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MODERN CONTROL SYSTEM THEORY AND HUMAN CONTROL FUNCTIONS

by R. W. Obermayer and F. A. Muckler

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BUNKER-RAMO CORPORATION
Canoga Park, Calif.
for

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SUMMARY

Within the past decade considerable changes have occurred in the theoretical framework underlying modern control theory synthesis and analysis. Until recently, the traditional approach to formal theory in guidance and control has assumed as a tool linear differential equations in the servomechanism concept. Based on certain mathematical developments, there has been a shift in theoretical interest from classical stability analysis to precise statements of system performance requirements and particularly specifications of optimal control systems. Modern control system theory, based primarily although not exclusively on the calculus of variations, provides formal, rigorous, and exact statements of system performance optimization.

Within the framework of current modern optimal control theory, the approaches of Pontryagin, Bellman, and Kalman are presented. Expansions to cover the case of adaptive control are examined. These developments are simply recorded, no attempt is made to extend, or expand, or present original theoretical developments. It is noted that there are a number of problems in the applications of these formal mathematical models including difficulties in defining the physical problem, computational requirements, introduction of feedback control, and proper selection of the performance index.

The major problem of interest in this report is the application of modern control theory to the case of manual control systems. First, and foremost, is the analytic and design problem of the allocation of control function either to man or to machine. In the past (and present) man's function in guidance and control has been allocated largely in haphazard and non-formal ways. Modern control theory may provide a formal structure from which rational allocations may be made.

It is also interesting that beginning with the framework of modern control theory certain direct implications can be drawn to the nature of the human's control task and crew station design. In some cases, for example, crew station displays are required that do not differ from previous display technology, but in other cases rather different tasks and hence display forms are suggested. Three specific cases are noted: (1) display of the state vector, (2) "quickened" displays, and (3) display of switching curves.

A difficult methodological problem in manual control system studies has been the selection of the appropriate performance index or indices. In the past, the experimenter has chosen his measure set primarily based on his personal criteria. Modern control theory provides a rather rigid set of requirements for the specification of the performance index which may influence considerably future performance measurement techniques.

In the general area of manual control system research, the most active single topic is the development of formal theoretical models for the description of human performance per se. Modern control theory has implications

here, and the distinction between the transfer function and state vector representations are discussed. One particularly interesting problem is that of the inverse optimal control problem.

DESIGN OF MANUAL CONTROL SYSTEMS

Modern Control Theory

Until recently, the traditional approach to formal theory in guidance and control has assumed as a tool linear differential equations within a framework of the servomechanism concept. However, stringent system performance requirements and radically increased hardware complexity have led theorists to attempt mathematical representations of control systems that cannot be reasonably and usefully approximated by linear techniques. Thus, modern control system theorists are becoming much more concerned with the formal theory of nonlinear systems. Further, modern control system theorists have begun to evaluate the conceptualization and construction of adaptive control systems.

Most important of all has been a shift in theoretical interest from stability analysis to precise statements of system performance requirements. All current major theoretical efforts are directed toward the definition of optimal control systems. Modern control system theory, based primarily although not exclusively on the calculus of variations, provides, formal, rigorous, and exact statements of system performance optimization.

Manual Control Modes

As a primary element in most past guidance and control systems, the human operator remains an important potential design component in future control systems. While over the past decade many guidance and control systems have been predominantly automatic, there is an increasing tendency to re-introduce man into control systems where automatic control techniques have been paramount.[19,20] Whatever course future design may take, it seems reasonable to assume that manual modes of control will continue to be of interest. Accepting this assumption, it is of value to trace any relationships between modern control system theory and the more particular problem of manual control. Among several, three areas may be noted:

1. While modern control theory establishes an exact framework for system optimization, none of the current theoretical variations specify the mechanisms by which system requirements are to be achieved. That is, the form of the "control function" may be theoretically defined, but the hardware mechanization of that function is not. In some cases, indeed, the physical control function may be quite difficult even to conceptualize. At any rate, in preliminary design one is free to consider a wide spectrum of control techniques ranging from automatic, semi-automatic to manual control

modes. Thus, one is led directly to the basic human factors problem of the allocation of human control function.

