RESEARCH INVESTIGATIONS OF BULKHEAD CYLINDRICAL JUNCTIONS EXPOSED TO COMBINED LOAD, CRYOGENIC TEMPERATURE AND PRESSURE

PART IV THEORETICAL ANALYSIS OF THE BUCKLING PROBLEM

by

Tsu–Te Wu
H. Wagner

Technical Report
Contract No. NAS8-5199
Prepared for
George C. Marshall Space Flight Center
NASA
Huntsville, Alabama
National Aeronautics and Space Administration
Washington 25, D. C.

Department of Engineering Science and Mechanics
Engineering and Industrial Experiment Station
University of Florida
Gainesville, Florida

August, 1964

RESEARCH INVESTIGATIONS OF BULKHEAD CYLINDRICAL JUNCTIONS EXPOSED TO COMBINED LOAD, CRYOGENIC TEMPERATURE
PART IV THEORETICAL ANALYSIS OF THE BUCKLING PROBLEM

by

Tsu-Te Wu
H. Wagner

Technical Report
Contract No. NAS8-5199
Prepared for
George C. Marshall Space Flight Center
NASA
Huntsville, Alabama
National Aeronautics and Space Administration
Washington 25, D. C.

Department of Engineering Science and Mechanics
Engineering and Industrial Experiment Station
University of Florida
Gainesville, Florida

August, 1964

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF TABLES

LIST OF FIGURES

NOTATIONS

a	Major axis of meridian ellipse or radius of circular cylinder
b	Minor axis of meridian ellipse or height of dome
t	Constant shell thickness
A_n, B_n	Coefficients of series expansion for radial deflection and stress function, respectively
A_1, A_2	Lamé parameters defined by equation (A7)*
R_1, R_2	Principal radii of curvature of the middle surface of a shell equal to kav^3 and kav, respectively
R_o	Radius of the parallel circle of a shell
k	$\frac{a}{b}$
ϕ, θ	Middle surface coordinates
v	$\frac{1}{[1 + (k^2 - 1)\sin^2\phi]^{1/2}}$

---

*The letter "A" refers to an equation of the Appendices.

E	Young's modulus
μ	Poisson's ratio
l	$\cos \phi_0$
m	number of buckle waves in the circumferential direction
P	Total axial tensile load per unit circumferential length at the shell equator
P_e	Axial tensile load per unit circumferential length at the shell equator due to liquid load and its inertial force
p	Internal pressure
q	Normal load per unit middle surface area
T_ϕ , T_θ , $T_{\phi\theta}$, $T_{\theta\phi}$	Additional force components per unit length in the buckled shell
T_{ϕ_0} , T_{θ_0}	Membrane force components per unit length prior to buckling
w	Normal buckling displacement of a meridian point
x	$\cos \phi$
ϕ_0	Altitude angle of the truncated edge (see Figure 2)
Φ	Stress function defined by equation (2)
$\nabla^2 = \Delta$	Laplacian operator defined by equation (A4)
D	Differential operator defined by equation (A5)

K_1, K_2	Principal curvature of middle surface, see equations (A10)
R	Resulting axial force (see Figure 4)

CHAPTER I

INTRODUCTION

The desire to reduce structural weight in missiles and space vehicles in order to increase payload has resulted in thin-walled shell structures subject to elastic and inelastic buckling as one primary mode of failure. In many instances an adequate buckling analysis or experimental information is difficult to obtain. Consequently, very crude but hopefully conservative approximations or idealizations are employed for analysis of the design. Often designers even avoid entirely particularly lightweight configurations because of the complete lack of experimental or theoretical information on potential instability problems. The results of the above situations may be either the choice of a design which may not be near optimum or, perhaps even worse, a very expensive static test or flight test failure resulting in costly delays and vehicle modifications.

This paper discusses a stability problem of a missile liquid propellant tank. Primarily, the buckling of missile shells consists of two factors:

(a) Internal pressure. In shallow shells of revolution (i.e. where the ratio radius of cylinder "a" over height of dome "b" is larger than $\sqrt{2}$), the internal pressure can produce circumferential compressive stresses of sufficient magnitude to cause elastic buckling of the knuckle part of the shell. In 1964, Adachi and Benicek [1]* made an experimental investigation on buckling of bulkheads with toroidal transitions between spherical caps and cylindrical walls under internal pressure.

(b) Axial tension. In deeper bulkheads of revolution (i.e. $a/b \leq \sqrt{2}$), the internal pressure acting on the concave surface produces predominantly tensile stresses without introducing the possibility of dome circumferential buckling. Then, the main cause of buckling of these kind of shells is axial tensile forces.

This paper, however, discusses buckling of the knuckle of the joint between a cylindrical shell and a

---

*Numbers in bracket refer to the list of references.

semielliptical bulkhead under axial tension. Since the ratio of the length of major to the length of minor axis of the shell considered is $\sqrt{2}$, internal pressure cannot produce circumferential compressive stresses to cause elastic buckling. The buckling of this type of shell is mainly due to axially tensile load (including liquid load and its inertial force). Viewed from testing patterns, the buckling of a shell of revolution under axial tension occurs primarily at the region where its curvature is largest. Thus, for simplicity, this type of shell may be replaced by a truncated semielliptical shell under axial tension to examine its buckling behavior by introducing adequate boundary conditions.

In 1963, Yao made the investigation on buckling of a truncated hemisphere under axial tension [2]. A similar basic idea is employed in this paper, but the problem discussed in this paper is quite different from that of Yao's paper.

In the analytic work, Vlasov's small deflection theory and Galerkin's method are used. The numerical execution was performed by means of an IBM 709 computer.

CHAPTER II

ANALYSIS

Examing buckling testing patterns, we know that a shell of revolution under axial tension, as shown in Figure 1, will buckle primarily at its knuckle part. Thus it may be possible to examine the buckling behavior of a truncated semielliptical shell of revolution (shown in Figure 2) instead of that of a complete semielliptical one by introducing adequate boundary conditions.

1. Basic Equations

The following geometric relations are introduced:

$$\begin{aligned} k &= \frac{a}{b} \\ v &= \frac{1}{[1 + (k^2 - 1)\sin^2\phi]^{\frac{1}{2}}} \\ R_1 &= kav^3 \\ R_2 &= kav \\ R_o &= kav\sin\phi \end{aligned} \tag{1}$$

(a) Equilibrium equation. Using the middle surface coordinates ϕ and θ, as shown in Figure 3, we know that the equilibrium condition in the direction normal to the surface for an elliptical shell element can be expressed by the following equation [3]

$$\frac{Et^3}{12(1-\mu^2)} \Delta\Delta(\omega) - D(\Phi) = q \tag{2}$$

where

$$\Delta = \frac{1}{k^2 a^2 V^6}\left\{\frac{\partial^2}{\partial\phi^2} + V^2\left[\cot\phi + 3(k^2-1)\sin\phi\cos\phi\right]\frac{\partial}{\partial\phi} + V^4 \csc^2\phi \frac{\partial^2}{\partial\theta^2}\right\} \tag{3}$$

$$D = \frac{1}{k^3 a^3 V^5 \sin\phi}\left\{\frac{\sin\phi}{V^2}\frac{\partial^2}{\partial\phi^2} + \left[\frac{\cos\phi}{V^2} + 3(k^2-1)\sin^2\phi\cos\phi\right]\frac{\partial}{\partial\phi} + \frac{1}{\sin\phi}\frac{\partial^2}{\partial\theta^2}\right\} \tag{4}$$

Φ = stress function

For derivation of these equations see Appendix A.

The additional force components in the buckled shell wall are given by

$$T_\phi = -\frac{1}{k^2 a^2 V^2}\left[\csc^2\phi \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\cot\phi}{V^2}\frac{\partial \Phi}{\partial \phi}\right] \tag{5}$$

$$T_\theta = -\frac{1}{k^2 a^2 V^6}\left[3V^2(k^2-1)\sin\phi\cos\phi\frac{\partial \Phi}{\partial \phi} + \frac{\partial^2 \Phi}{\partial \phi^2}\right] \tag{6}$$

$$T_{\phi\theta} = \frac{1}{k^2 a^2 V^4 \sin\phi}\left[\frac{\partial^2 \Phi}{\partial \phi \partial \theta} - V^2 \cot\phi \frac{\partial \Phi}{\partial \theta}\right] \tag{7}$$

For derivation of these equations see the Appendix B.

Equations (5), (6) and (7) approximately satisfy the equations of equilibrium for a shell element in both the meridional and the circumferential directions.

(b) Compatibility equation. The stress function Φ and the normal deflection ω are then related by the compatibility equation [3]

$$\frac{1}{E\,t}\Delta\Delta(\Phi) + D(\omega) = 0 \tag{8}$$

2. Normal Component of Stress Existing Prior to Buckling

In dealing with the problem of stability of a semielliptical shell of revolution, one must take into account the normal component of stress existing prior to buckling.

From equilibrium equations [4] (see Figure 4)

$$2\pi R_o T_{\phi_o} \sin\phi + R = 0 \tag{9}$$

$$\frac{T_{\phi_o}}{R_1} + \frac{T_{\theta_o}}{R_2} = q \tag{10}$$

$$2\pi R_o^* P^* - 2\pi a P + p\pi(a^2 - R_o^{*2}) = 0 \tag{11}$$

where $R = -\pi p(R_o^2 - R_o^{*2}) - 2\pi P^* R_o^*$

and relations (1), we obtain the following expressions for prebuckling forces

$$T_{\phi_o} = \left\{ \frac{P}{Vk} + \frac{pa}{2kV(k^2-1)}\left[1 - k^2V^2\right]\right\} \csc^2\phi \tag{12}$$

$$T_{\theta_o} = kaVp - \left\{ \frac{P}{V^3 k} + \frac{pa}{2kV^3(k^2-1)}\left[1 - k^2V^2\right]\right\} \csc^2\phi \tag{13}$$

During buckling, because of curvature changes, these finite membrane forces contribute a normal component. Principal curvature changes of an ellipsoidal revolutionary shell can be evaluated to be

$$K_1 = -\frac{1}{k^2 a^2 V^6}\left[3V^2(k^2-1)\sin\phi\cos\phi\frac{\partial w}{\partial\phi} + \frac{\partial^2 w}{\partial\phi^2}\right] \tag{14}$$

$$K_2 = -\frac{1}{k^2 a^2 V^4}\left[V^2\csc^2\phi\frac{\partial^2 w}{\partial\theta^2} + \cot\phi\frac{\partial w}{\partial\phi}\right] \tag{15}$$

For derivation of K_1 and K_2, see Appendix C. The normal component is

$$q = -K_1 T_{\phi_0} - K_2 T_{\theta_0} \tag{16}$$

Therefore, by substituting equations (12), (13), (14) and (15) into equation (16), and using relations (1) we obtain the following expression for the normal component:

$$q = \frac{P\csc^2\phi}{k^3 a^2 V^7}\left\{\frac{\partial^2 w}{\partial\phi^2} + \left[3V^2(k^2-1)\sin\phi\cos\phi - \cot\phi\right]\frac{\partial w}{\partial\phi} - V^2\csc^2\phi\frac{\partial^2 w}{\partial\theta^2}\right\} - \frac{p}{2k^3 a V^5}\left\{\cot^2\phi\frac{\partial^2 w}{\partial\phi^2} + \left[3V^2(k^2-1)\cos^2\phi - \cot^2\phi - 2k^2V^2\right]\cot\phi\frac{\partial w}{\partial\phi} - V^2\csc^2\phi(\cot^2\phi + 2k^2V^2)\frac{\partial^2 w}{\partial\theta^2}\right\} \tag{17}$$

3. Boundary Conditions

We can assume that the upper edge of the truncated semielliptical shell of revolution which displaces the complete semielliptical one is clamped and that its lower edge is simply supported, depending upon the following facts:

(a) The upper edge of the shell is actually stiffened by the presence of a skirt which supports the tank. (See Figure 5).

(b) The shell will buckle primarily at the region where its curvature is largest (see indicated buckling pattern in Figure 2), and buckling does not arrive at both edges of the truncated semielliptical shell of revolution .

These conditions may be expressed as

$$\begin{aligned} w = \frac{\partial w}{\partial \phi} = 0 \qquad & \text{at} \quad \phi = \frac{\pi}{2} \\ w = \frac{\partial^2 w}{\partial \phi^2} = 0 \qquad & \text{at} \quad \phi = \phi_0 \end{aligned} \tag{18}$$

$w = 0$ leading to

$$T_\phi = T_{\phi\theta} = 0 \qquad \text{at} \quad \phi = \frac{\pi}{2} \text{ and } \phi_0$$

or, for equations (5), (6) and (7), to

$$\Phi = \frac{\partial \Phi}{\partial \phi} = 0 \qquad \text{at} \quad \phi = \frac{\pi}{2} \text{ and } \phi_0 \tag{19}$$

4. The Method of Approximate Solutions

A possible method of solving the problem would be first to assume an arbitrary normal deflection function that satisfies the normal deflection boundary conditions (18); then equation (8) could be solved for the stress function Φ in terms of a particular solution involving ω and the general solution of the homogeneous equation, the constants of integration being determined by the stress function boundary conditions. Finally, the normal deflection function and the derived stress function can be substituted into equation (2), which then would be solved by means of the Galerkin method [5]. Because it is difficult to solve for the stress function in terms of the normal deflection function, however, this method of solution can be replaced by the equivalent process of choosing arbitrary functions that satisfy the appropriate boundary conditions for both the stress function Φ and the normal deflection function ω and solving both equations (2) and (8) by the Galerkin method.

Thus,

$$\int_{\phi_0}^{\frac{\pi}{2}}\int_0^{2\pi}\left[\frac{Et^3}{12(1-\mu^2)}\Delta\Delta(\omega^*)-D(\Phi^*)-q\right]\delta\omega^* R_1R_2\sin\phi\, d\theta\, d\phi=0 \qquad (20)$$

$$\int_{\phi_0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \left[\frac{1}{Et} \Delta\Delta(\Phi^*) + D(\omega^*) \right] \delta\Phi^* R_1 R_2 \sin\phi \, d\theta \, d\phi = 0 \tag{21}$$

where ω^* and Φ^* are proper functions subjected to the boundary conditions given by equations (18) and (19) and δ expresses the first variation.

For convenience, the following substitutions are introduced:

$$\begin{aligned} & x = \cos\phi, \qquad \ell = \cos\phi_0 \\ & \frac{\partial}{\partial\phi} = -(1-x^2)^{1/2} \frac{\partial}{\partial x} \\ & \frac{\partial^2}{\partial\phi^2} = (1-x^2)\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x} \\ & V = \frac{1}{[1+(k^2-1)(1-x^2)]^{1/2}} \quad \text{or} \quad V = \frac{1}{[k^2-(k^2-1)x^2]^{1/2}} \end{aligned} \tag{22}$$

The boundary conditions (18) and (19) then become

$$\left.\begin{aligned} \omega &= \frac{\partial\omega}{\partial x} = 0 \\ \Phi &= \frac{\partial\Phi}{\partial x} = 0 \end{aligned}\right\} \text{ at } x = 0 \tag{23}$$

and

$$\left.\begin{aligned} \omega &= (1-x^2)\frac{\partial^2\omega}{\partial x^2} - x\frac{\partial\omega}{\partial x} = 0 \\ \Phi &= \frac{\partial\Phi}{\partial x} = 0 \end{aligned}\right\} \text{ at } x = \ell \tag{24}$$

Substituting the relations (22) into equation (3), we have

$$\triangle = \frac{(1-X^2)[k^2-(k^2-1)X^2]^3}{k^2a^2}\left\{\frac{\partial^2}{\partial X^2} - \frac{2X}{(1-X^2)[k^2-(k^2-1)X^2]}[2k^2-1-2(k^2-1)X^2]\frac{\partial}{\partial X}\right.$$

$$\left. + \frac{1}{(1-X^2)^2[k^2-(k^2-1)X^2]^2}\frac{\partial^2}{\partial\theta^2}\right\} \tag{25}$$

