x = L_x$ . The membrane condition used are presented in Section IV, Equations (3) and (4).

NDTHP is the number of points along the flare-dome intersection at which compatibility and equilibrium are satisfied (Equations (1) and (2) of Section IV). The specific points are defined by the THDP.

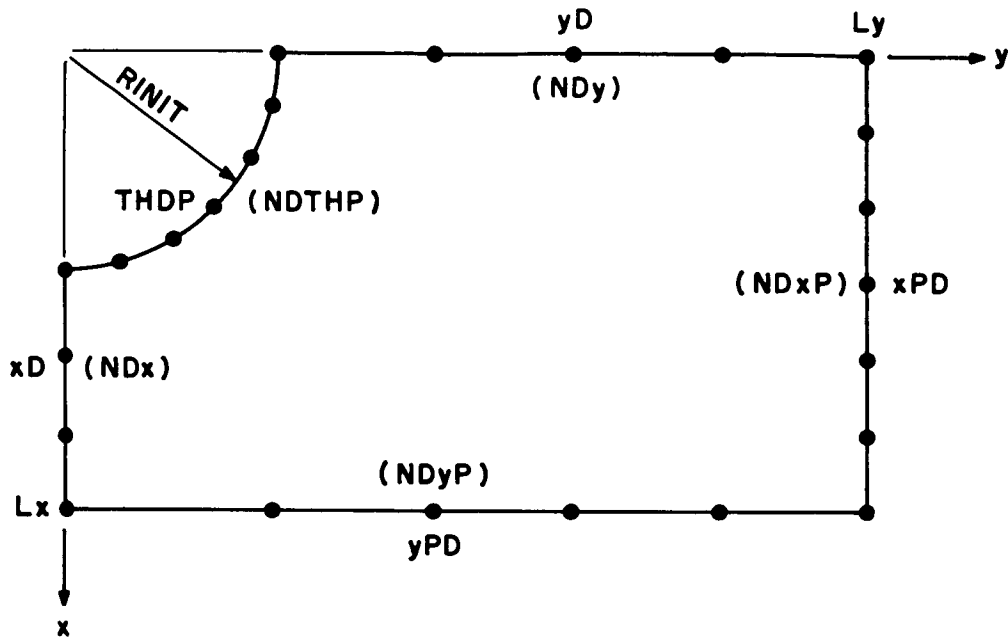


Figure 11. Points at Which Boundary Conditions are Matched

#### E. NOMENCLATURE AND PROGRAM FLOW CHARTS

The Input Nomenclature is presented in the following pages and Appendix IV contains the program Flow Charts. In addition, a separate simplified User's Manual, supplements this report.

#### INPUT NOMENCLATURE

<u>Physical Symbols</u>	<u>Dimensional Units</u>	<u>Program Symbols</u>	<u>Physical Description</u>
E	psi	EF, ED, ELAS	Modulus of elasticity; assumed the same for dome-flare-cylinder configuration
$\gamma$		NU, NUD	Poisson's ratio
p	psi	PSI	Internal pressure
$r_c$	in.	RC	Characteristic radius
$t_c$	in.	TC	Characteristic radius

<u>Physical Symbols</u>	<u>Dimensional Units</u>	<u>Program Symbols</u>	<u>Physical Description</u>
$\sigma_c$	psi	SIGC	Characteristic stress
$\bar{n}$		NBAR	Maximum Fourier index for flare

If NC option 2 = 1, use the following flare input geometry.  
(Reference Figure 9a)

$\bar{N}$		NB	Number of flare input coordinate points
$\bar{\bar{N}}$		NBB	Number of finite-difference intervals into which segment between successive input points is subdivided
$N_F$		NF	Number of flare finite difference intervals = $\bar{N} \cdot \bar{\bar{N}}$
a	in.	A, ACR	Radius of cylinder
$r'_k$	in.	RPK	Radii of cylinder-flare input points
$z'_k$	in.	ZPK	Vertical coordinate of cylinder-flare input points
$t'_o$	in.	TZP	Thickness at top of cylinder
$\omega'_o$	in. <sup>-1</sup>	OMZP	Curvature at top of cylinder
$\omega'_k$	in. <sup>-1</sup>	OMPK	Curvature at cylinder-flare input points

If NC Option 2 = 0, use the following flare input geometry.  
(Reference Figure 9b)

$\alpha$	in.	ALPHA	Horizontal semi-axis (ellipse) for flare geometry
$\beta$	in.	BETA	Vertical semi-axis (ellipse) for flare geometry
$z_F$	in.	ZF	Flare thickness parameter; adjusted so quadratic flare thickness approximates desired thickness variation
$S_c$	in.	SC	Cylinder length
$\bar{t}_D$	in.	TTDD	Flare thickness parameter
$L_x$	in.	LX	Dome half-span in x direction
$L_y$	in.	LY	Dome half-span in y direction

<u>Physical Symbols</u>	<u>Dimensional Units</u>	<u>Program Symbols</u>	<u>Physical Description</u>
$t_D$	in.	TD	Dome thickness
$1/R_x$	in. <sup>-1</sup>	RATIO X	Dome curvature in x direction
$1/R_y$	in. <sup>-1</sup>	RATIO Y	Dome curvature in y direction
I		ID	Maximum index on series which decays exponentially in y direction
J		JD	Maximum index on series which decays exponentially in x direction
NDY		NDY	Number of points matched along y axis
NDX		NDX	Number of points matched along x axis
NDYP		NDYP	Number of points matched along $y = L_y$
NDXP		NDXP	Number of points matched along $x = L_x$
NDTHP		NDTHP	Number of points matched along $r = r_i$
$r_i$	in.	RINIT	Radius of flare-dome intersection
$\Delta r$ (or $\Delta R$ )	in.	DELTAR	Radial increment for output stations for stress and deflection computation
$x_o$	in.	XO	Cartesian output stations
$\Delta x$	in.	DELTAX	Increment for cartesian output stations
$\Delta y$	in.	DELTAY	Increment for cartesian output stations
$\theta_o$	degrees	THETAO	Polar output stations
$\Delta\theta$	degrees	DTHETA	Polar increment for output stations
$\theta_\ell$	degrees	THLAST	Final angle for polar output stations

SECTION VI  
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APPENDIX I

FORMULAS FOR THE "a" COEFFICIENTS OF SECTION II

$$a_1 = [b + d\lambda^2(\omega_s - \omega_\theta)\omega_\theta]$$

$$a_2 = [\gamma b + b' + \gamma d\lambda^2(\omega_s - \omega_\theta)^2 + d'\lambda^2\omega_\theta(\omega_s - \omega_\theta) + d\lambda^2\omega_s'\omega_\theta]$$

$$a_3 = \gamma(\nu b' - \gamma b) - \nu b\omega_s\omega_\theta - \frac{b(1-\nu)}{2}\left(\frac{n}{\rho}\right)^2 + \nu\lambda^2[(\gamma d'\omega_s - \gamma^2 d\omega_s - d\omega_s^2\omega_\theta)(\omega_s - \omega_\theta) + \gamma d\omega_s'(2\omega_s - \omega_\theta) + \nu d\gamma^2\omega_s^2] - \lambda^2 d\left(\gamma^2 + \frac{(1-\nu)}{2}\left(\frac{n}{\rho}\right)^2\right)(3\omega_s^2 - 3\omega_\theta\omega_s + \omega_\theta^2)$$

$$a_4 = \frac{b(1+\nu)}{2}\frac{n}{\rho} + d\lambda^2\frac{n}{\rho}\left[\frac{1+\nu}{2}\omega_s\omega_\theta - \nu\omega_\theta^2\right]$$

$$a_5 = \left[\nu b' - \frac{(3-\nu)}{2}\gamma b\right]\frac{n}{\rho} + \nu\lambda^2\frac{n}{\rho}[d'\omega_\theta(\omega_s - \omega_\theta) + d\omega_\theta\omega_s'] + \lambda^2\left(\frac{n}{\rho}\right)\gamma d[-(1-\nu)\omega_s^2 - \left(\frac{1+5\nu}{2} - \nu^2\right)\omega_s\omega_\theta + 2\nu\omega_\theta^2]$$

$$a_5' = -\nu d\lambda^2\gamma(\omega_s - \omega_\theta)$$

$$a_6 = b(\omega_s + \nu\omega_\theta) - \nu\gamma\lambda^2[d'(\omega_s - \omega_\theta) + d\omega_s'] + d\lambda^2[(1+\nu)(\omega_s - \omega_\theta)(\omega_s\omega_\theta + \gamma^2) + (1-\nu^2)\gamma^2\omega_s + \frac{(3-\nu)}{2}\left(\frac{n}{\rho}\right)^2\omega_s - \frac{(1+\nu)}{2}\omega_\theta\left(\frac{n}{\rho}\right)^2]$$

$$a_7 = b'(\omega_s + \nu\omega_\theta) + \lambda^2(\omega_s\omega_\theta + \nu\frac{n^2}{\rho^2})[d'(\omega_s - \omega_\theta) + d\omega_s'] + b[\gamma(\omega_s - \omega_\theta) + \omega_s'] + (\text{continued})$$

$$a_7 (\text{continuation}) + d\lambda^2 \omega_\theta (\omega_s - \omega_\theta) \omega_s' \\ + d\gamma \lambda^2 [ (\omega_s - \omega_\theta)^3 - v \omega_s^2 (\omega_s - \omega_\theta) \\ - \left(\frac{n}{\rho}\right)^2 \left\{ (\omega_s - \omega_\theta) - \frac{(1-v)}{2} (\omega_s - \omega_\theta) \right. \\ \left. + (1-v^2) \omega_s + (1-v)(2\omega_s - \omega_\theta) \right\} ]$$

$$a_8 = -\lambda^2 \omega_\theta$$

$$a_9 = \lambda^2 [ \omega_s' - \gamma(1-v) \omega_s ]$$

$$a_{10} = -\frac{b}{2} \left(\frac{n}{\rho}\right) (1+v) - \lambda^2 d \omega_\theta \left(\frac{n}{\rho}\right) \left[ \frac{(1+v)}{2} \omega_s - v \omega_\theta \right]$$

$$a_{11} = -\left(\frac{n}{\rho}\right) \left[ \frac{b'(1-v)}{2} + b\gamma \right] - \left(\frac{n}{\rho}\right) \lambda^2 \omega_\theta \frac{(1-v)}{2} (d\omega_s' + d\omega_s) \\ - \left(\frac{n}{\rho}\right) \lambda^2 \gamma d\omega_s \left[ \frac{(3-v)}{2} \omega_s - v^2 \omega_\theta \right]$$

$$a_{12} = \frac{(1-v)}{2} [ b + d\lambda^2 (\omega_s^2 - 3\omega_\theta \omega_s + 3\omega_\theta^2) ]$$

$$a_{13} = \frac{(1-v)}{2} [ b' + d'\lambda^2 (\omega_s^2 - 3\omega_\theta \omega_s + 3\omega_\theta^2) \\ + d\lambda^2 \omega_s' (2\omega_s - 3\omega_\theta) + d\lambda^2 \gamma (2\omega_\theta - 3\omega_s) (\omega_s - 2\omega_\theta) ]$$

$$a_{14} = \frac{(1-v)}{2} [ b\omega_s \omega_\theta - b'\gamma - d'\lambda^2 \gamma (\omega_s^2 - 3\omega_s \omega_\theta + 3\omega_\theta^2) \\ + d\lambda^2 \gamma \omega_s' (3\omega_\theta - 2\omega_s) + d\lambda^2 \gamma^2 (3\omega_s - 2\omega_\theta) (\omega_s - 2\omega_\theta) \\ + d\lambda^2 \omega_s \omega_\theta (\omega_s^2 - 3\omega_s \omega_\theta + 3\omega_\theta^2) ] \\ - \left(\frac{n}{\rho}\right)^2 [ b + \lambda^2 d \omega_\theta (\omega_s - v^2 \omega_\theta) ]$$

$$a_{15} = \frac{(1-v)}{2} \left(\frac{n}{\rho}\right) \lambda^2 d [ 3\omega_\theta - \omega_s ]$$

$$\begin{aligned}
a_{16} &= \frac{(1-\nu)}{2} \left(\frac{n}{\rho}\right) \lambda^2 [d'(3\omega_\theta - \omega_s) - d\omega_s'] \\
&\quad + \left(\frac{n}{\rho}\right) \delta d \lambda^2 [(3\omega_s - 2\omega_\theta) - 2\nu(\omega_s - \omega_\theta) - \nu^2 \omega_\theta] \\
a_{17} &= \frac{(1-\nu)}{2} \left(\frac{n}{\rho}\right) \delta \lambda^2 [d'(\omega_s - 3\omega_\theta) + d\omega_s'] - \frac{b n}{\rho} (\nu \omega_s + \omega_\theta) \\
&\quad - 2(1-\nu) \left(\frac{n}{\rho}\right) d \delta \lambda^2 (\omega_s - \omega_\theta) - \left(\frac{n}{\rho}\right)^3 d \lambda^2 (\omega_s - \nu^2 \omega_\theta) \\
&\quad + \frac{n}{2\rho} d \lambda^2 \omega_s \omega_\theta [(3\omega_\theta - \omega_s) - \nu(\omega_s + \omega_\theta)] \\
a_{18} &= \nu \left(\frac{n}{\rho}\right) \lambda^2 \omega_\theta
\end{aligned}$$

$$a_{19}' = -\nu \lambda^2 \delta d (\omega_s - \omega_\theta)$$

$$\begin{aligned}
a_{19} &= -b(\omega_s + \nu \omega_\theta) - \nu \lambda^2 \delta d' (\omega_s - \omega_\theta) - (1-\nu) d \lambda^2 \omega_s \omega_\theta (\omega_s - \omega_\theta) \\
&\quad - \frac{d \lambda^2}{2} \left(\frac{n}{\rho}\right)^2 [(3\omega_s - \omega_\theta) - \nu(\omega_s + \omega_\theta)] \\
&\quad - d \lambda^2 \delta^2 [(2-\nu-\nu^2)\omega_s - (1-\nu)\omega_\theta] - \nu d \lambda^2 \delta \omega_s'
\end{aligned}$$

$$\begin{aligned}
a_{20} &= -b\delta(\nu \omega_s + \omega_\theta) - \lambda^2 d \delta [\nu \omega_s^3 - (3+\nu-2\nu^2)\omega_s^2 \omega_\theta + \omega_\theta^3] \\
&\quad - \lambda^2 \delta^2 d' [(2-\nu^2)\omega_s - \omega_\theta] - \frac{(1-\nu)}{2} \lambda^2 \left(\frac{n}{\rho}\right)^2 d' (3\omega_s - \omega_\theta) \\
&\quad - \lambda^2 \delta^3 d [2\omega_\theta - (3-\nu^2)\omega_s] - \lambda^2 d \omega_s' [(2-\nu^2) \delta^2 \\
&\quad + \frac{3(1-\nu)}{2} \left(\frac{n}{\rho}\right)^2] - \lambda^2 \delta d \left(\frac{n}{\rho}\right)^2 \left[\left(\frac{3}{2} + \frac{\nu}{2} - \nu^2\right) \omega_s \right. \\
&\quad \left. - \frac{(1+\nu)}{2} \omega_\theta\right]
\end{aligned}$$

$$a_{21} = \frac{(1-\nu)}{2} \lambda^2 d \left(\frac{n}{\rho}\right) (3\omega_\theta - \omega_s)$$

$$\begin{aligned}
a_{22} &= -\lambda^2 \delta d \left(\frac{n}{\rho}\right) \left[-\frac{(1-3\nu)}{2} \omega_s + \left(\frac{3}{2} - \frac{3}{2}\nu - \nu^2\right) \omega_\theta\right] \\
&\quad - \frac{(1-\nu)}{2} \lambda^2 \left(\frac{n}{\rho}\right) [-d'(3\omega_\theta - \omega_s) + d\omega_s']
\end{aligned}$$

$$\begin{aligned}
 a_{23} = & -b\left(\frac{n}{\rho}\right)(v\omega_s + \omega_\theta) + \lambda^2 d\left(\frac{n}{\rho}\right)\omega_s\omega_\theta \left[ \frac{(1-v)}{2}\omega_s + \left(\frac{3-v}{2} - v^2\right)\omega_\theta \right] \\
 & - \lambda^2 \delta d\left(\frac{n}{\rho}\right) \left[ \left(\frac{1+v}{2}\right)\omega_s + \left(\frac{3}{2} - \frac{3}{2}v - v^2\right)\omega_\theta \right] \\
 & - \lambda^2 \delta^2 d\left(\frac{n}{\rho}\right) \left[ \left(\frac{1-3v}{2} - v^2\right)\omega_s + \left(-\frac{3}{2} + \frac{3}{2}v + 2v^2\right)\omega_\theta \right] \\
 & - \lambda^2 \delta d\left(\frac{n}{\rho}\right)\omega_s' \frac{(1+v)}{2} - \lambda^2 d\left(\frac{n}{\rho}\right)^3 (\omega_s - v^2\omega_\theta)
 \end{aligned}$$

$$a_{24} = \lambda^2 d \left[ (1-v^2)\delta^2 + 2(1-v)\left(\frac{n}{\rho}\right)^2 \right]$$

$$\begin{aligned}
 a_{25} = & -\lambda^2 \delta d \left[ (1-v^2)(\delta^2 + 2\omega_s\omega_\theta) + 2(1-v)\left(\frac{n}{\rho}\right)^2 \right] \\
 & + \lambda^2 d' \left[ (1-v^2)\delta^2 + 2(1-v)\left(\frac{n}{\rho}\right)^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 a_{26} = & -b(\omega_s^2 + 2v\omega_s\omega_\theta + \omega_\theta^2) - d\lambda^2(\omega_s - \omega_\theta)\omega_\theta \left[ (1-v)\omega_s^2 \right. \\
 & \left. - \omega_s\omega_\theta - \omega_\theta^2 \right] - d\lambda^2\left(\frac{n}{\rho}\right)^2 \left[ 2(\omega_s - \omega_\theta)(v\omega_s + \omega_\theta) \right. \\
 & \left. - (3 - 2v - v^2)\omega_s\omega_\theta + (1-v^2)\left(\frac{n}{\rho}\right)^2 \right] \\
 & - d'\lambda^2\delta \left[ (\omega_s - \omega_\theta)(\omega_\theta + v\omega_s) + (3 - 2v - v^2)\left(\frac{n}{\rho}\right)^2 \right] \\
 & - \lambda^2 \delta d \left[ \delta(\omega_s - \omega_\theta) \left\{ (1-v)\omega_s - 2\omega_\theta \right\} + \omega_s' \left\{ (1-v)\omega_\theta + 2v\omega_s \right\} \right. \\
 & \left. - 2(2-v-v^2)\left(\frac{n}{\rho}\right)^2\delta \right]
 \end{aligned}$$

$$a_{27} = -\lambda^2$$

$$a_{28} = -\delta\lambda^2(2-v)$$

$$a_{29} = -\lambda^2 \left[ \omega_s^2 - (2-v)\omega_s\omega_\theta - v\left(\frac{n}{\rho}\right)^2 \right]$$

$$a_{30} = d\omega_\theta$$

$$a_{31} = d(\omega_s' + v\delta\omega_s)$$

$$a_{32} = v d \left( \frac{\eta}{\rho} \right) \omega_0$$

$$a_{33} = -d$$

$$a_{34} = -v d \gamma$$

$$a_{35} = -d \left[ u_s (\omega_s - \omega_0) - v \left( \frac{\eta}{\rho} \right)^2 \right]$$

$$a_{36} = 1$$

APPENDIX II

FORMULAS FOR THE "b" COEFFICIENTS OF SECTION II

$$b_1 = b + d\lambda^2(\omega_s - \omega_\theta)\omega_\theta$$

$$b_2 = \nu b\gamma + \nu\gamma d\lambda^2\omega_s(\omega_s - \omega_\theta)$$

$$b_3 = \nu b\left(\frac{n}{\rho}\right) + \nu d\lambda^2\left(\frac{n}{\rho}\right)\omega_\theta(\omega_s - \omega_\theta)$$

$$b_4' = -\nu\gamma d\lambda^2(\omega_s - \omega_\theta)$$

$$b_4 = b(\omega_s + \nu\omega_\theta) + d\lambda^2(\omega_s - \omega_\theta)(\omega_s\omega_\theta + \nu\left(\frac{n}{\rho}\right)^2)$$

$$b_4'' = \lambda^2(\omega_s - \omega_\theta)$$

$$b_5 = -\frac{(1-\nu)}{2}\left(\frac{n}{\rho}\right)(b + \lambda^2 d\omega_s\omega_\theta)$$

$$b_6 = \frac{(1-\nu)}{2}\left[b + d\lambda^2\left\{(\omega_s - \omega_\theta)^2 + \omega_\theta(2\omega_\theta - \omega_s)\right\}\right]$$

$$b_7 = -\gamma b_6$$

$$b_8 = \frac{(1-\nu)}{2}d\lambda^2\left(\frac{n}{\rho}\right)(3\omega_\theta - \omega_s)$$

$$b_9 = -\gamma b_8$$

$$b_{10}' = \gamma\nu d\lambda^2(\omega_s - \omega_\theta)$$

$$b_{10} = d\lambda^2\gamma^2\left[(2\omega_s - \omega_\theta) - \nu^2\omega_s\right] + \frac{(1-\nu)}{2}d\lambda^2\left(\frac{n}{\rho}\right)^2(3\omega_s - \omega_\theta)$$

$$b_{11} = -\frac{(1-\nu)}{2}d\lambda^2\left(\frac{n}{\rho}\right)(3\omega_\theta - \omega_s)$$

$$b_{12} = d\lambda^2\gamma\left(\frac{n}{\rho}\right)\left[\omega_s - \nu^2\omega_\theta + \frac{(1-\nu)}{2}(3\omega_\theta - \omega_s)\right]$$

$$b_{13} = -2(1-\nu) d \lambda^2 \left(\frac{n}{\rho}\right)^2 - d \lambda^2 \gamma^2 (1-\nu^2)$$

$$b_{14} = d \lambda^2 \gamma \left[ (\omega_0 + \nu \omega_s) (\omega_s - \omega_0) + (3 - 2\nu - \nu^2) \left(\frac{n}{\rho}\right)^2 \right]$$

$$b_{15} = \lambda^2$$

$$b_{16} = \lambda^2 \gamma (1-\nu)$$



APPENDIX III  
SHALLOW SHELL EQUATIONS

A. INTRODUCTION

A system of partial differential equations suitable for the analysis of thin, shallow shells was originally developed by Marguerre (Reference 12). The assumption of shallowness, which entails neglecting quantities such as  $\left(\frac{x}{R_x}\right)^2$  in comparison to unity, represents an essential simplification in the analysis of arbitrarily shaped shells. Although the use of a shallow shell theory approach offers attractive possibilities for the solution of engineering problems, standard reference texts, such as Flügge (Reference 9) and Timoshenko and Woinowsky-Krieger (Reference 13) do not include the particulars of Marguerre's investigation. Therefore, to insure completeness of the present report, the governing equations for a shallow, elliptic-paraboloidal shell will be developed in detail.

It should be noted that the elliptic paraboloid is capable of representing a variety of shapes provided that

- The region represented is shallow
- The curvatures are essentially constant
- The Gaussian curvature is positive.

The results obtained herein also apply to shells of negative Gaussian curvature, which could be shown by following a similar development using an hyperbolic paraboloid.

The geometric assumptions (these form the basis for the shallow shell theory), the constituent stress-strain relations, and the associated equilibrium equations (compatible with the shallowness assumptions) will be discussed in Subsections B, C, and D, respectively. Reduction of the resulting system of fourteen equations, involving the same number of unknowns, to one fourth order

partial differential equation for a complex deflection-stress function, will be presented in subsection E.

## B. GEOMETRICAL CONSIDERATIONS

### 1. Shell Geometry

The geometry of a shell surface is completely determined by specifying the equation of the surface. For the elliptic-paraboloidal shell, with respect to a centrally located Cartesian coordinate system (Figure 5), this surface is represented by

$$z = -\frac{1}{2} \left( \frac{x^2}{R_x} + \frac{y^2}{R_y} \right) = f(x, y) \quad (\text{A-1})$$

where  $R_x$  and  $R_y$  are constants. An arbitrary point on the middle surface is given by the vector

$$\hat{r} = xi + yj + f(x, y)k \quad (\text{A-2})$$

where  $i$ ,  $j$ , and  $k$  are unit vectors aligned in the  $x$ ,  $y$ , and  $z$  directions, respectively. The locations of all other field points are measured along a normal ( $\hat{n}$ ) to the median surface. Thus, a generic point in the shell is given by (Figure 5)

$$\hat{R}(x, y, \zeta) = \hat{r} + \zeta \hat{n} \quad (\text{A-3})$$

where  $\zeta$  is measured along the local normal.

A preliminary step in developing the field equations is the construction of a curvilinear coordinate system on the reference surface of the shell. Since this surface is given in Monge's form (Equation A-1), then the Cartesian coordinates  $x$  and  $y$  are taken as parameters. Thus, the curvilinear coordinate system consists of the parametric curves formed by the intersection of the surface with the planes  $x = \text{constant}$  and  $y = \text{constant}$ . Unit vectors, tangent to the coordinate curves are denoted as  $i_x$  and  $i_y$  (Figure A-1) and are defined as

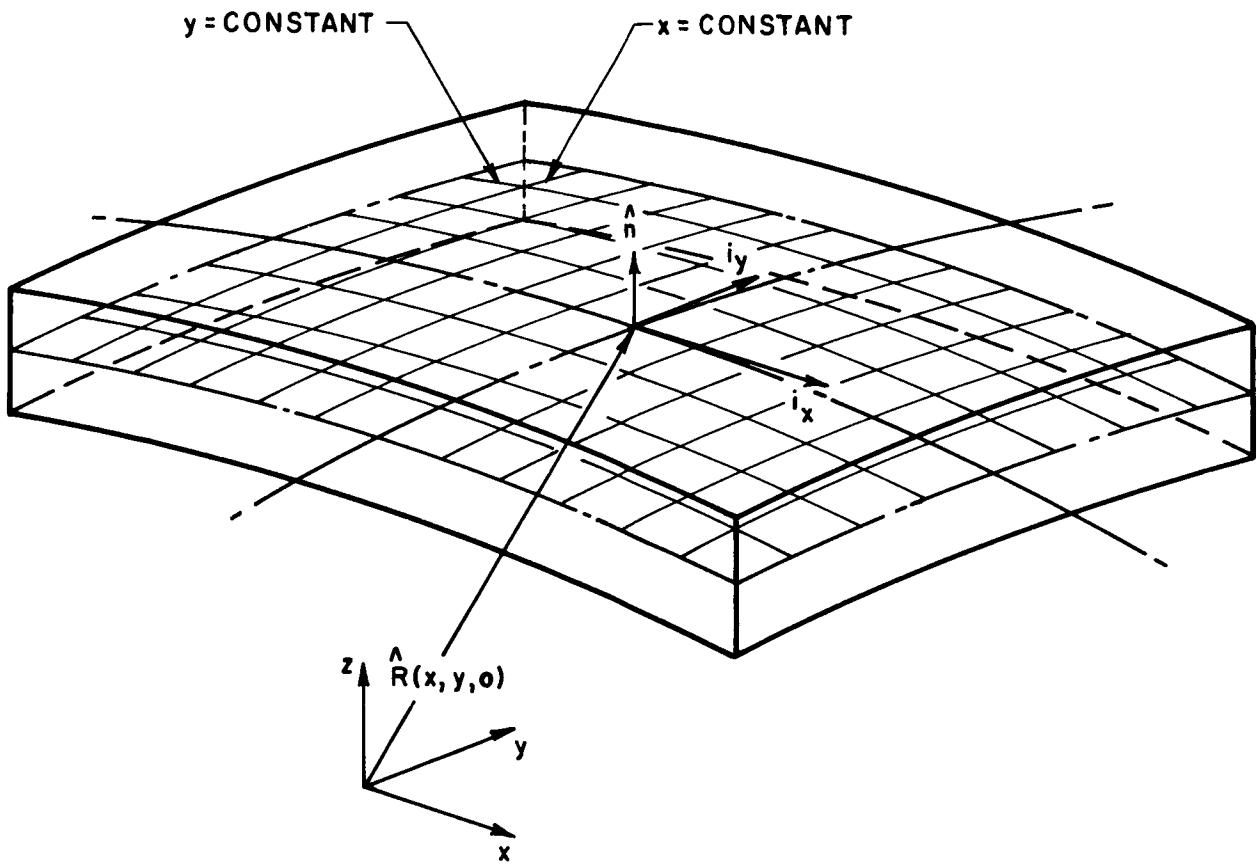


Figure A-1. Parametric Curves on an Elliptic-Paraboloidal Shell Surface

$$i_x = \frac{\frac{\partial \hat{R}}{\partial x}}{\left| \frac{\partial \hat{R}}{\partial x} \right|} \quad (A-4a)$$

$$i_y = \frac{\frac{\partial \hat{R}}{\partial y}}{\left| \frac{\partial \hat{R}}{\partial y} \right|}$$

in addition the unit normal is

$$\hat{n} = \frac{\frac{\partial \hat{R}}{\partial x} \times \frac{\partial \hat{R}}{\partial y}}{\left| \frac{\partial \hat{R}}{\partial x} \times \frac{\partial \hat{R}}{\partial y} \right|} \quad (A-4b)$$