2. If modern control theory requires a given control function to be performed, it will follow that certain specific types of control tasks will be required. Given a manual or semi-automatic control mode, specific tasks will be generated for the human controller. As will be shown, the nature of these tasks are often quite unusual when compared with conventional human controller tasks in past systems. Further, as should be expected, particular display-control configurations needed to perform these tasks will differ from past basic display and control designs.

3. Apart from the systems context, quantitative theory of the human per se is of interest. To date, the major theoretical advances in the formal specification of human controller performance has come from the describing function variation of conventional servomechanism analytic methods.[18] In the sense that modern control theory attempts to provide a theoretical framework for optimal, adaptive, and nonlinear systems, the question is how such techniques might apply to that most adaptive nonlinear, and sometimes optimal system - the human controller. In short, how adequately and usefully do these models serve to represent human control performance? Thomas[27] has explored this question at some length with respect to two techniques: Bellman's Dynamic Programming and Pontryagin's Maximum Principle. As will be noted, the general theoretical problem is the comparative evaluation of the state vector representation with the conventional transfer function approach.

Ultimately, the objective of modern control system theory is improved analysis and design of future control systems. Insofar as the human controller may play a role in the mechanization of these systems, it is important to explore the possible conceptual and formal bridges between modern control system theory and the allocation of control function whether it be automatic, manual, or - more probably - some combination of both.

MODERN OPTIMAL CONTROL THEORY

It will become quite apparent that modern control theory is not a clear, cohesive, and unified domain of structured knowledge. It is, rather, an evolving set of concepts and techniques with limited to very broad scope and application. One attribute found throughout, however, is that the mathematics is depressingly formidable to the uninitiated. Since the essence of modern control theory lies in the mathematical superstructure, at least some elementary notions and operations must be stated. It should be noted that these are indeed elementary with respect to the full complexity of this theoretical work. But it is only fair to add that the phenomena to be modelled - as we understand them - are in themselves complex.

A Conceptual Framework

The State Concept. The concept of state is fundamental to the modern description of dynamical systems; it is, however, essentially a primitive concept not susceptible to exact definition. Nevertheless, one can attempt to give a definition to help establish the role of the concept. Kalman,[10] for example, gives the following definition: "The state of a dynamical system is a minimal set of numbers which, specified at any given time, suffice to determine completely the future evolution of the system, provided the future forces acting on the system are known." In short, if one separates the object and its environment, the state of the object is a full set of descriptors pertaining to the object and when combined with knowledge about the environment determines the future behavior of the object.

The concept of state is essentially the same as employed in the classical Turing machine and in Shannon's information theory. The Turing machine is a discrete process where the output at time t and the state at time t+1 are determined by the state and input at time t. The state equations for such a machine are given by:

$$\begin{aligned} S_{t+1} &= f(S_t, u_t) \\ y_t &= g(S_t, u_t) \end{aligned} \qquad t = 0, 1, 2 \ldots$$

where S_t, u, and y are the state, the input, and the output respectively. Since continuous-time differential systems are of principal interest, the discrete equations may be generalized to.

$$\frac{dS}{dt} = \dot{S} = f\left[S(t), u(t)\right]$$

$$y(t) = g\left[S(t), u(t)\right]$$

For a first-order ordinary differential equation, $\dot{x} = Kx$, it may be seen that x is the state variable. For a second-order differential equation, $\ddot{x} + A\dot{x} + Bx = f(t)$, this can be rewritten in the following form letting $x_1 = x$; $x_2 = \dot{x}_1$:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Ax_2 - Bx_1 + f(t)$$

or, putting it into matrix notation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -B & -A \end{bmatrix} x \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\dot{x} = Kx + F(t) \qquad K = \begin{bmatrix} 0 & 1 \\ -B & -A \end{bmatrix} \qquad F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

where x^T is the transpose of x, i.e., has rows and columns interchanged. Here the state of the dynamical system is given by a two-element column matrix, or vector, x, composed of x_1 and x_2 - the position and velocity of the object. Thus, x is called the state vector. For an ordinary differential equation with constant coefficients, the state is, as one might expect, described by the same specifications that are needed as initial conditions to solve the differential equations. Here x will normally be taken as the state variable, and differential equations will be written as: $\dot{x} = f(x,u)$. It should be understood that x is generally a vector of many components, and the differential equation is actually a set of simultaneous differential equations.