Therefore,

$$\begin{aligned}
&k^4a^4\triangle\triangle \\
&= (1-X^2)^2[k^2-(k^2-1)X^2]^6\frac{\partial^4}{\partial X^4} \\
&+ 8(1-X^2)[k^2-(k^2-1)X^2]^5[3(k^2-1)X^3-(3k^2-1)X]\frac{\partial^3}{\partial X^3} \\
&+ [k^2-(k^2-1)X^2]^5\Big\{2(17k^2+12)X^2+10k^2-4 \\
&-2(k^2-1)(80X^4-25X^2-36X+3) \\
&+ \frac{24(k^2-1)^2X^2(1-X^2)^2+28(k^2-1)X^2(1-X^2)+4X^2}{k^2-(k^2-1)X^2}\Big\}\frac{\partial^2}{\partial X^2} \\
&+ [k^2-(k^2-1)X^2]^3\Big\{8(k^2-1)(1-X^2)X[3k^2-5(k^2-1)X^2][1+3k^2 \\
&-3(k^2-1)X^2] + 4X[2k^2+1-2(k^2-1)X^2][3k^2-19(k^2-1)X^2]\Big\}\frac{\partial}{\partial X} \\
&+ 2[k^2-(k^2-1)X^2]^4\frac{\partial^4}{\partial X^2\partial\theta^2} \\
&+ \frac{8[k^2-(k^2-1)X^2]^3[(k^2-1)X^3-k^2X]}{1-X^2}\frac{\partial^3}{\partial X\partial\theta^2} \\
&+ \frac{[k^2-(k^2-1)X^2]^3}{(1-X^2)^2}\Big\{2(1-3X^2)+\frac{4(k^2-1)X^2(1-X^2)}{k^2-(k^2-1)X^2}\Big\}\frac{\partial^2}{\partial\theta^2} +
\end{aligned}$$

$$+\frac{[k^2-(k^2-1)X^2]^2}{(1-X^2)^2}\frac{\partial^4}{\partial\theta^4} \tag{26}$$

Substituting the relations (22) into equation (4), we have

$$D=\frac{(1-X^2)[k^2-(k^2-1)X^2]^{7/2}}{k^3a^3}\left\{\frac{\partial^2}{\partial X^2}+\frac{[5(k^2-1)X^2-5k^2+3]X}{(1-X^2)[k^2-(k^2-1)X^2]}\frac{\partial}{\partial X}\right.$$

$$\left.+\frac{1}{(1-X^2)^2[k^2-(k^2-1)X^2]}\frac{\partial^2}{\partial\theta^2}\right\} \tag{27}$$

Substituting the relations (22) into equation (17), we have

$$q=\frac{P[k^2-(k^2-1)X^2]^{7/2}}{k^3a^2}\left\{\frac{\partial^2 w}{\partial X^2}-\frac{3(k^2-1)X}{k^2-(k^2-1)X^2}\frac{\partial w}{\partial X}-\frac{1}{(1-X^2)^2[k^2-(k^2-1)X^2]}\frac{\partial^2 w}{\partial\theta^2}\right\}$$

$$-\frac{p[k^2-(k^2-1)X^2]^{5/2}}{2k^3a}\left\{X^2\frac{\partial^2 w}{\partial X^2}-\frac{1}{k^2-(k^2-1)X^2}[3(k^2-1)X^2-2k^2]\frac{\partial w}{\partial X}\right.$$

$$\left.-\frac{1}{(1-X^2)[k^2-(k^2-1)X^2]}\left[\frac{X^2}{1-X^2}+\frac{2k^2}{k^2-(k^2-1)X^2}\right]\frac{\partial^2 w}{\partial\theta^2}\right\} \tag{28}$$

Suitable expressions for w^* and Φ^* which satisfy the boundary conditions (23) and (24) may be taken as

$$w^*=\sum_{n=2}^{N}A_nX^2(X-l)(1+a_nX^n)\cos m\theta \tag{29}$$

where $$a_n = \frac{4-5\ell^2}{(5+2n)\ell^{n+2}-(4+2n)\ell^n}$$

$$\Phi^* = \sum_{n=2}^{N} B_n X^n (X-\ell)^n \cos m\theta \qquad (30)$$

Therefore, applying the operator $\triangle\triangle$ [equation (26)] on the function ω^* [equation (28)], we obtain

$$\begin{aligned}
\triangle\triangle(\omega^*) = \frac{1}{k^4a^4}\sum_{n=2}^{N} A_n \Big\{ & (1-X^2)^2[k^2-(k^2-1)X^2]^6[(n+3)(n+2)(n+1)n a_n X^{n-1} \\
& -(n+2)(n+1)n(n-1)a_n \ell X^{n-2}] + 8(1-X^2)[k^2-(k^2-1)X^2]^5 \\
& \times[3(k^2-1)X^3-(3k^2-1)X][(n+3)(n+2)(n+1)a_n X^n \\
& -(n+2)(n+1)n a_n \ell X^{n-1}+6] + [k^2-(k^2-1)X^2]^5[10k^2-4 \\
& -2(k^2-1)(80X^4-25X^2-36X+3)+2(17k^2+12)X^2 \\
& + \frac{24(k^2-1)^2X^2(1-X^2)^2+28(k^2-1)X^2(1-X^2)+4X^2}{k^2-(k^2-1)X^2}][(n+3)(n+2)a_n X^{n+1} \\
& -(n+2)(n+1)a_n \ell X^n+6X-2\ell] + [k^2-(k^2-1)X^2]^3 \\
& \times\{8(k^2-1)(1-X^2)X[3k^2-5(k^2-1)X^2][1+3k^2-3(k^2-1)X^2] \\
& +4X[2k^2+1-2(k^2-1)X^2][3k^2-19(k^2-1)X^2]\}[(n+3)a_n X^{n+2} \\
& -(n+2)a_n \ell X^{n+1}+3X^2-2\ell X] - 2m^2[k^2-(k^2-1)X^2]^4 \\
& \times[(n+3)(n+2)a_n X^{n+1}-(n+2)(n+1)a_n \ell X^n+6X-2\ell] \\
& - \frac{8m^2[k^2-(k^2-1)X^2]^3[(k^2-1)X^3-k^2X]}{1-X^2}\,[(n+3)a_n X^{n+2}
\end{aligned}$$

$$- (n+2) a_n \ell x^{n+1} + 3x^2 - 2\ell x]$$

$$- \frac{m^2[k^2-(k^2-1)x^2]^3}{(1-x^2)^2}\left[2(1-3x^2) + \frac{4(k^2-1)x^2(1-x^2)}{k^2-(k^2-1)x^2}\right]x^2(x-\ell)$$

$$\cdot(1+a_n x^n) + \frac{m^4[k^2-(k^2-1)x^2]^2}{(1-x^2)^2}x^2(x-\ell)(1+a_n x^n)\Big\}\cos m\theta \qquad (31)$$

Applying the operator D (------) on Φ^* (------) we have

$$D(\Phi^*) = \sum_{n=2}^{N} B_n \frac{(1-x^2)[k^2-(k^2-1)x^2]^{7/2}}{k^3 a^3}\Big\{ n(n-1)[x^{n-2}(x-\ell)^n$$

$$+ \frac{2n}{n-1}x^{n-1}(x-\ell)^{n-1} + x^n(x-\ell)^{n-2}]$$

$$+ \frac{[5(k^2-1)x^2-5k^2+3]x}{(1-x^2)[k^2-(k^2-1)x^2]}[nx^{n-1}(x-\ell)+nx^n(x-\ell)^{n-1}]$$

$$- \frac{m^2}{(1-x^2)^2[k^2-(k^2-1)x^2]}x^n(x-\ell)^n\Big\}\cos m\theta \qquad (32)$$

Applying the operator $\triangle\triangle$ on Φ^* [equation (30)], we have

$$\triangle\triangle(\Phi^*) = \frac{1}{k^4 a^4}\sum_{n=2}^{N} B_n \Big\{(1-x^2)^2[k^2-(k^2-1)x^2]^6 n(n-1)(n-2)$$

$$\cdot[(n-3)x^{n-4}(x-\ell)^n + 4nx^{n-3}(x-\ell)^{n-1} + \frac{6n(n-1)}{n-2}x^{n-2}(x-\ell)^{n-2}$$

$$+4nx^{n-1}(x-\ell)^{n-3} + (n-3)x^n(x-\ell)^{n-4}] + 8(1-x^2)[k^2-(k^2-1)x^2]^5 \cdot$$

$$\cdot [3(k^2-1)X^3-(3k^2-1)X]\, n(n-1)(n-2)[X^{n-3}(X-\ell)^n+\frac{3n}{n-2}X^{n-2}(X-\ell)^{n-1}$$

$$+\frac{3n}{n-2}X^{n-1}(X-\ell)^{n-2}+X^n(X-\ell)^{n-3}]+[k^2-(k^2-1)X^2]^5$$

$$\cdot\Big\{2(17k^2+12)X^2+10k^2-4-2(k^2-1)(80X^4-25X^2-36X+3)$$

$$+\frac{24(k^2-1)^2X^2(1-X^2)^2+28(k^2-1)X^2(1-X^2)+4X^2}{k^2-(k^2-1)X^2}\Big\}n(n-1)[X^{n-2}(X-\ell)^n$$

$$+\frac{2n}{n-1}X^{n-1}(X-\ell)^{n-1}+X^n(X-\ell)^{n-2}]+[k^2-(k^2-1)X^2]^3$$

$$\cdot\Big\{8(k^2-1)(1-X^2)X[3k^2-5(k^2-1)X^2][1+3k^2-3(k^2-1)X^2]$$

$$+4X[2k^2+1-2(k^2-1)X^2][3k^2-19(k^2-1)X^2]\Big\}[X^{n-2}(X-\ell)^n$$

$$+\frac{2n}{n-1}X^{n-1}(X-\ell)^{n-1}+X^n(X-\ell)^{n-2}]-\frac{8m^2[k^2-(k^2-1)X^2]^3[(k^2-1)X^3-k^2X]}{1-X^2}$$

$$\cdot[nX^{n-1}(X-\ell)^n+nX^n(X-\ell)^{n-1}]-\frac{m^2[k^2-(k^2-1)X^2]^3}{(1-X^2)^2}$$

$$\cdot\Big\{2(1-3X^2)+\frac{4(k^2-1)X^2(1-X^2)}{k^2-(k^2-1)X^2}\Big\}X^n(X-\ell)^n$$

$$+\frac{m^4[k^2-(k^2-1)X^2]^2}{(1-X^2)^2}X^n(X-\ell)^n\Big\}\cos m\theta \tag{33}$$

Performing D on ω^*, one finds

$$D(\omega^*)=\sum_{n=2}^{N}A_n\frac{(1-X^2)[k^2-(k^2-1)X^2]^{7/2}}{k^3a^3}\Big\{(n+3)(n+2)a_nX^{n+1}$$

$$-(n+2)(n+1)a_n\ell X^n+6X-2\ell+\frac{[5(k^2-1)X^2-5k^2+3]X}{(1-X^2)[k^2-(k^2-1)X^2]}$$

$$\cdot[(n+3)a_nX^{n+2}-(n+2)a_n\ell X^{n+1}+3X^2-2\ell X]$$

$$-\frac{m^2}{(1-X^2)^2[k^2-(k^2-1)X^2]}X^2(X-\ell)(1+a_nX^n)\Big\}\cos m\theta \tag{34}$$

Substituting equation (29) into equation (28), we obtain

$$q = \sum_{n=2}^{N} A_n \frac{P[k^2-(k^2-1)X^2]^{7/2}}{k^3 a^2} \Big\{ (n+3)(n+2) a_n X^{n+1} - (n+2)(n+1) a_n \ell X^n$$

$$+ 6X - 2\ell - \frac{3(k^2-1)}{k^2-(k^2-1)X^2} \big[(n+3) a_n X^{n+3} - (n+2) a_n \ell X^{n+2}$$

$$+ 3X^3 - 2\ell X^2 \big] + \frac{m^2}{(1-X^2)[k^2-(k^2-1)X^2]} X^2 (X-\ell)(1+a_n X^n) \Big\} \cos m\theta$$

$$- \sum_{n=2}^{N} A_n \frac{p[k^2-(k^2-1)X^2]^{5/2}}{2k^3 a} \Big\{ (n+3)(n+2) a_n X^{n+3}$$

$$- (n+2)(n+1) a_n \ell X^{n+2} + 6X^3 - 2\ell X^2 - \frac{3(k^2-1)X^2 - 2k^2}{k^2-(k^2-1)X^2}$$

$$\cdot \big[(n+3) a_n X^{n+3} - (n+2) a_n \ell X^{n+2} + 3X^3 - 2\ell X^2 \big]$$

$$+ \frac{m^2}{(1-X^2)[k^2-(k^2-1)X^2]} \Big[\frac{X^2}{1-X^2} + \frac{2k^2}{k^2-(k^2-1)X^2} \Big]$$

$$\cdot X^2 (X-\ell)(1+a_n X^n) \Big\} \cos m\theta \qquad (35)$$

Substituting equations (31), (32) and (35), and relations (1) and (22) into equation (20), we obtain

$$\int_{\ell}^{0} \int_{0}^{2\pi} \Big[\frac{Et^3}{12(1-\mu^2) k a^3} \sum_{n=2}^{N} A_n \Big\{ (1-X^2)^2 [k^2-(k^2-1)X^2]^4 \big[(n+3)(n+2)(n+1) n a_n X^{n-1}$$

$$- (n+2)(n+1) n (n-1) a_n \ell X^{n-2} \big] + 8(1-X^2)[k^2-(k^2-1)X^2]^3 [3(k^2-1)X^3$$

$$- (3k^2-1)X] [(n+3)(n+2)(n+1) a_n X^n - (n+2)(n+1) n a_n \ell X^{n-1} + 6]$$

$$+ [k^2-(k^2-1)X^2]^3 [2(17k^2+12)X^2 - 2(k^2-1)(80X^4 - 25X^2 - 36X + 3)$$

$$+ 10k^2 - 4 + \frac{24(k^2-1)^2 X^2 (1-X^2)^2 + 28(k^2-1)X^2(1-X^2) + 4X^2}{k^2-(k^2-1)X^2}] \cdot$$

$$\cdot\,[(n+3)(n+2)a_n x^{n+1} - (n+2)(n+1)a_n \ell x^n + 6x - 2\ell]$$

$$+ [k^2-(k^2-1)x^2]\{8(k^2-1)(1-x^2)x[3k^2-5(k^2-1)x^2][1+3k^2-3(k^2-1)x^2]$$

$$+4x[2k^2+1-2(k^2-1)x^2][3k^2-19(k^2-1)x^2]\}[(n+3)a_n x^{n+2} - (n+2)a_n \ell x^{n+1}$$

$$+3x^2-2\ell x] - 2m^2[k^2-(k^2-1)x^2]^2[(n+3)(n+2)a_n x^{n+1} - (n+2)(n+1)a_n \ell x^n$$

$$+6x-2\ell] + \frac{8m^2 x[k^2-(k^2-1)x^2]^2}{1-x^2}[(n+3)a_n x^{n+2} - (n+2)a_n \ell x^{n+1} + 3x^2$$

$$-2\ell x] - \frac{m^2[k^2-(k^2-1)x^2]}{(1-x^2)^2}\left[2(1-3x^2) + \frac{4(k^2-1)x^2(1-x^2)}{k^2-(k^2-1)x^2}\right]x^2(x-\ell)$$

$$\cdot\,(1+a_n x^n) + \frac{m^4 x^2(x-\ell)(1+a_n x^n)}{(1-x^2)^2}\Big\} - \sum_{n=2}^{N} B_n \frac{(1-x^2)[k^2-(k^2-1)x^2]^{3/2}}{a^2}$$

$$\cdot\,\Big\{n(n-1)\left[x^{n-2}(x-\ell)^n + \frac{2n}{n-1}x^{n-1}(x-\ell)^{n-1} + x^n(x-\ell)^{n-2}\right]$$

$$+\frac{[5(k^2-1)x^2-5k^2+3]x}{(1-x^2)[k^2-(k^2-1)x^2]}[nx^{n-1}(x-\ell)^n + nx^n(x-\ell)^{n-1}]$$

$$-\frac{m^2 x^n(x-\ell)^n}{(1-x^2)^2[k^2-(k^2-1)x^2]}\Big\} - \sum_{n=2}^{N} A_n \frac{P[k^2-(k^2-1)x^2]^{3/2}}{a}$$

$$\cdot\,\Big\{(n+3)(n+2)a_n x^{n+1} - (n+2)(n+1)a_n \ell x^n + 6x - 2\ell - \frac{3(k^2-1)}{k^2-(k^2-1)x^2}$$

$$\cdot\,[(n+3)a_n x^{n+3} - (n+2)a_n \ell x^{n+2} + 3x^3 - 2\ell x^2] + \frac{m^2}{(1-x^2)^2[k^2-(k^2-1)x^2]}$$

$$\cdot\,x^2(x-\ell)(1+a_n x^n)\Big\} + \sum_{n=2}^{N} A_n \frac{p[k^2-(k^2-1)x^2]^{1/2}}{2}\Big\{[(n+3)(n+2)a_n x^{n+3}$$

$$-(n+2)(n+1)a_n \ell x^{n+2} + 6x^3 - 2\ell x^2] - \frac{[3(k^2-1)x^2-2k^2]}{k^2-(k^2-1)x^2}$$