Putting Equation A-3, with  $\zeta = 0$ , into Equations A-4 and making use of Equation A-1 results in

$$i_x = \frac{i - \frac{x}{R_x} k}{\left[ 1 + \left( \frac{x}{R_x} \right)^2 \right]^{\frac{1}{2}}}$$

$$i_y = \frac{j - \frac{y}{R_y} k}{\left[ 1 + \left( \frac{y}{R_y} \right)^2 \right]^{\frac{1}{2}}} \quad (A-5)$$

$$\hat{n} = \frac{\frac{x}{R_x} i + \frac{y}{R_y} j + k}{\left[ 1 + \left( \frac{x}{R_x} \right)^2 + \left( \frac{y}{R_y} \right)^2 \right]^{\frac{1}{2}}}$$

Considering the shell to be shallow, then  $\left(\frac{x}{R_x}\right)^2$  and  $\left(\frac{y}{R_y}\right)^2$  will be negligible

in comparison with unity and Equation A-5 becomes

$$i_x = i - \frac{x}{R_x} k$$

$$i_y = j - \frac{y}{R_y} k \quad (A-6)$$

$$n = \frac{x}{R_x} i + \frac{y}{R_y} j + k$$

The differential length of arc between two neighboring points, with position vectors  $\hat{R}(x, y, \zeta)$  and  $\hat{R}+d\hat{R}$ , respectively (Figure A-2) is

$$(ds)^2 = d\hat{R} \cdot d\hat{R} \quad (A-7a)$$

where

$$d\hat{R} = \frac{\partial \hat{R}}{\partial x} dx + \frac{\partial \hat{R}}{\partial y} dy + \frac{\partial \hat{R}}{\partial \zeta} d\zeta \quad (A-7b)$$

and to the order of approximation in the analysis

$$d\hat{R} = \left(i_x + \zeta \frac{\partial \hat{n}}{\partial x}\right) dx + \left(i_y + \zeta \frac{\partial \hat{n}}{\partial y}\right) dy + \hat{n} d\zeta \quad (A-7c)$$

Putting Equation A-7c into A-7a, and carrying out the indicated operations leads to

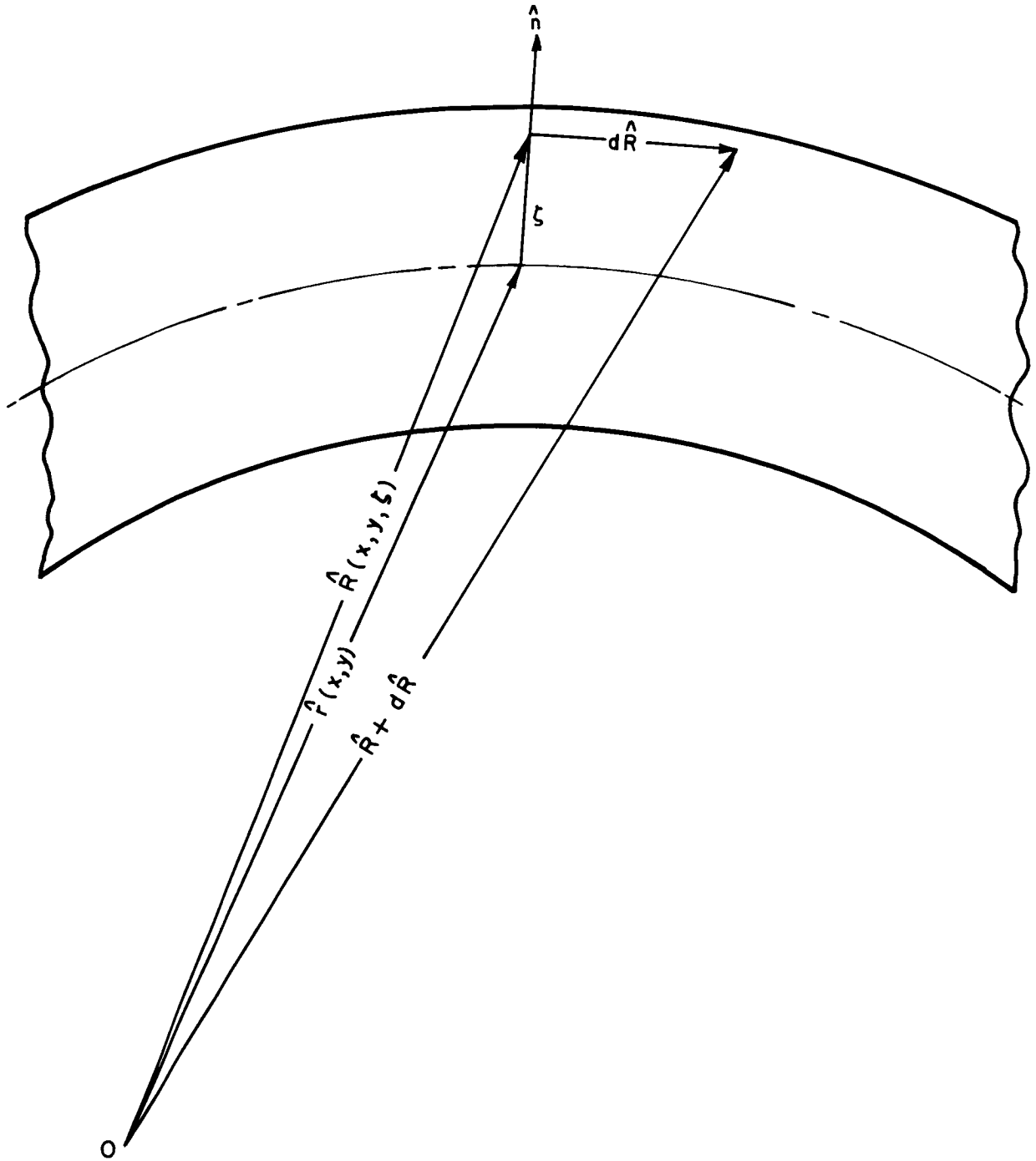


Figure A-2. Displacement of a Generic Point in the Shell

$$\begin{aligned}
(ds)^2 = & \left[ \left(1 + \frac{\zeta}{R_x}\right)^2 + \left(\frac{x}{R_x}\right)^2 \right] (dx)^2 + \left[ \left(1 + \frac{\zeta}{R_y}\right)^2 + \left(\frac{y}{R_y}\right)^2 \right] (dy)^2 \\
& + \left[ 1 + \left(\frac{x}{R_x}\right)^2 + \left(\frac{y}{R_y}\right)^2 \right] (d\zeta)^2 + \frac{2x\zeta}{R_x^2} dx d\zeta \\
& + \frac{2y\zeta}{R_y^2} dy d\zeta + \frac{2xy}{R_x R_y} dx dy
\end{aligned} \tag{A-7d}$$

After neglecting terms of the type  $\left(\frac{x}{R_x}\right)^2$ ,  $\left(\frac{xy}{R_x R_y}\right)$ , etc., when compared to unity, in accordance with the shallowness assumption, Equation A-7d reduces to

$$(ds)^2 = \left(1 + \frac{\zeta}{R_x}\right)^2 (dx)^2 + \left(1 + \frac{\zeta}{R_y}\right)^2 (dy)^2 + (d\zeta)^2 \tag{A-8a}$$

which is the first fundamental form of the shallow elliptic-paraboloidal shell. In addition, the principal radii of curvature,  $R_1$  and  $R_2$ , can be written as

$$\frac{1}{R_1} = \frac{1}{R_x \left(1 + \left[\frac{x}{R_x}\right]^2\right) \sqrt{1 + \left(\frac{x}{R_x}\right)^2 + \left(\frac{y}{R_y}\right)^2}} \tag{A-8b}$$

$$\frac{1}{R_2} = \frac{1}{R_y \left(1 + \left[\frac{y}{R_y}\right]^2\right) \sqrt{1 + \left(\frac{x}{R_x}\right)^2 + \left(\frac{y}{R_y}\right)^2}}$$

and, as a consequence of the shallowness assumption,  $R_x$  and  $R_y$  are approximately equal to the principal radii of curvature.

Equation A-8a is employed to determine the strain displacement components. The metric coefficients, i.e.,  $\left(1 + \frac{\zeta}{R_x}\right)$ ,  $\left(1 + \frac{\zeta}{R_y}\right)$ , and 1 are also required to define the stress and moment resultants.

## 2. Strain Displacement Relations

General expressions for the strain components in an arbitrarily shaped body have been derived by Wang (Reference 14). These expressions are referred to an orthogonal curvilinear coordinate system, where a differential length of arc is specified by

$$(ds)^2 = (A_1 d\xi_1)^2 + (A_2 d\xi_2)^2 + (A_3 d\xi_3)^2 \quad (\text{A-9})$$

The strain-displacement relations are

$$\epsilon_1 = \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} + \frac{u_3}{A_1 A_3} \frac{\partial A_1}{\partial \xi_3} \quad (\text{A-10a})$$

$$\epsilon_2 = \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} + \frac{u_3}{A_2 A_3} \frac{\partial A_2}{\partial \xi_3} + \frac{u_1}{A_2 A_1} \frac{\partial A_2}{\partial \xi_1} \quad (\text{A-10b})$$

$$\epsilon_3 = \frac{1}{A_3} \frac{\partial u_3}{\partial \xi_3} + \frac{u_1}{A_3 A_1} \frac{\partial A_3}{\partial \xi_1} + \frac{u_2}{A_3 A_2} \frac{\partial A_3}{\partial \xi_2} \quad (\text{A-10c})$$

$$\gamma_{12} = \frac{A_2}{A_1} \frac{\partial}{\partial \xi_1} \left( \frac{u_2}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \xi_2} \left( \frac{u_1}{A_1} \right) \quad (\text{A-10d})$$

$$\gamma_{13} = \frac{A_1}{A_3} \frac{\partial}{\partial \xi_3} \left( \frac{u_1}{A_1} \right) + \frac{A_3}{A_1} \frac{\partial}{\partial \xi_1} \left( \frac{u_3}{A_3} \right) \quad (\text{A-10e})$$

$$\gamma_{23} = \frac{A_3}{A_2} \frac{\partial}{\partial \xi_2} \left( \frac{u_3}{A_3} \right) + \frac{A_2}{A_3} \frac{\partial}{\partial \xi_3} \left( \frac{u_2}{A_2} \right) \quad (\text{A-10f})$$



where  $u_1$ ,  $u_2$ , and  $u_3$  are the displacements in the coordinate directions  $\xi_1$ ,  $\xi_2$ , and  $\xi_3$ , respectively. The notation used for extensional strains ( $\epsilon_1$ ,  $\epsilon_2$ ,  $\epsilon_3$ ) and shear strains ( $\gamma_{12}$ ,  $\gamma_{23}$ ,  $\gamma_{31}$ ) is in accordance with the usual convention of elasticity.

The strain displacement relations for a shallow elliptic paraboloidal shell can be obtained directly from Equations A-10 by formally identifying the metric coefficients in Equation A-8a with their counterparts in Equation A-9. Therefore, with

$$\begin{aligned}
 \xi_1 &= x \\
 \xi_2 &= y \\
 \xi_3 &= \zeta \\
 A_1 &= 1 + \frac{\zeta}{R_x} \\
 A_2 &= 1 + \frac{\zeta}{R_y} \\
 A_3 &= 1 \\
 u_1 &= U \\
 u_2 &= V \\
 u_3 &= w
 \end{aligned}
 \tag{A-11}$$

Equations A-10 become

$$\epsilon_x = \left( \frac{1}{1 + \frac{\zeta}{R_x}} \right) \left[ \frac{\partial U}{\partial x} + \frac{w}{R_x} \right]
 \tag{A-12a}$$

$$\epsilon_y = \left( \frac{1}{1 + \frac{\zeta}{R_y}} \right) \left[ \frac{\partial V}{\partial y} + \frac{w}{R_y} \right]
 \tag{A-12b}$$

$$\epsilon_{\zeta} = \frac{\partial w}{\partial \zeta} \quad (\text{A-12c})$$

$$\gamma_{xy} = \frac{1}{\left(1 + \frac{\zeta}{R_x}\right)} \frac{\partial V}{\partial x} + \frac{1}{\left(1 + \frac{\zeta}{R_y}\right)} \frac{\partial U}{\partial y} \quad (\text{A-12d})$$

$$\gamma_{x\zeta} = \left(1 + \frac{\zeta}{R_x}\right) \frac{\partial}{\partial \zeta} \left( \frac{U}{1 + \frac{\zeta}{R_x}} \right) + \frac{1}{\left(1 + \frac{\zeta}{R_x}\right)} \frac{\partial w}{\partial x} \quad (\text{A-12e})$$

$$\gamma_{y\zeta} = \frac{1}{\left(1 + \frac{\zeta}{R_y}\right)} \frac{\partial w}{\partial y} + \left(1 + \frac{\zeta}{R_y}\right) \frac{\partial}{\partial \zeta} \left( \frac{V}{1 + \frac{\zeta}{R_y}} \right) \quad (\text{A-12f})$$

These relations can be simplified by assuming that normals to the undeformed middle surface remain normal to the deformed middle surface, and are unextended. Mathematically, these assumptions require

$$\gamma_{x\zeta} = \gamma_{y\zeta} = \epsilon_{\zeta} = 0 \quad (\text{A-13})$$

which, in terms of the displacements (Equations A-12c, A-12e, and A-12f), lead to the constraints

$$\frac{\partial w}{\partial \zeta} = 0$$

$$\frac{\partial}{\partial \zeta} \left( \frac{U}{1 + \frac{\zeta}{R_x}} \right) + \left( \frac{1}{1 + \frac{\zeta}{R_x}} \right)^2 \frac{\partial w}{\partial x} = 0 \quad (\text{A-14})$$

$$\frac{\partial}{\partial \zeta} \left( \frac{V}{1 + \frac{\zeta}{R_y}} \right) + \left( \frac{1}{1 + \frac{\zeta}{R_y}} \right)^2 \frac{\partial w}{\partial y} = 0$$

The first of Equations A-14 specifies that

$$w = w(x, y) \quad (\text{A-15a})$$

and after integrating the remaining two equations with respect to  $\zeta$ , the displacements become

$$U(x, y, \zeta) = \left(1 + \frac{\zeta}{R_x}\right) u(x, y, 0) - \zeta \frac{\partial w}{\partial x} \quad (\text{A-15b})$$

$$V(x, y, \zeta) = \left(1 + \frac{\zeta}{R_y}\right) v(x, y, 0) - \zeta \frac{\partial w}{\partial y} \quad (\text{A-15c})$$

where  $u(x, y, 0)$  and  $v(x, y, 0)$  are the median surface displacements.

Substituting Equations A-15 into Equations A-12a, A-12b, and A-12d and assuming that the shell is sufficiently thin so that  $\frac{\zeta}{R_x}$  and  $\frac{\zeta}{R_y}$  are negligible in comparison to unity, yields the final form of the strain displacement relations for the elliptic-paraboloidal shell. These kinematic relations are

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{w}{R_x} - \zeta \frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{w}{R_y} - \zeta \frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2\zeta \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \quad (\text{A-16})$$

### C. STRESS-STRAIN RELATIONS

Stress-strain relations, suitable for thin shells, are

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x) \\ \tau_{xy} &= \frac{E}{2(1+\nu)} \gamma_{xy}\end{aligned}\tag{A-17}$$

where the normal stress,  $\sigma_\zeta$ , has been omitted, since it does not contribute to the analysis. However, the integral of  $\sigma_\zeta$  through the thickness of the shell is an essential feature of the problem, as it gives rise to the surface loading.

Substituting the strain displacement relations, Equations A-16, into Equations A-17 results in

$$\begin{aligned}\sigma_x &= \left( \frac{E}{1-\nu^2} \right) \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} + \left( \frac{1}{R_x} + \frac{\nu}{R_y} \right) w - \zeta \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ \sigma_y &= \left( \frac{E}{1-\nu^2} \right) \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} + \left( \frac{1}{R_y} + \frac{\nu}{R_x} \right) w - \zeta \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ \tau_{xy} &= \frac{E}{2(1+\nu)} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2\zeta \frac{\partial^2 w}{\partial x \partial y} \right]\end{aligned}\tag{A-18}$$

which are the requisite stress displacement relations for a thin, shallow, elliptic paraboloidal shell.

In accordance with the thinness assumption, i. e.,  $\frac{\zeta}{R_x}$  and  $\frac{\zeta}{R_y}$  are negligible in comparison with unity, the force and moment resultants are defined by

$$\begin{aligned}
N_x &= \int_{-t/2}^{t/2} \sigma_x d\zeta & M_x &= \int_{-t/2}^{t/2} \zeta \sigma_x d\zeta \\
N_y &= \int_{-t/2}^{t/2} \sigma_y d\zeta & M_y &= \int_{-t/2}^{t/2} \zeta \sigma_y d\zeta \\
N_{xy} &= \int_{-t/2}^{t/2} \tau_{xy} d\zeta & M_{xy} &= \int_{-t/2}^{t/2} \zeta \tau_{xy} d\zeta
\end{aligned} \tag{A-19}$$

with the associated sign convention shown in Figure 6. The transverse shear resultants,  $Q_x$  and  $Q_y$ , and the surface load  $p(x, y)$  are given by

$$\begin{aligned}
Q_x &= \int_{-t/2}^{t/2} \tau_{x\zeta} d\zeta \\
Q_y &= \int_{-t/2}^{t/2} \tau_{y\zeta} d\zeta \\
p(x, y) &= \int_{-t/2}^{t/2} \sigma_\zeta d\zeta
\end{aligned} \tag{A-20}$$

The force and moment resultants can be evaluated from Equations A-18, and are

$$N_x = A \left[ \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} + \left( \frac{1}{R_x} + \frac{\nu}{R_y} \right) w \right]$$

$$N_y = A \left[ \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} + \left( \frac{1}{R_y} + \frac{\nu}{R_x} \right) w \right]$$

$$N_{xy} = \frac{(1 - \nu)}{2} A \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]$$

(A-21)

$$M_x = -D \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right]$$

$$M_y = -D \left[ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right]$$

$$M_{xy} = - (1 - \nu) D \frac{\partial^2 w}{\partial x \partial y}$$

where as usual  $D = \frac{Et^3}{12(1 - \nu^2)}$  and  $A = \frac{Et}{1 - \nu^2}$ . Consequently, the stresses

can now be expressed in terms of the resultants and Equations A-18 will then take the form

$$\sigma_x = \frac{N_x}{t} + \frac{12 \zeta M_x}{t^3}$$

$$\sigma_y = \frac{N_y}{t} + \frac{12 \zeta M_y}{t^3}$$

(A-22)

$$\tau_{xy} = \frac{N_{xy}}{t} + \frac{12 \zeta M_{xy}}{t^3}$$

#### D. EQUILIBRIUM EQUATIONS

A consistent set of equilibrium equations and associated boundary conditions, compatible with the assumed kinematic and constituent stress-strain relations, will be developed by employing the Theorem of Minimum Potential Energy. The potential energy ( $\pi$ ) is given by

$$\pi = \pi_s - \pi_w \quad (\text{A-23})$$

where  $\pi_s$  is the strain energy and  $\pi_w$  is the work done by the boundary forces. Setting the first variation of the potential energy equal to zero will lead to the equilibrium conditions.

The strain energy functional is taken as

$$\pi_s = \frac{1}{2} \iiint \left\{ \sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy} \right\} d_x d_y d\zeta \quad (\text{A-24a})$$

and after putting Equations A-16 and A-18 into A-24a and integrating the resulting equation with respect to  $\zeta$  the strain energy becomes, in terms of the displacements,

$$\begin{aligned} \pi_s = & \frac{A}{2} \iint \left\{ \left( \frac{\partial u}{\partial x} + \frac{w}{R_x} \right)^2 + 2\nu \left( \frac{\partial u}{\partial x} + \frac{w}{R_x} \right) \left( \frac{\partial v}{\partial y} + \frac{w}{R_y} \right) + \left( \frac{\partial v}{\partial y} + \frac{w}{R_y} \right)^2 \right. \\ & \left. + \frac{(1-\nu)}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} dx dy + \frac{D}{2} \iint \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right. \\ & \left. + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} dx dy \end{aligned} \quad (\text{A-24b})$$

The work done by the boundary forces ( $\pi_w$ ) is given by

$$\pi_w = \int \int_{\zeta s} \bar{F} \cdot \bar{Z} ds d\zeta + \int \int p w dx dy \quad (\text{A-25a})$$

where  $\bar{F}$  is the force vector

$$\bar{F} = X\hat{i}_x + Y\hat{i}_y + Z\hat{i}_n \quad (\text{A-25b})$$

and the components are

$$\begin{aligned} X &= \sigma_x \cos(\ell, x) + \tau_{xy} \cos(\ell, y) \\ Y &= \tau_{xy} \cos(\ell, x) + \sigma_y \cos(\ell, y) \\ Z &= \tau_{x\zeta} \cos(\ell, x) + \tau_{y\zeta} \cos(\ell, y) \end{aligned} \quad (\text{A-25c})$$

In the above equations  $\ell$  is the outward drawn normal to the boundary curve in the reference surface of the shell. The displacement vector  $\bar{Z}$  is

$$\bar{Z} = U\hat{i}_x + V\hat{i}_y + w\hat{n} \quad (\text{A-25d})$$

or (from Equations A-15b and A-15c with  $\frac{\zeta}{R_x}$ ;  $\frac{\zeta}{R_y} \ll 1$ )

$$\bar{Z} = \left( u - \zeta \frac{\partial w}{\partial x} \right) \hat{i}_x + \left( v - \zeta \frac{\partial w}{\partial y} \right) \hat{i}_y + w \hat{n} \quad (\text{A-25e})$$

Substituting Equations A-25b, A-25c, and A-25e into Equation A-25a and integrating through the thickness results in

$$\begin{aligned} \pi_w = \int_s \left\{ \left[ N_x \cos(\ell, x) + N_{xy} \cos(\ell, y) \right] u \right. \\ + \left[ N_{xy} \cos(\ell, x) + N_y \cos(\ell, y) \right] v + \left[ Q_x \cos(\ell, x) + Q_y \cos(\ell, y) \right] w \\ - \left[ M_x \cos(\ell, x) + M_{xy} \cos(\ell, y) \right] \frac{\partial w}{\partial x} - \left[ M_{xy} \cos(\ell, x) \right. \\ \left. + M_y \cos(\ell, y) \right] \frac{\partial w}{\partial y} \left. \right\} ds + \iint p w \, dx dy \end{aligned} \quad (\text{A-26})$$



Before proceeding with the minimization process a portion of the work integral ( $\pi_w$ ) is recast in order to display the "Kirchhoff free edge condition." To this end, consider the identity

$$\begin{aligned}
 & \int \left\{ \left[ Q_x \cos(\ell, x) + Q_y \cos(\ell, y) \right] w - \left[ M_x \cos(\ell, x) + M_{xy} \cos(\ell, y) \right] \frac{dw}{dx} \right. \\
 & \quad \left. - \left[ M_{xy} \cos(\ell, x) + M_y \cos(\ell, y) \right] \frac{\partial w}{\partial y} \right\} ds = \\
 & \int \left\{ \left[ \left( Q_x + \frac{\partial M_{xy}}{\partial y} \right) \cos(\ell, x) + \left( Q_y + \frac{\partial M_{xy}}{\partial x} \right) \cos(\ell, y) \right] w \right. \\
 & \quad \left. - M_x \cos(\ell, x) \frac{\partial w}{\partial x} - M_y \cos(\ell, y) \frac{\partial w}{\partial y} \right. \\
 & \quad \left. - \frac{\partial}{\partial x} (M_{xy} w) \cos(\ell, y) - \frac{\partial}{\partial y} (M_{xy} w) \cos(\ell, x) \right\} ds \quad (A-27a)
 \end{aligned}$$

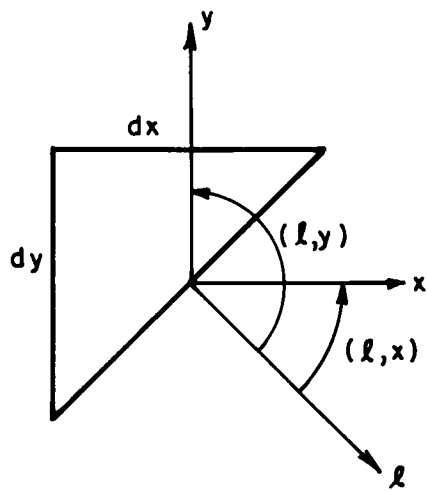
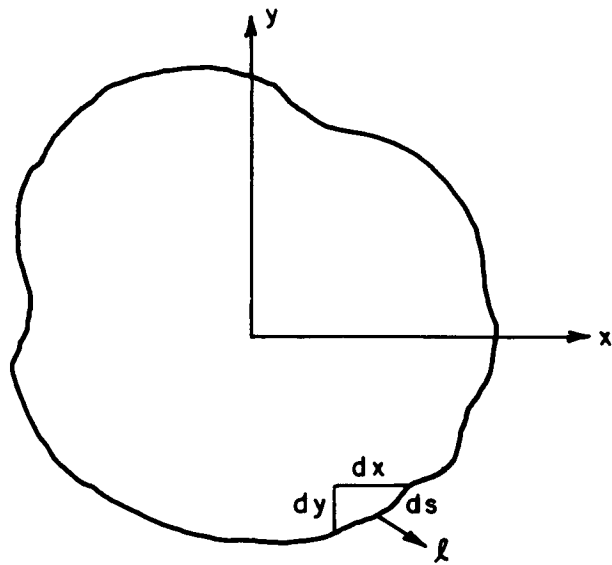
According to the sign convention shown in Figure 6 and the relations between  $dx$ ,  $dy$ , and  $ds$  (Figure A-3), which are

$$\begin{aligned}
 \cos(\ell, x) &= \frac{dy}{ds} \\
 \cos(\ell, y) &= -\frac{dx}{ds}
 \end{aligned} \quad (A-27b)$$

then

$$\begin{aligned}
 & \int \left\{ \frac{\partial}{\partial x} (M_{xy} w) \cos(\ell, y) + \frac{\partial}{\partial y} (M_{xy} w) \cos(\ell, x) \right\} ds = \\
 & = \int \left\{ -\frac{\partial (M_{xy} w)}{\partial x} dx + \frac{\partial (M_{xy} w)}{\partial y} dy \right\} = 0 \quad (A-27c)
 \end{aligned}$$

The vanishing of this integral follows from the evaluation of the contour integral over a rectangular path with the sign convention given in Figure 6.



$$\cos(l, x) = \frac{dy}{ds}$$

$$\cos(l, y) = -\frac{dx}{ds}$$

Figure A-3. Direction Cosines Along a Contour in the x, y Plane

Making use of the identity, Equation A-27a, with the associated simplification given by Equation A-27c, in A-26, the work integral reduces to

$$\begin{aligned} \pi_w = & \int \left\{ [N_x \cos(\ell, x) + N_{xy} \cos(\ell, y)] u + [N_{xy} \cos(\ell, x) + N_y \cos(\ell, y)] v \right. \\ & + \left[ \left( Q_x + \frac{\partial M_{xy}}{\partial y} \right) \cos(\ell, x) + \left( Q_y + \frac{\partial M_{xy}}{\partial x} \right) \cos(\ell, y) \right] w \\ & \left. - M_x \cos(\ell, x) \frac{\partial w}{\partial x} - M_y \cos(\ell, y) \frac{\partial w}{\partial y} \right\} ds + \iint pw \, dx dy \quad (A-28) \end{aligned}$$

Thus, the potential energy is composed of the difference between the strain energy, given by A-24b, and the work done by the boundary forces, given by A-28.

Minimization of the potential energy follows by application of the usual procedures in the Calculus of Variations. That is, the displacements  $u$ ,  $v$ , and  $w$  are replaced by the one parameter family of comparison functions  $u+\eta\delta u$ ,  $v+\eta\delta v$ , and  $w+\eta\delta w$  in Equation A-23. The potential energy is then expressed as a function of the parameter  $\eta$ , and the necessary condition for  $\pi$  to assume a stationary value is

that  $\frac{d\pi(\eta)}{d\eta} = 0$  as  $\eta$  approaches zero.