A dynamic system will ordinarily change state with time, and therefore one will be interested in specifying a particular state at a particular time. To simplify communication, the combination of state and time (x,t) is called a <u>phase</u>. Also, the total range that the state variable x may take is referred to as the <u>state space</u> or <u>phase space</u>. The mathematical methods using the state vector representation are commonly called state-space or state-variable techniques.

<u>Policy Space and Performance Index</u>. In control theory one is for the most part interested in finding a particular input (control, or control input) which will cause the state of the system to change in some desired manner. It is frequently a requirement to assign an appropriate control input to each possible state. The functional relation of input to state is called a <u>policy</u>; in control engineering, this relation is often called the <u>control law</u>.

Much of modern control theory assumes in advance that the system engineer can specify completely and quantitatively all system performance tradeoffs. Thus, he must be able to give an equation from which can be computed a single number, or performance index, rating the system. In flight control, for example, this may mean combining into one number the effect of error in maintaining a desirable trajectory, the amount of fuel used, control action, the time to reach terminal conditions, the error at the terminal state, etc. While the problem of specifying a performance index will be discussed in detail later, it is evident that one would normally have trouble defining such an index. However, it may seem reasonable for the theorist, developing a quantitative theory for optimal system performance, to expect the system designer to specify what optimal performance is.

The Fundamental Optimal Control Problem. Therefore, it is assumed that the object to be controlled (sometimes called the "plant") is (1) describable by a system of differential equations, (2) the initial state is given, (3) the control variables (u) are identified and any control limitations or constraints are specified, and (4) the performance index (J) is defined. For example:

$$\dot{x} = f(x, a, t)$$

$$x^T(0) = x^T{}_0 = \left[x_1 (0), \ x_2 (0), \ \ldots, x_n (0)\right]$$

$$u^T(t) = \left[u_1 , u_2 , \ldots u_n\right]$$

$$J = \int_{t0}^{tf} f \ (x , u , t) \, dt$$

is the minimum set of equations satisfying these four conditions.

The optimal control problem can then be stated: Given any initial phase, find a corresponding allowable control that transfers the controlled object to the desired region of the state space and for which the performance index is minimized. In short, find the optimal policy, the policy yielding the minimum performance index.

A number of variations of the fundamental control problem have been treated in the literature differing primarily in the nature of the desired state change and the specific performance index. A number of classical problems are shown in Table 1. Each of these problems can be formulated in terms of the fundamental control problem, and the specific techniques discussed below are generally applicable. (See Table 1, page 7).

Optimization Techniques

The problem of achieving a desired goal in an optimal fashion is obviously not a new one, and a large number of specific techniques have been developed: The techniques for finding maxima and minima taught with the differential calculus are perhaps the most familiar. Lagrange multipliers and the calculus of variations are classical approaches. Iterative search techniques exist for seeking the optimum using fast computers. There are many other techniques[17]; the entire topic of optimization techniques therefore is a broad one, and complete treatment is far beyond the scope of this paper. While the subject has long been popular, in the past decade remarkable advances have occurred in the mathematical treatment of system optimization problems. Russian interest, based on the work of Pontryagin[24] and others, initially far exceeded that in this country and probably still does.[16] Of a host of techniques, three will be presented in brief: Pontryagin's Maximum Principle, Bellman's Dynamic Programming and Kalman's solution for linear systems. These appear to represent the most powerful of existing techniques.

Table 1

Typical Control Problems

TERMINAL CONTROL	Bring the state of the system as close as possible to a given terminal state at a given terminal time.
MINIMAL-TIME CONTROL	Reach a terminal state in the shortest possible time.
REGULATOR PROBLEM	With the system in some initial phase, return the system to an equilibrium state so that some integral of the motion is minimized.
PURSUIT PROBLEM	Given a moving target, cause the controlled system to have the same phase trajectory in a finite time.
SERVOMECHANISM PROBLEM	Cause the phase of the controlled motion to be as close as possible to a desired state time history (a generalization of the regulator problem).
MINIMUM ENERGY CONTROL	Transfer the system from an initial phase to a final phase with a minimal expenditure of control energy.