$$\cdot\,[(n+3)a_n x^{n+3} - (n+2)a_n \ell x^{n+2} + 3x^3 - 2\ell x^2] + \frac{m^2}{(1-x^2)[k^2-(k^2-1)x^2]}$$

$$\cdot\left[\frac{x^2}{1-x^2} + \frac{2k^3}{k^2-(k^2-1)x^2}\right]x^2(x-\ell)(1+a_n x^n)\Big\}\Big]x^2(x-\ell)(1+a_j x^j)\,dx$$

$$\cdot\,\cos^2 m\theta\, d\theta = 0 \qquad (36)$$

Substituting equations (33) and (34), and relations (1) and (22), into equation (21), we obtain

$$
\begin{aligned}
&\int_{\ell}^{0}\int_{0}^{2\pi}\Big[\frac{1}{kaEt}\sum_{n=2}^{N} B_n\Big\{(1-X^2)^2[k^2-(k^2-1)X^2]^4\, n(n-1)(n-2)[(n-3)X^{n-4} \\
&\cdot(X-\ell)^n+4nX^{n-3}(X-\ell)^{n-1}+\frac{6n(n-1)}{n-2}X^{n-2}(X-\ell)^{n-2} \\
&+4nX^{n-1}(X-\ell)^{n-3}+(n-3)X^n(X-\ell)^{n-4}]+8(1-X^2)[k^2-(k^2-1)X^2]^3 \\
&\cdot[3(k^2-1)X^3-(3k^2-1)X]\,n(n-1)(n-2)[X^{n-3}(X-\ell)^n+\frac{3n}{n-2}X^{n-2}(X-\ell)^{n-1} \\
&+\frac{3n}{n-2}X^{n-1}(X-\ell)^{n-2}+X^n(X-\ell)^{n-3}]+[k^2-(k^2-1)X^2]^3\{2(17k^2+12)X^2 \\
&-2(k^2-1)(80X^4-25X^2-36X+3]+10k^2-4 \\
&+\frac{24(k^2-1)^2X^2(1-X^2)^2+28(k^2-1)X^2(1-X^2)+4X^2}{k^2-(k^2-1)X^2}\Big\}\,n(n-1)[X^{n-2}(X-\ell)^n \\
&+\frac{2n}{n-1}X^{n-1}(X-\ell)^{n-1}+X^n(X-\ell)^{n-2}]+[k^2-(k^2-1)X^2]\{8(k^2-1) \\
&\cdot(1-X^2)X[3k^2-5(k^2-1)X^2][1+3k^2-3(k^2-1)X^2]+4X[2k^2+1 \\
&-2(k^2-1)X^2][3k^2-19(k^2-1)X^2]\}[nX^{n-1}(X-\ell)^n+nX^n(X-\ell)^{n-1}] \\
&-2m^2[k^2-(k^2-1)X^2]^2\, n(n-1)[X^{n-2}(X-\ell)^n+\frac{2n}{n-1}X^{n-1}(X-\ell)^{n-1} \\
&+X^n(X-\ell)^{n-2}]+\frac{8m^2X[k^2-(k^2-1)X^2]^2}{1-X^2}[nX^{n-1}(X-\ell)^n+nX^n(X-\ell)^{n-1}] \\
&-\frac{m^2[k^2-(k^2-1)X^2]}{(1-X^2)^2}\Big\{2(1-3X^2)+\frac{4(k^2-1)X^2(1-X^2)}{k^2-(k^2-1)X^2}\Big\}X^n(X-\ell)^n \\
&+\frac{m^4X^n(X-\ell)^n}{(1-X^2)^2}\Big\}+\sum A_n(1-X^2)[k^2-(k^2-1)X^2]^{3/2}\{(n+3)(n+2)a_nX^{n+1} \\
&-(n+2)(n+1)a_n\ell X^n+6X-2\ell+\frac{[5(k^2-1)X^2-5k^2+3]X}{(1-X^2)[k^2-(k^2-1)X^2]}[(n+3)a_nX^{n+2} \\
&-(n+2)a_n\ell X^{n+1}+3X^2-2\ell X]-\frac{m^2}{(1-X^2)^2[k^2-(k^2-1)X^2]}X^2(X-\ell)(1+a_nX^n)\}\Big] \\
&\cdot X^i(X-\ell)^i\,dX\cos^2 m\theta\, d\theta=0 \qquad (37)
\end{aligned}
$$

By dropping $\int_0^{2\pi} \cos^2 m\theta \, d\theta \neq 0$, equations (36) and (37) become

$$\sum_{n=2}^{N} A_n \frac{Et^3}{12(1-\mu^2)kq^3} \int_0^{l} \Big\{ (1-x^2)^2 [k^2-(k^2-1)x^2]^4 [(n+3)(n+2)(n+1)n a_n x^{n-1}$$

$$-(n+2)(n+1)n(n-1)a_n l x^{n-2}] + 8(1-x^2)[k^2-(k^2-1)x^2]^3 [3(k^2-1)x^3$$

$$-(3k^2-1)x][(n+3)(n+2)(n+1)a_n x^n - (n+2)(n+1)n a_n l x^{n-1} + 6]$$

$$+[k^2-(k^2-1)x^2]^3 [\, 2(17k^2+12)x^2 \; -2(k^2-1)(80x^4-25x^2-36x+3)$$

$$+10k^2-4+\frac{24(k^2-1)^2x^2(1-x^2)^2+28(k^2-1)x^2(1-x^2)+4x^2}{k^2-(k^2-1)x^2}][(n+3)(n+2)$$

$$\cdot a_n x^{n+1} - (n+2)(n+1)a_n l x^n + 6x - 2l] + [k^2-(k^2-1)x^2]\{8(k^2-1)(1-x^2)$$

$$\cdot x[3k^2-5(k^2-1)x^2][1+3k^2-3(k^2-1)x^2] + 4x[2k^2+1-2(k^2-1)x^2]$$

$$\cdot [3k^2-19(k^2-1)x^2]\}[(n+3)a_n x^{n+2} - (n+2)a_n l x^{n+1} + 3x^2 - 2lx]$$

$$-2m^2[k^2-(k^2-1)x^2]^2[(n+3)(n+2)a_n x^{n+1} - (n+2)(n+1)a_n l x^n + 6x - 2l]$$

$$+\frac{8m^2x[k^2-(k^2-1)x^2]^2}{1-x^2}[(n+3)a_n x^{n+2} - (n+2)a_n l x^{n+1} + 3x^2 - 2lx]$$

$$-\frac{m^2[k^2-(k^2-1)x^2]}{(1-x^2)^2}[2(1-3x^2)+\frac{4(k^2-1)x^2(1-x^2)}{k^2-(k^2-1)x^2}]x^2(x-l)(1+a_n x^n)$$

$$+\frac{m^4x^2(x-l)(1+a_n x^n)}{(1-x^2)^2}\Big\} x^2(x-l)(1+a_j x^j)\,dx$$

$$-\sum_{n=2}^{N}\frac{B_n}{q^2}\int_0^{l}(1-x^2)[k^2-(k^2-1)x^2]^{3/2}\Big\{n(n-1)[x^{n-2}(x-l)^n+\frac{2n}{n-1}x^{n-1}(x-l)^{n-1}$$

$$+x^n(x-l)^{n-2}]+\frac{[5(k^2-1)x^2-5k^2+3]x}{(1-x^2)[k^2-(k^2-1)x^2]}[nx^{n-1}(x-l)^n+nx^n(x-l)^{n-1}]$$

$$-\frac{m^2}{(1-x^2)[k^2-(k^2-1)x^2]}x^n(x-l)^n\Big\}x^2(x-l)(1+a_j x^j)\,dx$$

$$-\sum_{n=2}^{N}\frac{A_n P}{a}\int_0^{l}[k^2-(k^2-1)x^2]^{3/2}\Big\{[(n+3)(n+2)a_n x^{n+1} - (n+2)(n+1)a_n l x^n$$

$$+6x-2l]-\frac{3(k^2-1)}{[k^2-(k^2-1)x^2]}[(n+3)a_n x^{n+3} - (n+2)a_n l x^{n+2} + 3x^3 - 2lx^2] +$$

$$+ \frac{m^2}{(1-x^2)^2[k^2-(k^2-1)x^2]} x^2(x-\ell)(1+a_n x^n)\Big\} x^2(x-\ell)(1+a_j x^j)\,dx$$

$$+\sum_{n=2}^{N} \frac{A_n p}{2} \int_0^{\ell} [k^2-(k^2-1)x^2]^{1/2} \Big\{ [(n+3)(n+2)a_n x^{n+3} - (n+2)(n+1)a_n \ell x^{n+2}$$

$$+ 6x^3 - 2\ell x^2] - \frac{[3(k^2-1)x^2-2k^2]}{k^2-(k^2-1)x^2} [(n+3)a_n x^{n+3} - (n+2)a_n \ell x^{n+2}$$

$$+3x^3 - 2\ell x^2] + \frac{m^2}{(1-x^2)[k^2-(k^2-1)x^2]} \left[\frac{x^2}{1-x^2} + \frac{2k^2}{k^2-(k^2-1)x^2}\right] x^2$$

$$\cdot (x-\ell)(1+a_n x^n)\Big\} x^2(x-\ell)(1+a_j x^j)\,dx = 0 \tag{38}$$

and

$$\sum_{n=2}^{N} B_n \frac{1}{kaEt} \int_0^{\ell} \Big\{ (1-x^2)^2 [k^2-(k^2-1)x^2]^4 n(n-1)(n-2)[(n-3)x^{n-4}(x-\ell)^n$$

$$+4n x^{n-3}(x-\ell)^{n-1} + \frac{6n(n-1)}{n-2} x^{n-2}(x-\ell)^{n-2} + 4n x^{n-1}(x-\ell)^{n-3}$$

$$+(n-3)x^n(x-\ell)^{n-4}] + 8(1-x^2)[k^2-(k^2-1)x^2]^3[3(k^2-1)x^3-(3k^2-1)x]$$

$$\cdot n(n-1)(n-2)\left[x^{n-3}(x-\ell)^n + \frac{3n}{n-2} x^{n-2}(x-\ell)^{n-1} + \frac{3n}{n-2} x^{n-1}(x-\ell)^{n-2}\right.$$

$$\left. + x^n(x-\ell)^{n-3}\right] + [k^2-(k^2-1)x^2]^3 \Big\{ 2(17k^2+12)x^2 - 2(k^2-1)(80x^4 - 25x^2$$

$$-36x+3) + 10k^2 - 4 + \frac{24(k^2-1)^2x^2(1-x^2)^2 + 28(k^2-1)x^2(1-x^2) + 4x^2}{k^2-(k^2-1)x^2} \Big\}$$

$$\cdot n(n-1)\left[x^{n-2}(x-\ell)^n + \frac{2n}{n-1} x^{n-1}(x-\ell)^{n-1} + x^n(x-\ell)^{n-2}\right]$$

$$+[k^2-(k^2-1)x^2]\Big\{ 8(k^2-1)(1-x^2)x[3k^2-5(k^2-1)x^2][1+3k^2-3(k^2-1)x^2]$$

$$+4x[2k+1-2(k^2-1)x^2][3k^2-19(k^2-1)x^2] \Big\} [n x^{n-1}(x-\ell)^n$$

$$+ n x^n (x-\ell)^{n-1}] - 2m^2[k^2-(k^2-1)x^2]^2 n(n-1)[x^{n-2}(x-\ell)^n$$

$$+ \frac{2n}{n-1} x^{n-1}(x-\ell)^{n-1} + x^n(x-\ell)^{n-2}] + \frac{8m^2 x[k^2-(k^2-1)x^2]^2}{1-x^2}$$

$$\cdot [n x^{n-1}(x-\ell)^n + n x^n(x-\ell)^{n-1}] - \frac{m^2[k^2-(k^2-1)x^2]}{(1-x^2)^2} \Big\{ 2(1-3x^2) +$$

$$+\frac{4(k^2-1)X^2(1-X^2)}{k^2-(k^2-1)X^2}\Big\}X^n(X-\ell)^n + \frac{m^4X^n(X-\ell)^n}{(1-X^2)^2}\Big\}X^j(X-\ell)^j dX$$

$$+\sum_{n=2}^{N} A_n \int_0^{\ell} (1-X^2)[k^2-(k^2-1)X^2]^{3/2}\Big\{[(n+3)(n+2)a_nX^{n+1}-(n+2)(n+1)a_n\ell X^n$$

$$+6X-2\ell] + \frac{[5(k^2-1)X^2-5k^2+3]X}{(1-X^2)[k^2-(k^2-1)X^2]}[(n+3)a_nX^{n+2}-(n+2)a_n\ell X^{n+1}+3X^2$$

$$-2\ell X] - \frac{m^2}{(1-X^2)^2[k^2-(k^2-1)X^2]}X^2(X-\ell)(1+a_nX^n)\Big\}X^j(X-\ell)^j dX = 0$$

(39)

By integrating equations (38) and (39), we get the following equations:

$$\sum_{n=2}^{N}\Big\{\Big[\frac{Et^3}{12(1-\mu^2)ka^3}f_{nj} - \frac{P}{a}h_{nj} + \frac{p}{2}g_{nj}\Big]A_n - \frac{g_{nj}}{a^2}B_n\Big\} = 0 \qquad (40)$$

$$\sum_{n=2}^{N}\Big(d_{nj}A_n + \frac{e_{nj}}{kaEt}B_n\Big) = 0 \qquad (41)$$

where $j = 2, 3, 4 \cdots\cdots N$

Introducing the relation $P = P_e + \frac{pa}{2}$ (where P_e = external axial load) into equations (38), we obtain the following equations

$$\sum_{n=2}^{N}\Big\{\Big[\frac{Et^3}{12(1-\mu^2)ka^3}f_{nj} - \frac{P_e}{a}h_{nj} + \frac{p}{2}(g_{nj}-h_{nj})\Big]A_n - \frac{g_{nj}}{a^2}B_n\Big\} = 0 \qquad (42)$$

$$\sum_{n=2}^{N} \left(d_{nj} A_n + \frac{e_{nj}}{kaEt} B_n \right) = 0 \tag{43}$$

where $j = 2, 3, 4 ----- N$

For f_{nj}, h_{nj}, q_{nj}, g_{nj}, d_{nj} and e_{nj}, see Appendix D.

For a nontrivial solution for A_n and B_n, the determinant of their coefficients in equations (42) and (43) must vanish, yielding an algebraic equation in the following form

$$f_1(m) P_e + f_2(m) p = f_3(m) \tag{44}$$

where $f_i(m)$ $(i = 1, 2, 3)$ are rational functions of m.

Equation (44) can be solved for P_e, i.e.

$$P_e = \psi(p, m)$$

The minima of P_e for various p are the critical values of the axially tensile loads (P_e).

CHAPTER III

APPLICATION OF A SPECIAL PROBLEM

The general characteristics of the special semielliptical shell of revolution made of aluminum alloy 2219 are (see Figure 1):

$$a = 198 \text{ in}$$

$$b = 140 \text{ in}$$

$$t = 0.163 \text{ in}$$

$$h = 68 \text{ in}$$

$$l = \cos \phi_0$$

$$= 0.647$$

$$E = 10.6 \times 10^6 \text{ lb/in}^2$$

$$\mu = 0.33$$

By substituting the above values into equations (42) and (43) and taking one term in the summation (n = 2) we find that the determinant of the coefficients of equations (42) and (43) yields

$$(1.6139\times10^{-4} + 0.8357\times10^{-5}m^2 - 0.9543\times10^{-7}m^4 - 0.3388\times10^{-9}m^6)P_e$$
$$-(0.00405 + 2.0894\times10^{-3}m^2 - 2.3857\times10^{-5}m^4 - 0.847\times10^{-7}m^6)p$$
$$+0.7889 - 0.06371m^2 + 0.00087m^4 + 0.37596\times10^{-6}m^6$$
$$+0.1933\times10^{-10}m^8 = 0 \tag{45}$$

Putting internal pressure p equal to 0, 5, 15, 25, 35, 45 and 55^{psi} respectively, we get the buckling loads P_e for various number of waves (m) (see Tables 1 through 5).

The minima of these buckling loads are the critical tensile loads for internal pressure 0, 5, 15, 25, 35, 45 and 55^{psi} respectively.

The pertinent numerical execution was performed by means of an IBM 709 computer.

CHAPTER IV

CONCLUSION AND DISCUSSION

It can be observed from Figure 5 that the critical load of tension increases with the increase of internal pressure. This fact indicates that internal pressure will decrease the degree of buckling of this type of shell.

Figure 7 shows that there is an approximate linear relation between critical tension load and internal pressure in this particular problem. From the equation (44) we know that, for $m = m_o$, a linear relation between P_e and p holds. m_o indicates the argument for which P_e is minimal for a given p. Actually, for all p the number m_o is practically the same, thus giving the above mentioned linear relation shown in Figure 7.