Making the appropriate substitution in Equations A-24b and A-28 and performing the indicated operations results in

$$\begin{aligned} A \iint \left\{ \left( \frac{\partial u}{\partial x} + \frac{w}{R_x} \right) \left( \frac{\partial \delta u}{\partial x} + \frac{\delta w}{R_x} \right) + \nu \left( \frac{\partial u}{\partial x} + \frac{w}{R_x} \right) \left( \frac{\partial \delta v}{\partial y} + \frac{\delta w}{R_y} \right) \right. \\ \left. + \nu \left( \frac{\partial \delta u}{\partial x} + \frac{\delta w}{R_x} \right) \left( \frac{\partial v}{\partial y} + \frac{w}{R_y} \right) + \left( \frac{\partial v}{\partial y} + \frac{w}{R_y} \right) \left( \frac{\partial \delta v}{\partial y} + \frac{\delta w}{R_y} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{(1-\nu)}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left( \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) \} dx dy + D \iint \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \delta w}{\partial x^2} \right. \\
& + \nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \delta w}{\partial y^2} + \nu \frac{\partial^2 \delta w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \delta w}{\partial y^2} + 2(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \delta w}{\partial x \partial y} \left. \right\} dx dy \\
& - \int_s \left\{ \left[ \bar{N}_x \cos(\ell, x) + \bar{N}_{xy} \cos(\ell, y) \right] \delta u + \left[ \bar{N}_{xy} \cos(\ell, x) + \bar{N}_y \cos(\ell, y) \right] \delta v \right. \\
& + \left[ \left( \bar{Q}_x + \frac{\partial \bar{M}_{xy}}{\partial y} \right) \cos(\ell, x) + \left( \bar{Q}_y + \frac{\partial \bar{M}_{xy}}{\partial x} \right) \cos(\ell, y) \right] \delta w \\
& \left. - \bar{M}_x \cos(\ell, x) \frac{\partial \delta w}{\partial x} - \bar{M}_y \cos(\ell, y) \frac{\partial \delta w}{\partial y} \right\} ds - \iint p \delta w dx dy = 0 \quad (A-29)
\end{aligned}$$

where the bar notation on the force and moment resultants denotes prescribed edge quantities. Recalling the force and moment resultant-displacement relations (Equations A-21), permits Equation A-29 to be put into the form

$$\begin{aligned}
& \iint \left\{ N_x \frac{\partial \delta u}{\partial x} + N_y \frac{\partial \delta v}{\partial y} + N_{xy} \left( \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) + \left( \frac{N_x}{R_x} + \frac{N_y}{R_y} - p \right) \delta w \right. \\
& \left. - M_x \frac{\partial^2 \delta w}{\partial x^2} - M_y \frac{\partial^2 \delta w}{\partial y^2} - 2 M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} \right\} dx dy \\
& - \int_s \left\{ \left[ \bar{N}_x \cos(\ell, x) + \bar{N}_{xy} \cos(\ell, y) \right] \delta u + \left[ \bar{N}_{xy} \cos(\ell, x) + \bar{N}_y \cos(\ell, y) \right] \delta v \right. \\
& + \left[ \left( \bar{Q}_x + \frac{\partial \bar{M}_{xy}}{\partial y} \right) \cos(\ell, x) + \left( \bar{Q}_y + \frac{\partial \bar{M}_{xy}}{\partial x} \right) \cos(\ell, y) \right] \delta w - \bar{M}_x \cos(\ell, x) \frac{\partial \delta w}{\partial x} \\
& \left. - \bar{M}_y \cos(\ell, y) \frac{\partial \delta w}{\partial y} \right\} ds = 0 \quad (A-30)
\end{aligned}$$

Employing the identity

$$C \frac{\partial B}{\partial \xi} = \frac{\partial CB}{\partial \xi} - B \frac{\partial C}{\partial \xi} \quad (\text{A-31a})$$

and then making use of Green's theorem

$$\iint_R \left[ \frac{\partial C}{\partial x} + \frac{\partial B}{\partial y} \right] dR = \int_s (Cdy - Bdx) \quad (\text{A-31b})$$

with the relations A-27b, Equation A-31b becomes

$$\int_s (Cdy - Bdx) = \int \left( C \cos(\ell, x) + B \cos(\ell, y) \right) ds \quad (\text{A-31c})$$

and the final form of Equation A-29 can be written as

$$\begin{aligned} & \iint \left\{ -\delta u \left[ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right] - \delta v \left[ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right] \right. \\ & \quad \left. - \delta w \left[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{N_x}{R_x} - \frac{N_y}{R_y} + p \right] \right\} dx dy \\ & + \int_s \left\{ \left[ N_x \cos(\ell, x) + N_{xy} \cos(\ell, y) - \left( \bar{N}_x \cos(\ell, x) + \bar{N}_{xy} \cos(\ell, y) \right) \right] \delta u \right. \\ & \quad + \left[ N_{xy} \cos(\ell, x) + N_y \cos(\ell, y) - \bar{N}_{xy} \cos(\ell, x) + \bar{N}_y \cos(\ell, y) \right] \delta v \\ & \quad - \left[ (M_x - \bar{M}_x) \cos(\ell, x) \frac{\partial \delta w}{\partial x} + (M_y - \bar{M}_y) \cos(\ell, y) \frac{\partial \delta w}{\partial y} \right] \\ & \quad + \left[ \left( \frac{\partial M_x}{\partial x} + \frac{2 \partial M_{xy}}{\partial y} - \bar{Q}_x - \frac{\partial \bar{M}_{xy}}{\partial y} \right) \cos(\ell, x) \right. \\ & \quad \left. + \left( \frac{2 \partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - \bar{Q}_y - \frac{\partial \bar{M}_{xy}}{\partial x} \right) \cos(\ell, y) \right] \delta w \left. \right\} ds = 0 \quad (\text{A-32}) \end{aligned}$$

Since the domain of integration is arbitrary, then satisfaction of Equation A-32 requires the vanishing of the integrands. The area integrals, lead to the field equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (\text{A-33a})$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (\text{A-33b})$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{2 \partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{N_x}{R_x} - \frac{N_y}{R_y} + p = 0 \quad (\text{A-33c})$$

and the contour integral stipulates the associated boundary conditions. A consistent set must include

$$u \quad \text{or} \quad N_x \cos(\ell, x) + N_{xy} \cos(\ell y) \quad (\text{A-34a})$$

$$v \quad \text{or} \quad N_{xy} \cos(\ell, x) + N_y \cos(\ell y) \quad (\text{A-34b})$$

$$w \quad \text{or} \quad \left( Q_x + \frac{\partial M_{xy}}{\partial y} \right) \cos(\ell x) + \left( Q_y + \frac{\partial M_{xy}}{\partial x} \right) \cos(\ell y) \quad (\text{A-34c})$$

$$\frac{\partial w}{\partial x} \quad \text{or} \quad M_x \quad (\text{A-34d})$$

$$\frac{\partial w}{\partial y} \quad \text{or} \quad M_y \quad (\text{A-34e})$$

In addition, the arbitrariness of the boundaries leads to the field equations for the transverse shear resultants

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \quad (\text{A-35a})$$

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \quad (\text{A-35b})$$

The formulation is now completed and the coupled linear system of fourteen partial differential equations for the fourteen unknowns,  $u$ ,  $v$ ,  $w$ ,  $\epsilon_x$ ,  $\epsilon_y$ ,  $\gamma_{xy}$ ,  $N_x$ ,  $N_y$ ,  $N_{xy}$ ,  $M_x$ ,  $M_y$ ,  $M_{xy}$ ,  $Q_x$ , and  $Q_y$ , consists of Equations A-16, A-21, A-33, and A-35, subject to the boundary conditions listed in Equations A-34.

#### E. COMPLEX DEFLECTION-STRESS FUNCTION

The governing equations for the stresses and deflections (which are fourteen in number) in a thin, shallow elliptic paraboloidal shell can be reduced to one fourth-order partial differential equation in terms of a complex deflection-stress function. To this end, one may define the auxiliary stress function,  $F$ , which is related to the force resultants by

$$\begin{aligned} N_x &= \frac{\partial^2 F}{\partial y^2} \\ N_{xy} &= - \frac{\partial^2 F}{\partial x \partial y} \\ N_y &= \frac{\partial^2 F}{\partial x^2} \end{aligned} \quad (\text{A-36})$$

and satisfies Equations A-33a and A-33b identically. Then, making use of the last three relations of Equations A-21 in A-33c in conjunction with Equation A-36 results in

$$D \nabla^4 W + \nabla_k^2 F = p(x, y) \quad (\text{A-37})$$

where

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ \nabla_k^2 &= \frac{1}{R_x} \frac{\partial^2}{\partial y^2} + \frac{1}{R_y} \frac{\partial^2}{\partial x^2} \end{aligned} \quad (\text{A-38})$$

$$\nabla^4 F - (1 - \nu^2) A \nabla_k^2 W = 0 \quad (\text{A-39})$$

The system of Equations A-37 and A-39 for the deflection and stress function will be reduced to a single fourth-order partial differential equation by introducing the complex function  $\varphi$ , defined by

$$\varphi = w + \frac{iF}{[(1 - \nu^2) AD]^{1/2}} \quad (\text{A-40})$$

where now  $i = \sqrt{-1}$ . Substituting from Equations A-40 into Equations A-37 and A-39 leads to the controlling equation for  $\varphi$ , which is

$$\nabla^4 \varphi - i \delta^2 \nabla_k^2 \varphi = \frac{p(x,y)}{D} \quad (\text{A-41a})$$

with

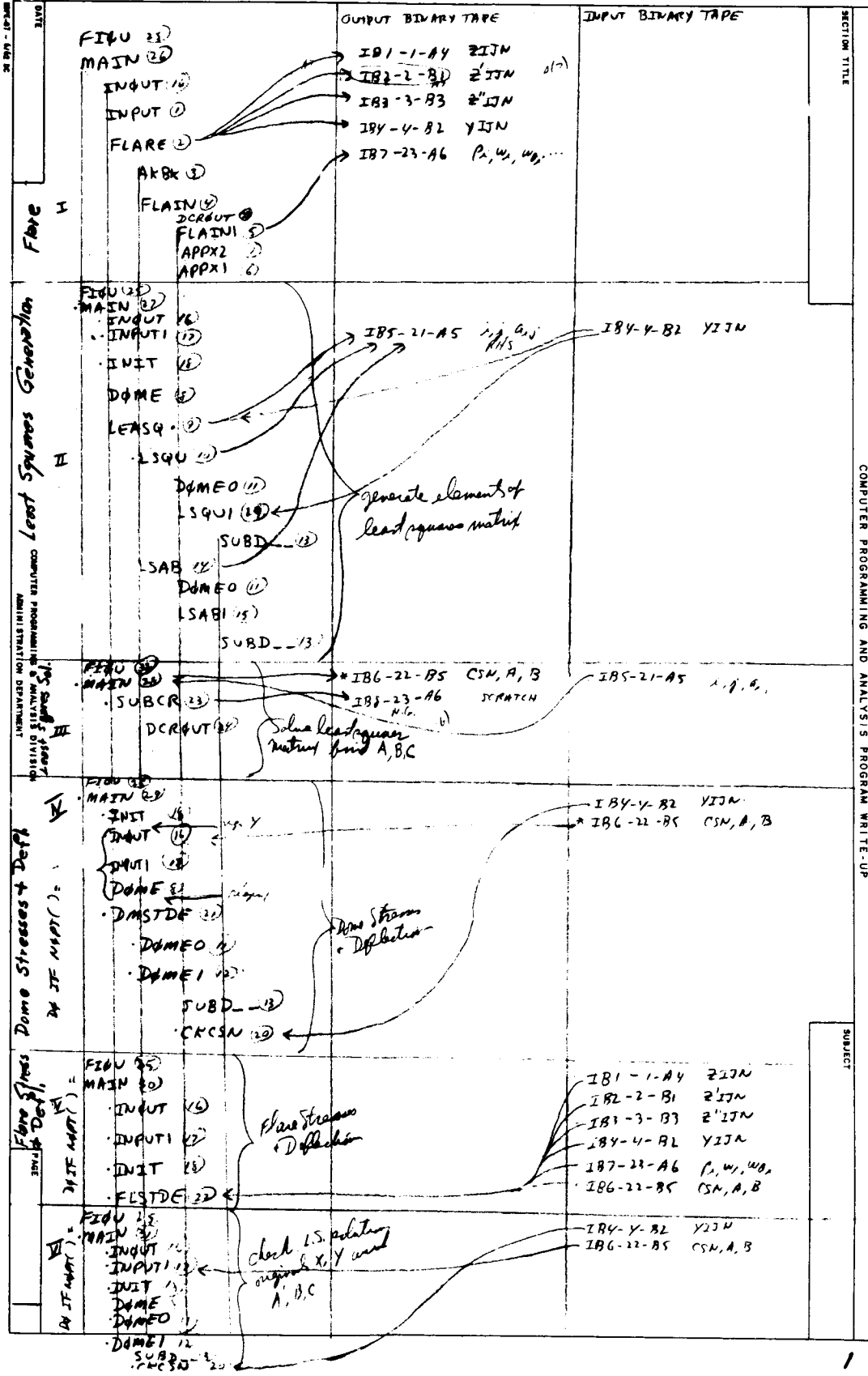
$$\delta^2 = \frac{(1 - \nu^2)A}{D} \quad (\text{A-41b})$$

Thus, the solution to a problem involving a thin, shallow elliptic-paraboloidal shell reduces to the solution of Equation A-41a subject to the admissible boundary conditions (Equations A-34). With  $\varphi$  determined, all the field quantities can be evaluated by direct application of the interconnecting relations, Equations A-40, A-36, A-21 and A-35.



APPENDIX IV  
PROGRAM FLOW CHARTS

SECTION TITLE		SUBJECT	
COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP		61HG23-FLARED NOZZLE - C3442	
PAGE		PAGE	
1	GENERAL LAY OUT OF PARTS I-III WITH POSITIVE ARCS - INPUT-OUTPUT TABLES	45	SUBROUTINE DOMEI #11
2	FLAGS FOR PRINT OUT UC(1)	46	SUBD12-3A DESCRIPTION #13.1-13.2
3	OPTIONAL NOPT(1)	47	SUBD12-#13.1 SUBD34-#13.2
4	INPUT SYMBOLS - INPUT1, INPUT2	48	SUBD56-#13.3 SUBD78-#13.4
5	INPUT FORMAT INPUT #1	49	SUBD90-#13.5 SUBD92-#13.6
6	"	50	SUBDDE-#13.7 SUBDFF-#13.8
7	" TO PART III (CANDS)	51	SUBDHI-#13.9 SUBDJK-#13.10
8	INPUT #17	52	SUBDLM-#13.11 SUBDNP-#13.12
9	PART II - FLOW TABLE DESCRIPTION	53	SUBDPQ-#13.13 SUBDPS-#13.14
10	ZONE EQUATIONS FOR LEAST SQ.	54	SUBDTU-#13.15 SUBDVW-#13.16
11	DIAGRAM OF LEAST SQUARES BEHAVIOR	55	SUBDXY-#13.17 SUBDZ-#13.18
12	INPUT TABLE DESCRIPTION OF FLARE	56	SUBDIA-#13.19 SUBDJA-#13.20
13	SUBROUTINES INPUT#8, INPUT#16	57	SUBD3A-#13.21
14	" INPUT #17	58	MAIN #21 PART III
15	MAIN #26, INPUT #16, INPUT #1	59	SUBROUTINE SUBCR #23
16	SUBROUTINE INPUT #1	60	MAIN #29 DOME STRESSES - DEF PART II
17	" FLARE #2	61	SUBROUTINE DMSTDE #21
18	"	62	" DOMEI #12
19	"	63	"
20	"	64	"
21	"	65	"
22	" AREA #3	66	"
		67	SUBROUTINE DOMEI #13
		68	"
		69	" CRCSL # 20
		70	MAIN #30 FLARE STRESSES - DEF PART II
		71	SUBROUTINE FLSTDE #22
		72	"
		73	"
		74	MAIN #31 CHECK L.S. SOL. DOME #9 PART II
		75	"
		76	"
		77	"
		78	"
		79	"
		80	"
		81	"
		82	"
		83	"
		84	"
		85	"
		86	"
		87	"
		88	"



SECTION TITLE

COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP

SUBJECT

DATE	NO(I)		NO(I)		NO(I)		NO(I)		SECTION TITLE
	1		37	DAME1	73		109		COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP I = 1, 144
	2		38	DAME1	74	APPX2	110		
	3		39	DAME1	75	APPX1	111	FLSTDE	
	4		40	DAME1	76	DASTDE	112	FLSTDE	
	5		41	DAME1	77	DASTDE	113	FLSTDE	
	6		42	DAME1	78	DASTDE	114	FLSTDE	
	7		43	DAME1	79		115	FLSTDE	
	8		44	DAME1	80		116	FLSTDE	
	9		45	DAME1	81		117	FLSTDE	
	10		46	DAME1	82		118	FLSTDE	
	11	DAME	47	DAME1	83		119	FLSTDE	
	12	DAME	48	DAME1	84		120	FLSTDE	
	13	DAME	49	DAME1	85	LEASQ	121	SUBDVW	
	14	DAMEO	50		86	LSQUI	122	SUBDXY	
	15	DAMEO	51		87	LSQUI	123	SUBDZ	
	16	DAME	52		88	LSQUI	124	SUBD1A	
	17	DAME	53		89		125	SUBD2A	
	18		54		90	LSQUI	126	SUBD3A	
	19		55		91	LSQUI	127		
	20	SUBD12	56		92		128		
	21	SUBD34	57		93	LSAB	129		
	22	SUBD56	58		94	LSAB1	130		
	23	SUBD78	59		95	LSAB1	131		
	24	SUBD9A	60	FLARE	96	LEASQ	132		
	25	SUBDBC	61	AKBK	97		133		
	26	SUBDDE	62	FLARE	98	LSQU LHM, AC, RASLS(1)	134		
	27	SUBDFG	63	FLARE	99	LSQU	135		
	28	SUBDHI	64	FLARE	100		136		
	29	SUBDJK	65	FLARE	101		137		
	30	SUBDKM	66	FLARE	102		138		
	31	SUBDNQ	67	FLARE	103		139		
	32	SUBDPQ	68		104		140	CHCSN	
	33		69	FLADU1	105		141	CHCSN	
	34	DAME1, SUBDRS, SUBDTU	70	FLAZN	106		142	CHCSN	
	35	DAME1	71		107		143	CHCSN	
	36	DAME1	72	FLAZN	108		144	LSQU	

COMPUTER PROGRAMMING & ANALYSIS DIVISION  
ADMINISTRATION DEPARTMENT

PAGE 1 OF 2  
DATE 1/15/54



FORM 1 - 6/65 DC

DATE	PHYSICAL SYMBOLS	PROGRAM SYMBOLS	PHYSICAL DESCRIPTION INPUT #17	PHYSICAL SYMBOLS	PROGRAM SYMBOLS	PHYSICAL DESCRIPTION INPUT #1	SECTION TITLE
	Lx	LX	Half Length of Dome - x Direction	E	EF	Flare Modulus of Elasticity	INPUT SYMBOLS AND EXPLANATIONS FOR INPUT #17, INPUT #1
	Ly	LY	" " " " - y	ν	NU	Poisson's Ratio	
	t <sub>0</sub>	TD	Dome Thickness	p <sub>int</sub>	PSI	Internal Pressure	
	1/R <sub>x</sub>	RATIX	Curvature - x Direction	r <sub>c</sub>	RC	Characteristic Radius	
	1/R <sub>y</sub>	RATIDY	" " " " - y	t <sub>c</sub>	TC	Thickness	
	E	ED-ELAS	Dome Modulus of Elasticity	σ <sub>0</sub>	SIGC	" stress	
	A	AD	Dome Stretching Stiffness		NENET		
	D	DD	Dome Bending		NBAR		
	ν	NUD-NU	Poisson's Ratio	n̄	NB		
	N <sub>F</sub>	NF	Number of Flare Difference Pts	n̄	NBB		
		UNIT		a	A	Flare Radius at 0 station	
	ψ°	PSID	Angle Flare tangent makes with dome	κ <sub>R</sub>	RPK( )		
	p <sub>int</sub>	PSI	Internal Pressure	ε <sub>R</sub>	EPK( )		
	r <sub>c</sub>	RC	Characteristic Radius	t <sub>0</sub>	TTP		
	t <sub>c</sub>	TC	" thickness	t <sub>R</sub>	TRK( )		
	σ <sub>0</sub>	SIGC	" stress	ψ°	PSID	Angle flare tangent makes with dome	
	t <sub>0</sub>	TTP	Flare Thickness at 0 station	α	ALPHA	Ellipse semi-major axis	
	a	ACR	" Radius at "	β	BETA	" " - minor "	
	R <sub>i</sub>	RINIT	Radius of Flare-Dome Junction	T <sub>F</sub>	TF	Flare Thickness	
	ΔR	DELTA R	Dome Station at which Dome-Dome Junction		SC	Length of Cylinder Above Flare	
	X <sub>0</sub>	XO	PART II ONLY IF NENET(A) ≠ 0 MUSTDCL	t <sub>0</sub>	TO		
	Δx	DELTA X	" " " " " " " " " " " "		TRDD		
	Δy	DELTA Y	" " " " " " " " " " " "	ε <sub>F</sub>	EF		
	θ <sub>0</sub>	THETAO	" Dome station at which Dome-Dome Junction		AF		
	θ <sub>0</sub>	THETAO	station		DF		
	θ <sub>cut</sub>	THLAST	cut	m	M	No. Diff. Stations in Cylinder	
		XD( )		N <sub>F</sub>	N <sub>F</sub>	No. Flare Diff. Stations	
		YD( )		E	CREPS	FRISON FOR CROUT ROUTINE	
		XPD( )		w <sub>0</sub>	DMZP		
		YPD( )		w <sub>R</sub>	DMPK		
		THD( )					
		THDP( )					
	A <sub>0</sub>	AOLS					
	A <sub>1</sub>	A1LS( )					
	B <sub>0</sub>	BOLS					
	B <sub>1</sub>	B1LS( )					
	G <sub>1</sub>	CSN( )					

COMPUTER PROGRAMMING & ANALYSIS DIVISION  
ADMINISTRATION DEPARTMENT

COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP

SUBJECT

INPUT #1      FORMAT      PROB.      PAGE 1 OF 2

SUBMITTED BY EXT.		CHARGE NO.	DATE							
OPERATION NOTES		WORD 1	WORD 2	WORD 3	WORD 4	WORD 5	WORD 6	WORD 7	WORD 8	
1	Col 1-72									
2	FORMAT (72I) PRINT FLAGS									
3	FORMAT (72I) " " "									
4	FORMAT (72I) OPTION FLAGS									
5	FORMAT (15I5) INPUT INDICATOR	1								
6	FORMAT (7E10.0)	EF								
7	FORMAT (5I5) INPUT INDICATOR	2								
8	FORMAT (7E10.0)	3								
9	FORMAT (15I5)	AC								
10	FORMAT (7E10.0)	4								
11	FORMAT (15I5)	HEAT								
12	"	5								
13	"	6								
14	"	7								
15	"	8								
16	FORMAT (7E10.0)	APK(1)								
17	FORMAT (15I5)	APK(2)								
18	FORMAT (7E10.0)	APK(1)								
19	FORMAT (15I5)	APK(2)								
20	FORMAT (7E10.0)	TPK(1)								
21	FORMAT (15I5)	TPK(2)								
22	FORMAT (7E10.0)	TPK(1)								
23	FORMAT (15I5)	TPK(2)								
24	FORMAT (7E10.0)	PSID								
25	"	TF								
26	FORMAT (15I5)	TF								
27	FORMAT (7E10.0)	TF								
28	FORMAT (15I5)	TF								
29	"	TF								
30	"	TF								





CARD INPUT TO PART III CALLED FROM MAIN # 28

SUBMITTED BY EXT.	OPERATION NOTES	CHARGE NO.								PAGE DATE								
		WORD 1	WORD 2	WORD 3	WORD 4	WORD 5	WORD 6	WORD 7	WORD 8	WORD 1	WORD 2	WORD 3	WORD 4	WORD 5	WORD 6	WORD 7	WORD 8	
	1 UNCL(1) AT 1, IS PRINT (OFF)	9	20111213	18151617	1815202122	24252627282930	3132333435363738	394041424344454647	48495051525354555657	58596061626364656667	68697071727374757677	78798081828384858687	88899091929394959697	98990001020304050607	108109110112113114115116117	126127128129130131132133134135136137	144145146147148149150151152153154155156157	164165166167168169170171172173174175176177
	2 E.g. EASY FORMAT (26/10, I10)																	
	3 USED IN SUBCF # 23																	
	4																	
	5 See from CHART MAIN # 28																	
	6																	
	7																	
	8																	
	9																	
	10																	
	11																	
	12																	
	13																	
	14																	
	15																	
	16																	
	17																	
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INPUT # 17 FORMAT

SUBMITTED BY EXT.	OPERATION NOTES	CHARGE NO.								PAGE OF DATE							
		WORD 1	WORD 2	WORD 3	WORD 4	WORD 5	WORD 6	WORD 7	WORD 8	WORD 1	WORD 2	WORD 3	WORD 4	WORD 5	WORD 6	WORD 7	WORD 8
1	CO/ 1-72																
2	FORMAT (2,1) PRINT FLAGS	MC(I)	IF, 72														
3	FORMAT (2,1) " "	NC(I)	IF 73, 74, 75														
4	FORMAT (2,1) OPTION FLAGS	NAPT(I)	I=1, 72														
5	FORMAT (2E10.0)	LX	LY	TD													
6	"	ED	AD	DD													
7	"	NDY	MDX	NDYA													
8	"	MF	MINIT	PO													
9	"	PST	RC	TC													
10	"	RINIT	OR														
11	" (DMSTDE) NPT6970	XO	OX	BY													
12	" (DMSTDE)	PO	OP	BLAST													
13	"	XD(1)	XD(2)														
14	"	YD(1)	YD(2)														
15	"	X'D(1)	X'D(2)														
16	"	Y'D(1)	Y'D(2)														
17	"	OD(1)	OD(2)														
18	"	OD(1)	OD(3)														
19	FORMAT (2E15.0)	AOLS	AOLS(1)														
20	"	AOLS(4)															
21	"	BOLS	BOLS(1)														
22	"	BOLS(4)															
23	"	CSM(1,1)	CSM(1,2)														
24	"	CSM(2,1)	CSM(2,2)														
25	"																
26	"																
27	"	CSM(E+1,1)	CSM(E+1,2)														
28																	
29																	
30																	











CARD NO	TEXT	MARKS	DESCRIPTION
1	READ IN : TITLE(I) I=1,12		
2	NC(I) I=1,12		
3	NC(I) I=13,144		
4	NAPT(I) I=1,12		
5	LX, LY, TD, RATIOX, RATIOY, ID, JD		
6	ED, AD, DB, J		
7	NBY, NDX, NBYE, NDXP, NDB, NDB', N		
8	NE, NMIT, VS		
9	PI, RC, TC, CE, TO, AIC		
10	RMIT, OR		
11	XO, OX, OY		USED IN DOME STRESSES + DEFLECTIONS (DMSTDE #21) OPTION IF NAPT(I) #0
12	BO, OB, OBY		USED IN DOME STRESSES + DEFLECTIONS (DMSTDE #21)
13	YO(A) A=1, NDX	21	X'S USED TO GENERATE EQUATIONS IN LMM=16 Q4
14	YO(A) A=1, NBY	21	Y'S USED TO GENERATE EQUATIONS IN LMM=15 Q4
15	XB(A) A=1, NDXP	21	X'S USED TO GENERATE EQUATIONS IN LMM=10,12,14,15, (SUX, SY, MY, NY)
16	YB(A) A=1, NDXP	21	Y'S USED TO GENERATE EQUATIONS IN LMM=9,11,13,17; (RXP, QX, MX, NX)
17	BO(A) A=1, NDB		
18	OB(A) A=1, NDB		
19	AOSS, AILS(A) A=1, ID		BO'S USED TO GENERATE EQUATIONS IN LMM=1-8; (UD, VD, WD, QR, NR, NRD, QR, NR)
20	BOSS, BZIS(A) A=1, ID		GENERATED IN PART III SOLUTIONS TO LEAST SQUARES
21	CSW(A, XS) A=1, Y, A=1, Z=1		INPUT TO CHECK DOME EQUATIONS