Pontryagin Maximum Principle

General Form. Given a system of differential equations with a specified initial state, constraints that a control must satisfy, and a performance index (J) with the following matrix form:

$$\dot{x}(t) = f(x, u, t); \qquad x(0) = x_0$$

$$J = \int_{t_0}^{t_f} f^0 (x, u, t)\, dt$$

Pontryagin's Maximum Principle provides as a necessary condition that a specific control, $u^*(t)$, is optimum. The Maximum Principle only gives necessary conditions, not sufficient conditions; the optimal control, therefore, satisfies the Maximum Principle, but not all controls satisfying the Maximum Principle are optimum.

The Maximum Principle requires that a system of auxiliary variables be formed, sometimes called Lagrange multipliers, that are defined in terms of the following:

$$\Psi^T = (\Psi_0, \Psi_1, \ldots, \Psi_n)$$

$$\dot{\Psi} = - \left[\frac{\partial f}{\partial x}\right]^T \Psi$$

With the state variables and the auxiliary variables, a new function, H, is formed:

$$H(\Psi, x, u) = \Psi^T \dot{x} = \Psi^T f$$

The state variables and the auxiliary variables form a Hamiltonian system,

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial \Psi_i}; \quad \frac{d\Psi_i}{dt} = -\frac{\partial H}{\partial x^i} \qquad i = 0, 1, \ldots, n.$$

the function, H, is sometimes referred to as the Hamiltonian.

The function H involves the control variable, $u(t)$. The Maximum Principle states that in order that the specific control, $u^*(t)$, be an optimal control, it is necessary that the function H corresponding to u(t) attains its maximum for this control for all time:

$$H = (\Psi, x, u^*) = M$$

The Maximum Principle is shown in the Appendix with matrix forms expanded.

Rather than giving a direct solution to the optimal control problem the Maximum Principle produces the result in terms of the solution of another set of differential equations. Whether or not this system of equations can be solved depends upon the existence of initial conditions for the differential equations and the extent to which the equations in the state variable and auxiliary variable are uncoupled. Often, only initial and final values for the state variable equations, yielding a two-point boundary value problem that may be complex. It is possible that no solution exists, or equally perplexing, that an infinite number of solutions exist. However, the basic utility of the Maximum Principle approach should not be overlooked: Even though a complex system of equations may be encountered, the explicit maximization of the function H is a positive step forward from the implicit minimization of the performance index. Further, often the <u>form</u> of the control law may be derived without actually solving the differential equations.

<u>Bang-bang Control.</u> A class of problems of some practical importance occurs when the control variable, u(t), enters the system equations in a linear manner. The principal matrix equations are:

$$\dot{x} = f(x, t) + g(x, t)\ u(t) \qquad x(0) = x^0$$

$$J = \int_{t_o}^{t_f} f^0(x,t)\ dt$$

The control, u(t), is taken to be bounded, and, when normalized, the constraint may be expressed as:

$$|u(t)| \leq 1$$

Forming the function H as defined in the Maximum Principle:

$$H = \sum_{i=0}^{n} \Psi_i \left[f^i(x, t) + g^i(x, t)\ u\right] = A(\Psi, x, t) + B(\Psi, x, t)\ u$$

it will be seen that H is maximized when maximum control effort is expended; that is, when $u^*(t) = +1$ or $u^*(t) = -1$:

$$u^*(t) = \text{Sign } B(\Psi, x, t)$$

This type of control has been popularly termed "bang-bang control."

It was just noted that the Maximum Principle may specify the form of control law sufficiently for complete problem solution without solving the auxiliary set of equations. This is, in fact, the case with bang-bang control when the performance criteria is to achieve a new state in minimal time.[14,15] For a dynamical system such as an inertial space vehicle, the trajectory of the vehicle for the case of full constant control thrust can be easily specified.[8] On the phase plane (position plotted against velocity), the paths for bang-bang control will be segments of parabolas. If the control objective is to move to the origin of the phase plane (i.e., bringing the system to rest at some fixed position), there is only one parabolic path that will intersect the origin - as shown in Figure 1. The control task then becomes one of exerting full control force in one direction until a point on the path intersecting the origin is achieved. The reverse control force is then used until the vehicle comes to rest.

The solution to this type of problem is given in terms of the switching curve; in this case, as shown in Figure 1, it is the parabolic curve intersecting the origin of the phase plane. One only needs to know when to change the direction of control. For systems described by higher order differential equations where a higher dimensional space and not just a plane is necessary to specify system phase, the result is more complicated. For example, with a third order system, one must change control direction when a switching surface is intersected, stay in this surface until another switching curve is intersected, and then follow the final switching curve to the origin.