The subject problem is an interesting one, as it is a particular example that an elastic system is buckled by tension, although the real cause of the instability is the compressive hoop stress. Certainly, there remain many areas for future studies, among them determination of the buckling load by large deflection theory and consideration of initial imperfections.

TABLE 1

BUCKLING FORCES FOR INTERNAL PRESSURE $p = 0^{psi}$

m	P_e lb/in
10	11782
20	4157
30	2953
40	2343
50	2029
60	1871
70	1804
80	1794
90	1823
100	1883
110	1966

TABLE 2

BUCKLING FORCES FOR INTERNAL PRESSURE $p = 5^{psi}$

m	P_e lb/in
10	13640
20	5413
30	4203
40	3593
50	3279
60	3121
70	3054
80	3044
90	3073
100	3133
110	3216

TABLE 3

BUCKLING FORCES FOR INTERNAL PRESSURE $p = 15^{psi}$

m	P_e lb/in
10	17357
20	7923
30	6705
40	6094
50	5779
60	5621
70	5554
80	5544
90	5573
100	5633
110	5716

TABLE 4

BUCKLING FORCES FOR INTERNAL PRESSURE $p = 25^{psi}$

m	P_e lb/in
10	21074
20	10434
30	9206
40	8594
50	8279
60	8121
70	8054
80	8044
90	8073
100	8133
110	8216

TABLE 5

BUCKLING FORCES FOR INTERNAL PRESSURE $p = 35^{psi}$

m	P_e lb/in
10	24791
20	12944
30	11707
40	11094
50	10779
60	10621
70	10554
80	10544
90	10573
100	10632
110	10716

TABLE 6

BUCKLING FORCES FOR INTERNAL PRESSURE $p = 45^{psi}$

m	P_e lb/in
10	28507
20	15455
30	14208
40	13594
50	13279
60	13121
70	13054
80	13044
90	13073
100	13133
110	13216

TABLE 7

BUCKLING FORCES FOR INTERNAL PRESSURE p = 55^{psi}

m	P_e lb/in
10	32224
20	17966
30	16709
40	16094
50	15779
60	15621
70	15554
80	15544
90	15573
100	15633
110	15716