READ FROM CARDS IF NAPT(I) #1  
 " TAPE 106 " " =0  
 OMIT READ IN IF NAPT(I) #1



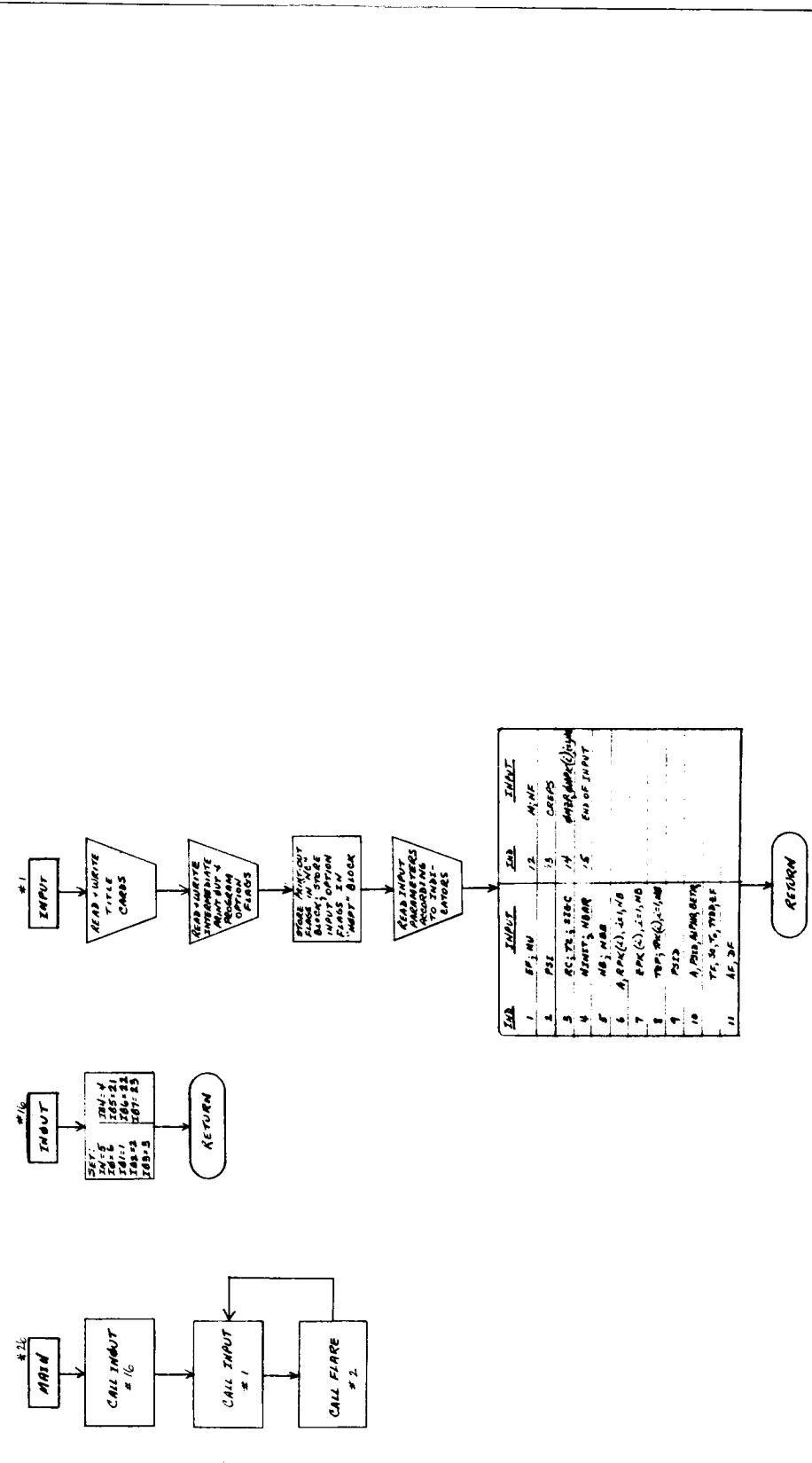
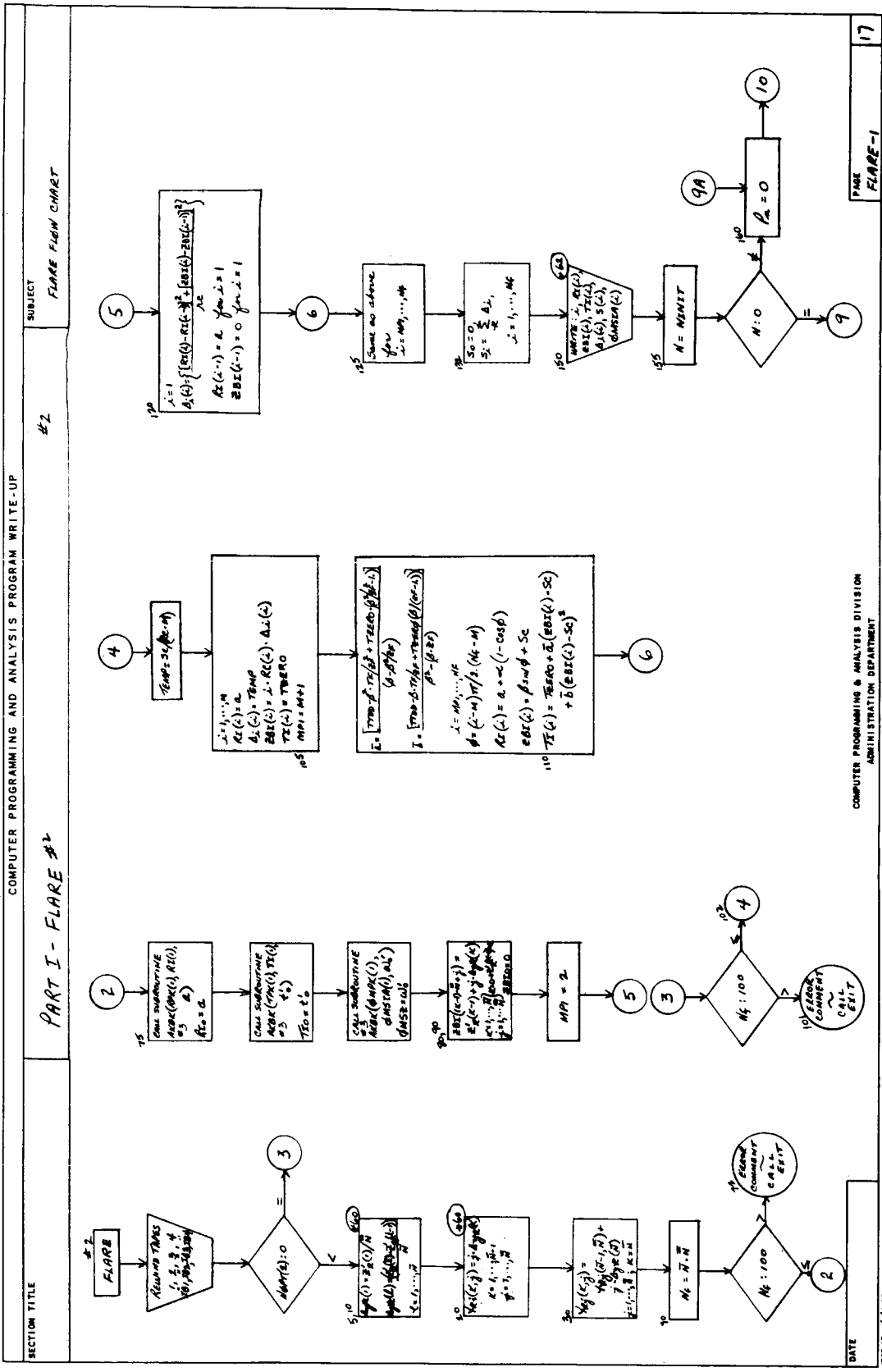


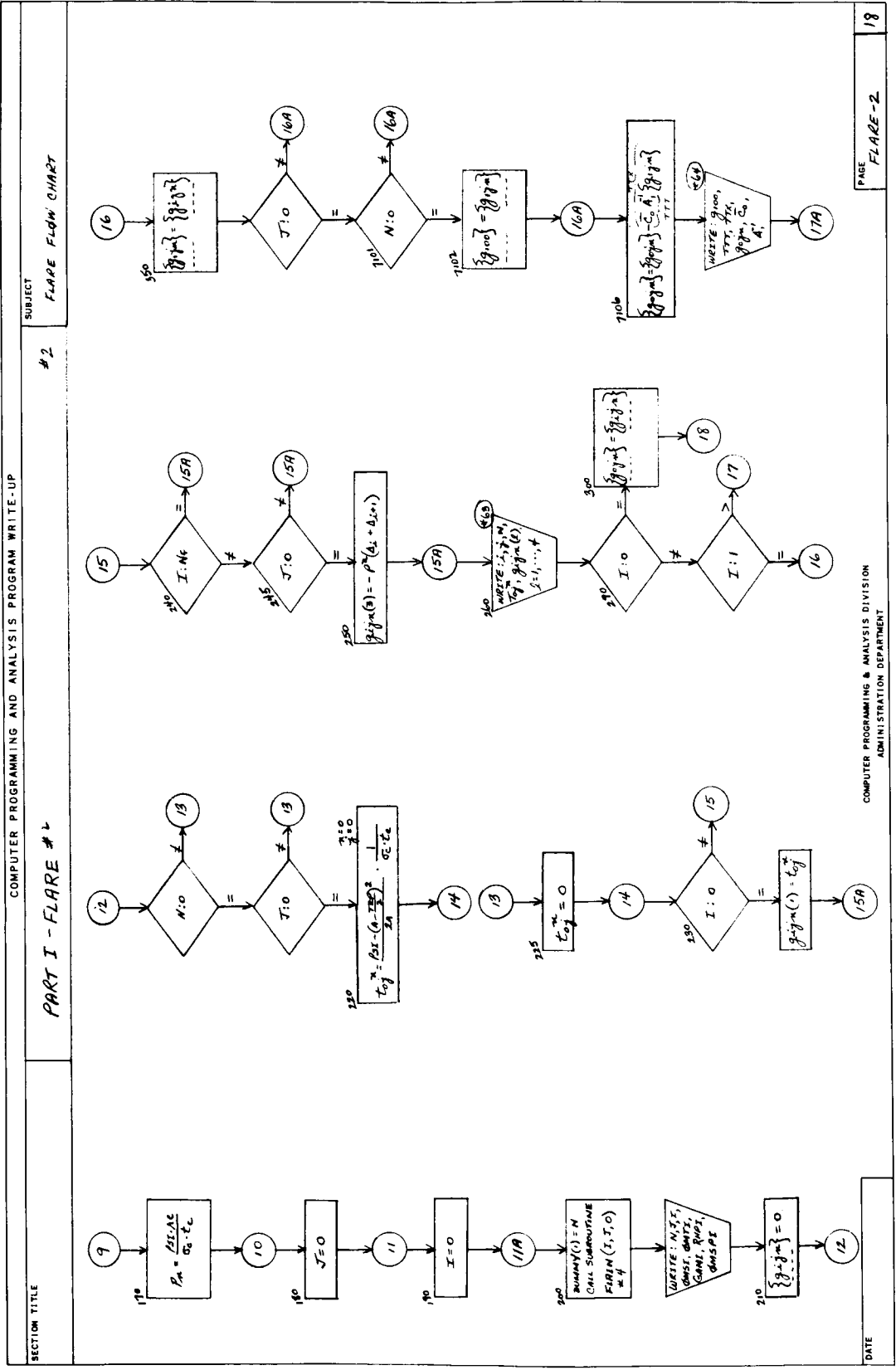
FIG 3  
 MAIN 4  
 INPUT 1  
 FLARE 3  
 AKBK 3  
 FLAIN 4  
 FLAINI 5  
 APPX2 7  
 APPX1 6

CARD INPUT - READ BY SUBROUTINE INPUT 1\*

CARD NO.	BLOCK	DESCRIPTION
1	TELE(I), I=1,...,12	TITLE CARD-BCD
2	NC(I), I=1,...,72	PRINT OUT OPTIONS - INTEGER BLANK OR 1
3	NC(I), I=73,...,144	"
4	NOPT(I), I=1,...,72	PROGRAM FLOW OPTIONS - INTEGER BLANK OR 1
THE REMAINING CARDS ARE READ IN UNDER OPTION - A CARD CONTAINING-		
THE OPTION NUMBER IN COLUMNS 1-5 MUST PRECEED EACH INPUT CARD OR		
OPTRM NO.	GROUP OF CARDS	
1	EF, NU	F, V - FLOATING PT.
2	PSI	PSI - FLOATING PT.
3	RC, TC, SECC	RC, IC, SE - FLOATING PT.
4	NEMIT, NBAR	NEMIT, NBAR - INTEGER
5	NB, NAB	N, N - INTEGER
6	A, RPK(I), I=1,...,NB	A, PAR(I), I=1,...,N - FLOATING PT.
7	ZPK(I), I=1,...,NB	ZPK(I), I=1,...,N - FLOATING PT.
8	TRP, TRK(I), I=1,...,NB	TRP(I), TRK(I), I=1,...,N - FLOATING PT.
9	PSID	W - FLOATING PT.
10	A, PSID, ALPHA, BETA, TC, TREQ, TRD, ZF	A, NU, X, P, TC, SC, T, C, D, Z, ZF - FLOATING PT.
11	AF, DF	GF, DG - FLOATING PT.
12	M, NF	M, NS - INTEGER
13	CREPS	C for subroutine DCROUT - FLOATING PT.
14	QMPK, QMPK(I), I=1,...,NB	QPO, QPK(I), I=1,...,N
15	RETURN FROM SUBROUTINE TO MAIN PROGRAM	

\* SEE ALSO INPUT #1 FORMAT

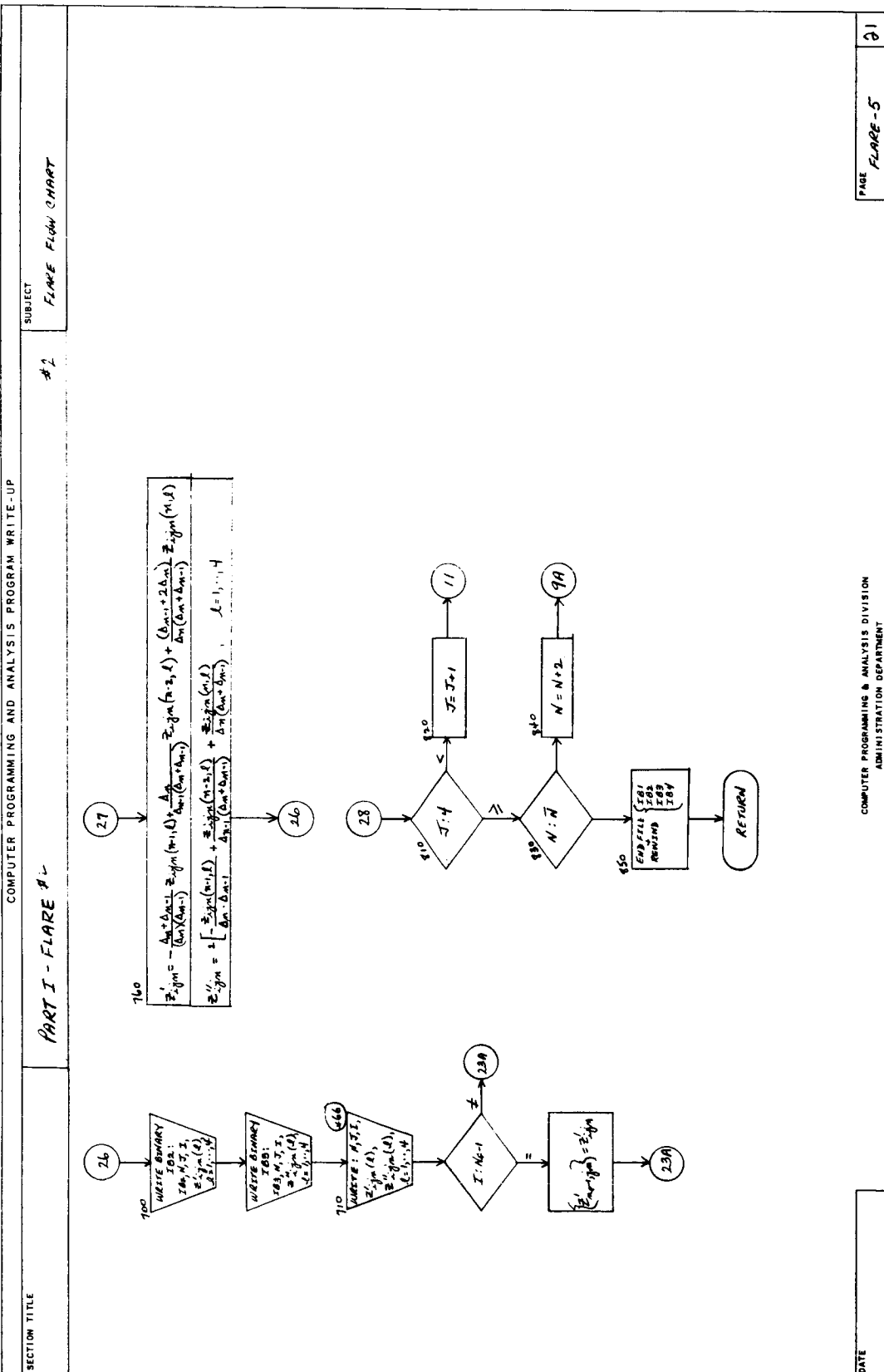




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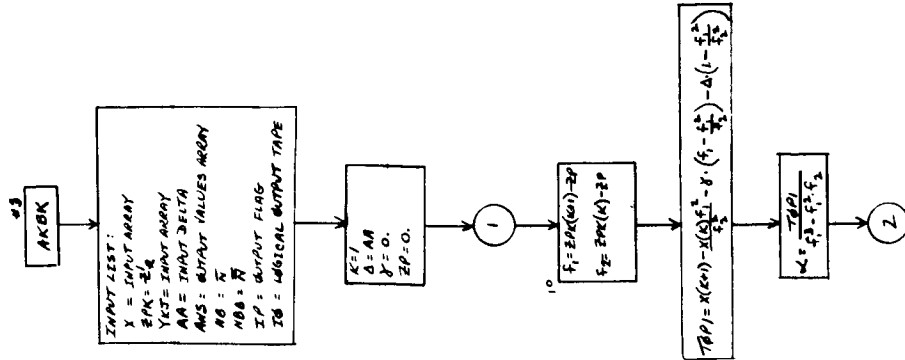
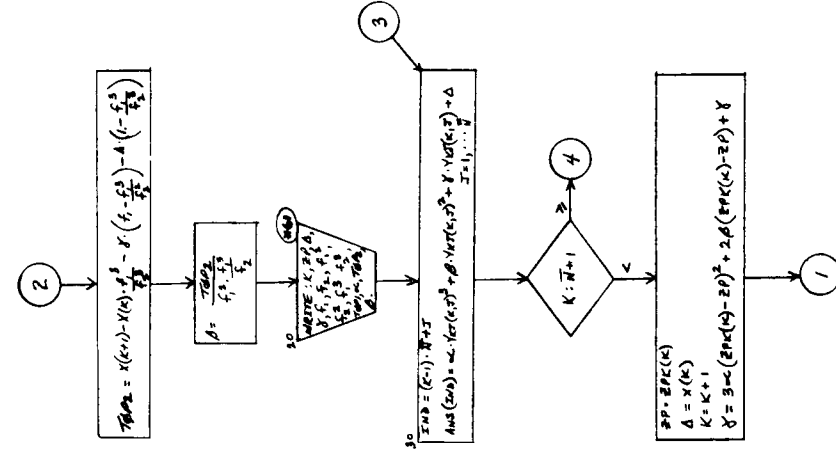
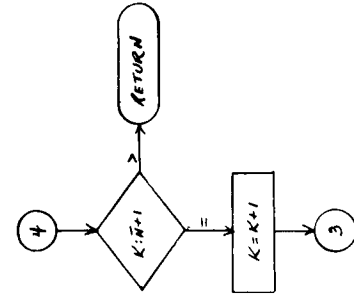
DATE: \_\_\_\_\_

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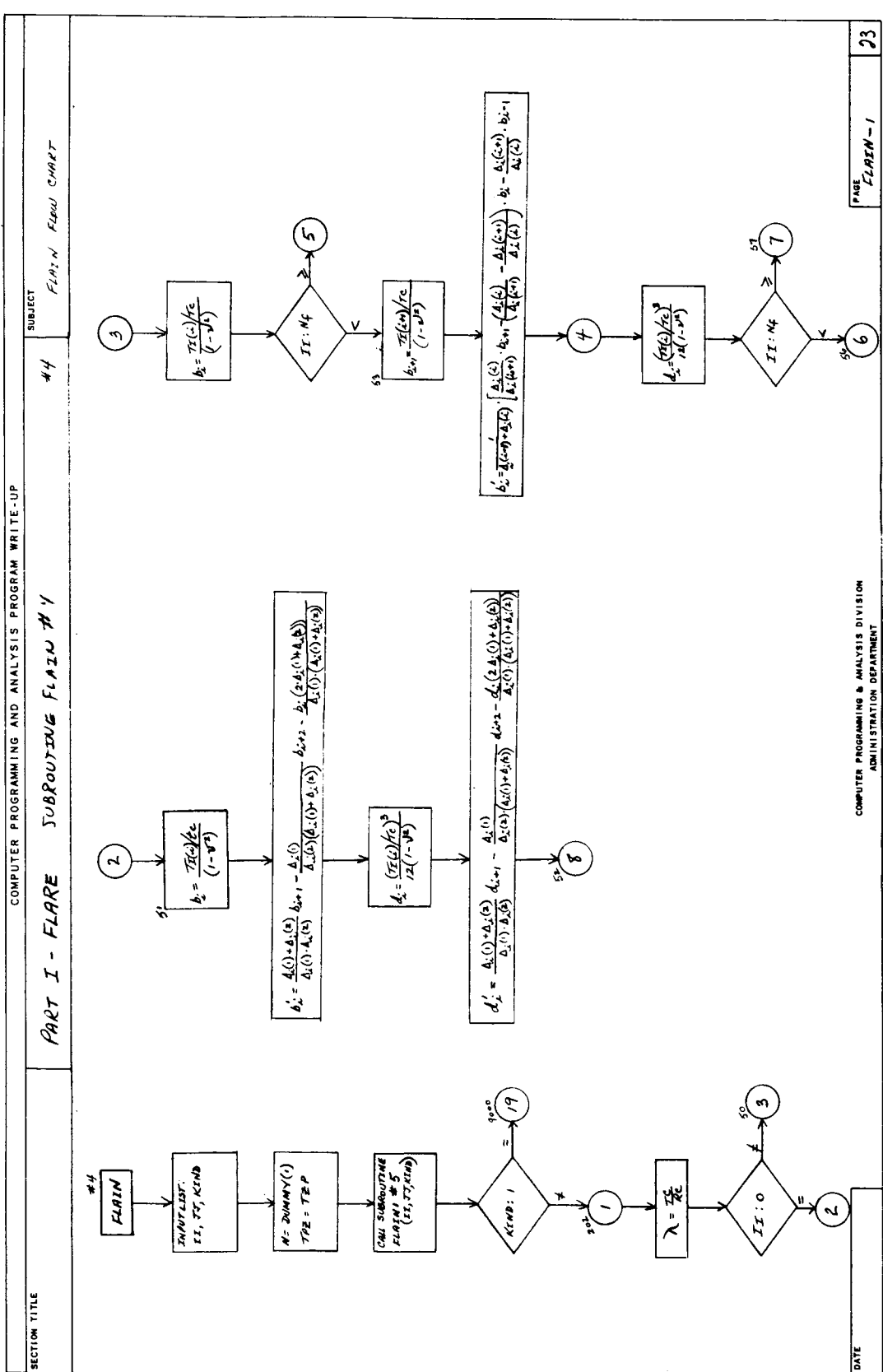
DPF-37 - 6/62 DC

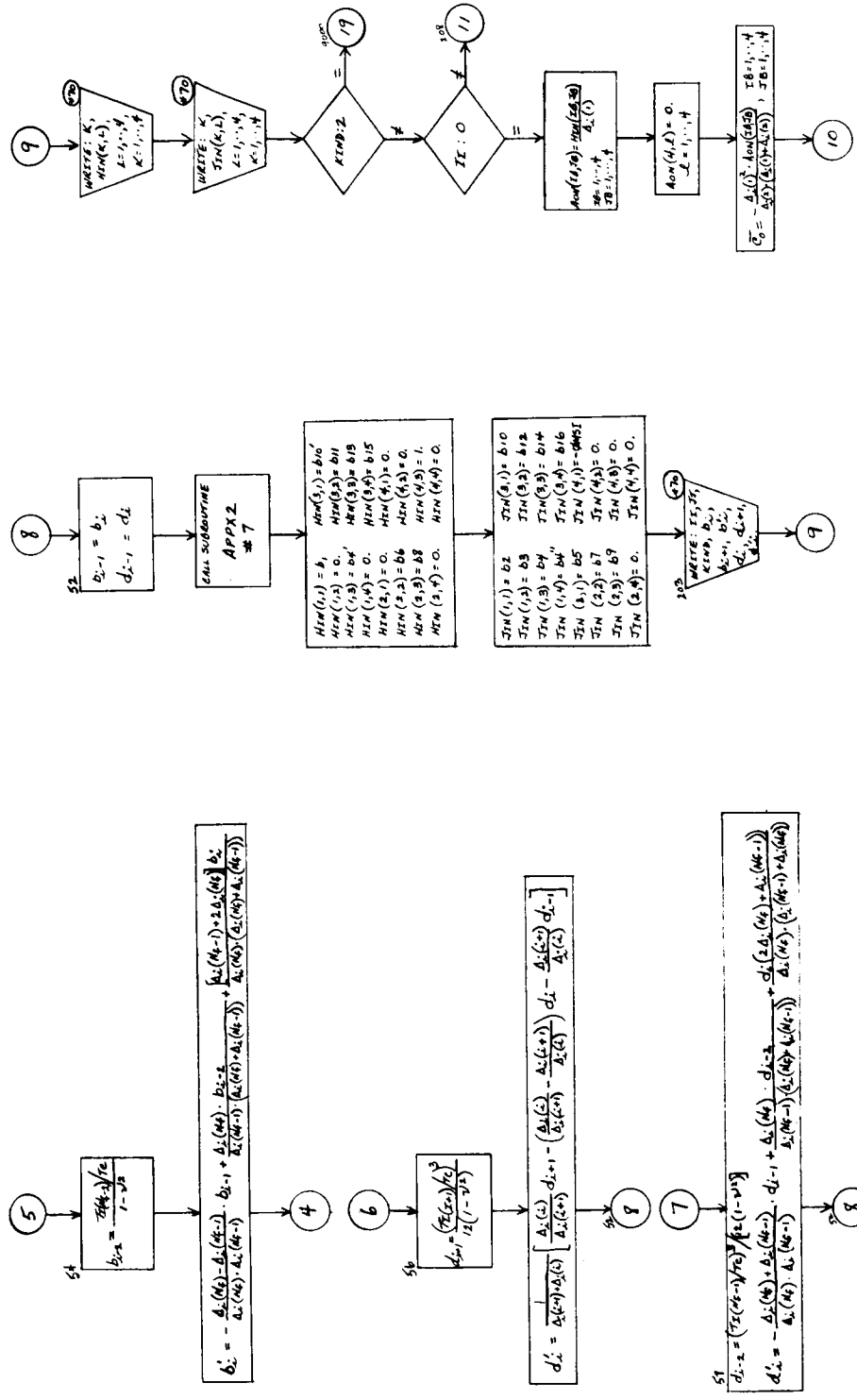
PAGE: **FLARE-5**

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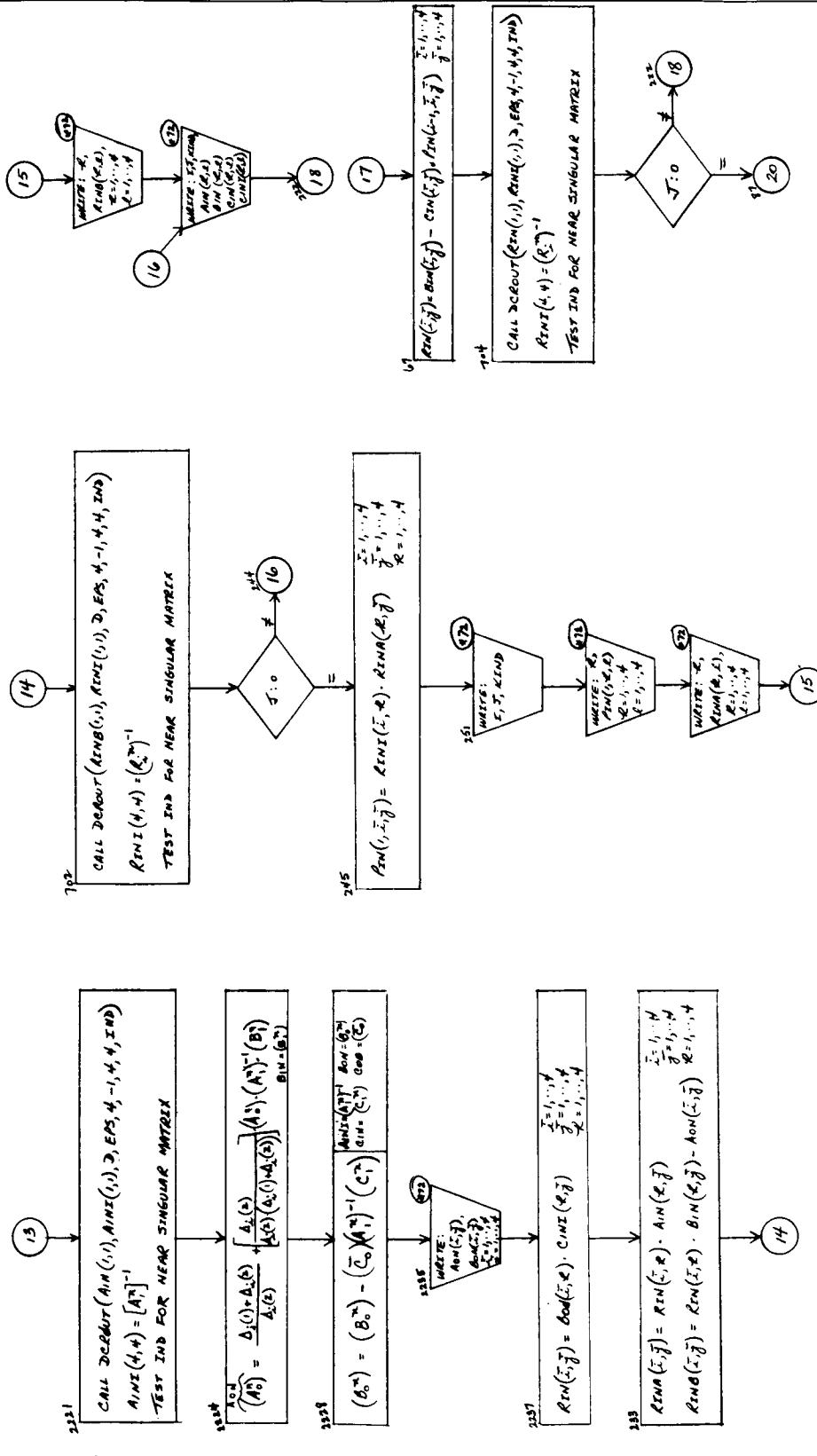