Singular Control. Unfortunately, there are cases of linear optimization problems where bang-bang control does not apply. Sometimes, the switching function, B, becomes zero over some finite time interval. During the interval when B is identically zero, the Hamiltonian function ceases to be an explicit function of the control variable, $u(t)$, and the usual procedure of selecting $u^*(t)$ so as to maximize H seems to break down.[9]

Control during intervals when $B = 0$ is termed singular control. Singular control does not have to be bang-bang; it may be variable control such as a linear feedback of the state variables. Thus, the optimal solution of linear optimization problems may be bang-bang, or some combination of bang-bang control mixed with variable control. Instead of a unique solution, bang-bang control becomes, then, a candidate for the optimum. Each practical problem must be closely examined for possibly superior combinations of control techniques.

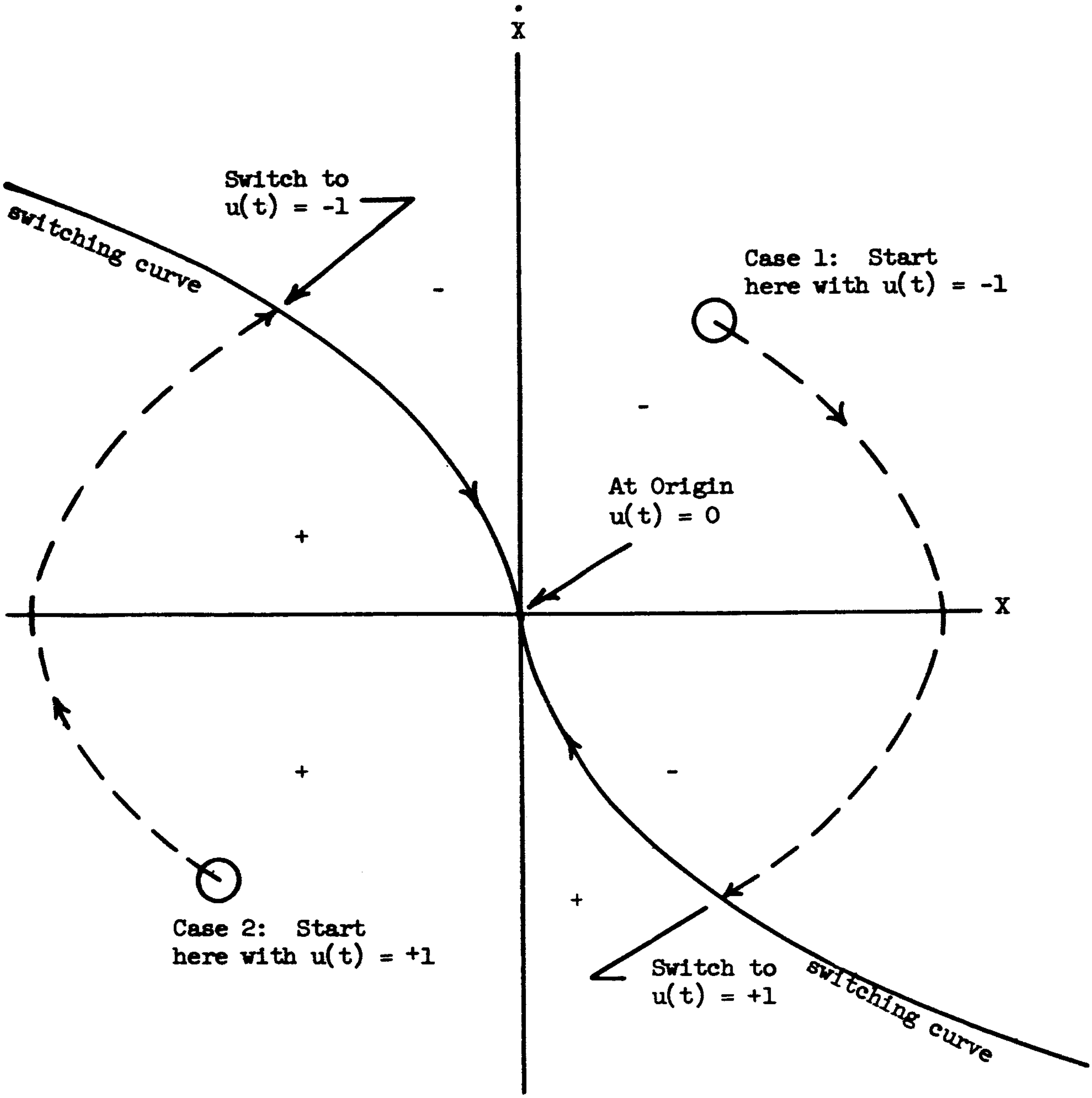


Figure 1. Phase-plane presentation of bang-bang control

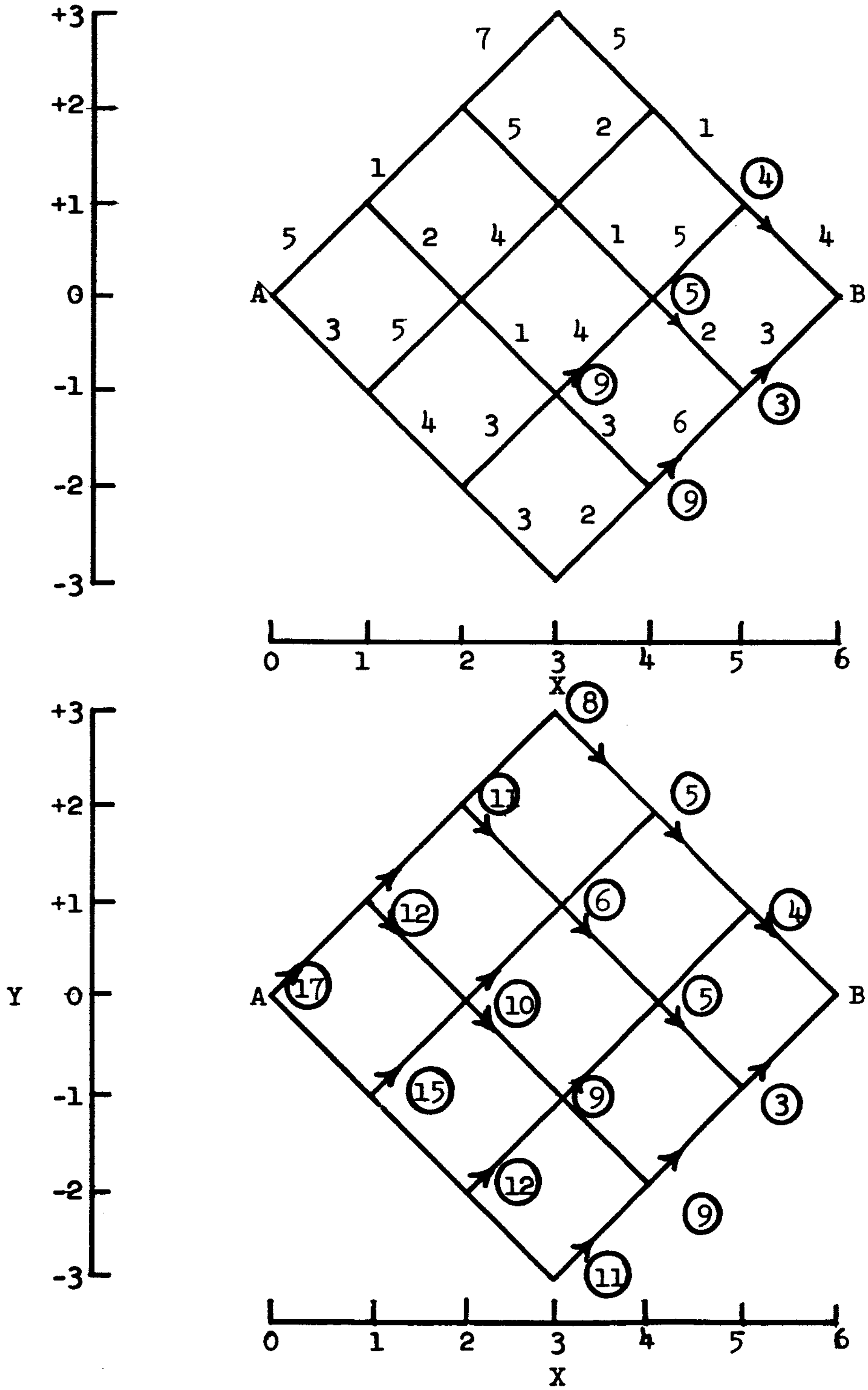
Bellman's Dynamic Programming

Principle of Optimality. Dynamic programming is a discrete process designed especially for use on digital computers. In order to derive Bellman's computational algorithm, the continuous time problem must be converted to discrete event form. In discrete form, the problem of finding the control $u(t)$ becomes that of finding the values of $u(t_0)$, $u(t_0 + \Delta)$, ..., $u(t_0 + n\Delta)$, and so forth. Rather than attempting to find all the values of control simultaneously, dynamic programming formulates a sequential decision process, finding in order one control value at a time until the entire control function is known.

Bellman bases his technique primarily on the Principle of Optimality (Bellman and Dreyfus, 1962): "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." The Principle of Optimality holds whether one proceeds forward or backward in time. Therefore, it is permissible to attempt to find the last control decision first and move backward to the first control decision. This procedure, in fact, is perhaps more common than computing from the first control decision to the last.