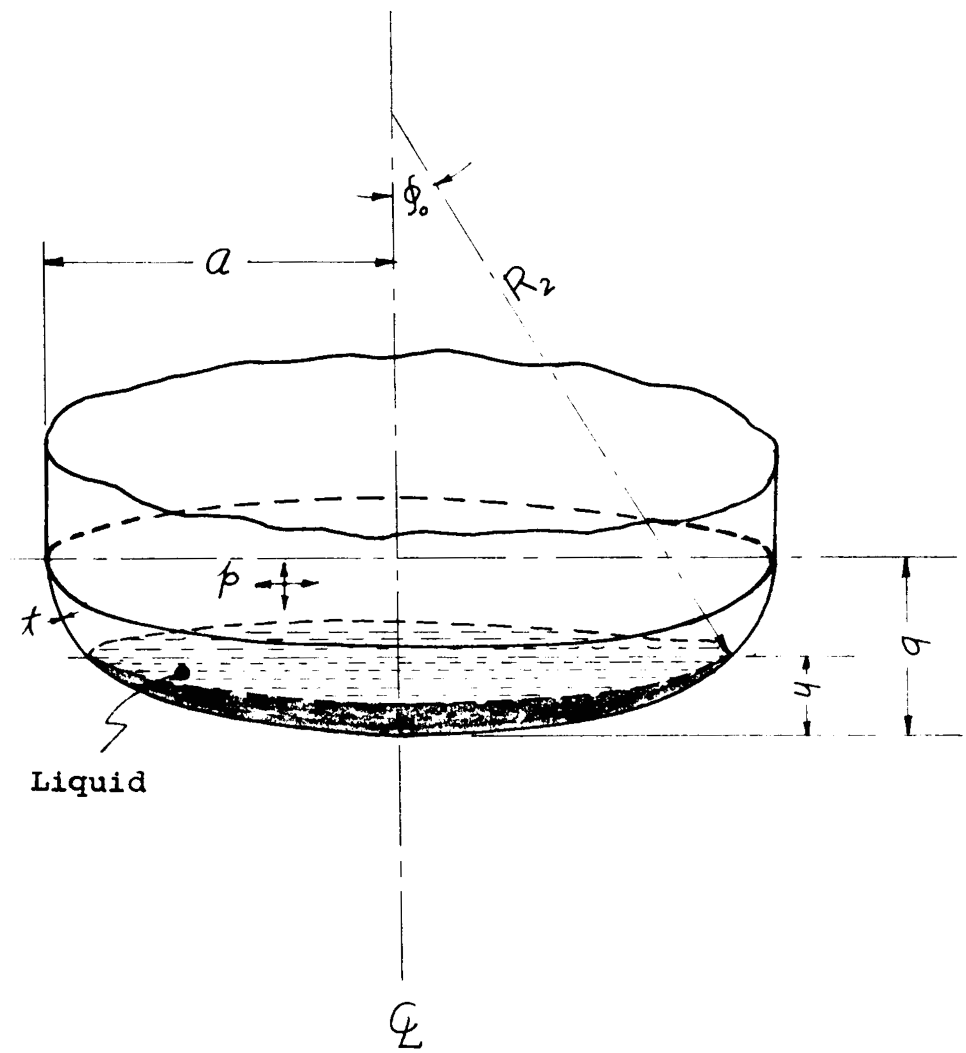

Figure 1. General sketch

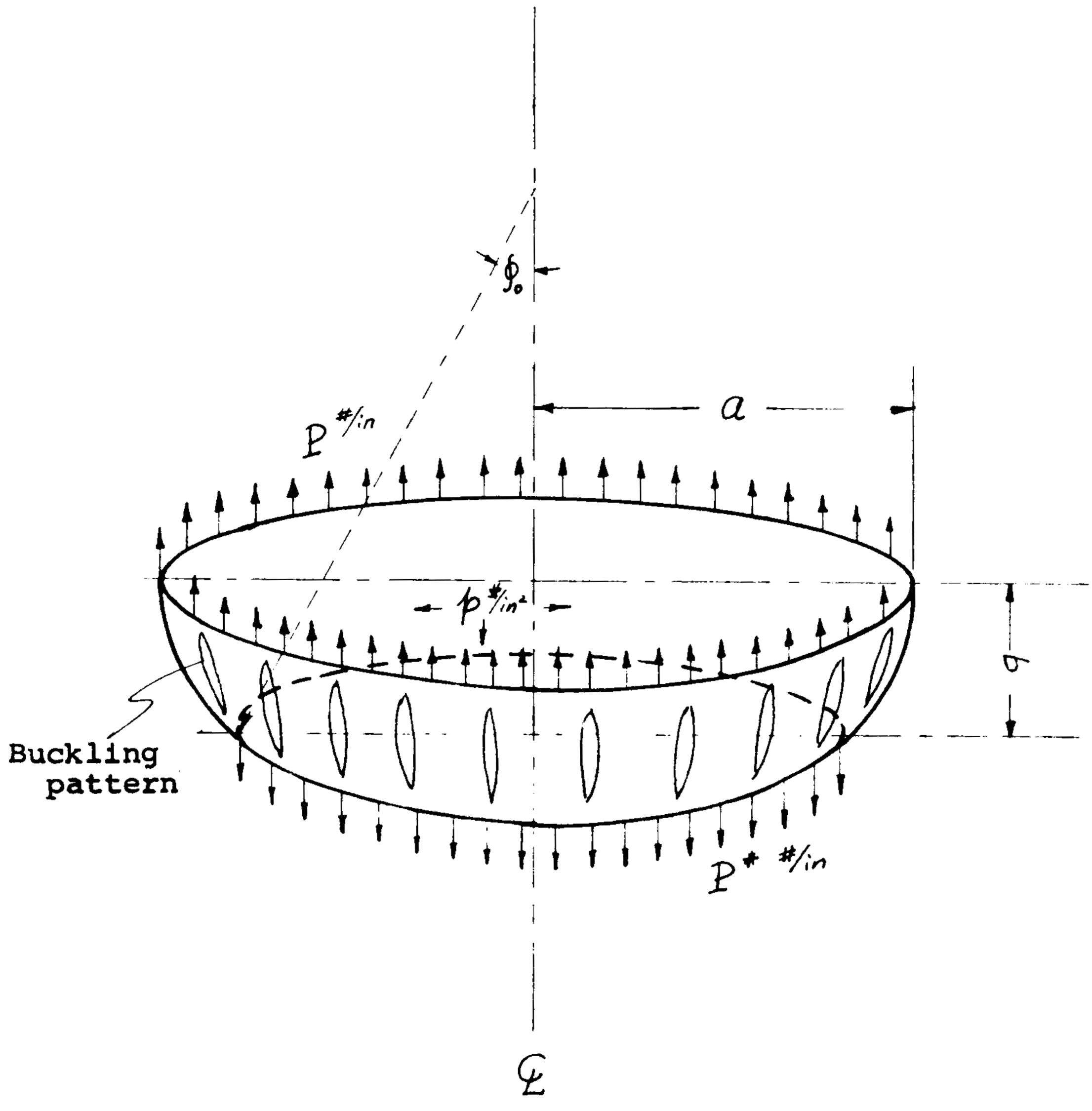

Figure 2. Truncated shell geometry and load

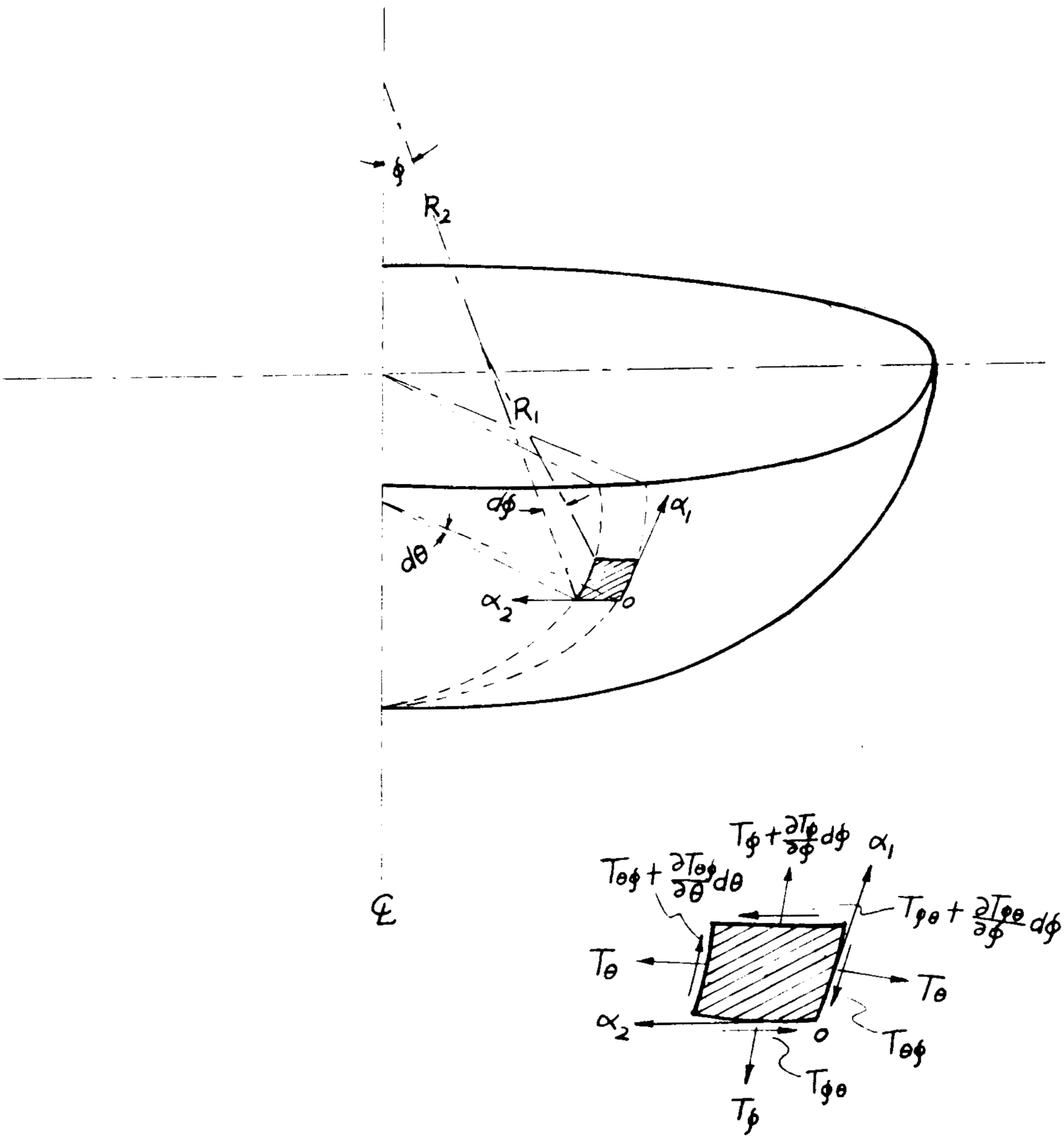

Figure 3. Element of the middle surface

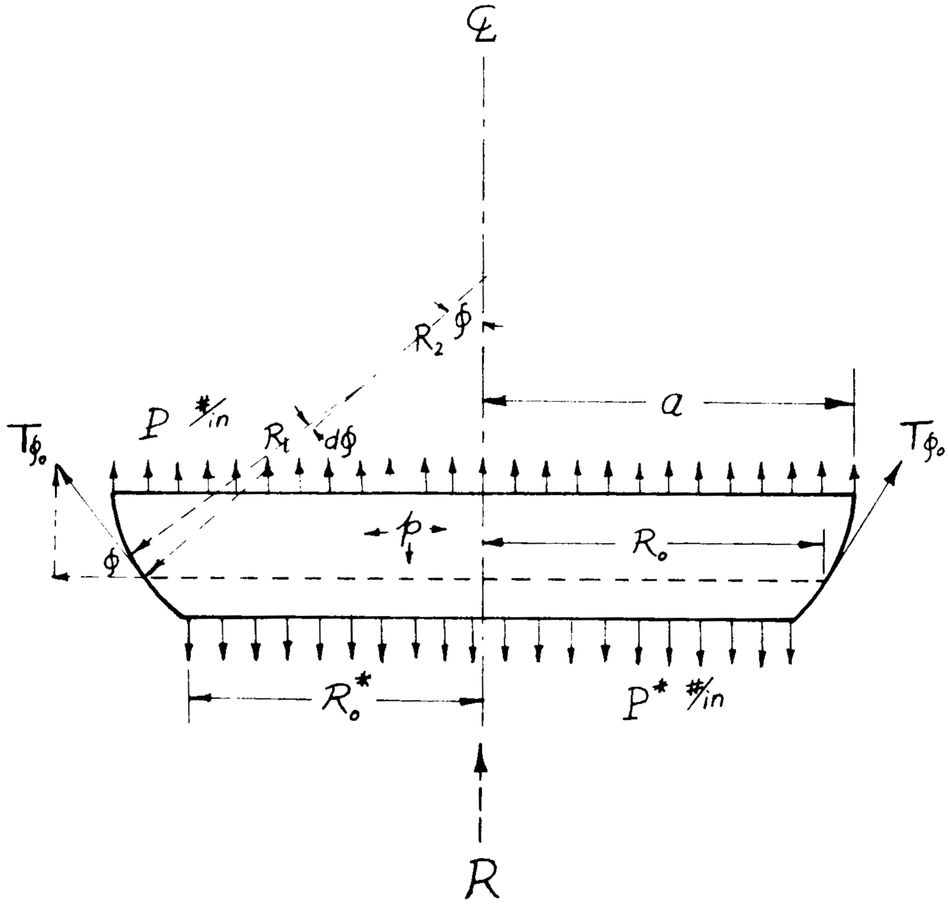

Figure 4. Sketch of equilibrium conditions

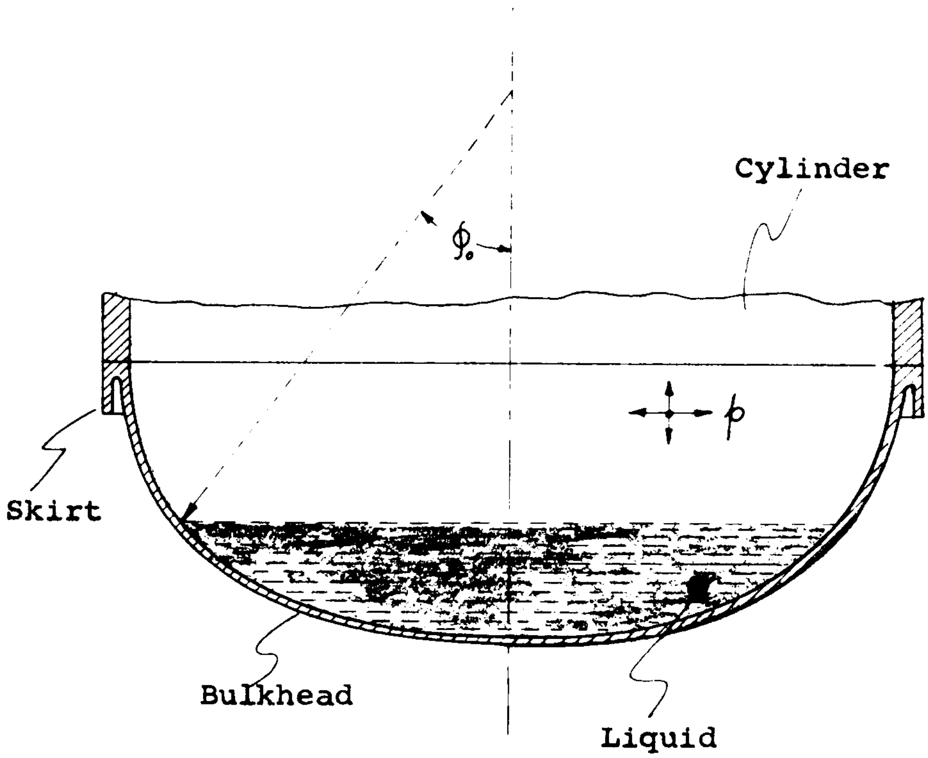

Figure 5. Sketch of the upper-end support of the bulkhead

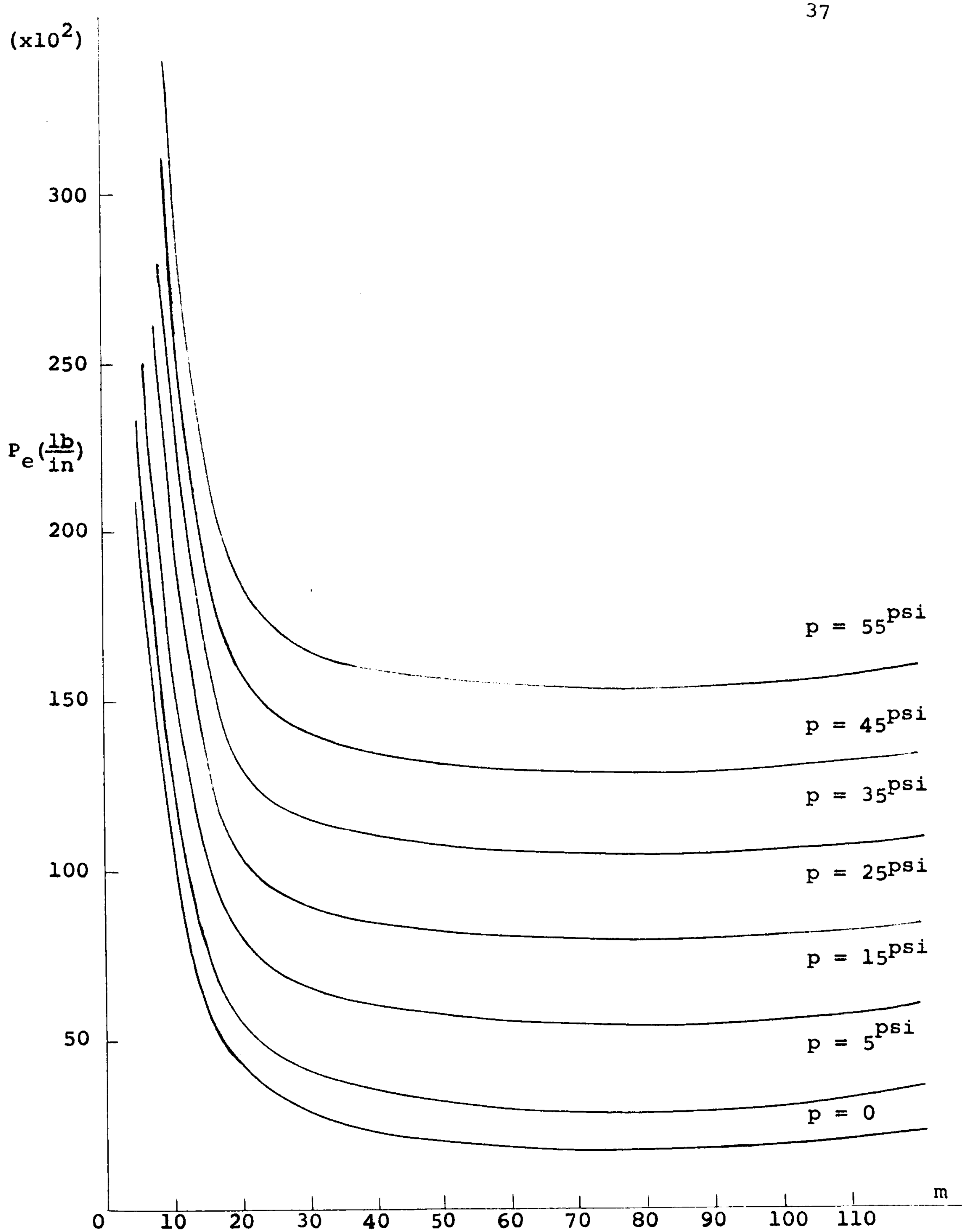

Figure 6. Buckling axial force vs number of circumferential buckling waves for various internal pressures

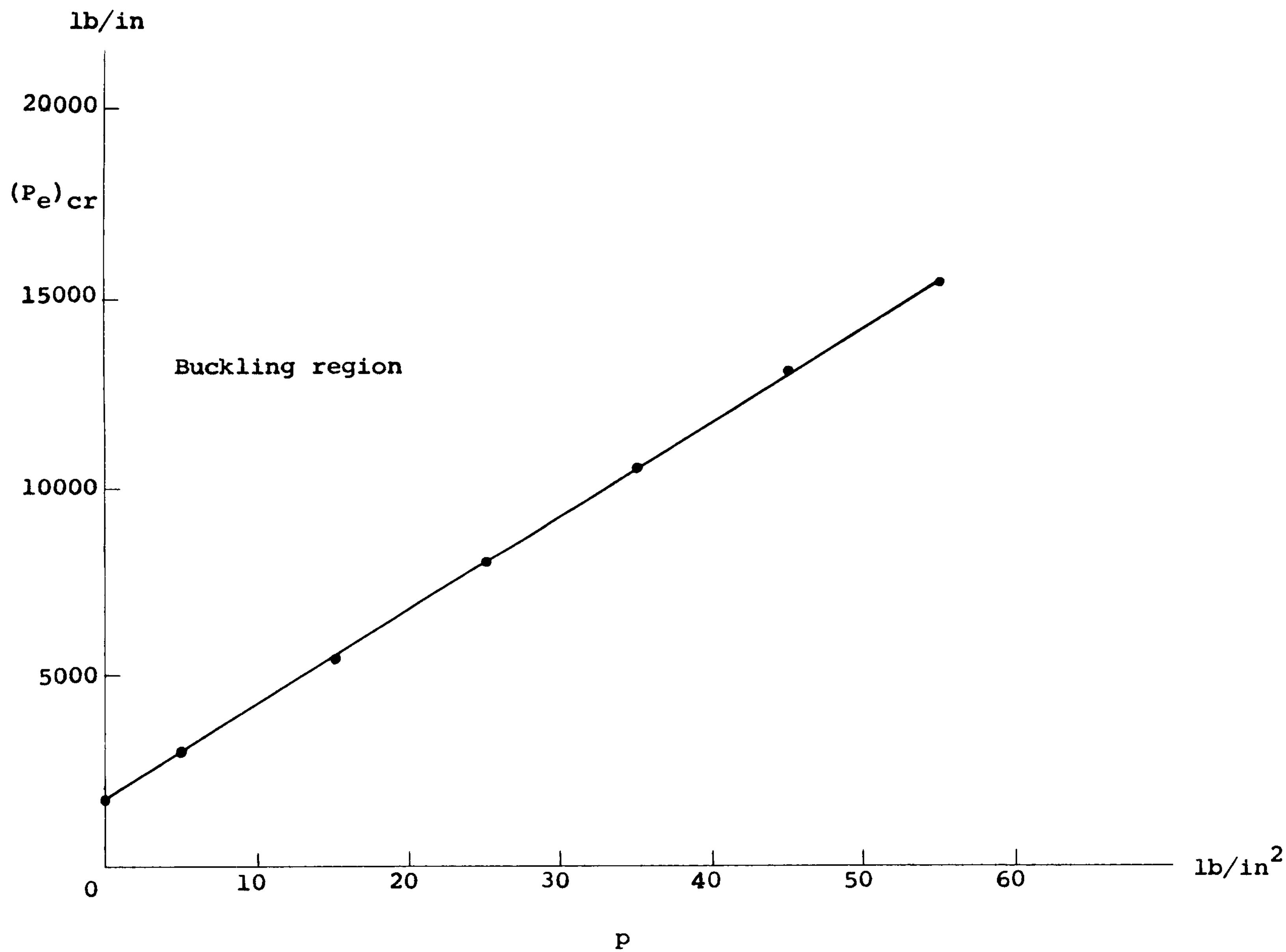

Figure 7. Critical axial force $(P_e)_{cr}$ vs internal pressure p

LIST OF REFERENCES

1. Adachi, G., and Benicek, M., "Buckling of Torispherical Shells Under Internal Pressure," (SESA paper No. 884, May, 1964).

2. Yao, J. C., "Buckling of a Truncated Hemisphere Under Axial Tension," (AIAA Journal, October, 1963).

3. Novozhilov, V. V., "The Theory of Thin Shells," trans. from Russian by P. G. Lowe (P. Noordhoff, Netherlands, 1959), pp. 85, 87, 88, 264.

4. Timoshenko, S., and Gere, J. M., "Theory of Elastic Stability," (McGraw-Hill Book Co. Inc., New York, 1961).

5. Wang, C. J., "Applied Elasticity," (McGraw-Hill Book Co. Inc., New York, 1963), p. 162.

APPENDICES

APPENDIX A

DERIVATION OF DIFFERENTIAL OPERATORS $\triangle$ AND D

From reference [3], pp. 264, 87, we have the following differential operators for revolutionary shells

$$\triangle = \frac{1}{R_1^2}\frac{\partial^2}{\partial\phi^2} + \left(\frac{\cot\phi}{R_1 R_2} - \frac{1}{R_1^3}\frac{dR_1}{d\phi}\right)\frac{\partial}{\partial\phi} + \frac{1}{R_2^2\sin\phi}\frac{\partial^2}{\partial\theta^2} \qquad \text{(A1)}$$

$$D = \frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial\phi}\left[\frac{1}{R_2}\frac{A_2}{A_1}\frac{\partial}{\partial\phi}\right] + \frac{\partial}{\partial\theta}\left[\frac{1}{R_1}\frac{A_1}{A_2}\frac{\partial}{\partial\theta}\right]\right\} \qquad \text{(A2)}$$

Applying these operators to an elliptical shell of revolution

$$\begin{aligned} R_1 &= kaV^3 \\ R_2 &= kaV \\ A_1 &= kaV^3 \\ A_2 &= kaV\sin\phi \end{aligned} \qquad \text{(A3)}$$

We arrive at

$$\Delta = \frac{1}{k^2 a^2 V^6}\left\{\frac{\partial^2}{\partial\phi^2} + V^2[\cot\phi + 3(k^2-1)\sin\phi\cos\phi]\frac{\partial}{\partial\phi} + V^4\csc^2\phi\frac{\partial^2}{\partial\theta^2}\right\} \tag{A4}$$

$$D = \frac{1}{k^3 a^3 V^5 \sin\phi}\left\{\frac{\sin\phi}{V^2}\frac{\partial^2}{\partial\phi^2} + \left[\frac{\cos\phi}{V^2} + 3(k^2-1)\sin^2\phi\cos\phi\right]\frac{\partial}{\partial\phi} + \frac{1}{\sin\phi}\frac{\partial^2}{\partial\theta^2}\right\} \tag{A5}$$

APPENDIX B

DERIVATION OF ADDITIONAL FORCE COMPONENTS IN THE BUCKLED SHELL (T_ϕ , T_θ and $T_{\phi\theta}$)

From reference [3], p. 87, we have

$$T_\phi = -\frac{1}{A_2}\frac{\partial}{\partial\theta}\left(\frac{1}{A_2}\frac{\partial\Phi}{\partial\theta}\right) - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\phi}\frac{1}{A_1}\frac{\partial\Phi}{\partial\phi}$$

$$T_\theta = -\frac{1}{A_1}\frac{\partial}{\partial\phi}\left(\frac{1}{A_1}\frac{\partial\Phi}{\partial\phi}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\theta}\frac{1}{A_2}\frac{\partial\Phi}{\partial\theta}$$

$$T_{\phi\theta} = \frac{1}{A_1 A_2}\left(\frac{\partial^2\Phi}{\partial\phi\partial\theta} - \frac{1}{A_1}\frac{\partial A_1}{\partial\theta}\frac{\partial\Phi}{\partial\phi} - \frac{1}{A_2}\frac{\partial A_2}{\partial\phi}\frac{\partial\Phi}{\partial\theta}\right) \qquad \text{(A6)}$$

The Lamé parameters for elliptical revolutionary shells are determined by the expressions

$$A_1 = R_1 = kaV^3, \qquad A_2 = R_2 \sin\phi = kaV\sin\phi \qquad \text{(A7)}$$

and the Codazzi-Gauss conditions [1] reduce to the relation

$$\frac{dR_2\sin\phi}{d\phi} = \frac{dA_2}{d\phi} = R_1\cos\phi \qquad \text{(A8)}$$

Thus we arrive at .

$$T_\phi = -\frac{1}{k^2a^2V^2}\left[\csc^2\phi\,\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\cot\phi}{V^2}\frac{\partial\Phi}{\partial\phi}\right]$$

$$T_\theta = -\frac{1}{k^2a^2V^6}\left[3V^2(k^2-1)\sin\phi\cos\phi\,\frac{\partial\Phi}{\partial\phi} + \frac{\partial^2\Phi}{\partial\phi^2}\right]$$

$$T_{\phi\theta} = \frac{1}{k^2a^2V^4\sin\phi}\left[\frac{\partial^2\Phi}{\partial\phi\,\partial\theta} - V^2\cot\phi\,\frac{\partial\Phi}{\partial\theta}\right] \tag{A9}$$

APPENDIX C

DERIVATION OF THE PARAMETERS OF THE CHANGES OF CURVATURE (K_1 AND K_2)

Rejecting the displacements **U** and **V**, in the formulae for the parameters of the changes of curvature, one finds [3]

$$K_1 = -\frac{1}{A_1}\frac{\partial}{\partial\phi}\left(\frac{1}{A_1}\frac{\partial w}{\partial\phi}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\theta}\frac{1}{A_2}\frac{\partial w}{\partial\theta}$$

$$K_2 = -\frac{1}{A_2}\frac{\partial}{\partial\theta}\left(\frac{1}{A_2}\frac{\partial w}{\partial\theta}\right) - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\phi}\frac{1}{A_1}\frac{\partial w}{\partial\phi} \qquad \text{(A10)}$$

Substituting the following expressions

$$A_1 = kaV^3, \qquad A_2 = kaV\sin\phi$$

into the above equations and using the relation [3]

$$\frac{dA_2}{d\phi} = kaV^3\cos\phi \qquad \text{(A11)}$$

one has

$$K_1 = -\frac{1}{k^2a^2V^6}\left[3V^2(k^2-1)\sin\phi\cos\phi\frac{\partial w}{\partial\phi} + \frac{\partial^2 w}{\partial\phi^2}\right]$$

$$K_2 = -\frac{1}{k^2a^2V^4}\left[V^2\csc^2\phi\frac{\partial^2 w}{\partial\theta^2} + \cot\phi\frac{\partial w}{\partial\phi}\right] \qquad \text{(A12)}$$