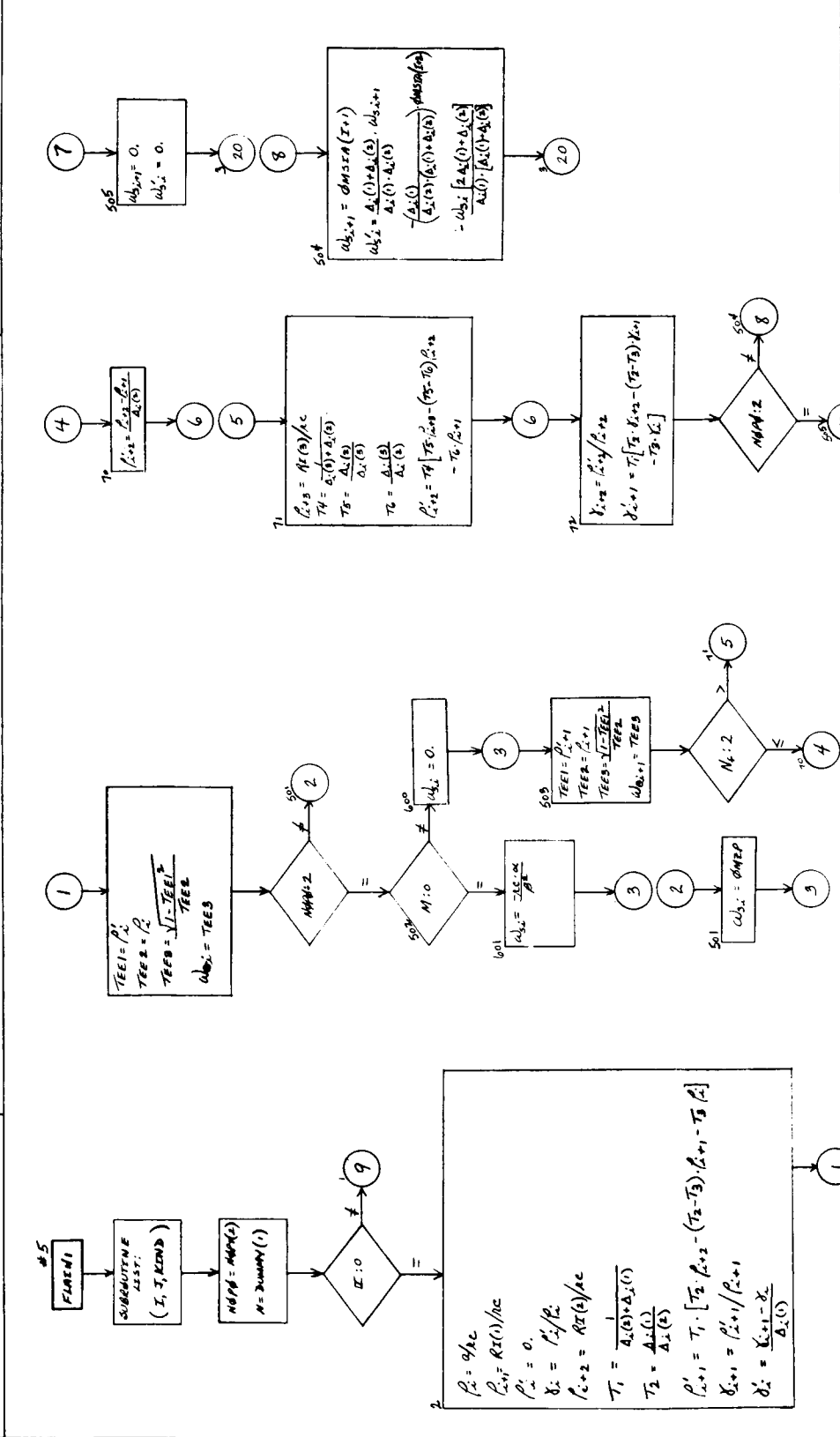


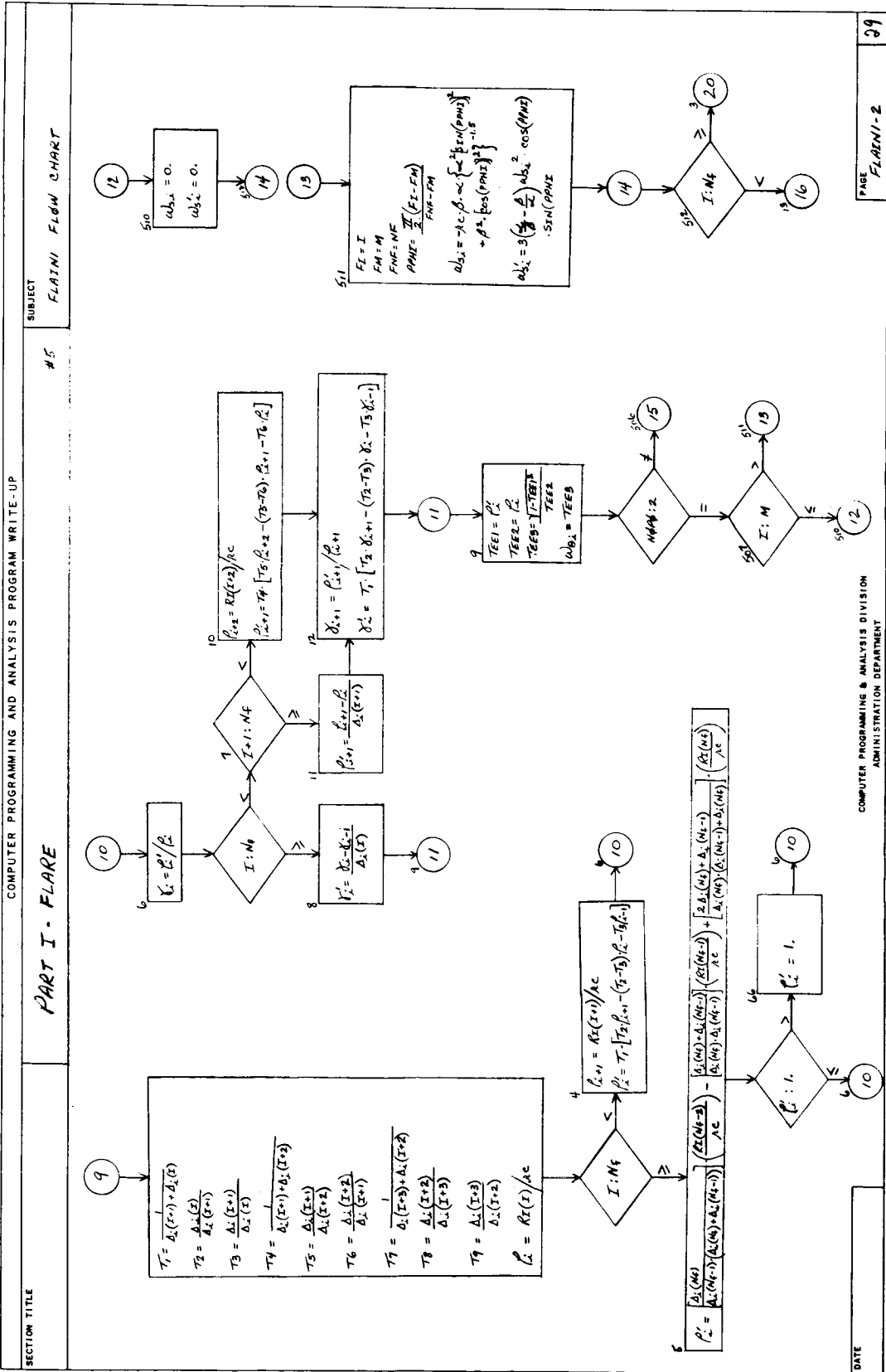






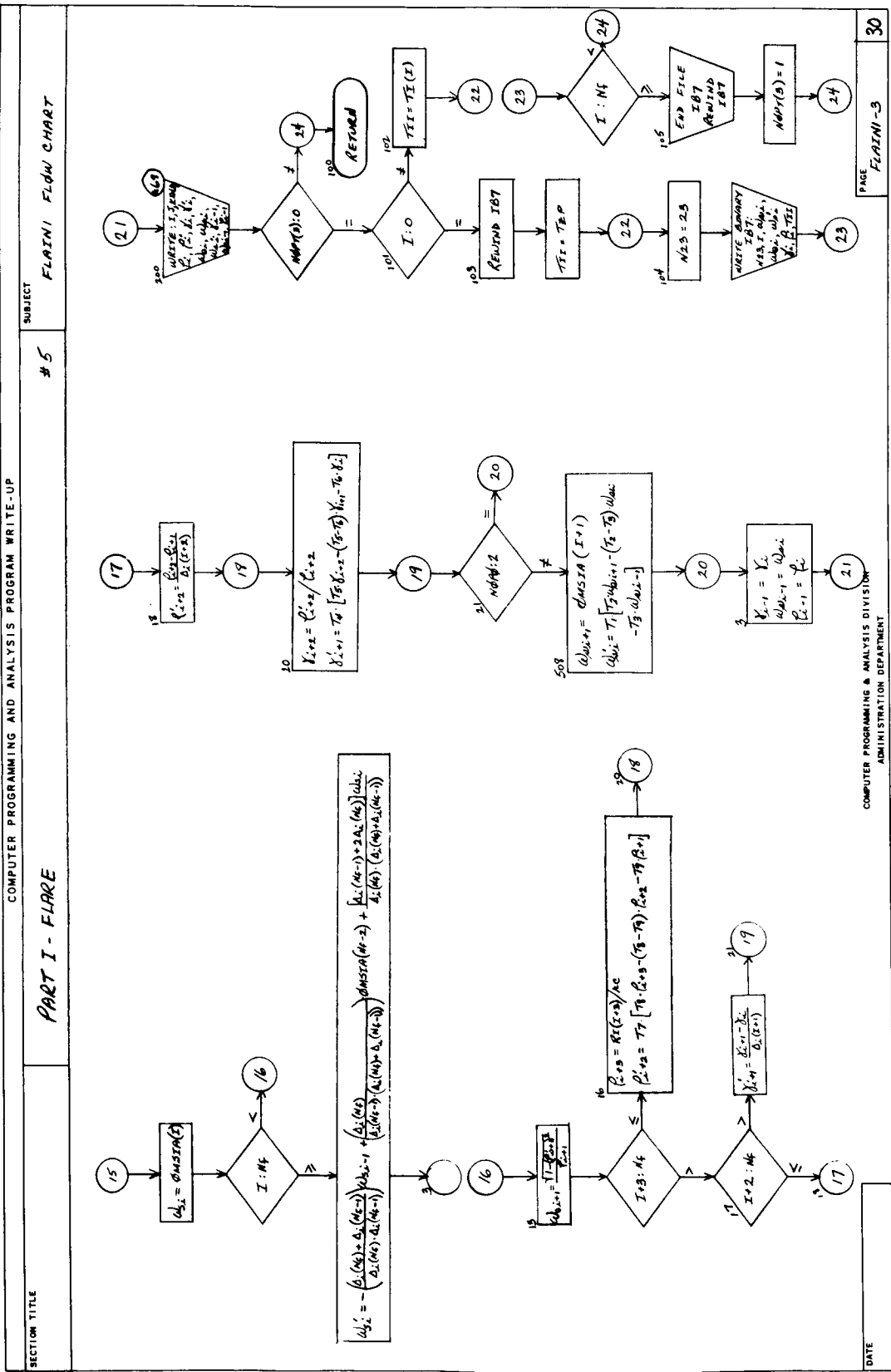






510  $\omega_{b,1} = 0.$   
 $\omega_{b,2} = 0.$

511  
 $FI = I$   
 $FM = M$   
 $PM = \frac{I(FI - FM)}{FM - FI}$   
 $\omega_{b,1} = -10 \cdot \beta \cdot \omega \cdot \left\{ \frac{2 \cdot \beta \cdot \sin(\beta \cdot \omega \cdot \tau)}{\beta \cdot \omega \cdot \tau} - 1.5 \right\}$   
 $\omega_{b,2} = 3 \left( \frac{\beta}{\omega} - \frac{\omega}{\beta} \right) \omega_{b,1}^2 \cdot \cos(\beta \cdot \omega \cdot \tau)$   
 $\cdot \sin(\beta \cdot \omega \cdot \tau)$

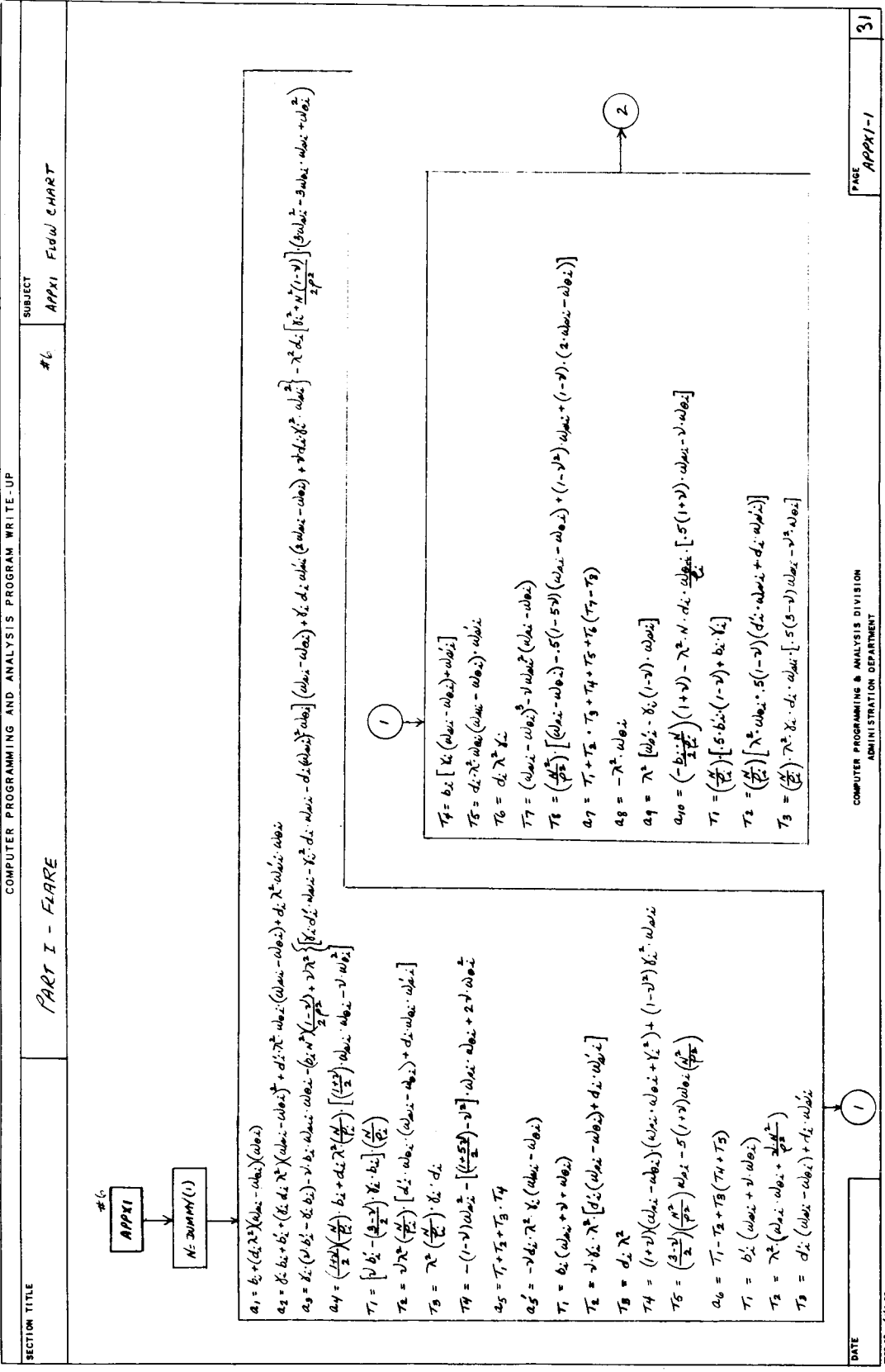


**22** 
$$u_{2s} = \frac{\beta_1(N_4) + \beta_2(N_4 - 1)}{\beta_1(N_4) + \beta_2(N_4 - 1)} u_{2s-1} + \frac{\beta_1(N_4) + 2\beta_2(N_4)}{\beta_1(N_4) + \beta_2(N_4 - 1)} \text{GAMMA}(N_4 - 2) + \frac{\beta_1(N_4 - 1) + 2\beta_2(N_4)}{\beta_1(N_4) + \beta_2(N_4 - 1)} u_{2s}$$

**23** 
$$u_{2s} = \frac{\beta_1(N_4) + \beta_2(N_4 - 1)}{\beta_1(N_4) + \beta_2(N_4 - 1)} u_{2s-1} + \frac{\beta_1(N_4) + 2\beta_2(N_4)}{\beta_1(N_4) + \beta_2(N_4 - 1)} \text{GAMMA}(N_4 - 2) + \frac{\beta_1(N_4 - 1) + 2\beta_2(N_4)}{\beta_1(N_4) + \beta_2(N_4 - 1)} u_{2s}$$

**24** 
$$u_{2s} = \frac{\beta_1(N_4) + \beta_2(N_4 - 1)}{\beta_1(N_4) + \beta_2(N_4 - 1)} u_{2s-1} + \frac{\beta_1(N_4) + 2\beta_2(N_4)}{\beta_1(N_4) + \beta_2(N_4 - 1)} \text{GAMMA}(N_4 - 2) + \frac{\beta_1(N_4 - 1) + 2\beta_2(N_4)}{\beta_1(N_4) + \beta_2(N_4 - 1)} u_{2s}$$

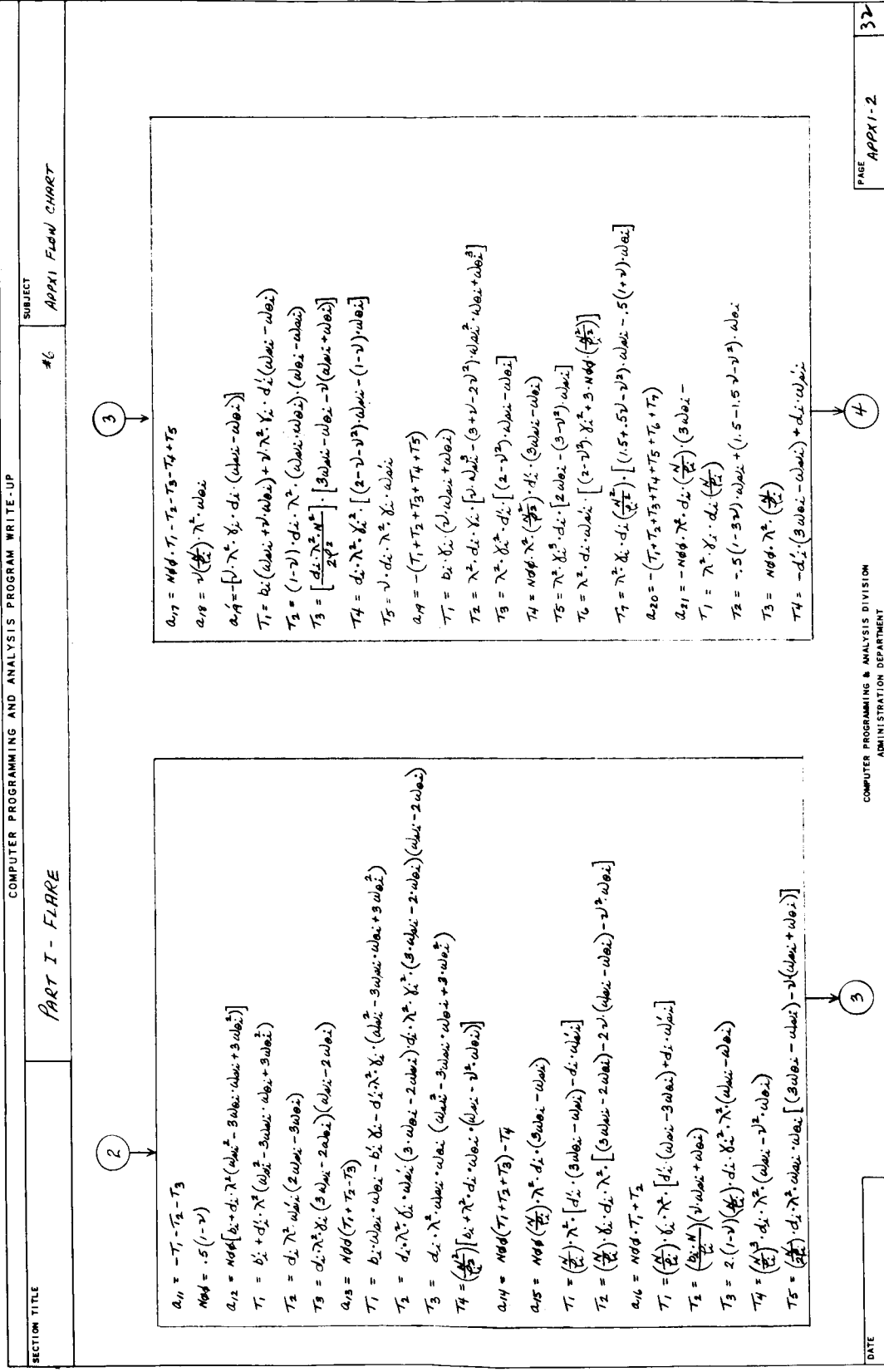




DATE: \_\_\_\_\_

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$$\begin{aligned}
 a_{11} &= -T_1 - T_2 - T_3 \\
 N06 &= .5(-v) \\
 a_{12} &= N06[b_1 + d_1 \cdot \lambda^2 \cdot \omega_{01}^2 - 3\omega_{01} \cdot \omega_{02} + 3\omega_{01}^2] \\
 T_1 &= b_1^2 + d_1^2 \cdot \lambda^2 \cdot \omega_{01}^2 - 3\omega_{01} \cdot \omega_{02} + 3\omega_{01}^2 \\
 T_2 &= d_1 \cdot \lambda^2 \cdot \omega_{01} \cdot (2\omega_{01} - 3\omega_{02}) \\
 T_3 &= d_1 \cdot \lambda^2 \cdot \omega_{01} \cdot (3\omega_{01} - 2\omega_{02}) \cdot (\omega_{01} - 2\omega_{02}) \\
 a_{13} &= N06(T_1 + T_2 + T_3) \\
 T_1 &= b_1 \cdot \lambda^2 \cdot \omega_{01} \cdot \omega_{02} - b_1^2 \cdot \lambda^2 \cdot \omega_{01}^2 - 3\omega_{01} \cdot \omega_{02} + 3\omega_{01}^2 \\
 T_2 &= d_1 \cdot \lambda^2 \cdot \omega_{01} \cdot (3\omega_{01} - 2\omega_{02}) \cdot d_1 \cdot \lambda^2 \cdot \omega_{01} \cdot (3\omega_{01} - 2\omega_{02}) \cdot (\omega_{01} - 2\omega_{02}) \\
 T_3 &= d_1 \cdot \lambda^2 \cdot \omega_{01} \cdot \omega_{02} \cdot (\omega_{01}^2 - 3\omega_{01} \cdot \omega_{02} + 3\omega_{01}^2) \\
 T_4 &= \left(\frac{d_1^2}{2\lambda^2}\right) [b_1 + \lambda^2 \cdot d_1 \cdot \omega_{01}] \cdot (\omega_{01} - 2\omega_{02}) \\
 a_{14} &= N06(T_1 + T_2 + T_3) - T_4 \\
 a_{15} &= N06\left(\frac{d_1}{\lambda^2}\right) \cdot \lambda^2 \cdot d_1 \cdot (3\omega_{01} - \omega_{02}) \\
 T_1 &= \left(\frac{d_1}{\lambda^2}\right) \cdot \lambda^2 \cdot [d_1 \cdot (3\omega_{01} - \omega_{02}) - d_1 \cdot \omega_{01}] \\
 T_2 &= \left(\frac{d_1}{\lambda^2}\right) \cdot d_1 \cdot \lambda^2 \cdot [3\omega_{01} - 2\omega_{02}] - 2v \cdot (\omega_{01} - \omega_{02}) - v^2 \cdot \omega_{01} \\
 a_{16} &= N06 \cdot T_1 + T_2 \\
 T_1 &= \left(\frac{d_1}{\lambda^2}\right) \cdot \lambda^2 \cdot [d_1 \cdot (\omega_{01} - 3\omega_{02}) + d_1 \cdot \omega_{01}] \\
 T_2 &= \left(\frac{d_1}{\lambda^2}\right) \cdot v \cdot \omega_{01} + \omega_{02} \\
 T_3 &= 2 \cdot (1-v) \cdot \left(\frac{d_1}{\lambda^2}\right) \cdot d_1 \cdot \lambda^2 \cdot (\omega_{01} - \omega_{02}) \\
 T_4 &= \left(\frac{d_1}{\lambda^2}\right) \cdot d_1 \cdot \lambda^2 \cdot (\omega_{01} - v)^2 \cdot \omega_{01} \\
 T_5 &= \left(\frac{d_1}{\lambda^2}\right) \cdot d_1 \cdot \lambda^2 \cdot \omega_{01} \cdot \omega_{02} \cdot [3\omega_{01} - \omega_{02}] - v(\omega_{01} + \omega_{02})
 \end{aligned}$$

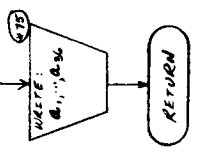
$$\begin{aligned}
 a_{17} &= N06 \cdot T_1 - T_2 - T_3 - T_4 + T_5 \\
 a_{18} &= v \cdot \left(\frac{d_1}{\lambda^2}\right) \cdot \lambda^2 \cdot \omega_{01} \\
 a_{19} &= -v \cdot \lambda^2 \cdot \omega_{01} \cdot d_1 \cdot (\omega_{01} - \omega_{02}) \\
 T_1 &= b_1 \cdot (\omega_{01} + v \cdot \omega_{02}) + v \cdot \lambda^2 \cdot \omega_{01} \cdot d_1 \cdot (\omega_{01} - \omega_{02}) \\
 T_2 &= (1-v) \cdot d_1 \cdot \lambda^2 \cdot (\omega_{01} \cdot \omega_{02}) \cdot (\omega_{01} - \omega_{02}) \\
 T_3 &= \left[\frac{d_1 \cdot \lambda^2 \cdot \omega_{01}^2}{2\lambda^2}\right] \cdot [3\omega_{01} - \omega_{02}] - v(\omega_{01} + \omega_{02}) \\
 T_4 &= d_1 \cdot \lambda^2 \cdot \omega_{01}^2 \cdot [(2-v) \cdot v^2] \cdot \omega_{01} - (1-v) \cdot \omega_{01} \\
 T_5 &= v \cdot d_1 \cdot \lambda^2 \cdot \omega_{01} \cdot \omega_{02} \\
 a_{19} &= -(T_1 + T_2 + T_3 + T_4 + T_5) \\
 T_1 &= b_1 \cdot \omega_{01} \cdot (v \cdot \omega_{01} + \omega_{02}) \\
 T_2 &= \lambda^2 \cdot d_1 \cdot \omega_{01} \cdot [v \cdot \omega_{01}^2 - (3+v-2v^2) \cdot \omega_{01} \cdot \omega_{02} + \omega_{02}^2] \\
 T_3 &= \lambda^2 \cdot \omega_{01}^2 \cdot d_1 \cdot [(2-v)^2] \cdot \omega_{01} - \omega_{01} \\
 T_4 &= N06 \cdot \lambda^2 \cdot \left(\frac{d_1}{\lambda^2}\right) \cdot d_1 \cdot (3\omega_{01} - \omega_{02}) \\
 T_5 &= \lambda^2 \cdot \omega_{01}^2 \cdot d_1 \cdot [2\omega_{01} - (3-v)^2] \cdot \omega_{01} \\
 T_6 &= \lambda^2 \cdot d_1 \cdot \omega_{01} \cdot [(2-v)^2] \cdot \omega_{01} + 3 \cdot N06 \cdot \left(\frac{d_1}{\lambda^2}\right) \\
 T_7 &= \lambda^2 \cdot \omega_{01} \cdot d_1 \cdot \left(\frac{d_1^2}{\lambda^2}\right) \cdot [(1.5+5v-v^2) \cdot \omega_{01} - .5(1+v) \cdot \omega_{02}] \\
 a_{20} &= -(T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7) \\
 a_{21} &= -N06 \cdot \lambda^2 \cdot d_1 \cdot \left(\frac{d_1}{\lambda^2}\right) \cdot (3\omega_{01} - \omega_{02}) \\
 T_1 &= \lambda^2 \cdot \omega_{01} \cdot d_1 \cdot \left(\frac{d_1}{\lambda^2}\right) \\
 T_2 &= -.5(1-3v) \cdot \omega_{01} + (1.5-1.5v-v^2) \cdot \omega_{02} \\
 T_3 &= N06 \cdot \lambda^2 \cdot \left(\frac{d_1}{\lambda^2}\right) \\
 T_4 &= -d_1 \cdot (3\omega_{01} - \omega_{02}) + d_1 \cdot \omega_{01}
 \end{aligned}$$