An Example. The fundamentals of dynamic programming are perhaps most easily understood in terms of a concrete example. Figure 2 shows a simple trajectory problem where we wish to find one path, moving from right to left, that minimizes the sum of numbers (the "cost") encountered while going from A to B. The numbers next to the lines are the costs incurred in taking any given path. For convenience, a coordinate system is used specifying the junctions with (0, 0) and (6,0) the coordinates of A and B, respectively.

At each junction there are at most two alternatives. Moving backwards from the end of the trajectory (B) to the beginning (A), each junction is labelled with the minimal cost of achieving the terminal point. The optimal direction is marked with an arrow. For example, at junctions (5,+1) and (5,-1) movement to B the choice is necessarily the costs of 3 and 4. Moving back to junctions (4,+1), (4,0), and (4,-1), the direction of least cost is again selected. At (4,0) a choice exists between a cost of 5+4 and 2+3; obviously one selects the downward direction with a sum cost of 5 (marking the direction with an arrow). At junctions (4,+1) and (4,-1) there is only one possibility with costs of 1+4 and 6+3. Still moving backward toward the initial point A, at junction (3,-1) the choice is between a cost of 4+5 and 3+9; obviously, one would therefore select the upward direction, again marking the choice with an arrow. Thus, at every junction, the decision of optimum path is made simply by comparing two sums and selecting the smaller.



<u>Legend</u>

Uncircled numbers = cost of path segment

Circled numbers = cost from junction to B

⟶ = direction of optimal travel

Figure 2. An example of dynamic programming[3]

When the entire matrix has been completed, and all arrows have been drawn for the minimum cost path from A to B, it is, of course, possible to now start at the beginning (A) and follow the arrows to the terminal state (B). Further, if some disturbance were to suddenly displace the system into any unplanned state, the information is still available to decide on a new optimal path regardless of the state the system assumed.