APPENDIX D

COEFFICIENTS f_{nj}, h_{nj}, q_{nj}, g_{nj}, d_{nj} and e_{nj}

1. Part I

$$
\begin{aligned}
f_{nj} = {} & a_n a_j \ell^{n+j+15}\Big[5m^4 \frac{(n+j+12)!\,2!}{(n+j+15)!} - n(n-3)(n+2)(n+3)(k^2-1)^4 \\
& \cdot \frac{(n+j+13)!}{(n+j+15)!} + (n^2-n+4)(n+1)(n+2)(k^2-1)^4 \frac{(n+j+12)!}{(n+j+14)!}\Big] \\
& + a_n a_j \ell^{n+j+13}\Big\{\big[2n(n+1)(5k^2-1) - 4(n+1)(5k^2-2) + 8(2k^2-1)\big] \\
& \cdot (n+2)(n+3)(k^2-1)^3 \frac{(n+j+11)!}{(n+j+13)!} - 2\big[n(n-1)(5k^2-1) + 2(n+1)(5k^2-2)\big] \\
& \cdot (n+1)(n+2)(k^2-1)^3 \frac{(n+j+10)!}{(n+j+12)!} + 4m^4 \frac{(n+j+10)!\,2!}{(n+j+13)!}\Big\} \\
& + a_n a_j \ell^{n+j+11}\Big\{(n+2)(n+3)(k^2-1)^2\big[4(n+1)(10k^4-8k^2+1) \\
& - n(n+1)(5k^4-10k^2+1) - 4(6k^4-6k^2+1)\big]\frac{(n+j+9)!}{(n+j+11)!} \\
& + (n+1)(n+2)(k^2-1)^2\big[n(n-1)(15k^4-10k^2+1) + 4(6k^4-6k^2+1)\big] \\
& \cdot \frac{(n+j+8)!}{(n+j+10)!} + 3m^4 \frac{(n+j+8)!\,2!}{(n+j+11)!}\Big\} + a_n a_j \ell^{n+j+9}\Big\{(n+3)(k^2-1) \\
& \cdot \big[4n(n+1)(n+2)k^2(5k^4-5k^2+1) - 4(n+1)(n+2)k^2(10k^4-12k^2+3) \\
& + 8(n+2)k^2(k^2-1)(2k^2-1) + 2m^2(n+2)(k^2-1) - 4m^2(k^2-1)\big]\frac{(n+j+7)!}{(n+j+9)!} \\
& + (n+2)(k^2-1)\big[4m^2(k^2-1) - 4n(n+1)(n-1)k^2(5k^4-5k^2+1)
\end{aligned}
$$

$$- 8(n+1)k^2(k^2-1)(2k^2-1) - 2m^2(n+1)(k^2-1)\Big] \frac{(n+j+6)!}{(n+j+8)!} + 2m^4 \frac{(n+j+6)!\,2!}{(n+j+9)!}$$

$$+ a_n a_j \lambda^{n+j+7}\Big\{(n+3)k^2\Big[4(n+1)(n+2)k^2(k^2-1)(5k^2-3)$$

$$- n(n+1)(n+2)k^2(15k^4-20k^2+6) - 4(n+2)k^2(k^2-1)^2 - 4m^2(n+2)(k^2-1)$$

$$+ 4m^2(k^2-1)\Big]\frac{(n+j+5)!}{(n+j+7)!} + (n+2)k^2\Big[n(n+1)(n-1)k^2(15k^4-20k^2+6)$$

$$+ 4(n-1)k^2(k^2-1)^2 + 4m^2(n+1)(k^2-1) - 4m^2(k^2-1)\Big]\frac{(n+j+4)!}{(n+j+6)!}$$

$$+ m^4\frac{(n+j+4)!\,2!}{(n+j+7)!}\Big\} + a_n a_j \lambda^{n+j+5}\Big\{(n+2)(n+3)a_n a_j k^2\Big[2n(n+1)$$

$$\cdot (k^6+2k^2-2) - 4(n+1)k^4(k^2-1) + 2m^2k^2\Big]\frac{(n+j+3)!}{(n+j+5)!}$$

$$- 2(n+1)(n+2)k^2\Big[n(n-1)(k^6+2k^2-2) + m^2k^2\Big]\frac{(n+j+2)!}{(n+j+4)!}\Big\}$$

$$+ a_n a_j n(n+1)(n+2)k^8 \lambda^{n+j+3}\Big[(n-1)\frac{(n+j)!}{(n+j+2)!} - (n+3)\frac{(n+j+1)!}{(n+j+3)!}\Big]$$

$$+ a_n \lambda^{n+15}\Big\{(n+1)(n+2)(n^2-n+4)(k^2-1)^4\frac{(n+12)!}{(n+14)!}$$

$$- n(n+2)(n+3)(n-3)(k^2-1)^4\frac{(n+13)!}{(n+15)!} + 5m^4\frac{(n+12)!\,2!}{(n+15)!}$$

$$+ a_n\lambda^{n+13}\Big\{(n+2)(n+3)(k^2-1)^3\Big[2n(n-1)(5k^2-1) - 4(n+1)(5k^2-2)$$

$$+ 8(2k^2-1)\Big]\frac{(n+11)!}{(n+13)!} - 2(n+1)(n+2)(k^2-1)^3\Big[n(n-1)(5k^2-1)$$

$$+ 4(2k^2-1)\Big]\frac{(n+10)!}{(n+12)!} + 4m^4\frac{(n+10)!\,2!}{(n+13)!}\Big\} + a_n\lambda^{n+11}\Big\{3m^4\frac{(n+8)!\,2!}{(n+11)!}$$

$$+ (n+1)(n+2)(k^2-1)^2\Big[n(n-1)(15k^4-10k^2+1) + 4(6k^4-6k^2+1)\Big]\frac{(n+8)!}{(n+10)!}$$

$$-(n+2)(n+3)(k^2-1)^2\Big[n(n+1)(15k^4-10k^2+1) + 4(n+1)(10k^4-12k^2+1)$$

$$+ 4(6k^4-6k^2+1)\Big]\frac{(n+9)!}{(n+11)!}\Big\} + a_n\lambda^{n+9}\Big\{(n+3)(k^2-1)\Big[4n(n+1)$$

$$\cdot (n+2)k^2(5k^4-5k^2+1) - 4(n+1)(n+2)k^2(10k^4-12k^2+3)$$

$$+8(n+2)k^2(k^2-1)(2k^2-1)+2n(k^2-1)m^2]+2m^4\frac{(n+6)!\,2!}{(n+9)!}$$

$$+2(n+2)(k^2-1)[2n(n+1)(n-1)k^2(5k^4-5k^2+1)-4(n+1)k^2(k^2-1)$$

$$\cdot(2k^2-1)-m^2(n+1)(k^2-1)+2m^2(k^2-1)]\frac{(n+6)!}{(n+8)!}\Big\}$$

$$+a_n\ell^{n+7}\Big\{4(n+2)k^2(k^2-1)[(n+1)k^2(k^2-1)+nm^2]\frac{(n+4)!}{(n+6)!}$$

$$-(n+3)k^2[n(n+1)(n+2)k^2(15k^4-20k^2+6)+4(n+1)(n+2)k^2(k^2-1)$$

$$\cdot(5k^2-3)+4(n+2)k^2(k^2-1)^2+4(n+1)(k^2-1)m^2]\frac{(n+5)!}{(n+7)!}$$

$$+m^4\frac{(n+4)!\,2!}{(n+7)!}\Big\}+a_n\ell^{n+5}\Big\{2(n+2)(n+3)k^2[n(n+1)(k^6+2k^2-2)$$

$$-2(n+1)k^4(k^2-1)+m^2k^2]\frac{(n+3)!}{(n+5)!}-2m^2(n+1)(n+2)a_nk^4\frac{(n+2)!}{(n+4)!}\Big\}$$

$$-n(n+1)(n+2)(n+3)a_nk^8\frac{(n+1)!}{(n+3)!}\ell^{n+3}-4n(n+1)(n+2)a_j(k^2-1)^4$$

$$\cdot\frac{(j+13)!}{(j+15)!}\ell^{j+16}+a_j\ell^{j+15}\Big\{8(k^2-1)^4\frac{(j+12)!}{(j+14)!}+5m^4\frac{(j+12)!\,2!}{(j+15)!}$$

$$-24(k^2-1)^4\frac{(j+13)!}{(j+15)!}\Big\}+4n(n+1)(n+2)a_j(k^2-1)^3(5k^2-2)\frac{(j+11)!}{(j+13)!}\ell^{j+14}$$

$$+a_j\ell^{j+13}\Big\{16(k^2-1)^3(2k^2-1)[3\frac{(j+11)!}{(j+13)!}-(k^2-1)^3(2k^2-1)\frac{(j+10)!}{(j+12)!}]$$

$$+4m^4\frac{(j+10)!\,2!}{(j+13)!}-4n(n+1)(n+2)a_j(k^2-1)^2(10k^4-8k^2+1)$$

$$\cdot\frac{(j+9)!}{(j+11)!}\ell^{j+12}+a_j\ell^{j+11}\Big\{8(k^2-1)^2(6k^4-6k^2+1)\frac{(j+8)!}{(j+10)!}$$

$$+3m^4\frac{(j+8)!\,2!}{(j+11)!}-24(k^2-1)^2(6k^4-6k^2+1)\frac{(j+9)!}{(j+11)!}\Big\}$$

$$+4n(n+1)(n+2)a_jk^2(k^2-1)(10k^4-12k^2+3)\frac{(j+7)!}{(j+9)!}\ell^{j+10}$$

$$+a_j\ell^{j+9}\Big\{48k^2(2k^2-1)(k^2-1)^2\frac{(j+7)!}{(j+9)!}+4(k^2-1)^2[m^2-4k^2(2k^2-1)]$$

$$\cdot\frac{(j+6)!}{(j+8)!}+2m^4\frac{(j+6)!\,2!}{(j+9)!}\Big\}-4n(n+1)(n+2)a_jk^4(k^2-1)(5k^2-3)$$

$$\cdot \frac{(j+5)!}{(j+7)!}\,\ell^{j+8} + a_j\,\ell^{j+7}\Big\{8k^4(k^2-1)^2\frac{(j+4)!}{(j+6)!} - 12k^2(k^2-1)$$

$$\cdot \left[m^2+2k^2(k^2-1)\right]\frac{(j+5)!}{(j+7)!} + m^4\frac{(j+4)!\,2!}{(j+7)!}\Big\} + 4n(n-1)(n+2)a_j$$

$$\cdot k^6(k^2-1)\frac{(j+3)!}{(j+5)!}\,\ell^{j+6} + 4m^2a_jk^4\,\ell^{j+5}\left[3\frac{(j+3)!}{(j+5)!} - \frac{(j+2)!}{(j+4)!}\right]$$

$$-4n(n+1)(n+2)a_n(k^2-1)^4\frac{13!}{15!}\,\ell^{16} + \ell^{15}\Big[8(k^2-1)^4\frac{12!}{14!}$$

$$+5m^4\frac{12!}{15!} - 24(k^2-1)^4\frac{13!}{15!}\Big] + 4n(n+1)(n+2)a_n(k^2-1)^3(5k^2-2)$$

$$\cdot \ell^{14}\frac{11!}{13!} + 16(k^2-1)^3(2k^2-1)\,\ell^{13}\left[3\frac{11!}{13!} - \frac{10!}{12!}\right]$$

2. Part II

$$h_{nj} = -\frac{5m^2a_na_j(k^2-1)^3}{16k^5}\frac{(n+j+18)!\,2!}{(n+j+21)!}\,\ell^{n+j+21} - \frac{m^2a_na_j(k^2-1)^2(7k^2-2)}{8k^5}$$

$$\cdot \frac{(n+j+16)!\,2!}{(n+j+19)!}\,\ell^{n+j+9} - m^2a_na_j\frac{(k^2-1)}{16k^5}(51k^4-14k^2+3)$$

$$\cdot \frac{(n+j+14)!\,2!}{(n+j+17)!}\,\ell^{n+j+17} + \frac{(20k^6+25k^4-6k^2+1)}{8k^5}\frac{(n+j+12)!\,2!}{(n+j+15)!}$$

$$\cdot \ell^{n+j+15} + \frac{a_na_j}{16k^5}\,\ell^{n+j+13}\Big\{3(n+13)(k^2-1)^4\frac{(n+j+11)!}{(n+j+13)!}$$

$$+ m^2(35k^6+35k^4-7k^2+1)\frac{(n+j+10)!\,2!}{(n+j+13)!} - 3(n+2)(k^2-1)^4$$

$$\cdot \frac{(n+j+10)!}{(n+j+12)!}\Big\} + \frac{a_na_j}{16k^3}\,\ell^{n+j+11}\Big\{(n+2)(n-5)(k^2-1)^3\frac{(n+j+8)!}{(n+j+10)!}$$

$$-2m^2(9k^2-1)(k^2-1)\frac{(n+j+8)!\,2!}{(n+j+11)!} - (n+3)(n-4)(k^2-1)^3\frac{(n+j+9)}{(n+j+11)!}\Big\}$$

$$+\frac{a_na_j}{8k}\,\ell^{n+j+9}\Big\{3(n+2)\left[n+1-4(k^2-1)^2\right]\frac{(n+j+6)!}{(n+j+8)!}$$

$$-4m^2(5k^2-1)\frac{(n+j+6)!\,2!}{(n+j+9)!} - 3(n-2)(n+3)(k^2-1)^2\frac{(n+j+7)!}{(n+j+9)!}\Big\}$$

$$+a_na_jk\,\ell^{n+j+7}\Big\{3n(n+3)(k^2-1)\frac{(n+j+5)!}{(n+j+7)!} - 3(n-1)(n+2)(k^2-1)$$

$$\cdot \frac{(n+j+4)!}{(n+j+6)!} + 2m^2\frac{(n+j+4)!\,2!}{(n+j+7)!}\Big\} + (n+2)a_na_jk^3\,\ell^{n+j+5}$$

$$\cdot \left\{ (n+1)\frac{(n+j+2)!}{(n+j+4)!} - (n+3)\frac{(n+j+3)!}{(n+j+5)!} \right\} - \frac{5m^2 a_n (k^2-1)^3}{16k^5}\frac{(n+18)!\,2!}{(n+21)!}\ell^{n+21}$$

$$- \frac{m^2 a_n (k^2-1)^2 (7k^2-2)}{8k^5}\frac{(n+16)!\,2!}{(n+19)!}\ell^{n+19} - \frac{m^2 a_n (k^2-1)(51k^4-14k^2+3)}{16k^5}$$

$$\cdot \frac{(n+14)!\,2!}{(n+17)!}\ell^{n+17} + \frac{m^2 a_n (20k^6+25k^4-6k^2+1)}{8k^5}\frac{(n+12)!\,2!}{(n+15)!}\ell^{n+15}$$

$$+ \frac{a_n}{16k^5}\ell^{n+13}\left\{ 3(n+3)(k^2-1)^3\frac{(n+11)!}{(n+13)!} - 3(n+2)(k^2-1)^4\frac{(n+10)!}{(n+12)!} \right.$$

$$\left. + m^2(35k^6+35k^4-7k^2+1)\frac{(n+10)!\,2!}{(n+13)!} \right\} + \frac{a_n}{16k^3}\ell^{n+11}$$

$$\cdot \left\{ (n-2)(n-5)(k^2-1)^3\frac{(n+8)!}{(n+10)!} - 2m^2(9k^2-1)(k^2-1)\frac{(n+8)!\,2!}{(n+11)!} \right.$$

$$\left. - (n+3)(n-4)(k^2-1)^3\frac{(n+9)!}{(n+11)!} \right\} + \frac{a_n}{8k}\ell^{n+9}\left\{ 3(n+2)(k^2-1) \right.$$

$$\cdot [(n+1)(k^2-1)-4]\frac{(n+6)!}{(n+8)!} - 4m^2(5k^2-1)\frac{(n+6)!\,2!}{(n+9)!} - 3(n-2)(n+3)$$

$$\left. \cdot (k^2-1)^2\frac{(n+7)!}{(n+9)!} \right\} + \frac{a_n k}{2}\ell^{n+7}\left\{ 3(n+3)[(n+2)(k^2-1)-2]\frac{(n+5)!}{(n+7)!} \right.$$

$$\left. + 3(n+2)[2(k^2-1)-(n+1)]\frac{(n+4)!}{(n+6)!} + 2m^2\frac{(n+4)!\,2!}{(n+7)!} \right\}$$

$$+ (n+2)a_n k^3 \ell^{n+5}\left\{ (n+1)\frac{(n+2)!}{(n+4)!} - (n+3)\frac{(n+3)!}{(n+5)!} \right\}$$

$$- \frac{5m^2 a_j (k^2-1)^3}{16k^5}\frac{(j+18)!\,2!}{(j+21)!}\ell^{j+21} - \frac{m^2 a_j (k^2-1)^2(7k^2-2)}{8k^5}\frac{(j+16)!\,2!}{(j+19)!}\ell^{j+19}$$

$$- \frac{m^2 a_j (k^2-1)(51k^2-14k^2+3)}{16k^5}\frac{(j+14)!\,2!}{(j+17)!}\ell^{j+17} + \frac{m^2 a_j (20k^6+25k^4-6k^2+1)}{8k^5}$$