4

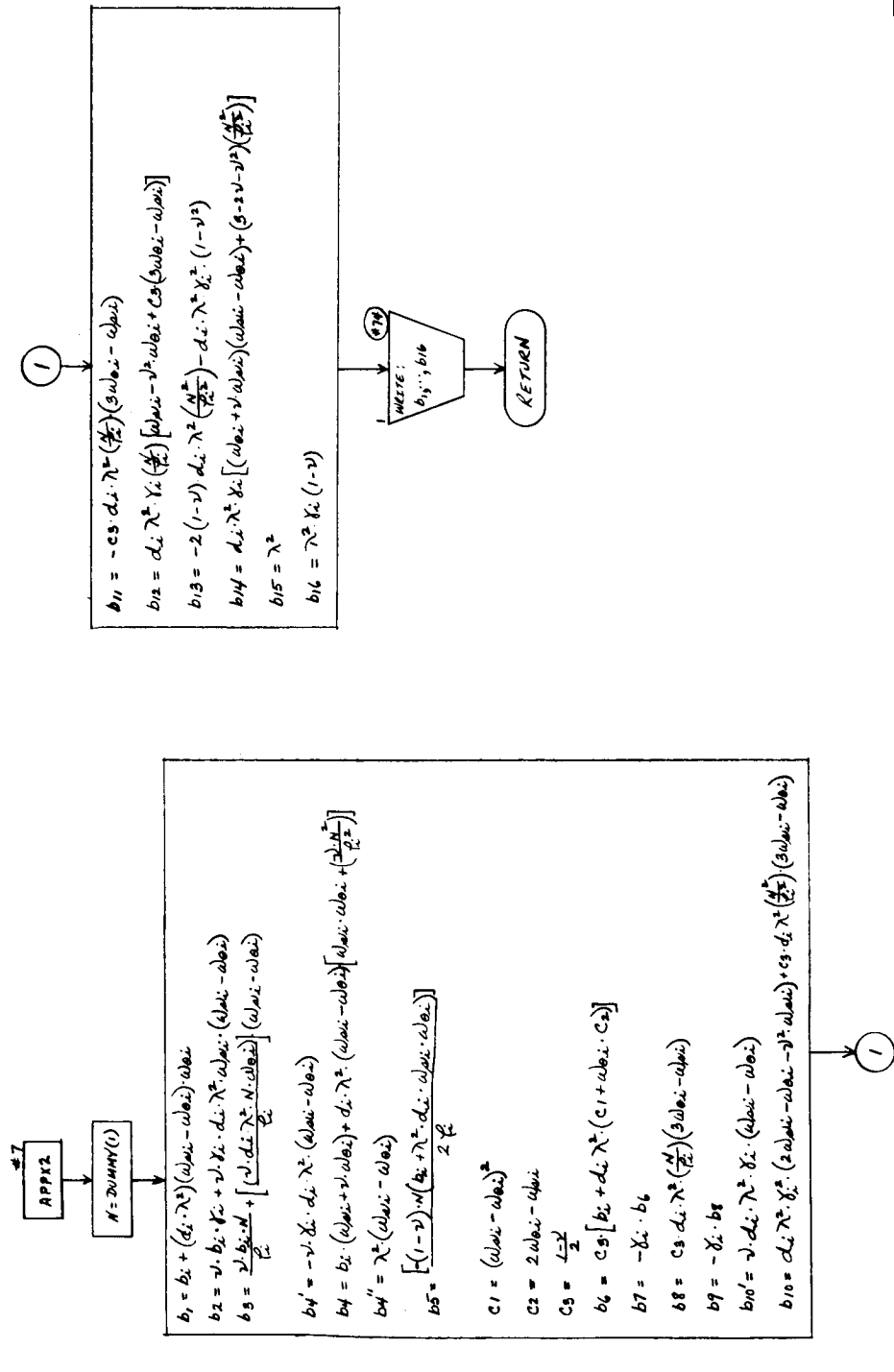
$a_{212} = -(T_1 \cdot T_2 + T_3 \cdot T_4)$   
 $T_1 = d_i \left(\frac{d_i}{\lambda_i}\right) (v^2 \cdot \omega_{1i} + \omega_{2i})$   
 $T_2 = [-\lambda_i^2 \cdot d_i \cdot \left(\frac{d_i}{\lambda_i}\right) \cdot \omega_{1i} \cdot \omega_{2i}] \cdot \{N \cdot \omega_{1i} \cdot \omega_{2i} + [5(3-v) \cdot v]^2 \cdot \omega_{2i}\}$   
 $T_3 = \lambda_i^2 \cdot d_i \cdot d_i \cdot \left(\frac{d_i}{\lambda_i}\right)$   
 $T_4 = .5(1+v) \cdot \omega_{1i} + (1.5 - 1.5v - v^2) \cdot \omega_{2i}$   
 $T_5 = \lambda_i^2 \cdot d_i^2 \cdot d_i \cdot \left(\frac{d_i}{\lambda_i}\right)$   
 $T_6 = [.5(1-3v) - v^2] \cdot \omega_{1i} + (1.5 + 1.5v + 2v^2) \cdot \omega_{2i}$   
 $T_7 = \lambda_i^2 \cdot d_i \cdot d_i \cdot \left(\frac{d_i}{\lambda_i}\right) \cdot \omega_{1i} \cdot 5(1+v)$   
 $T_8 = \lambda_i^2 \cdot d_i \cdot \left(\frac{d_i}{\lambda_i}\right) (\omega_{1i} - v^2 \cdot \omega_{2i})$   
 $a_{213} = -(T_1 \cdot T_2 + T_3 \cdot T_4 + T_5 \cdot T_6 + T_7 \cdot T_8)$   
 $a_{214} = \lambda_i^2 \cdot d_i \cdot [(1-3v) \cdot \lambda_i^2 + 2(1-v) \cdot \left(\frac{d_i}{\lambda_i}\right)]$   
 $T_1 = \lambda_i^2 \cdot d_i \cdot d_i \cdot \left[ -v^2 \cdot \lambda_i^2 + 2 \cdot \omega_{1i} \cdot \omega_{2i} + 2(1-v) \cdot \left(\frac{d_i}{\lambda_i}\right) \right]$   
 $T_2 = \lambda_i^2 \cdot d_i \cdot [(1-3v) \cdot \lambda_i^2 + 2(1-v) \cdot \left(\frac{d_i}{\lambda_i}\right)]$   
 $a_{215} = T_3 - T_1$   
 $T_1 = d_i \cdot (\omega_{1i}^2 + 2v \cdot \omega_{1i} \cdot \omega_{2i} + \omega_{2i}^2)$   
 $T_2 = d_i \cdot \lambda_i^2 \cdot (\omega_{1i} - \omega_{2i}) \cdot \omega_{2i} \cdot [(1-v) \cdot \omega_{1i}^2 - \omega_{1i} \cdot \omega_{2i} - \omega_{2i}^2]$   
 $T_3 = d_i \cdot \lambda_i^2 \cdot \left(\frac{d_i}{\lambda_i}\right)$   
 $T_4 = 2(\omega_{1i} - \omega_{2i}) \cdot (v \cdot \omega_{1i} + \omega_{2i}) - (3 - 2v - v^2) \cdot \omega_{1i} + (1-v) \cdot \left(\frac{d_i}{\lambda_i}\right)$   
 $T_5 = d_i \cdot \lambda_i^2 \cdot d_i$   
 $T_6 = (\omega_{1i} - \omega_{2i}) \cdot (\omega_{1i} + v \cdot \omega_{2i}) + (3 - 2v - v^2) \cdot \left(\frac{d_i}{\lambda_i}\right)$   
 $T_7 = \lambda_i^2 \cdot d_i \cdot d_i \cdot \left\{ \lambda_i^2 \cdot (\omega_{1i} - \omega_{2i}) \cdot [(1-v) \cdot \omega_{1i} - 2 \cdot \omega_{2i}] + \omega_{1i} \cdot [(1-v) \cdot \omega_{1i} + 2v \cdot \omega_{2i}] - 2(2-v-v^2) \cdot \left(\frac{d_i}{\lambda_i}\right) \cdot d_i \right\}$   
 $a_{216} = -(T_1 + T_2 + T_3 \cdot T_4 + T_5 \cdot T_6 + T_7)$   
 $a_{217} = -\lambda_i^2$

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$a_{218} = -d_i \cdot \lambda_i^2 \cdot (2-v)$   
 $a_{219} = -\lambda_i^2 \cdot [\omega_{1i}^2 - (2-v) \cdot \omega_{1i} \cdot \omega_{2i} - v \cdot \left(\frac{d_i}{\lambda_i}\right)]$   
 $a_{220} = d_i \cdot \omega_{2i}$   
 $a_{221} = d_i \cdot (\omega_{1i} + v) \cdot d_i \cdot \omega_{2i}$   
 $a_{222} = v \cdot d_i \cdot \left(\frac{d_i}{\lambda_i}\right) \cdot \omega_{2i}$   
 $a_{223} = -d_i$   
 $a_{224} = -v \cdot d_i \cdot d_i$   
 $a_{225} = -d_i \cdot [(\omega_{1i} - \omega_{2i}) - v \cdot \left(\frac{d_i}{\lambda_i}\right)]$   
 $a_{226} = 1$



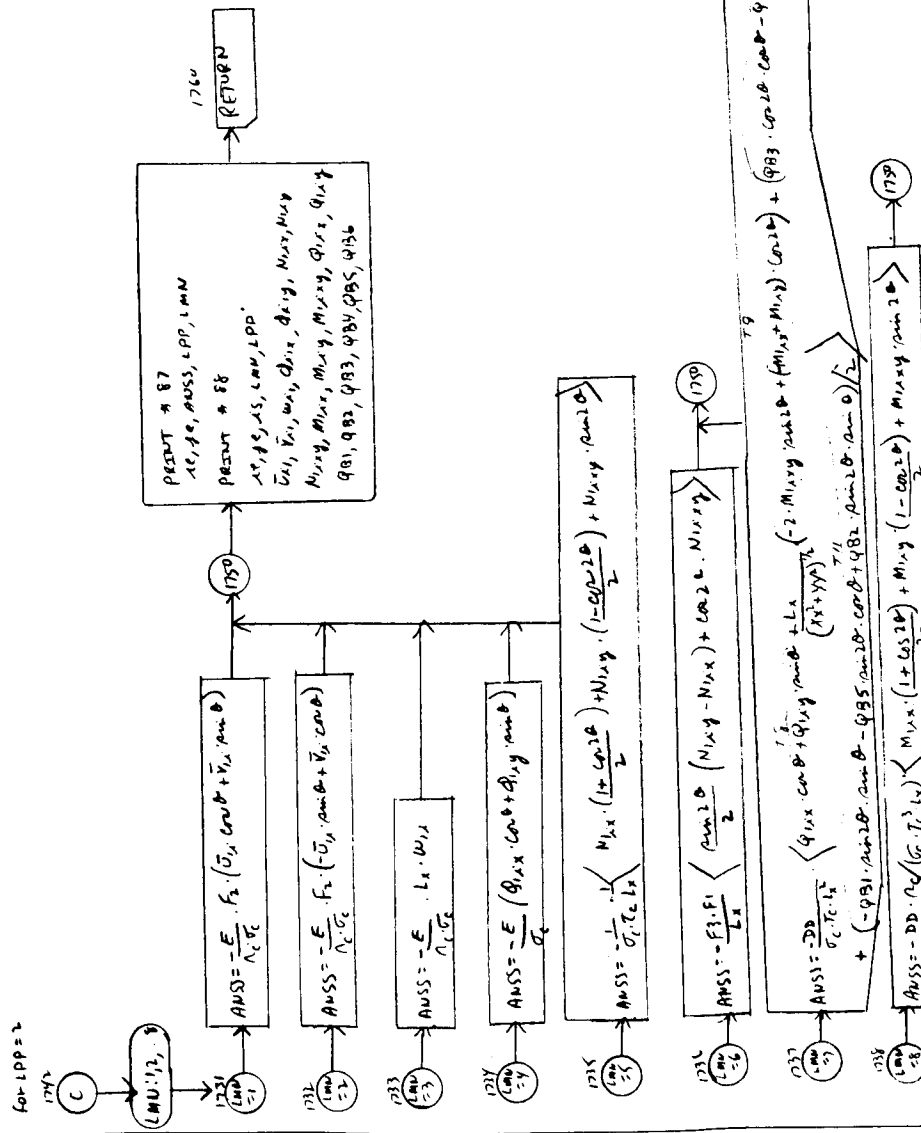
COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP







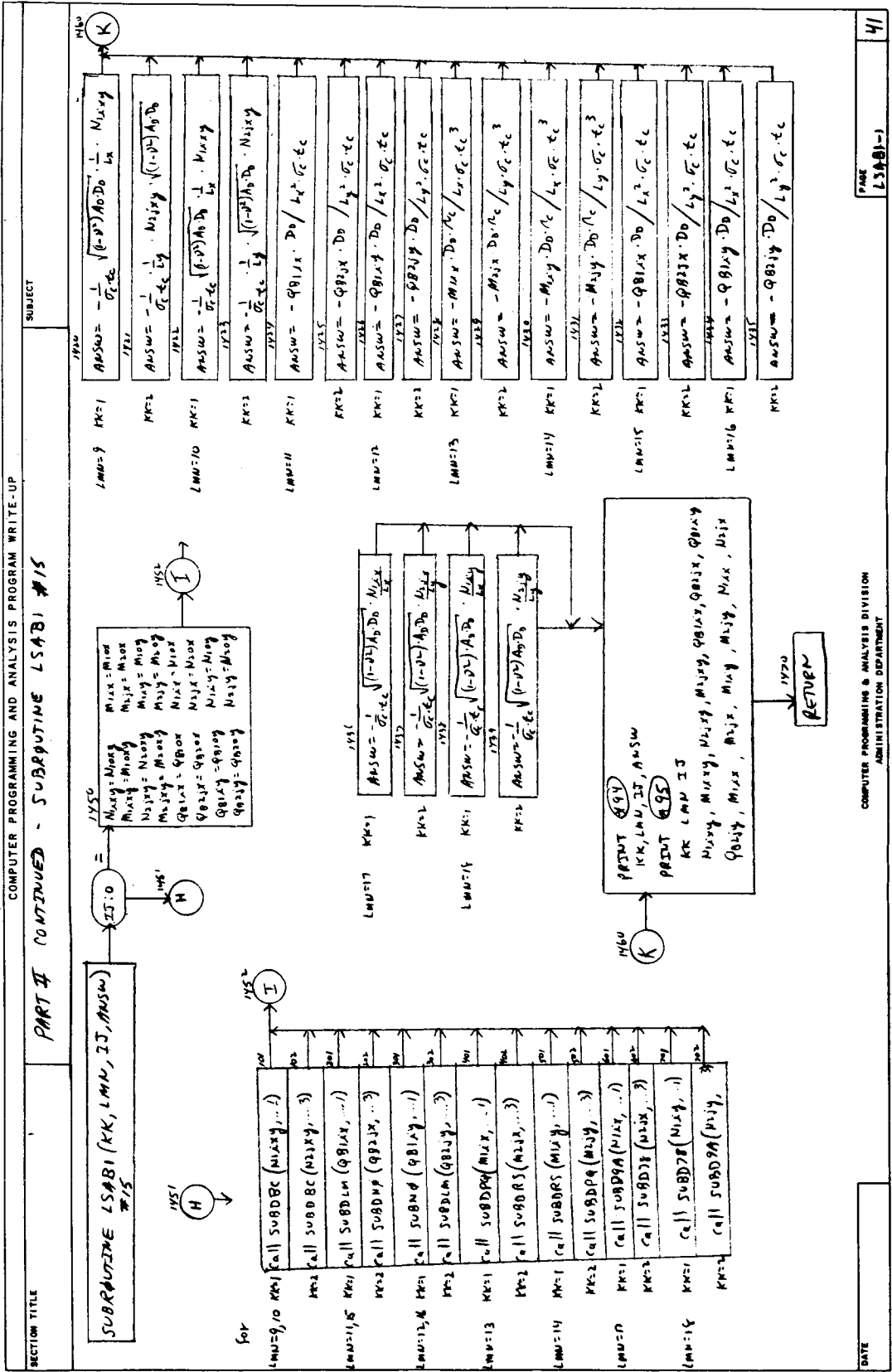












```

DO 9 FOR I=1,20
  DX2 = (1-D7) * AD * Lx / DD
  DY2 = (1-D7) * AD * Y / DD
  ALIX(I) = 1.7E
  MIX = 4 * (1 / (Lx * RATIOX)) * ALIX(I) * (Lx * RATIOX - Lx * RATIOX) / DX2
  AII(I) = [ 1/2 + 1/2 * (1 + MIX^2)^1/2 ]^1/2
  BII(I) = -MIX / (2 * AII(I))
  * (D)
  
```

```

DO 9 FOR J=1,20
  ALIY(J) = 0.7E
  MYS = 4 * (1 / (Ly * RATIOY)) * ALIY(J) * (Ly * RATIOY - Ly * RATIOY) / DY2
  CSI(J) = [ 1/2 + 1/2 * (1 + MYS^2)^1/2 ]^1/2
  DSI(J) = -MYS / (2 * CSI(J))
  * (D)
  
```

```

SUBROUTINE DAME #8
  DX2 = (1-D7) * AD * Lx / DD
  DY2 = (1-D7) * AD * Y / DD
  * (D)
  
```

```

DO 9 FOR I=1,20
  T1 = (ALIX(I)^2 - DX2 * BII(I) * Lx * RATIOX / 2)
  COEF = -1 IF T1 < 0
  COEF = 1 IF T1 > 0
  TERM4 = [ 1 + AII(I)^2 ]
  TERM5 = 2 * (1 / (Lx * RATIOX)) * ALIX(I)^2 - DY2 * BII(I)
  T2 = [ 1 + (DY2)^2 * TERM4 / TERM5 ]^1/2
  T3 = (ALIX(I)^2 + DY2 * BII(I) * Lx * RATIOX / 2)
  COEF2 = -1 IF T3 < 0
  COEF2 = 1 IF T3 > 0
  TERM4 = (1 - AII(I))^2
  TERM5 = 2 * (1 / (Lx * RATIOX)) * ALIX(I)^2 + DY2 * BII(I)
  TT2 = (1 + (DY2)^2 * TERM4 / TERM5)^1/2
  P(1,I) = [ 1/2 * T1 * (1 + COEF * TT2) ]^1/2
  P(2,I) = [ 1/2 * T3 * (1 + COEF2 * TT2) ]^1/2
  Q(1,I) = [ 1/2 * T1 * (1 - COEF * TT2) ]^1/2
  Q(2,I) = [ 1/2 * T3 * (1 - COEF2 * TT2) ]^1/2
  * (D)
  
```

NEXT PAGE  
A

D4 23  $i=1, JD$  for  $R=1$

$T7 = [P(B_{i+1}, r)^2 + Q(B_{i+1}, r)^2] \cdot [Q(B_{i+1}, r)^2 - 3 \cdot Q(B_{i+1}, r) \cdot P(B_{i+1}, r)]$

$T8 = [P(B_{i+1}, r)^2 - 3 \cdot Q(B_{i+1}, r) \cdot P(B_{i+1}, r)] \cdot [P(B_{i+1}, r) \cdot P(B_{i+1}, r) + Q(B_{i+1}, r) \cdot Q(B_{i+1}, r)]$

$T9 = [P(B_{i+1}, r)^2 - 3 \cdot Q(B_{i+1}, r) \cdot P(B_{i+1}, r)] \cdot [P(B_{i+1}, r) \cdot Q(B_{i+1}, r) - P(B_{i+1}, r) \cdot Q(B_{i+1}, r)]$

$T10 = [P(B_{i+1}, r)^2 + (Q(B_{i+1}, r))^2] \cdot [P(B_{i+1}, r)^2 - 3 \cdot (Q(B_{i+1}, r)) \cdot P(B_{i+1}, r)]$

$T11 = [Q(B_{i+1}, r)^2 - 3 \cdot Q(B_{i+1}, r) \cdot P(B_{i+1}, r)] \cdot [P(B_{i+1}, r) \cdot Q(B_{i+1}, r) - P(B_{i+1}, r) \cdot Q(B_{i+1}, r)]$

$T12 = [P(B_{i+1}, r)^2 - 3 \cdot (Q(B_{i+1}, r)) \cdot P(B_{i+1}, r)] \cdot [P(B_{i+1}, r) \cdot P(B_{i+1}, r) + Q(B_{i+1}, r) \cdot Q(B_{i+1}, r)]$

$BL(B_{i+1}, r) = (T7 - T8 - T9) / (T10 - T11 - T12)$

$T13 = [P(B_{i+1}, r) \cdot P(B_{i+1}, r) + Q(B_{i+1}, r) \cdot Q(B_{i+1}, r)] / [P(B_{i+1}, r)^2 + Q(B_{i+1}, r)^2]$

$T14 = [P(B_{i+1}, r) \cdot Q(B_{i+1}, r) - P(B_{i+1}, r) \cdot Q(B_{i+1}, r)] / [P(B_{i+1}, r)^2 + Q(B_{i+1}, r)^2]$

$AL(B_{i+1}, r) = -T13 + T14 \cdot BL(B_{i+1}, r)$

$T15 = [P(B_{i+1}, r) \cdot Q(B_{i+1}, r) - P(B_{i+1}, r) \cdot Q(B_{i+1}, r)] / [P(B_{i+1}, r)^2 + Q(B_{i+1}, r)^2]$

$T16 = [P(B_{i+1}, r) \cdot P(B_{i+1}, r) + Q(B_{i+1}, r) \cdot Q(B_{i+1}, r)] / [P(B_{i+1}, r)^2 + Q(B_{i+1}, r)^2]$

$BL(B_{i+1}, r) = T15 - T16 \cdot BL(B_{i+1}, r)$

\* (6) (7)

NEXT PAGE

8

D4 16 for  $\theta = 1, JD$

$T4 = (ALTY(\theta))^2 - DY2 \cdot DS(\theta) \cdot LY \cdot RATIO(\theta)$

COEF1 = 1 IF T4 < 0

COEF1 = -1 IF T4 > 0

TEMP4 = (1 + CT(\theta))<sup>2</sup>

TEMP5 = [2 \cdot (LY \cdot RATIO(\theta)) \cdot ALTY(\theta) - DY2 \cdot DS(\theta)]<sup>2</sup>

T5 = [1 + (DY2)<sup>2</sup> \cdot TEMP4 / TEMP5]<sup>1/2</sup>

T6 = ALTY(\theta) - DY2 \cdot DS(\theta) \cdot LY \cdot RATIO(\theta)

COEF3 = -1 IF T6 < 0

COEF3 = 1 IF T6 > 0

TEMPY = (-COEF3)<sup>2</sup>

TEMPSS = [2 \cdot (LY \cdot RATIO(\theta)) \cdot ALTY(\theta) + DY2 \cdot DS(\theta)]<sup>2</sup>

T7S = [1 + (DY2)<sup>2</sup> \cdot TEMPY / TEMPSS]<sup>1/2</sup>

P(3, \theta) = [1/2 \cdot T4 \cdot (1 + COEF1 \cdot T5)]<sup>1/2</sup>

P(4, \theta) = [1/2 \cdot T6 \cdot (1 + COEF3 \cdot T7S)]<sup>1/2</sup>

Q(3, \theta) = [1/2 \cdot T4 \cdot (-1 + COEF1 \cdot T5)]<sup>1/2</sup>

Q(4, \theta) = [1/2 \cdot T6 \cdot (-1 + COEF3 \cdot T7S)]<sup>1/2</sup>

\* (8)

31

RETURN

for  $l=3$   
 do 24  $l=1,2,3$

$$T7 = [P(0_{+1}, l)^2 + \phi(0_{+1}, l)^2] \cdot [\phi(0_{+1}, l)^3 - 3 \cdot \phi(0_{+1}, l) \cdot (P(0_{+1}, l))^2]$$

$$T8 = [Q(0_{+1}, l)^3 - 3 \cdot \phi(0_{+1}, l) \cdot P(0_{+1}, l)^2] \cdot [P(0_{+1}, l) \cdot \phi(0_{+1}, l) + \phi(0_{+1}, l) \cdot \phi(0_{+1}, l)]$$

$$T9 = [P(0_{+1}, l)^3 - 3 \cdot \phi(0_{+1}, l) \cdot P(0_{+1}, l)^2] \cdot [P(0_{+1}, l) \cdot \phi(0_{+1}, l) - P(0_{+1}, l) \cdot \phi(0_{+1}, l)]$$

$$T10 = [P(0_{+1}, l)^2 + \phi(0_{+1}, l)^2] \cdot P(0_{+1}, l)^3 - 3 \cdot (\phi(0_{+1}, l))^2 \cdot P(0_{+1}, l)$$

$$T11 = [\phi(0_{+1}, l)^3 - 3 \cdot \phi(0_{+1}, l) \cdot (P(0_{+1}, l))^2] \cdot [P(0_{+1}, l) \cdot \phi(0_{+1}, l) - P(0_{+1}, l) \cdot \phi(0_{+1}, l)]$$

$$T12 = [P(0_{+1}, l)^3 - 3 \cdot \phi(0_{+1}, l) \cdot P(0_{+1}, l)^2] \cdot [P(0_{+1}, l) \cdot \phi(0_{+1}, l) + \phi(0_{+1}, l) \cdot \phi(0_{+1}, l)]$$

$$BL(0_{+1}, l) = (T7 - T8 - T9) / (T10 - T11 - T12)$$

$$T13 = [P(0_{+1}, l) \cdot P(0_{+1}, l) + \phi(0_{+1}, l) \cdot \phi(0_{+1}, l)] / [P(0_{+1}, l)^2 + \phi(0_{+1}, l)^2]$$

$$T14 = [P(0_{+1}, l) \cdot \phi(0_{+1}, l) - P(0_{+1}, l) \cdot \phi(0_{+1}, l)] / [P(0_{+1}, l)^2 + \phi(0_{+1}, l)^2]$$

$$AL(0_{+1}, l) = -T13 + T14 \cdot BL(0_{+1}, l)$$

$$T15 = [P(0_{+1}, l) \cdot \phi(0_{+1}, l) - P(0_{+1}, l) \cdot \phi(0_{+1}, l)] / [P(0_{+1}, l)^2 + \phi(0_{+1}, l)^2]$$

$$T16 = [P(0_{+1}, l) \cdot P(0_{+1}, l) + \phi(0_{+1}, l) \cdot \phi(0_{+1}, l)] / [P(0_{+1}, l)^2 + \phi(0_{+1}, l)^2]$$

$$BL(0_{+1}, l) = T15 - T16 \cdot BL(0_{+1}, l)$$

\* (6) (7)



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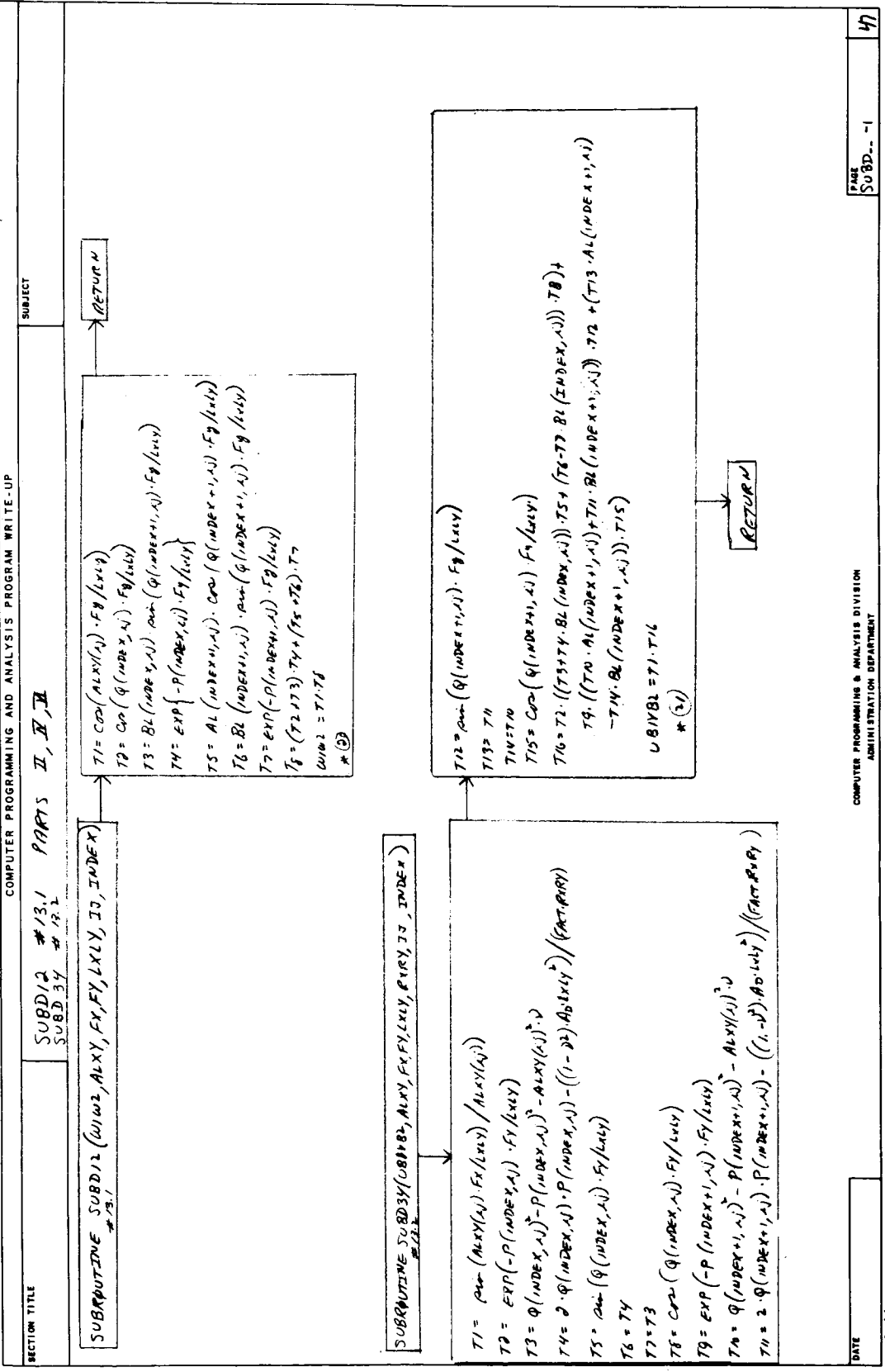
DATE	#	PARTS II, II, II	INDEX = 1	INDEX = 3	SECTION TITLE
	13.1	SUBD10 (W1W2, ALXY, FX, FY, LXLY, IJ, INDEX)	W1X	W2Y	SUBROUTINES SUBD12-SUBD34 COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP 13.1-13.21 PARTS II, I, II
	13.2	SUBD34 (UB1UB2, ALXY, FX, FY, LXLY, RXRY, IJ, INDEX)	U1X	U2Y	
	13.3	SUBD56 (VB1VB2, ALXY, FX, FY, LXLY, RXRY, IJ, INDEX)	V1X	V2Y	
	13.4	SUBD78 (N1N2, ALXY, FX, FY, LXLY, IJ, INDEX)	N1XY	N2JX	
	13.5	SUBD9A (N1XN2Y, ALXY, FX, FY, LXLY, IJ, INDEX)	N1XY	N2JY	
	13.6	SUBDBC (N1ZXY, ALXY, FX, FY, LXLY, IJ, INDEX)	N1XY	N2JXY	
	13.7	SUBDDE (PH1XY, ALXY, FX, FY, LXLY, IJ, INDEX)	Q1XX	Q2JY	
	13.8	SUBDFG (Q1XQ2Y, ALXY, FX, FY, LXLY, IJ, INDEX)	Q1XX	Q2JY	
	13.9	SUBDHI (Q1YQ2X, ALXY, FX, FY, LXLY, IJ, INDEX)	Q1XY	Q2JX	
	13.10	SUBDJK (PH1YX, ALXY, FX, FY, LXLY, IJ, INDEX)	Q1XY	Q2JX	
	13.11	SUBDLM (QBARXY, ALXY, FX, FY, LXLY, IJ, INDEX)	Q̄1XX	Q̄2JY	
	13.12	SUBDNP (QBARXY, ALXY, FX, FY, LXLY, IJ, INDEX)	Q̄1XY	Q̄2JX	
	13.13	SUBDPQ (M1XN2Y, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XX	M2JY	
	13.14	SUBDRS (M1YMLX, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XY	M2JX	
	13.15	SUBDTU (M1SIXY, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XY	M2JXY	
	13.16	SUBDKW (M1XN2Y, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XXDY 985	M2JYDX 981	
	13.17	SUBDXY (M1XN2Y, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XXDX 982	M2JYDY 982	
	13.18	SUBDZ (M1YMX, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XYDX 986	M2JYDY 981	
	13.19	SUBDIA (M1YMX, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XYDY 981	M2JYDX 987	
	13.20	SUBD2A (M1YMX, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XYDY 984	M2JYDX 989	
	13.21	SUBD3A (M1YMX, ALXY, FX, FY, LXLY, IJ, INDEX)	M1XYDX 983	M2JYDY 980	
					SUBJECT