Computational Requirements. Dynamic programming requires the sequential comparison of many alternatives, and in fact requires the solution of the optimization process for every possible initial state. While it may not be readily apparent, there is a considerable saving over computing and comparing the cost of all possible trajectories. The process of direct enumeration expands as the power of the number of alternatives to be compared, but the complexity of dynamic programming is approximately proportional to the number of alternatives. As Bellman and Dreyfus[3] state: "It is the principle of optimality that furnishes the key. This principle tells us that having chosen some initial x_N, we do not then examine all policies involving that particular choice of x_N, but rather only those policies which are optimal for an N-1 state process with resources $x - x_N$. In this way, we keep operations essentially additive rather than multiplicative. The time required for a twenty-stage process is now almost precisely twice the time required for a ten-stage process."

Dynamic programming is a computer algorithm that does not depend on the linearity of the system equations, the stochastic nature of the variables, or the nature of the performance index. The algorithm can be carried out given only appropriate tables of numbers. It is, therefore, a very general technique. The main problem is that very large computer storage capacities may be required for even relatively simple problems. The storage requirement is particularly aggravated with high dimensional problems (when the numerical tables become high dimensional volumes of numbers). On the other hand, whenever constraints on the control problem are known and given, the storage problem is simplified. Whenever functional relationships may be given, mathematical techniques may be used rather than the primitive computer search technique, relieving some of the requirement for storing some tables of numbers.