$$\cdot \frac{(j+12)!\,2!}{(j+15)!}\ell^{j+15} + \frac{a_j}{16k^5}\ell^{j+13}\left\{ 9(k^2-1)^4\frac{(j+11)!}{(j+13)!} - 6(k^2-1)^4 \right.$$

$$\left. \cdot \frac{(j+10)!}{(j+12)!} - m^2(35k^6+35k^4-7k^2+1)\frac{(j+10)!\,2!}{(j+13)!} \right\} + \frac{a_j}{8k^3}\ell^{j+11}$$

$$\cdot \left\{ 6(k^2-1)^3\frac{(j+9)!}{(j+11)!} - 5(k^2-1)^3\frac{(j+8)!}{(j+10)!} - m^2(9k^4-10k^2+1)\frac{(j+8)!\,2!}{(j+11)!} \right\}$$

$$+ \frac{a_j}{4k}\ell^{j+9}\left\{ 9(k^2-1)^2\frac{(j+7)!}{(j+9)!} - 9(k^2-1)^2\frac{(j+6)!}{(j+8)!} - 2m^2(5k^2-1)\frac{(j+6)!\,2!}{(j+9)!} \right\}$$

$$+ a_j k \ell^{j+7}\left\{ 3(k^2-1)\frac{(j+4)!}{(j+6)!} - m^2\frac{(j+4)!\,2!}{(j+7)!} \right\} + 2a_j k^3 \ell^{j+5}$$

$$\cdot\left\{\frac{(j+2)!}{(j+4)!}-3\frac{(j+3)!}{(j+5)!}\right\}-\frac{5m^2(k^2-1)^3}{16k^5}\frac{18!\,2!}{21!}\lambda^{21}-\frac{m^2(k^2-1)^2(7k^2-2)}{8k^5}$$

$$\cdot\frac{16!\,2!}{19!}\lambda^{19}-\frac{m^2(k^2-1)(51k^4-14k^2+3)}{16k^5}\frac{14!\,2!}{17!}\lambda^{17}+\frac{m^2(20k^6+25k^4-6k^2+1)}{8k^5}$$

$$\cdot\frac{12!\,2!}{15!}\lambda^{15}+\frac{1}{16k^5}\lambda^{13}\left\{9(k^2-1)^4\frac{11!}{13!}-6(k^2-1)^4\frac{10!}{12!}\right.$$

$$\left.+m^2(35k^6+35k^4-7k^2+1)\frac{10!\,2!}{13!}\right\}+\frac{1}{8k^3}\lambda^{11}\left\{6(k^2-1)^3\frac{9!}{11!}\right.$$

$$\left.-5(k^2-1)^3\frac{8!}{10!}-m^2(9k^2-1)(k^2-1)\frac{8!\,2!}{11!}\right\}+\frac{1}{4k}\lambda^9\left\{9(k^2-1)^2\frac{7!}{9!}\right.$$

$$\left.-2m^2(5k^2-1)\frac{6!\,2!}{9!}+3(k^2-1)^2\frac{6!}{8!}\right\}+\lambda^7\left\{3k(k^2-1)\frac{4!}{6!}\right.$$

$$\left.-\frac{3(k^2-1)^2}{k}\frac{6!}{8!}+m^2\frac{4!\,2!}{7!}\right\}+2k^3\lambda^5\left[\frac{2!}{4!}-3\frac{3!}{5!}\right]$$

3. Part III

$$q_{nj}=-5m^2a_na_j\frac{(k^2-1)^3}{64k^9}(43k^2-63)\frac{(n+j+20)!\,2!}{(n+j+23)!}\lambda^{n+j+23}$$

$$-\frac{5}{64}m^2a_na_j\frac{(k^2-1)^2}{k^9}\left[(k^2-1)(119k^2-63)+4(10k^2-1)k^2\right]\frac{(n+j+18)!\,2!}{(n+j+21)!}\lambda^{n+j+21}$$

$$-m^2a_na_j\frac{(k^2-1)}{64k^9}\left[(k^2-1)(167k^4-182k^2+63)+4k^2(79k^4-54k^2+15)\right]$$

$$\cdot\frac{(n+j+16)!\,2!}{(n+j+19)!}\lambda^{n+j+19}-\frac{m^2a_na_j}{16k^9}\left[16(k^2-1)(167k^6-349k^4+245k^2\right.$$

$$\left.-63)+k^2(140k^6-98k^4+48k^2-30)\right]\frac{(n+j+14)!\,2!}{(n+j+17)!}+\frac{a_na_j}{16k^7}\lambda^{n+j+15}$$

$$\cdot\left\{(n+2)\frac{15(k^2-1)^4}{16k^7}\frac{(n+j+12)!}{(n+j+14)!}-15(n+3)(k^2-1)^4\frac{(n+j+13)!}{(n+j+15)!}\right.$$

$$-\frac{m^2}{16k^7}\left[(105k^6-63k^4+27k^2-5)+16k^9(1091k^8-2708k^6\right.$$

$$\left.\left.+2970k^4-1540k^2+315)\right]\frac{(n+j+12)!\,2!}{(n+j+15)!}\right\}+\frac{a_na_j}{16k^7}\lambda^{n+j+13}$$

$$\cdot\left\{(n+3)(n-26)k^2(k^2-1)^3\frac{(n+j+11)!}{(n+j+13)!}-(n+2)(n-27)k^2(k^2-1)^3\frac{(n+j+10)!}{(n+j+12)!}\right.$$

$$\left.+m^2(27k^6-153k^4+129k^2-105)\frac{(n+j+10)!\,2!}{(n+j+13)!}\right\}+\frac{a_na_j}{8k^5}\lambda^{n+j+11}$$

$$\cdot \left\{(n+3)(k^2-1)^2[(n+2)k^2-18]\frac{(n+j+9)!}{(n+j+11)!} + (n+2)(k^2-1)^2\right.$$

$$\cdot [18-(n+1)k^2]\frac{(n+j+8)!}{(n+j+10)!} - 2m^2[2k^2(5k^2-1)+(31k^4-38k^2+15)]$$

$$\left.\cdot \frac{(n+j+8)!\,2!}{(n+j+11)!}\right\} + \frac{a_n a_j}{2k^3}\ell^{n+j+9}\left\{(n+3)(n-6)k^2(k^2-1)\frac{(n+j+7)!}{(n+j+9)!}\right.$$

$$\left.-(n+2)(n-7)k^2(k^2-1)\frac{(n+j+6)!}{(n+j+8)!} - 2m^2(5k^2-2)\frac{(n+j+6)!\,2!}{(n+j+9)!}\right\}$$

$$+ a_n a_j k\,\ell^{n+j+7}\left\{(n+2)(n+3)\frac{(n+j+4)!}{(n+j+6)!} - (n+3)(n+4)\frac{(n+j+5)!}{(n+j+7)!}\right.$$

$$\left.-\frac{2m^2}{k^2}\frac{(n+j+4)!\,2!}{(n+j+7)!}\right\} - \frac{5m^2a_n(k^2-1)^3(83k^2-63)}{64k^9}\frac{(n+20)!\,2!}{(n+23)!}\ell^{n+23}$$

$$-\frac{5m^2a_n(k^2-1)^2}{64k^9}[4k^2(10k^2-1)+(k^2-1)(119k^2-63)]\frac{(n+18)!\,2!}{(n+21)!}\ell^{n+21}$$

$$-m^2a_n\frac{(k^2-1)}{64k^9}[4k^2(79k^4-54k^2+15)+(k^2-1)(167k^4-182k^2$$

$$+63)]\frac{(n+16)!\,2!}{(n+19)!}\ell^{n+19} - \frac{m^2a_n}{64k^9}[4k^2(140k^6-98k^4+48k^2-30)$$

$$+(k^2-1)(167k^6-349k^4+245k^2-63)]\frac{(n+14)!\,2!}{(n+17)!}\ell^{n+17}$$

$$+\frac{a_n}{128k^9}\ell^{n+15}\left\{120(n+2)k^2(k^2-1)^4\frac{(n+12)!}{(n+14)!} - 120(n+3)k^2\right.$$

$$\cdot(k^2-1)^4\frac{(n+13)!}{(n+15)!} - m^2(1931k^8-3212k^6+3186k^4-1580k^2$$

$$\left.+315)\frac{(n+12)!\,2!}{(n+15)!}\right\} + \frac{a_n}{16k^5}\ell^{n+13}\left\{(n+3)(n-26)(k^2-1)^3\frac{(n+11)!}{(n+13)!}\right.$$

$$-(n+2)(n-27)(k^2-1)^3\frac{(n+10)!}{(n+12)!} - \frac{2m^2(132k^6-195k^4+138k^2-105)}{k^2}$$

$$\left.\cdot\frac{(n+10)!\,2!}{(n+13)!}\right\} + \frac{a_n}{8k^5}\ell^{n+11}\left\{(n+2)(k^2-1)^2[18-(n+1)k^2]\right.$$

$$\cdot\frac{(n+8)!}{(n+10)!} + (n+3)(k^2-1)^2[(n+2)k^2-18]\frac{(n+9)!}{(n+11)!}$$

$$\left.-2m^2(41k^4-38k^2+5)\frac{(n+8)!\,2!}{(n+11)!}\right\} + \frac{a_n}{2k^3}\ell^{n+9}$$

$$\cdot\left\{(n+3)(n-6)k^2(k^2-1)\frac{(n+7)!}{(n+9)!} - 2m^2(3k^2-1)\frac{(n+6)!\,2!}{(n+9)!}\right.$$

$$- (n+2)(n-7)k^2(k^2-1)\frac{n+6}{(n+8)!}\Big\} + a_n \rho^{n+7}\Big\{(n+2)(n+3)k\frac{(n+4)!}{(n+6)!}$$

$$- \frac{2m^2}{k}\frac{(n+4)!2!}{(n+7)!} - (n+3)(n+4)k\frac{(n+5)!}{(n+7)!}\Big\} - \frac{5a_j m^2(83k^2-63)(k^2-1)^3}{64k^9}$$

$$\cdot \frac{(j+20)!\,2!}{(j+23)!}\rho^{j+23} - \frac{5a_j m^2(k^2-1)^2}{64k^9}[4k^2(10k^2-1)+(k^2-1)$$

$$\cdot (119k^2-63)]\frac{(j+18)!\,2!}{(j+21)!}\rho^{j+21} - \frac{m^2 a_j(k^2-1)}{64k^9}[4k^2(79k^4-54k^2$$

$$+15)+(k^2-1)(167k^4-182k^2+63)]\frac{(j+16)!\,2!}{(j+19)!}\rho^{j+19}$$

$$- \frac{m^2 a_j}{64k^9}[4k^2(140k^6-98k^4+48k^2-30)+(k^2-1)(167k^6-349k^4$$

$$+245k^2-637)]\frac{(j+14)!\,2!}{(j+7)!}\rho^{j+17} + \frac{a_j}{64k^9}\rho^{j+15}\Big\{120k^2(k^2-1)^4$$

$$\cdot \frac{(j+12)!}{(j+14)!} - m^2(1511k^8-2960k^6+3078k^4-1560k^2+315)$$

$$\cdot \frac{(j+12)!\,2!}{(j+15)!} - 180k^2(k^2-1)^4\frac{(j+13)!}{(j+15)!}\Big\} + \frac{a_j}{8k^7}\rho^{j+13}\Big\{55k^2$$

$$\cdot (k^2-1)^3\frac{(j+10)!}{(j+12)!} - 39k^2(k^2-1)^3\frac{(j+11)!}{(j+13)!} - m^2(132k^2-195k^4$$

$$+138k^2-105)\frac{(j+10)!\,2!}{(j+13)!}\Big\} + \frac{a_j}{4k^5}\rho^{j+11}\Big\{3(k^2-9)(k^2-1)^2$$

$$\cdot \frac{(j+9)!}{(j+11)!} - (k^2-1)^2(k^2-36)\frac{(j+8)!}{(j+10)!} - m^2(41k^4-40k^2+15)$$

$$\cdot \frac{(j+8)!\,2!}{(j+11)!}\Big\} + \frac{a_j}{k^3}\rho^{j+9}\Big\{7k^2(k^2-1)\frac{(j+6)!}{(j+8)!} - 9k^2(k^2-1)\frac{(j+7)!}{(j+9)!}$$

$$- m^2(3k^2-1)\frac{(j+6)!\,2!}{(j+9)!}\Big\} + 2a_j k\rho^{j+7}\Big\{3\frac{(j+4)!}{(j+6)!} - 6\frac{(j+5)!}{(j+7)!}$$

$$- \frac{m^2}{k^2}\frac{(j+4)!\,2!}{(j+7)!}\Big\} - \frac{5m^2(k^2-1)^3}{64k^9}(68k^2-63)\frac{20!\,2!}{23!}\rho^{23}$$

$$- \frac{5m^2(k^2-1)^2}{64k^9}[4k^2(10k^2-1)+(k^2-1)(119k^2-63)]\frac{18!\,2!}{21!}\rho^{21}$$

$$- \frac{m^2(k^2-1)}{64k^9}[k^2(79k^4-54k^2+15)+(k^2-1)(167k^4-182k^2+63)]$$

$$\cdot \frac{16!\,2!}{19!}\rho^{19} - \frac{m^2}{64k^9}[4k^2(140k^6-98k^4+48k^2-30)$$

$$+ (k^2-1)(167k^6 - 349k^4 + 245k^2 - 63)] \frac{14!\,2!}{17!} \ell^{17}$$

$$+ \frac{1}{64k^9} \ell^{15} [120k^2(k^2-1)^4 \frac{12!}{14!} - 180k^2(k^2-1)^4 \frac{13!}{15!}$$

$$- m^2(1511k^8 - 2960k^6 + 3078k^4 - 1560k^2 + 315) \frac{12!\,2!}{15!} \}$$

$$+ \frac{\ell^{13}}{8k^7} [27k^2(k^2-1)^3 \frac{10!}{12!} - 39k^2(k^2-1)^3 \frac{11!}{13!} - m^2(132k^6 - 195k^4$$

$$+ 138k^2 - 105) \frac{10!\,2!}{13!} \} - \frac{1}{4k^5} \ell^{11} [25k^2(k^2-1)^2 \frac{9!}{11!}$$

$$+ (k^2-1)^2(k^2-18) \frac{8!}{10!} + m^2(41k^4 - 40k^2 + 15) \frac{8!\,2!}{11!}] + \frac{1}{k^3} \ell^9$$

$$\cdot [7k^2(k^2-1) \frac{6!}{8!} - 9k^2(k^2-1) \frac{7!}{9!} - m^2k^2(5k^2-2) \frac{6!\,2!}{9!}]$$

$$+ \frac{2}{k} \ell^7 [3k^2 \frac{4!}{8!} - 6k^2 \frac{5!}{7!} - m^2 \frac{4!\,2!}{7!}]$$