COMPUTER PROGRAMMING & ANALYSIS DIVISION  
ADMINISTRATION DEPARTMENT

PAGE  
SUBD-

	INDEX=1	INDEX=3
ALXY	ALIX	ALY
FX	XX	YY
FY	YY	XX
LXLY	LX	LY
RXRY	RX	RY





DATE

COMPUTER PROGRAMMING & ANALYSIS DIVISION  
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SUBD-- -1  
47

COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP

SECTION TITLE: SUBDS 6 #133  
SUBD 78 #134

SUBJECT:

SUBROUTINE SUBDS6 (V810B2, ALXY, FY, LXLY, FRY, IT, INDEX)  
#133

```

FACT1 = (1 - 2^1) * A0^2 * LXLY^2 / (FXY * D0^2)
T1 = COS(ALXY(LX)) * FY / LXLY
T1 = T1 / ALXY(LX)
T2 = EXP(-P(INDEX, LX)) * FY / LXLY
T3 = P(INDEX, LX) * Q(INDEX, LX)
T4 = ALXY(LX) * P(INDEX, LX) * V
T5 = Q(INDEX, LX) * FACT1
T6 = 2 * (1 + 2) * ALXY(LX) * P(INDEX, LX)
T7 = Q(INDEX, LX) * 3 - P(INDEX, LX) * V
T8 = Q(INDEX, LX) * ALXY(LX) * V
T9 = P(INDEX, LX) * FACT1
T10 = 2 * (1 + 2) * ALXY(LX) * Q(INDEX, LX)
    
```

```

T11 = 2 * P(INDEX, LX) * FY / LXLY
T12 = Q(INDEX, LX) * 3 - Q(INDEX, LX) * P(INDEX, LX)
T13 = T8
T14 = T9
T15 = T10
T16 = T3
T17 = T4
T18 = T5
T19 = T6
T20 = COS(Q(INDEX, LX)) * FY / LXLY
T21 = T2 * ((T3 + T4 + T5 - T6) * (T7 - T8 + T9 + T10) * BL(INDEX, LX)) * T11
    + ((T12 - T13 + T14 + T15) - (T16 + T17 + T18 - T19) * BL(INDEX, LX)) * T20
T22 = EXP(-P(INDEX, LX)) * FY / LXLY
T23 = P(INDEX, LX) * 3 - P(INDEX, LX) * Q(INDEX, LX)
T24 = ALXY(LX) * P(INDEX, LX) * V
    
```

SUBROUTINE SUBD78 (MM, INDEX, ALXY, FY, LXLY, T3, INDEX)  
#134

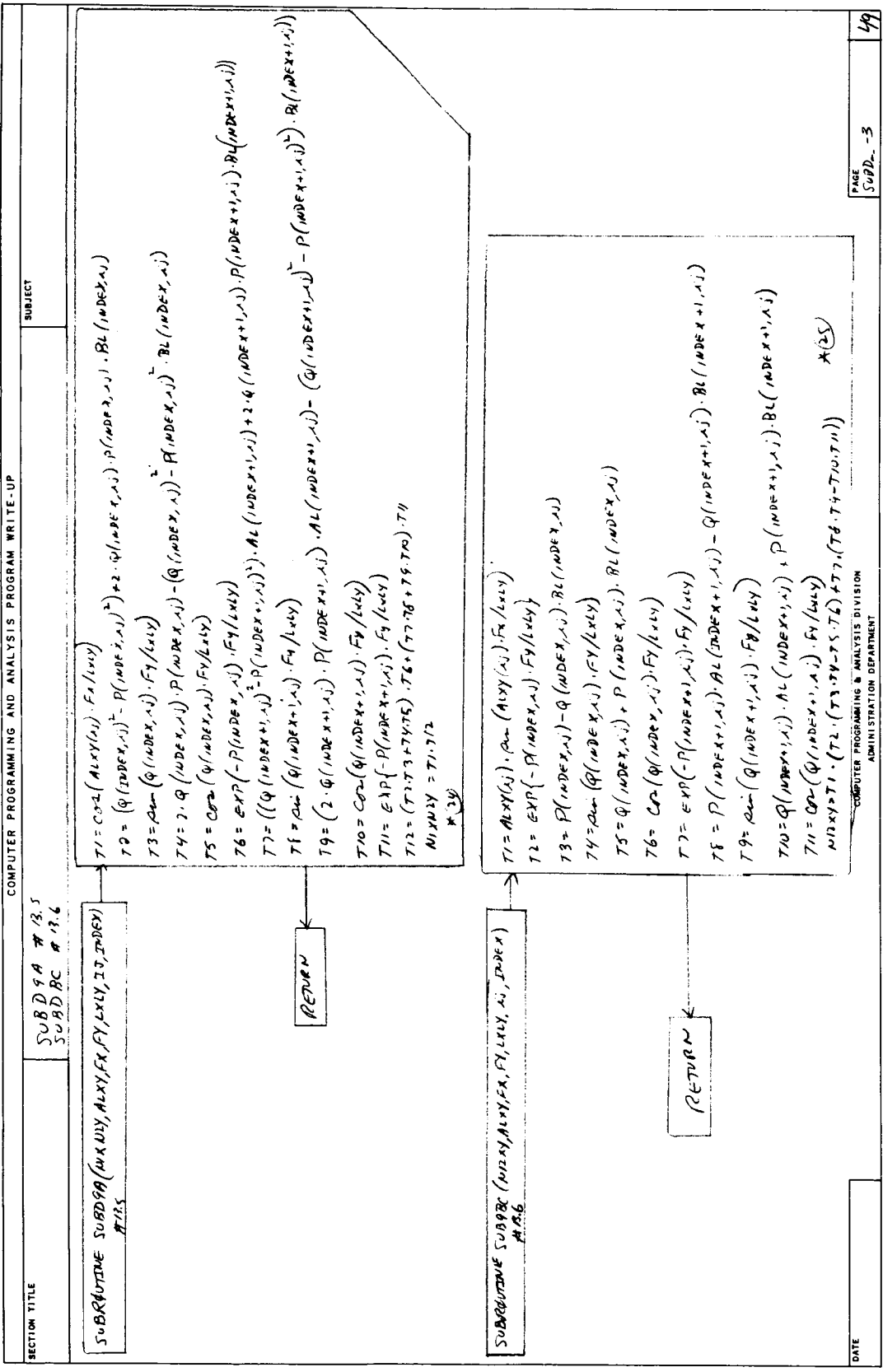
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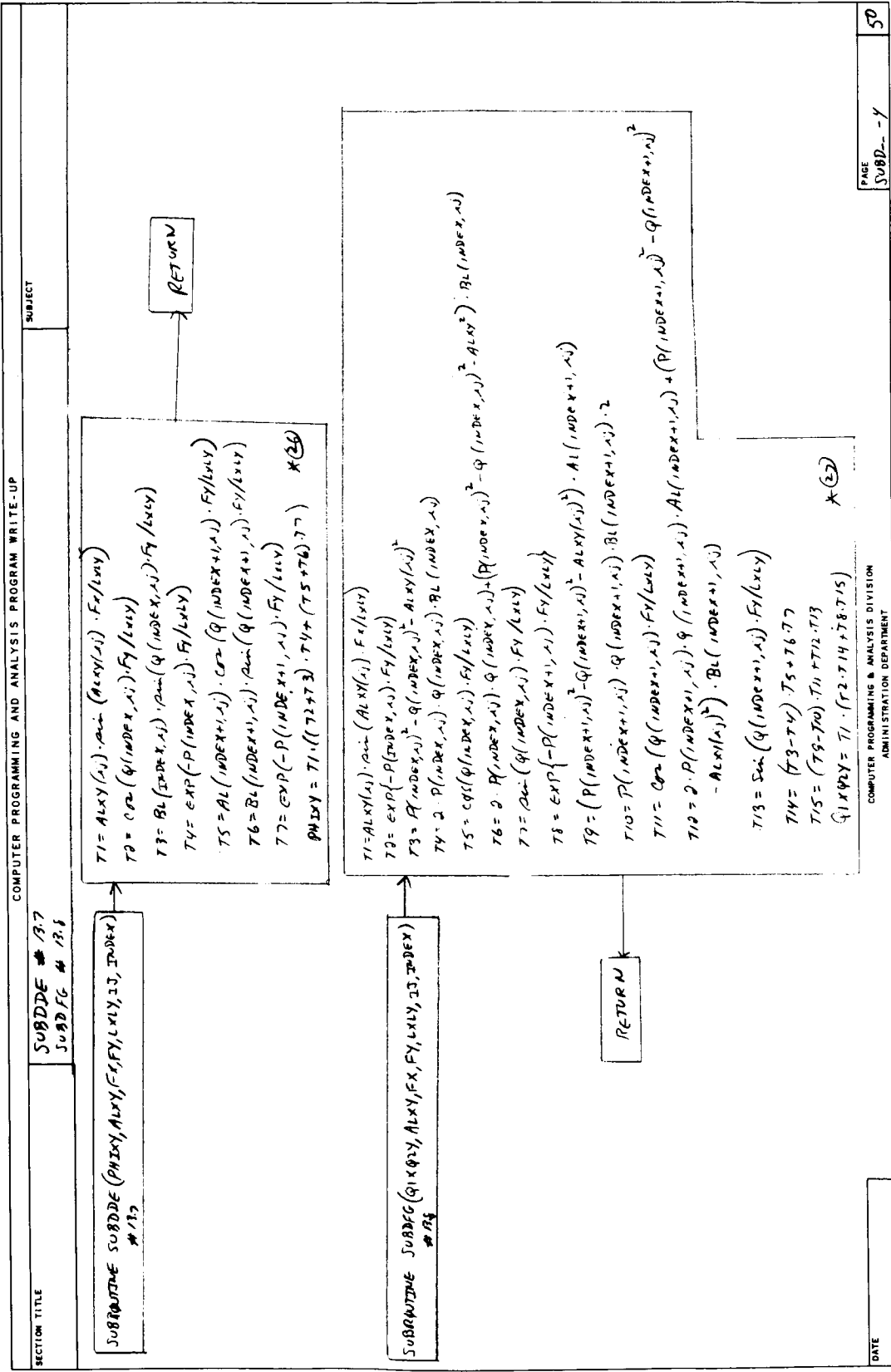
T1 = -ALXY(LX) * COS(ALXY(LX)) * FY / LXLY
T2 = 2 * P(INDEX, LX) * Q(INDEX, LX) * FY / LXLY + BL(INDEX, LX) * COS(Q(INDEX, LX) * FY / LXLY)
T3 = EXP(-P(INDEX, LX)) * FY / LXLY
T4 = -ALXY(LX) * P(INDEX, LX) * Q(INDEX, LX) * FY / LXLY + BL(INDEX, LX) * FY / LXLY
T5 = EXP(-P(INDEX, LX)) * FY / LXLY
T6 = T2 - T3 + T4 - T5
MM, INDEX = T1, T6
    
```

```

T735 = Q(INDEX, LX) * FACT1
T736 = 2 * (1 + 2) * ALXY(LX) * P(INDEX, LX)
T737 = Q(INDEX, LX) * 3 - P(INDEX, LX) * Q(INDEX, LX)
T738 = Q(INDEX, LX) * ALXY(LX) * V
T739 = P(INDEX, LX) * FACT1
T740 = 2 * (1 + 2) * ALXY(LX) * Q(INDEX, LX)
T741 = 2 * P(INDEX, LX) * Q(INDEX, LX) * FY / LXLY
T742 = T22 * ((T33 + T34 + T35 - T36) * AL(INDEX, LX)
    + (T37 - T38 + T39 + T40) * BL(INDEX, LX)) * T31
    + (T32 - T33 + T34 + T35) * AL(INDEX, LX)
    - (T36 + T37 + T38 - T39) * BL(INDEX, LX) * T40
V810B2 = T1 * (T2 + T4)
    
```

RETURN

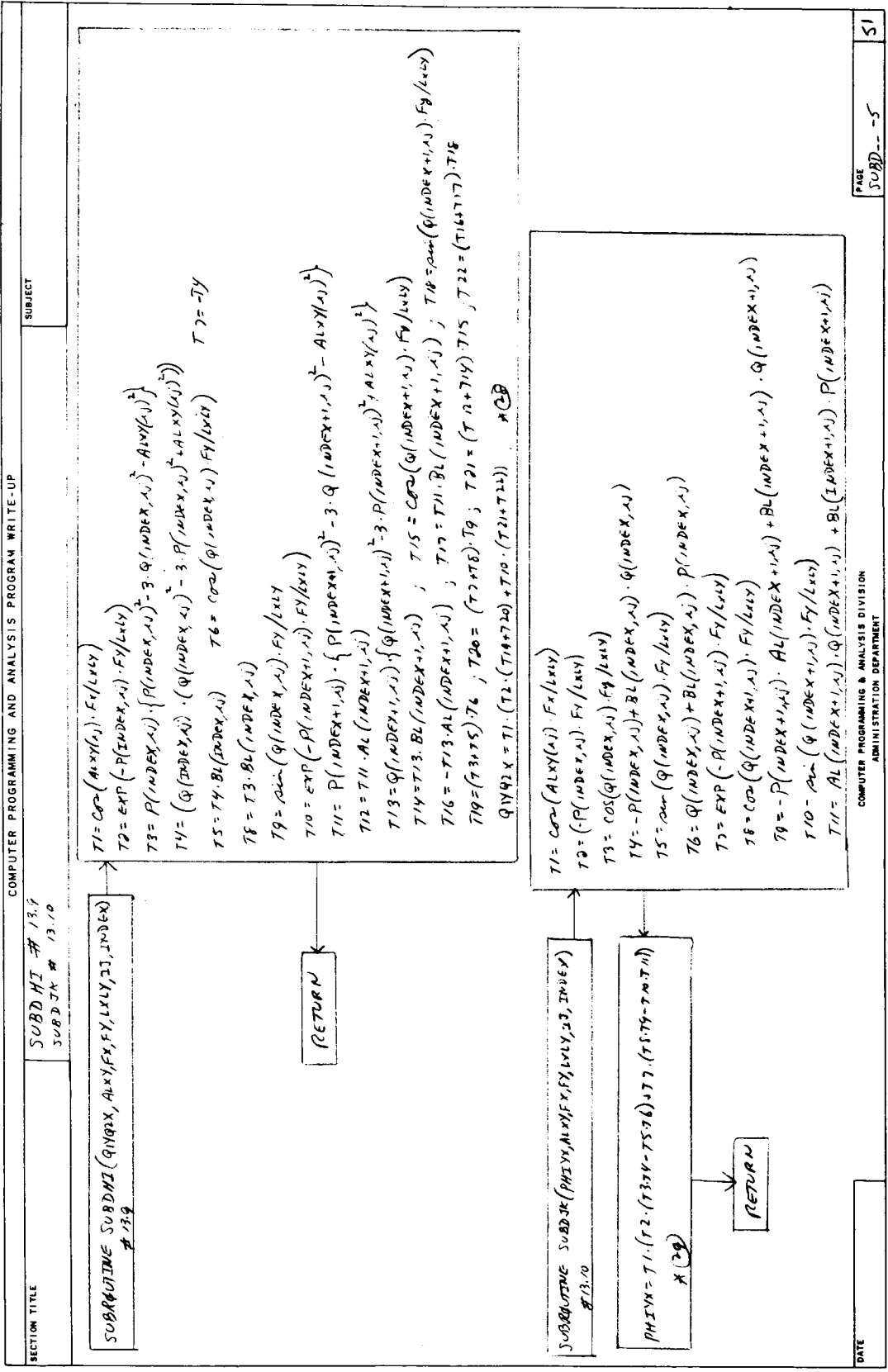


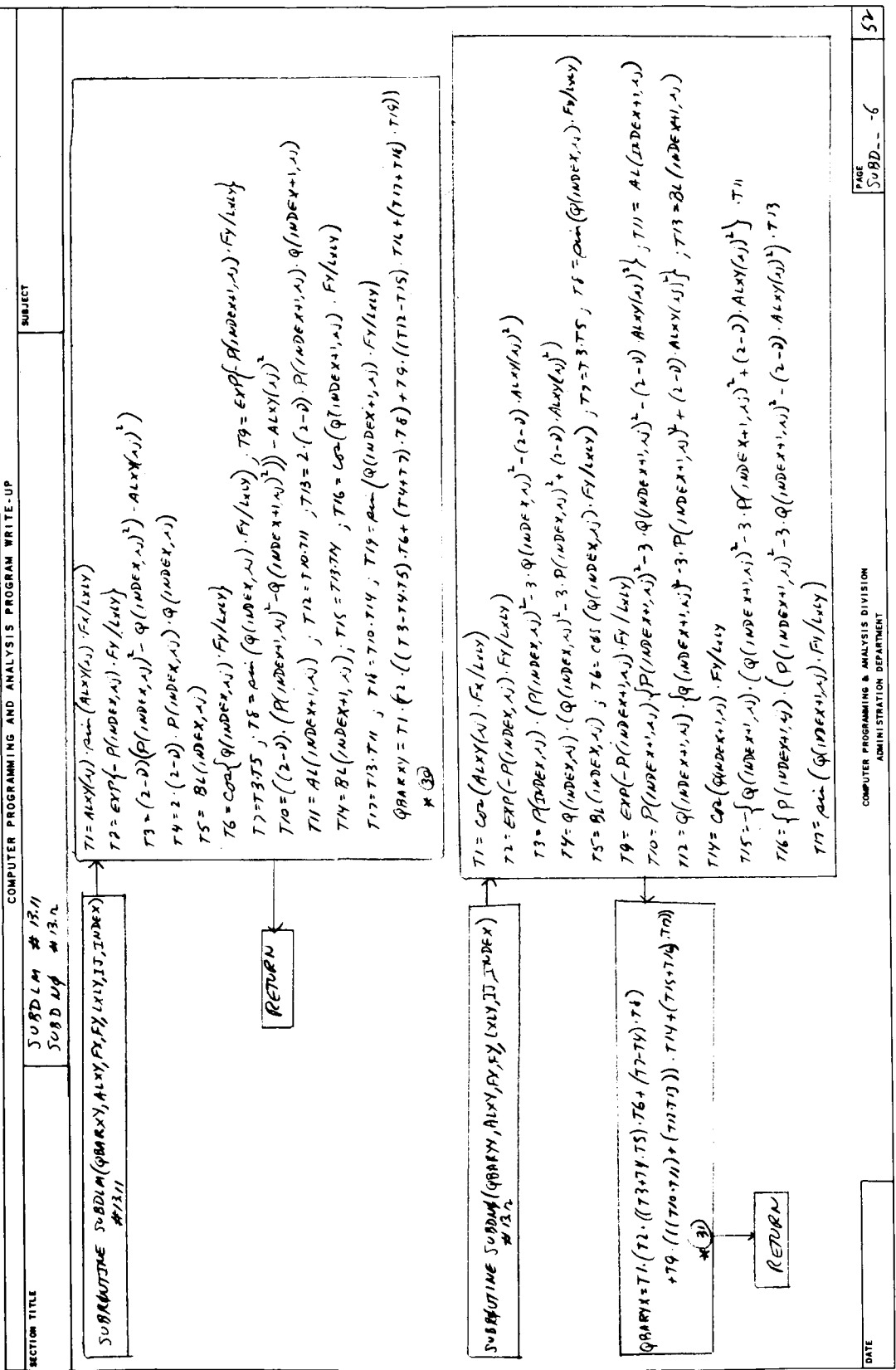


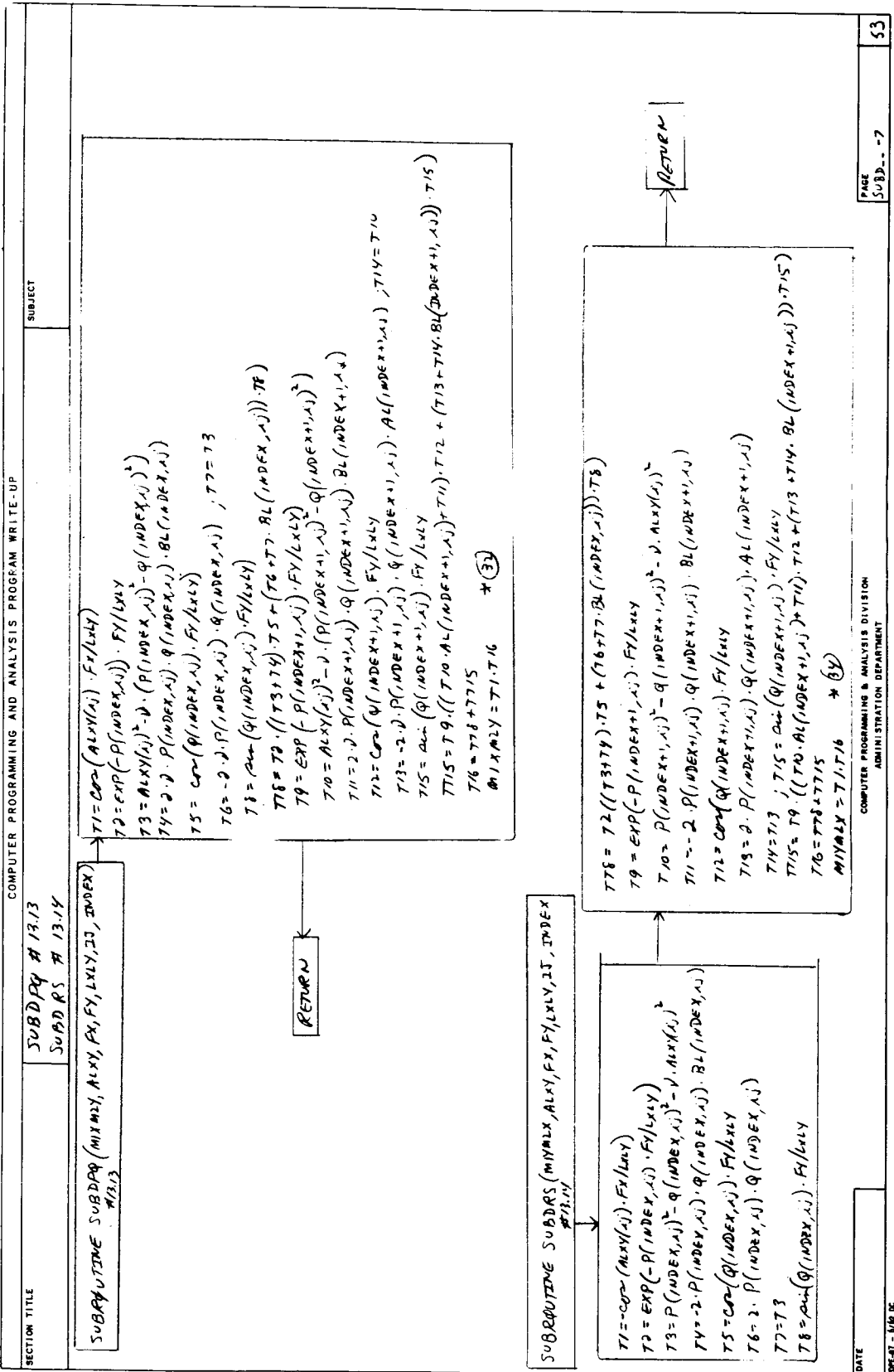
DATE

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ADMINISTRATION DEPARTMENT

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SUBD--Y



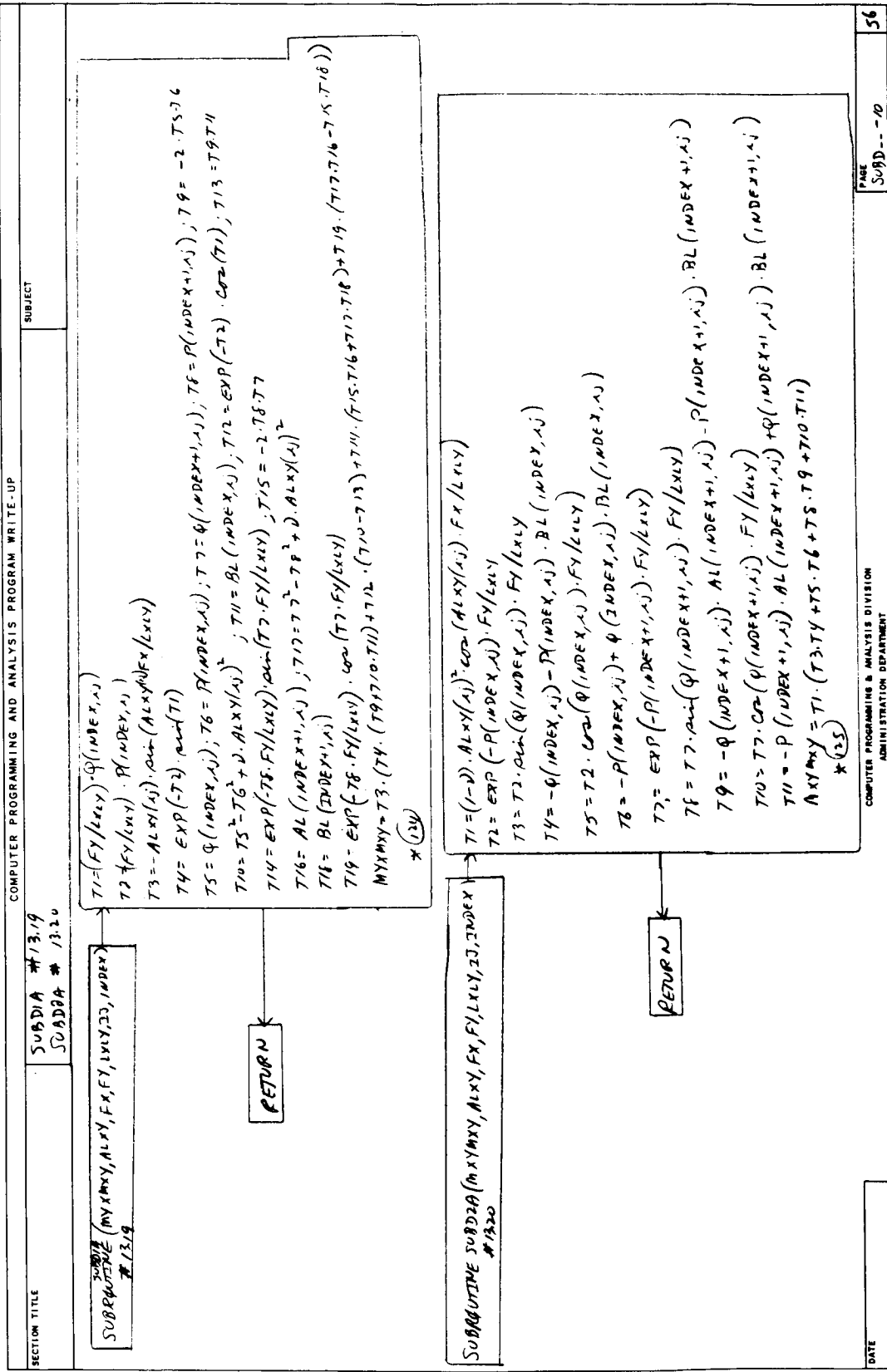




SECTION TITLE		COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP	
SUBDTU #13.15		SUBJECT	
SUBDTU #13.16			
SUBROUTINE SUBDTU(MIJKY, ALXY, FX, FY, LELY, IJ, INDEX) #13.15		$T1 = (ALXY(IJ) \cdot \sin(ALXY(IJ)) \cdot FX / LELY) \cdot (1 - U)$ $T2 = EXP(-P(INDEX, IJ)) \cdot FY / LELY$ $T3 = -P(INDEX, IJ) + Q(INDEX, IJ) \cdot BL(INDEX, IJ)$ $T4 = \cos(Q(INDEX, IJ)) \cdot FY / LELY$ $T5 = Q(INDEX, IJ) + P(INDEX, IJ) \cdot BL(INDEX, IJ)$ $T6 = \sin(Q(INDEX, IJ)) \cdot FY / LELY$ $T7 = EXP(-P(INDEX+1, IJ)) \cdot FY / LELY$ $T8 = -P(INDEX+1, IJ) \cdot AL(INDEX+1, IJ) + Q(INDEX+1, IJ) \cdot BL(INDEX+1, IJ)$ $T9 = \cos(Q(INDEX+1, IJ)) \cdot FY / LELY$ $T10 = Q(INDEX+1, IJ) \cdot AL(INDEX+1, IJ) + P(INDEX+1, IJ) \cdot BL(INDEX+1, IJ)$ $T11 = \sin(Q(INDEX+1, IJ)) \cdot FY / LELY$ $MIJKY = T1 \cdot (T2 \cdot (T3 \cdot T4 - T5 \cdot T6) + T7 \cdot (T8 \cdot T9 - T10 \cdot T11)) \cdot X(3Y)$	
RETURN			
SUBROUTINE SUBDTU(MIJKY, ALXY, FX, FY, LELY, IJ, INDEX) #13.16		$T1 = \cos(ALXY(IJ)) \cdot FX / LELY$ $T2 = EXP(-P(INDEX, IJ)) \cdot FY / LELY$ $T3 = -Q(INDEX, IJ) \cdot (ALXY(IJ))^2 + U \cdot Q(INDEX, IJ)^2 - 3 \cdot U \cdot P(INDEX, IJ)^2$ $T4 = -P(INDEX, IJ) \cdot (ALXY(IJ))^2 - U \cdot P(INDEX, IJ)^2 + 3 \cdot U \cdot Q(INDEX, IJ)^2$ $T5 = EXP(-P(INDEX, IJ)) \cdot FY / LELY \cdot \cos(Q(INDEX, IJ)) \cdot FY / LELY$ $T6 = T4$ $T7 = -T3 \cdot BL(INDEX, IJ)$ $T8 = EXP(-P(INDEX+1, IJ)) \cdot FY / LELY \cdot \sin(Q(INDEX+1, IJ)) \cdot FY / LELY$ $T9 = -Q(INDEX+1, IJ) \cdot (ALXY(IJ))^2 + U \cdot Q(INDEX+1, IJ)^2 - 3 \cdot U \cdot P(INDEX+1, IJ)^2$	
RETURN			
SUBROUTINE SUBDTU(MIJKY, ALXY, FX, FY, LELY, IJ, INDEX) #13.16		$T10 = T9 \cdot AL(INDEX+1, IJ)$ $T11 = -P(INDEX+1, IJ) \cdot (ALXY(IJ))^2 - U \cdot P(INDEX+1, IJ)^2 + 3 \cdot U \cdot Q(INDEX+1, IJ)^2$ $T12 = T11 \cdot BL(INDEX+1, IJ)$ $T13 = EXP(-P(INDEX+1, IJ)) \cdot FY / LELY \cdot \cos(Q(INDEX+1, IJ)) \cdot FY / LELY$ $T14 = T11 \cdot AL(INDEX+1, IJ)$ $T15 = -T9 \cdot BL(INDEX+1, IJ)$ $MIJKY = T1 \cdot (T2 \cdot (T3 + T4 \cdot BL(INDEX, IJ)) + T5 \cdot (T6 + T7)) + T8 \cdot (T9 + T12) + T13 \cdot (T14 + T15) \cdot X(2Y)$	
RETURN			







COMPUTER PROGRAMMING AND ANALYSIS PROGRAM WRITE-UP

SECTION TITLE

SUBDSA #1321

SUBJECT

SUBROUTINE SUBDSA (MXYMYX, ALXY, FX, FY, LILY, JJ, INDEX) #1321