4. Part IV

$$(-1)^j g_{nj} = m^2 a_n \frac{(k^2-1)^6}{16k^{11}} \frac{(j+n+14)!\,(j+1)!}{(n+2j+16)!} \ell^{n+2j+16} + m^2 a_n \frac{3(k^2-1)^5}{16k^9}$$

$$\cdot \frac{(n+j+12)!\,(j+1)!}{(n+2j+14)!} \ell^{n+2j+14} + a_n \frac{(k^2-1)^4}{16k^7} \ell^{n+2j+12}$$

$$\cdot \left\{ 9m^2 \frac{(n+j+10)!\,(j+1)!}{(n+2j+12)!} - 3(n+2)k^2 \frac{(n+j+10)!\,j!}{(n+2j+11)!} \right.$$

$$\left. + 3(n+3)k^2 \frac{(n+j+11)!\,j!}{(n+2j+12)!} \right\} + a_n \ell^{n+2j+10} \left\{ \frac{(n+3)(k^2-1)^3}{16k^5} \right.$$

$$\cdot [3(k^2+1) - (n+2)k^2] \frac{(n+j+9)!\,j!}{(n+2j+10)!} + \frac{(n+2)(k^2-1)^3}{16k^5}$$

$$\cdot [(n+1)k^2 - 3(k^2+1)] \frac{(n+j+8)!\,j!}{(n+2j+9)!} - m^2 \frac{9(k^2-1)^3}{16k^5}$$

$$\left. \cdot \frac{(j+n+8)!\,(j+1)!}{(n+2j+10)!} \right\} + a_n \ell^{n+2j+8} \left\{ \frac{(n+3)(k^2-1)^2}{16k^3} [6a_n(3k^2+1) \right.$$

$$- (n+2)(5k^2+1)] \frac{(n+j+7)!\,j!}{(n+2j+8)!} + (n+2) \frac{(k^2-1)^2}{16k^3} [(n+1)(5k^2+1)$$

$$- 6(3k^2+1)\Big] \frac{(n+j+6)!\,j!}{(n+2j+7)!} - m^2 \frac{11(k^2-1)^2}{8k^3} \frac{(n+j+6)!\,(j+1)!}{(n+2j+8)!} \Big\}$$

$$+ a_n \ell^{n+2j+6} \Big\{ \frac{3(n+3)(k^2-1)}{8k^2} \big[(n+2)k(5k^2-1) - 12(k^2-1)\big]$$

$$+ \frac{3(n+2)(k^2-1)}{8k} \big[4(3k^2-1) - (n+1)(5k^2-1)\big] \frac{(n+j+4)!\,j!}{(n+2j+5)!}$$

$$+ \frac{3m^2(k^2-1)}{2k} \frac{(n+j+4)!\,(j+1)!}{(n+2j+6)!} \Big\} + a_n \ell^{n+2j+4} \Big\{ \frac{(n+3)k}{2}$$

$$\cdot \big[6(k^2-1) - (n+2)(5k^2-3)\big] \frac{(n+j+3)!\,j!}{(n+2j+4)!} + \frac{(n+2)k}{2}$$

$$\cdot \big[(n+1)(5k^2-3) - 6(k^2-1)\big] \frac{(n+j+2)!\,j!}{(n+2j+3)!} - m^2 k \frac{(n+j+2)!\,(j+1)!}{(n+2j+4)!} \Big\}$$

$$+ (n+2)\, a_n k^3 \ell^{n+2j+2} \Big\{ (n+3) \frac{(n+j+1)!\,j!}{(n+2j+1)!} - (n+1) \frac{(n+j)!\,j!}{(n+2j+1)!} \Big\}$$

$$+ m^2 \frac{(k^2-1)^6}{16k^{11}} \frac{(j+14)!\,(j+1)!}{(2j+16)!} \ell^{2j+16} + \frac{3m^2(k^2-1)^5}{16k^9}$$

$$\cdot \frac{(j+12)!\,(j+1)!}{(2j+14)!} \ell^{2j+14} + \frac{3(k^2-1)^4}{16k^7} \Big[3k^2 \frac{(j+11)!\,j!}{(2j+12)!}$$

$$- 2k^2 \frac{(j+10)!\,j!}{(2j+11)!} + 3m^2 \frac{(j+10)!\,(j+1)!}{(2j+12)!} \Big] \ell^{2j+12}$$

$$+ \ell^{2j+10} \Big\{ \frac{3(k^2+3)(k^2-1)^3}{16k^5} \frac{(j+9)!\,j!}{(2j+10)!} - \frac{(2k^2+3)(k^2-1)^3}{8k^5} \frac{(j+9)!\,j!}{(2j+10)!}$$

$$- \frac{9m^2(k^2-1)^3}{16k^5} \frac{(j+8)!\,(j+1)!}{(2j+10)!} \Big\} + \frac{1}{8k^3} \ell^{2j+8} \Big\{ 6(2k^2+1)(k^2-1)^2$$

$$\cdot \frac{(j+7)!\,j!}{(2j+8)!} - (13k^2+5)(k^2-1)^2 \frac{(j+6)!\,j!}{(2j+7)!} - 11m^2(k^2-1)^2$$

$$\cdot \frac{(j+6)!\,(j+1)!}{(2j+8)!} \Big\} - \frac{1}{4k} \ell^{2j+6} \Big\{ 9(k^2-1)^2 \frac{(j+5)!\,j!}{(2j+6)!}$$

$$+ 3(k^2-1)(k^2+3) \frac{(j+4)!\,j!}{(2j+5)!} \Big\} + k\, \ell^{2j+4} \Big\{ (11k^2-9) \frac{(j+2)!\,j!}{(2j+3)!}$$

$$- 6(k^2-6) \frac{(j+3)!\,j!}{(2j+4)!} \Big\} + 2k^3 \ell^{2j+2} \Big\{ 3 \frac{(j+1)!\,j!}{(2j+2)!} - \frac{j!^2}{(2j+1)!} \Big\}$$

5. Part V

$$
\begin{aligned}
(-1)^j d_{nj} = & -\ell^{n+2j+10} \left\{ \frac{a_n (k^2-1)^3}{16k^3 (n+2j+10)!} \left[(n+3)(n+2)(n+j+9)!\,j! - 3(n+3) \right.\right. \\
& \cdot \frac{(k^2+1)}{k^2} (n+j+9)!\,j! + \frac{9m^2}{k^2} (j+n+8)!\,(j+1)! \Big] - (n+2) a_n \\
& \cdot \frac{(k^2-1)^3}{16k^3} \frac{(n+j+8)!\,j!}{(n+2j+9)!} \left[(n-2)k^2 - 3 \right] \Big\} + \ell^{n+2j+8} \left\{ \frac{a_n (k^2-1)^2}{16k^3} \right. \\
& \cdot \frac{1}{(n+2j+8)!} \Big[6(n+3)(3k^2+1)(n+j+7)!\,j! - (n+3)(n+2)(5k^2+1) \\
& \cdot (n+j+7)!\,j! - 22m^2 (n+j+6)!\,(j+1)! \Big] + (n+2) a_n \frac{(k^2-1)^2}{16k^3} \\
& \cdot \frac{(n+j+6)!\,j!}{(n+2j+7)!} \left[(5n-13)k^2 + n - 5 \right] \Big\} + \ell^{n+2j+6} \left\{ \frac{a_n (k^2-1)}{8k^2} \right. \\
& \cdot \frac{1}{(n+2j+6)!} \Big[3(n+3)(n+2)(5k^2-1)\, k\, (n+j+5)!\,j! \\
& - 36(n+3)(k^2-1)(n+j+5)!\,j! + 12m^2 k (n+j+4)!\,(j+1)! \Big] \\
& + 3(n+2) a_n \frac{(k^2-1)}{8k} \frac{(n+j+4)!\,j!}{(n+2j+5)!} \left[(7-5n)k^2 + n - 3 \right] \Big\} \\
& + \ell^{n+2j+4} \left\{ \frac{(n+3) a_n k}{2} \frac{(n+j+3)!\,j!}{(n+2j+4)!} \left[6(k^2-1) - (n+2)(5k^2-3) \right] \right. \\
& - m^2 a_n k \frac{(n+j+2)!\,(j+1)!}{(n+2j+4)!} + \frac{(n+2) a_n k}{2} \frac{(n+j+2)!\,j!}{(n+2j+3)!} \\
& \cdot \left[(n+1)(5k^2-3) - 6(k^2-1) \right] \Big\} + \ell^{n+2j+2} (n+2) a_n k^3 \\
& \cdot \left\{ (n+3) \frac{(n+j+1)!\,j!}{(n+2j+2)!} - (n+1) \frac{(n+j)!\,j!}{(n+2j+1)!} \right\} + \ell^{n+2j+12} \\
& \cdot \frac{3a_n (k^2-1)^4}{16k^7} \left\{ k^2 (n+3) \frac{(n+j+11)!\,j!}{(n+2j+12)!} - k^2 (n+2) \frac{(n+j+10)!\,j!}{(n+2j+11)!} \right. \\
& + 3m^2 \frac{(n+j+10)!\,(j+1)!}{(n+2j+12)!} \Big\} + m^2 a_n \frac{3(k^2-1)^5}{16k^9} \frac{(n+j+12)!\,(j+1)!}{(n+2j+14)!} \\
& \cdot \ell^{n+2j+14} + m^2 a_n \frac{(k^2-1)^6}{16k^{11}} \frac{(j+n+14)!\,(j+1)!}{(n+2j+16)!} \ell^{n+2j+16} \\
& + m^2 \frac{(k^2-1)^6}{16k^{11}} \frac{(j+4)!\,(j+1)!}{(2j+16)!} \ell^{2j+16} + m^2 \frac{3(k^2-1)^5}{16k^9} \frac{(j+12)!\,(j+1)!}{(2j+14)!}
\end{aligned}
$$

$$\cdot\, \ell^{2j+14} + \frac{3(k^2-1)^4}{16k^7}\, \ell^{2j+12} \Big\{ 3k^2 \frac{(j+11)!\, j!}{(2j+12)!} - 2k^2 \frac{(j+10)!\, j!}{(2j+11)!}$$

$$+ 3m^2 \frac{(j+10)!\,(j+1)!}{(2j+12)!} \Big\} + \frac{(k^2-1)^3}{16k^5}\, \ell^{2j+10} \Big\{ (3k^2+9) \frac{(j+9)!\, j!}{(2j+10)!}$$

$$- (4k^2+6) \frac{(j+8)!\, j!}{(2j+9)!} - 9m^2 \frac{(j+8)!\,(j+1)!}{(2j+10)!} \Big\} + \frac{(k^2-1)^2}{8k^3}\, \ell^{2j+8}$$

$$\cdot \Big\{ 3(4k^2+2) \frac{(j+7)!\, j!}{(2j+8)!} - (13k^2+5) \frac{(j+6)!\, j!}{(2j+7)!} - 11m^2 \frac{(j+6)!\,(j+1)!}{(2j+8)!} \Big\}$$

$$- \frac{3(k^2-1)}{4k}\, \ell^{2j+6} \Big[3(k^2-1) \frac{(j+5)!\, j!}{(2j+6)!} + (k^2+3) \frac{(j+4)!\, j!}{(2j+5)!} \Big]$$

$$+ k\, \ell^{2j+4} \Big\{ (11k^2-9) \frac{(j+2)!\, j!}{(2j+3)!} - 6(k^2-6) \frac{(j+3)!\, j!}{(2j+4)!} \Big\}$$

$$+ 2k^3\, \ell^{2j+2} \Big[3 \frac{(j+1)!\, j!}{(2j+2)!} - \frac{j!^2}{(2j+1)!} \Big]$$

6. Part VI

$$(-1)^{n+j} e_{nj} = \frac{\ell^{2n+2j+9}}{(2n+2j+9)!} \Big\{ n(n-1)(n-2)(n-3)(k^2-1)^4 (n+j+12)!\,(n+j-4)!$$

$$+ 4n(n-1)^2(n-2)(k^2-1)^4 (n+j+11)!\,(n+j-3)! + 2n(n-1)$$

$$\cdot (3n^2-9n+2)(k^2-1)^4 (n+j+10)!\,(n+j-2)! - 4n^2(n^2-6n$$

$$+7)(k^2-1)^4 (n+j+9)!\,(n+j-1)! + [n(n-1)(n-3)(n-6)(k^2-1)^4$$

$$+ 5m^4](n+j+8)!\,(n+j)! \Big\} + \frac{\ell^{2n+2j+7}}{(2n+2j+7)!} \Big\{ [2n(n-1)(k^2-1)^3$$

$$\cdot (35nk^2 - 5n^2k^2 - 54k^2 + n^2 - 9n + 16) + 4m^4](n+j+6)!$$

$$\cdot (n+j)! + 4n^2(k^2-1)^3 [2(n-1)(n-2)(5k^2-1) - 3(n-1)(5k^2-2)$$

$$+ 4(2k^2-1)](n+j+7)!\,(n+j-1)! + 4n(n-1)(k^2-1)^2$$

$$\cdot [3n(k^2-1)(5k^2-2) - 2(2k^2-1) - 3n(n-1)(k^2-1)(5k^2-1)]$$

$$\cdot (n+j+8)!\,(n+j-2)! + 4n(n-1)(n-2)(k^2-1)^3[(2n-1)(5k^2-1)$$

$$+1](n+j+9)!\,(n+j-3)! - 2n(n-1)(n-2)(n-3)(k^2-1)^3(5k^2-1)$$

$$\cdot (n+j+10)!\,(n+j-4)!\Big\} + \frac{\ell^{2n+2j+5}}{(2n+2j+5)!}\Big\{\Big[n(n-1)(k^2-1)^2$$

$$\cdot [4(6k^4-6k^2+1) - 4(n-2)(10k^4-8k^2+1) + (n-2)(n-3)$$

$$\cdot (15k^4-10k^2+1)] + 3m^4\Big](n+j+4)!\,(n+j)! + 4n^2(k^2-1)^2$$

$$\cdot (15nk^4-14nk^2+2n-12k^4+16k^2-3)(n+j+5)!\,(n+j-1)!$$

$$+2n(n-1)(k^2-1)^2[2(6k^4-4k^2+1) - 6n(10k^4-8k^2+1)$$

$$+3n(n-1)(15k^4-10k^2+1)](n+j+6)!\,(n+j-2)! + 4n(n-1)$$

$$\cdot (n-2)(k^2-1)^2[(10-15n)k^4+(10n-8)k^2+1-n](n+j+7)!$$

$$\cdot (n+j-3)! + n(n-1)(n-2)(n-3)(k^2-1)^2(15k^4-10k^2+1)$$

$$\cdot (n+j+8)!\,(n+j-4)!\Big\} + \frac{\ell^{2n+2j+3}}{(2n+2j+3)!}\Big\{\Big[4n(n-1)k^2(k^2-1)$$

$$\cdot [2(k^2-1)(2k^2-1) + (n-2)(10k^4-12k^2+3) - (n-2)(n-3)$$

$$\cdot (5k^4-5k^2+1)] + 2m^4 + 4m^2n(k^2-1)^2\Big\}(n+j+2)!\,(n+j)!$$

$$+4n(k^2-1)[4(n-1)(n-2)nk^2(5k^4-5k^2+1) - 3(n-1)nk^2$$

$$\cdot (10k^4-12k^2+3) + 4n(k^2-1)k^2(2k^2-1) - m^2n(k^2-1)$$

$$+m^2(k^2-1)](n+j+3)!\,(n+j-1)! + 2n(n-1)(k^2-1)$$

$$\cdot [6nk^2(10k^4-12k^2+3) - m^2(k^2-1) - 4k^2(k^2-1)(2k^2-1)$$

$$-12n(n-1)k^2(5k^4-5k^2+1)](n+j+4)!\,(n+j-2)!$$

$$+4n(n-1)(n-2)k^2(k^2-1)[5(n-2)k^4+(12-5n)k^2+n-3]$$

$$\cdot (n+j+5)!\,(n+j-3)! - n(n-1)(n-2)(n-3)k^2(k^2-1)(5k^4-5k^2$$

$$+1)(n+j+6)!\,(n+j-4)! \Big\} + \frac{\ell^{2n+2j+1}}{(2n+2j+1)!}\Big\{\Big[nk^2[4m^2$$

$$\cdot (n-1)(k^2-1) - 4k^2m^2(k^2-1) + 4(n-1)k^2(k^2-1)^2 - 4(n-1)$$

$$\cdot (n-2)k^2(k^2-1)(5k^2-3) + (n-1)(n-2)(n-3)k^2(15k^4-20k^2$$

$$+6)] + m^4\Big](n+j)!^2 + 4k^2n\,[n(n-1)(n-2)k^2(15k^4$$

$$-20k^2+6) + 3n(n-1)(k^2-1)k^2(5k^2-3) - 2n(k^2-1)^2k^2$$

$$-2m^2n(k^2-1) - k^2(k^2-1)m^2](n+j+1)!\,(n+j-1)! + 2n(n-1)$$

$$\cdot k^2\,[3n(n-1)k^2(15k^4-20k^2+6) - 30nk^2(k^2-1)(5k^2-3)$$

$$+2k^2(k^2-1)^2 + 2m^2(k^2-1)](n+j+2)!\,(n+j-2)! + 4n(n-1)$$

$$\cdot (n-2)k^4\,[(k^2-1)(5k^2-3) - n(15k^4-20k^2+6)](n+j+3)!$$

$$\cdot (n+j-3)! + n(n-1)(n-2)(n-3)k^4(15k^4-20k^2+6)$$

$$\cdot (n+j+4)!\,(n+j-4)! + \frac{\ell^{2n+2j-1}}{(2n+2j-1)!}\Big\{2n(n-1)k^2$$

$$\cdot [6n(k^2-1)k^4 - (n-2)(n-3)(k^6+2k^2-2) - 6n(n-1)$$

$$\cdot (k^6+2k^2-2) - 2m^2k^2](n+j)!\,(n+j-2)! + 4n^2k^2$$

$$\cdot [(n-1)(n-2)(k^6+2k^2-2) - 3(n-1)(k^2-1)k^4 - m^2k^2](n+j-1)!^2$$

$$+4n(n-1)(n-2)k^2[n(k^6+2k^2-2) - k^4(k^2-1)](n+j+1)!\,(n+j-3)!$$

$$-2n(n-1)(n-2)(n-3)k^2(k^6+2k^2-2)(n+j+2)!\,(n+j-4)!\Big\}$$

$$+\frac{\ell^{2n+2j-3}}{(2n+2j-3)!}\,2n(n-1)k^{8}\big[(n-2)(n-3)(n+j-4)!\,(n+j)!$$

$$-4n(n-2)(n+j-3)!\,(n+j-1)!+3n(n-1)(n+j-2)!^{2}\big]$$