```

T1 = (P-Q) . ALXY(AJ) . SIN(ALXY(AJ)) . EX/LILY
T2 = EXP(-P(INDEX, AJ)) . FY/LILY
T3 = T2 . (1 + Q(INDEX, AJ)) . FY/LILY
T4 = 2 . P(INDEX, AJ) . Q(INDEX, AJ) + (P(INDEX, AJ) - Q(INDEX, AJ)) . BL(INDEX, AJ)
T5 = T2 . (COS(Q(INDEX, AJ)) . FY/LILY
T6 = P(INDEX, AJ) - Q(INDEX, AJ) - 2 . P(INDEX, AJ) . Q(INDEX, AJ) . BL(INDEX, AJ)
T7 = EXP(-P(INDEX+1, AJ)) . FY/LILY
T8 = T7 . SIN(Q(INDEX+1, AJ)) . FY/LILY
T9 = 2 . P(INDEX+1, AJ) . Q(INDEX+1, AJ) . AL(INDEX+1, AJ) + (P(INDEX+1, AJ) - Q(INDEX+1, AJ)) . BL(INDEX+1, AJ)
T10 = T7 . (COS(Q(INDEX+1, AJ)) . FY/LILY)
T11 = (P(INDEX+1, AJ) - Q(INDEX+1, AJ)) . AL(INDEX+1, AJ) - 2 . P(INDEX+1, AJ) . Q(INDEX+1, AJ) . BL(INDEX+1, AJ)
MXYMYX = T1 . (T3 . TY + T5 . T6 + T8 . T9 + T10 . T11)

```

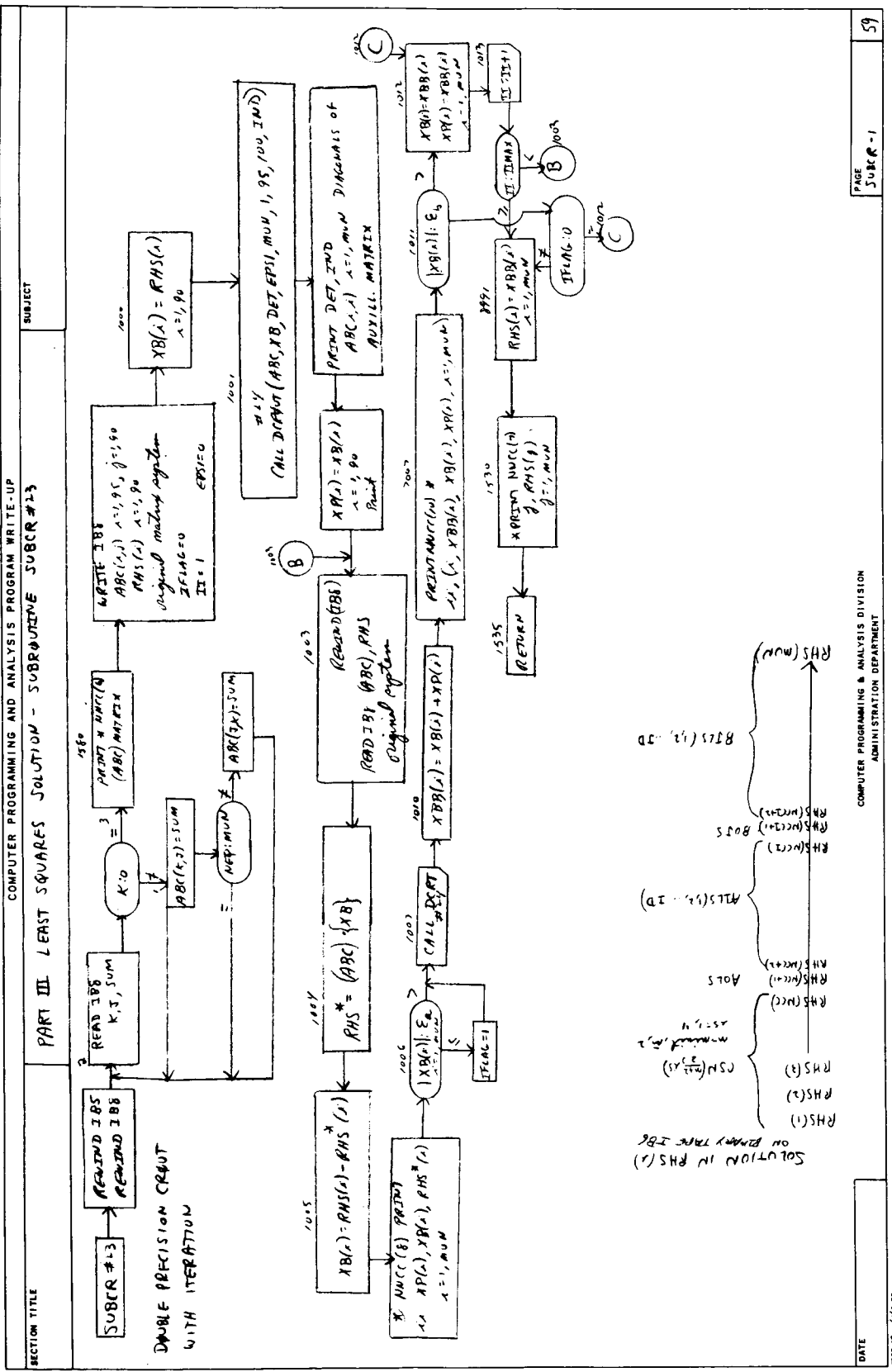
\*136

RETURN

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COMPUTER PROGRAMMING & ANALYSIS DIVISION  
ADMINISTRATION DEPARTMENT





SUBJECT

PART II DOME STRESSES + DEFLECTIONS - MAIN # 1

SECTION TITLE

```

MAIN 01
INPUT(IN,I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,I15,I16,I17,I18,I19,I20) /6
INPUT 02
INIT 03
DOME 04
DMSTDE 05
DAMEO 06
DAMEI 07
SUBD-- 08
RACSM 09 → USES IRY-4-B2 - {YESIN}

```

DATE

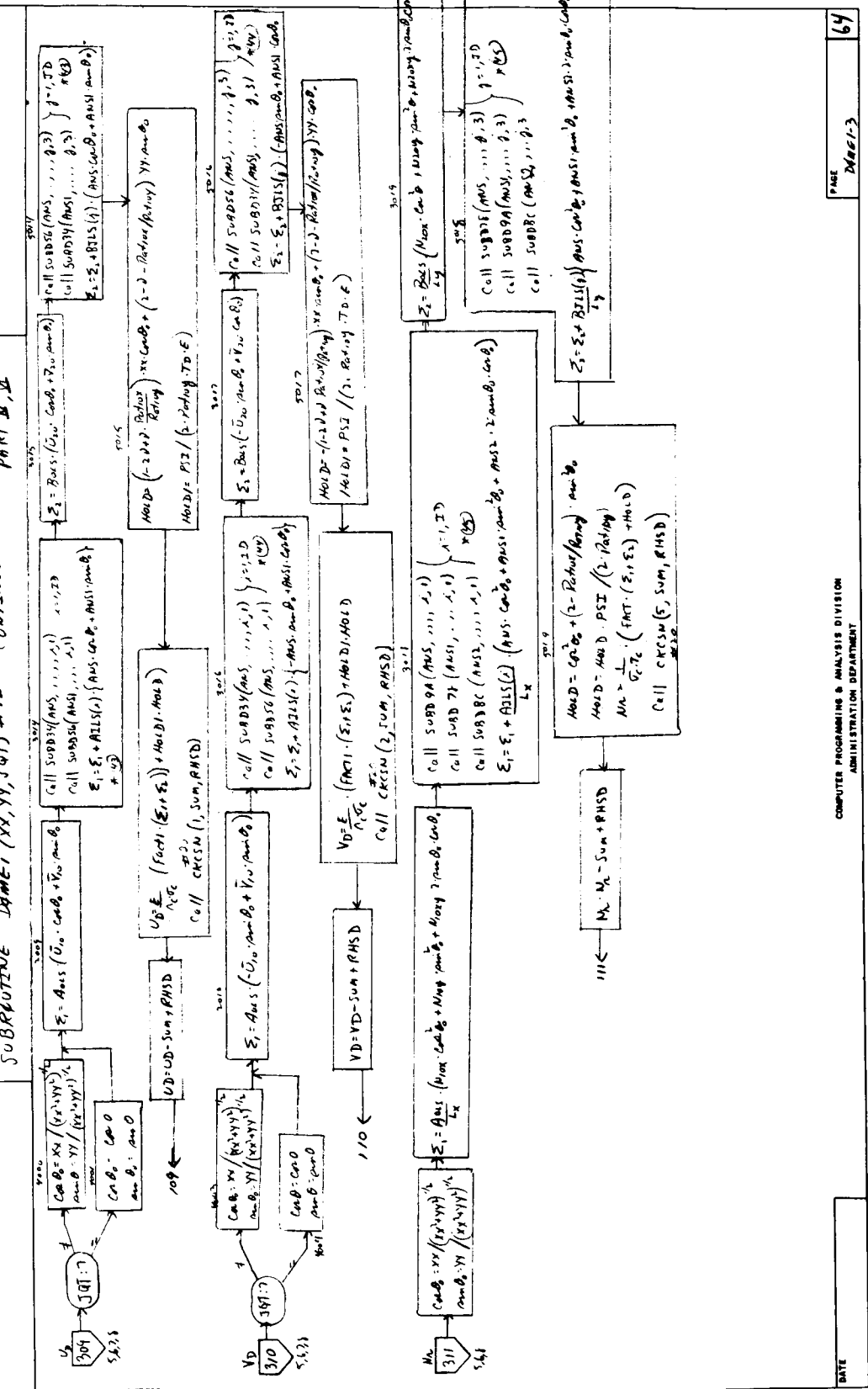
SP4-7 - 6/62 DC













SUBROUTINE GAME1 (XX, YY, JQT) #12

$\bar{Q}_n = \frac{B_{015}}{L_4} \left\{ \begin{array}{l} Q_{001} \cdot C_{010} + Q_{010} \cdot C_{001} + L_4 \cdot (P_1 + V_1)^2 \\ - S \cdot M_{101} \cdot C_{010} \cdot C_{001} - S \cdot M_{102} \cdot C_{010} \cdot C_{001} + S \cdot M_{103} \cdot C_{010} \cdot C_{001} \end{array} \right\}$   
 $\Sigma_1 = \frac{A_{015}}{L_4} \left\{ \begin{array}{l} Q_{001} \cdot C_{010} + Q_{010} \cdot C_{001} + L_4 \cdot (P_1 + V_1)^2 \\ - S \cdot M_{101} \cdot C_{010} \cdot C_{001} - S \cdot M_{102} \cdot C_{010} \cdot C_{001} + S \cdot M_{103} \cdot C_{010} \cdot C_{001} \end{array} \right\}$

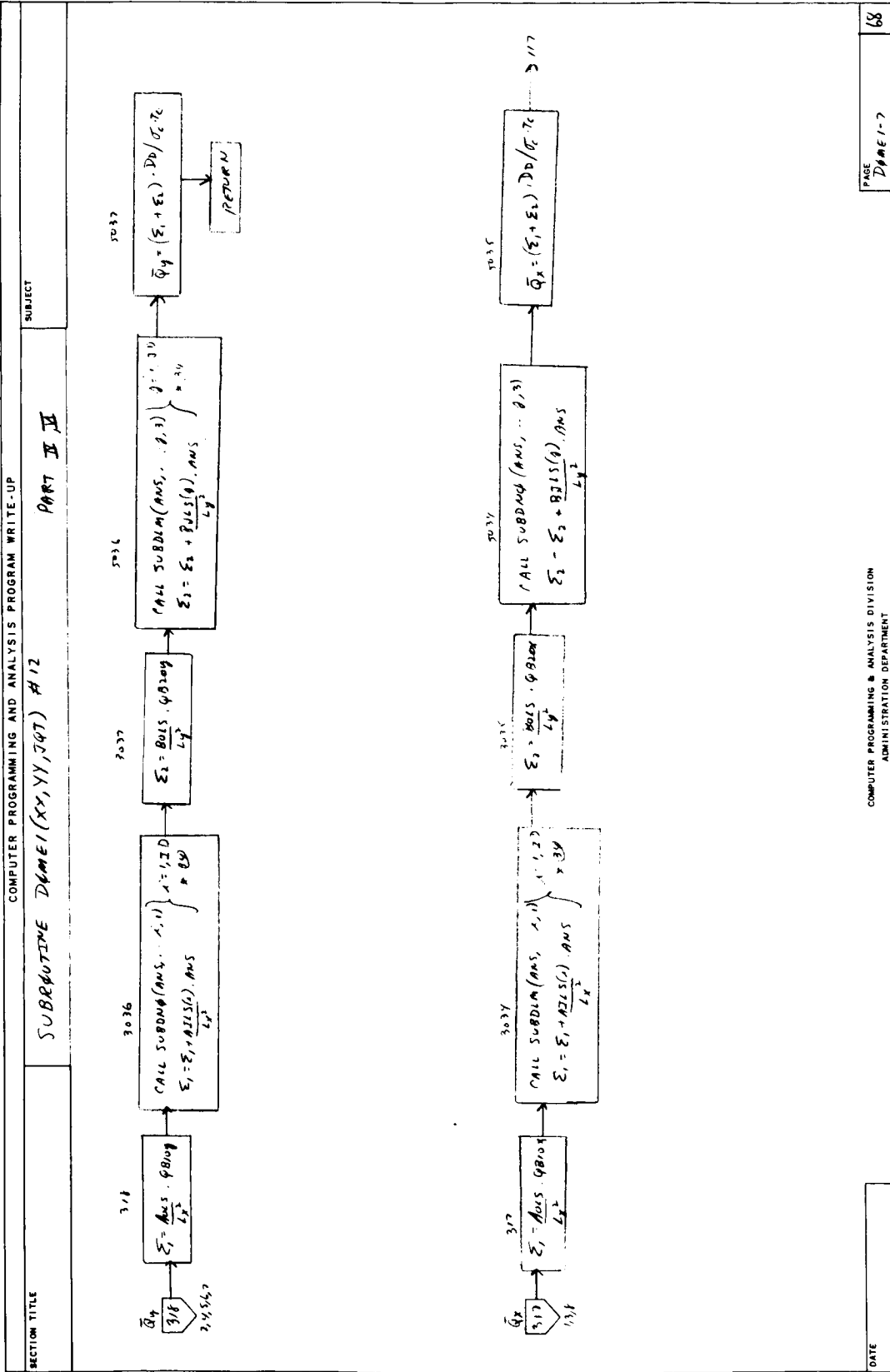
3026  
 Call SUBFG (ANS, ... J, 3)  
 Call SUBDHI (ANS1, ... J, 3)  
 Call SUBRS (ANS2, ... J, 3)  
 Call SWAPQ (ANS3, ... J, 3)  
 Call SWBDTU (ANS4, ... J, 3)  
 Call SWBDA (ANS5, ... J, 3)  
 Call SWBDA (ANS6, ... J, 3)  
 Call SWBDA (ANS7, ... J, 3)  
 Call SWBDA (ANS8, ... J, 3)  
 Call SWBDA (ANS9, ... J, 3)  
 Call SWBDA (ANS10, ... J, 3)

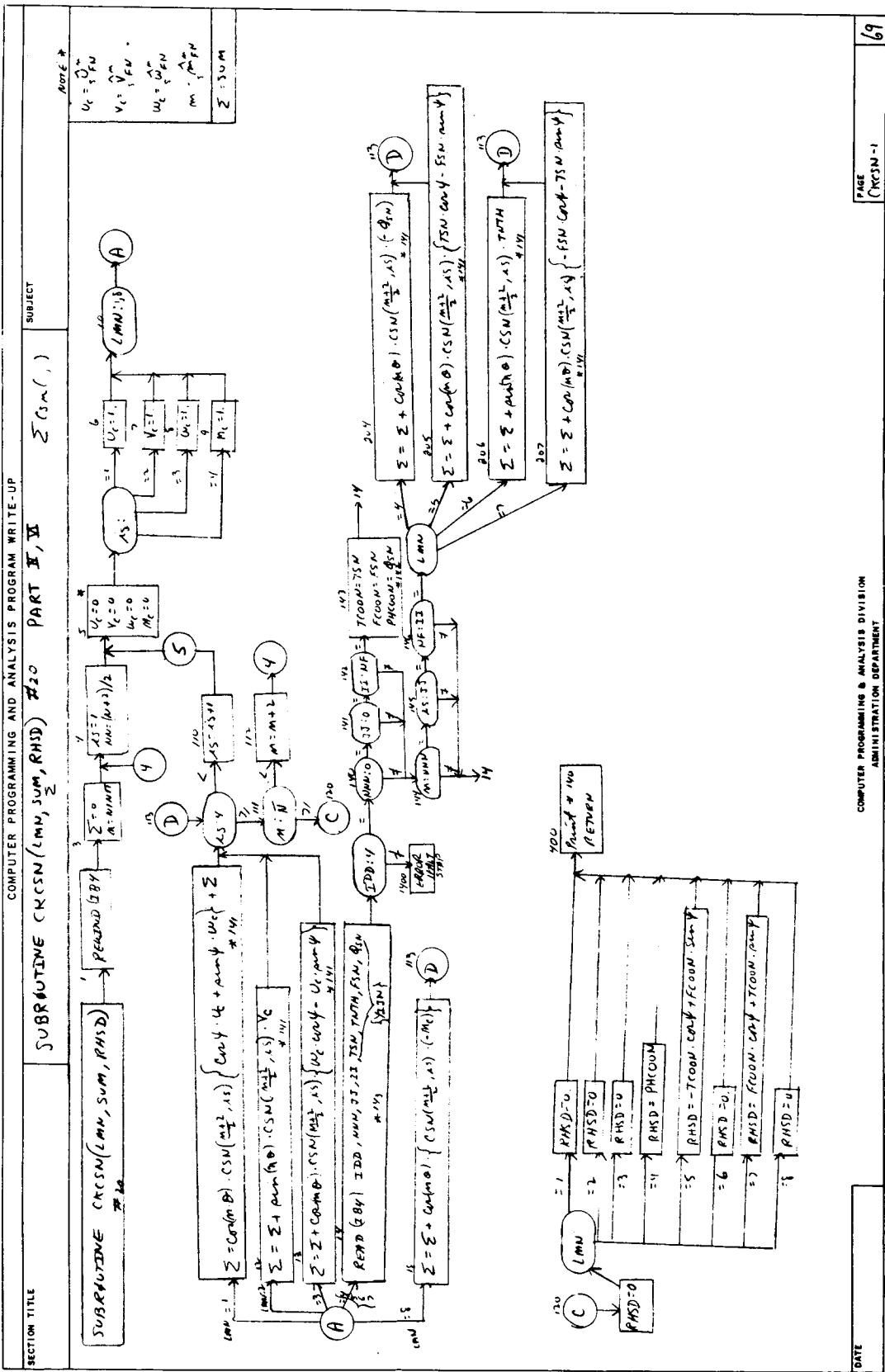
$\Sigma_2 = \frac{B_{015}}{L_4} \left\{ \begin{array}{l} Q_{001} \cdot C_{010} + Q_{010} \cdot C_{001} + L_4 \cdot (P_1 + V_1)^2 \\ - S \cdot M_{101} \cdot C_{010} \cdot C_{001} - S \cdot M_{102} \cdot C_{010} \cdot C_{001} + S \cdot M_{103} \cdot C_{010} \cdot C_{001} \end{array} \right\}$

3027  
 Call SUBDHI (ANS, ... J, 3)  
 FG (ANS1, ... J, 3)  
 PQ (ANS2, ... J, 3)  
 RS (ANS3, ... J, 3)  
 TU (ANS4, ... J, 3)  
 ZA (ANS5, ... J, 3)  
 XY (ANS6, ... J, 3)  
 VW (ANS7, ... J, 3)  
 IA (ANS8, ... J, 3)  
 Z (ANS9, ... J, 3)

$\Sigma_2 = \Sigma_1 + \frac{D_{15}(g)}{L_4} \left\{ \begin{array}{l} ANS \cdot C_{010} + ANS1 \cdot C_{001} + L_4 \cdot (P_1 + V_1)^2 \\ - ANS6 \cdot C_{010} \cdot C_{001} + S \cdot ANS7 \cdot C_{010} \cdot C_{001} - S \cdot ANS8 \cdot C_{010} \cdot C_{001} + S \cdot ANS9 \cdot C_{010} \cdot C_{001} \end{array} \right\}$   
 $\bar{Q}_n = \frac{D_0 \cdot (\Sigma_1 + \Sigma_2)}{\sigma_1 \cdot \tau_2}$   
 CALL CRESN(7, SUM, RMSD)  
 $\bar{Q}_n = \bar{Q}_n - SUM + RMSD$







COMPUTER PROGRAMMING & ANALYSIS DIVISION  
 ADMINISTRATION DEPARTMENT

DATE: \_\_\_\_\_

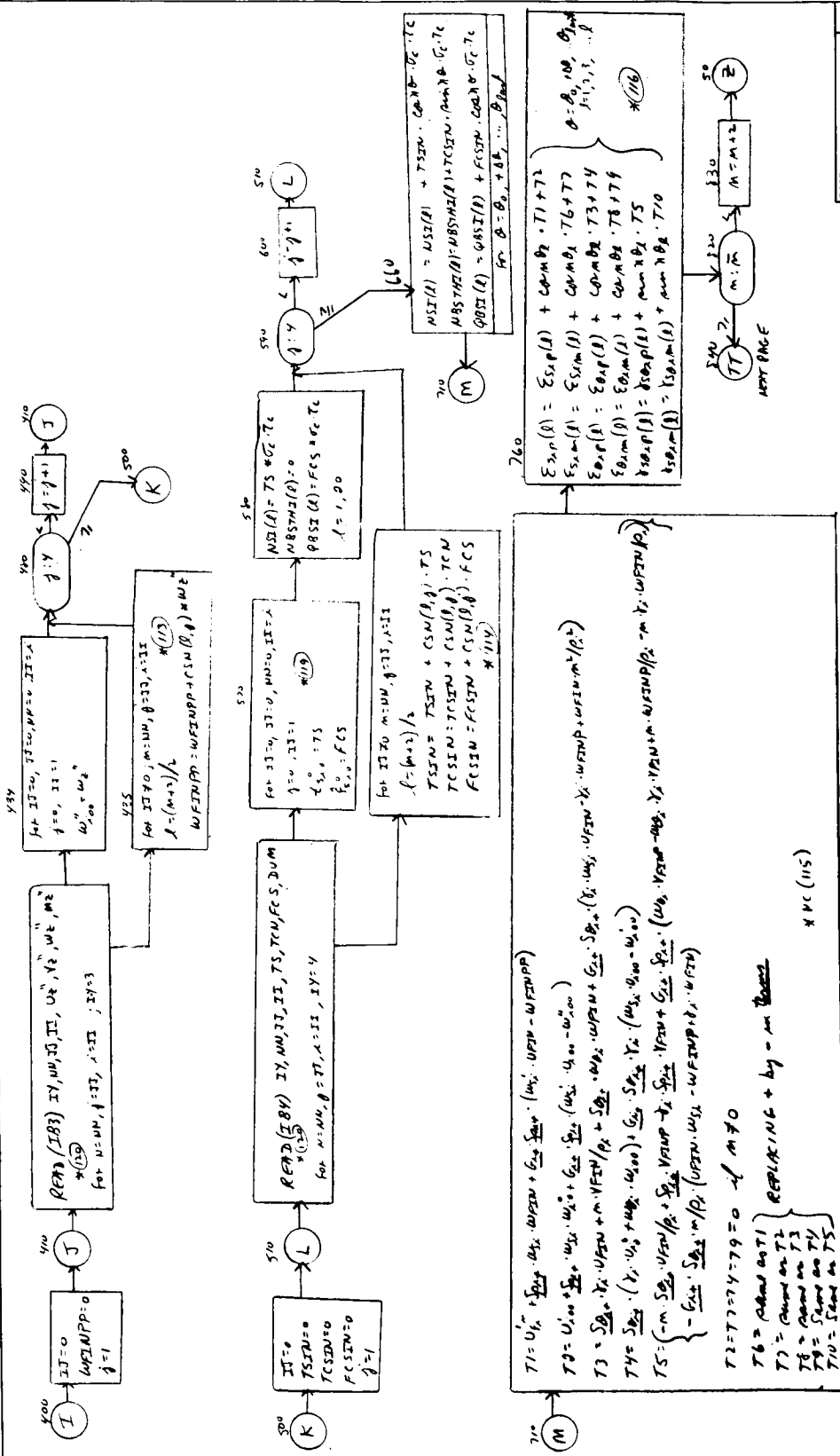
BP-47 - 6/66 DC







PART I CONTINUED FLSTDE #22



PART II FLSTDE #22 CONTINUED

SUBJECT

SECTION TITLE