Kalman's Solution for Linear Systems

Kalman's[11] result applies to linear systems of any order, with possibly time-variable coefficients, and a quadratic performance index of the form:

$$\dot{x} = A(t)\, x + G(t)\, u$$

$$J = \tfrac{1}{2} x^T S x + \tfrac{1}{2} \int_{t_o}^{t_f} \left[x^T Q\, x + u^T R u \right] dt$$

The matrices S, Q, and R must be symmetric, and the inverse of R must exist. For any system of this quite general form, Kalman asserts that the optimal control function is a linear feedback of the state vector.[11]

$$u^* = -R^{-1} G^T P x$$

Here the matrix $P(t)$ satisfies a matrix Riccati differential equation:

$$\dot{P} = P G R^{-1} G^T P - P A - A^T P - Q; \; P(t_f) = S$$

<u>The Automatic Synthesis Program (ASP)</u>. Solution of this type of problem is facilitated by the existence of a computer program (ASP) that is capable of giving optimal feedback for high order control systems (15-30). The designer must provide the numerical inputs for matrices R, S, Q, A(t), and G(t), but the computer will automatically print out the linear time-varying feedback gains.[12]

The ASP is a valuable tool that has achieved success in a number of applications.[29, 25, 13] In addition to solving problems in optimal control design, the program can perform stability analysis of linear systems and synthesis of optimal filters in statistical design. Therefore, it is possible to incorporate linear feedback in a noisy system with incomplete feedback of the state vector.

<u>Re-entry Trajectory Control</u>. Use of Kalman's technique might be better understood through a typical application.[13] Kovatch designed a controller for a lifting body re-entry vehicle. With an on-board computer storing a precomputed nominal trajectory and a set of precomputed optimal feedback gains, the controller operated on the difference between the state variables and the nominal values. The nonlinear equations of motion of the space vehicle were linearized about the nominal trajectory to produce a set of linear differential equations with time-varying coefficients. A quadratic performance index was used, weighting the terminal error, the deviation from the nominal, and the amount of control used. Kalman's optimization procedure, programmed in the ASP, derived the optimal control law as a linear combination of the deviations in the state variables. Kovatch concluded that "...using Kalman's optimal control procedure and the Automatic Synthesis Program one can obtain optimal linear feedback gains for a complex control problem." He further pointed out it would be possible, and desirable, to investigate, with modifications of the ASP, the effect of random perturbations, atmospheric noise, and measurement of noises on the operation of the control system.

Adaptive Control

Defining Adaptive Control. One of the problems that has perplexed workers in adaptive control theory is finding a suitable definition to guide and to constrain their activities. Truxal[30] describes an adaptive control system as, "A control system which is designed with an adaptive view." That is, an adaptive system cannot be identified by appearance or performance, but one must know something about the way it was designed. A feedback control system is adaptive in the sense that it may operate well in a changing environment, but Kalman[10] objects and states that, "Such a system may be more properly called insensitive or invariant, rather than adaptive."

Kalman[10] has given the following definition of an adaptive control system: "A control system is adaptive if it is capable of changing its control law as a result of measured changes of the control object and its environment and in such a way as to operate at all times in an optimal or nearly optimal fashion." Under this definition, the operation of any adaptive control system will depend upon two types of data: (1) measurements of the state variables of the control object which are used to determine the instantaneous values of control and (2) measurements defining the equations of motion of the control object and its environment, data which are used to determine an appropriate optimum control law. Kalman defines the first type of data as the "dynamic state" and the second type as the "learning state".

Adaptive Linear Systems. Assuming a control object describable by linear dynamical equations - which is ordinarily necessary in order to assure an explicit mathematical solution - the following matrix differential equations generally result:

$$\dot{x}(t) = Fx(t) + Gu(t) + Jw(t)$$

$$y(t) = Hx(t)$$

$$z(t) = y(t) + v(t)$$

where x is a column matrix of state variables, u is a matrix of control inputs, w is a matrix of system disturbances, y is a matrix of system outputs, v is noise or errors in determining measurements of the state indicated as z. This type of system is shown schematically in Figure 3.

In order that control of this linear system be adaptive, two dynamical processes must take place simultaneously: (1) based on the control law existing at a given moment, a control action is derived from estimates of the state variables of the control object, and (2) the control law is changed as needed to maintain optimum control based on estimates of the structural characteristics of the control object and its environment. If the measurement

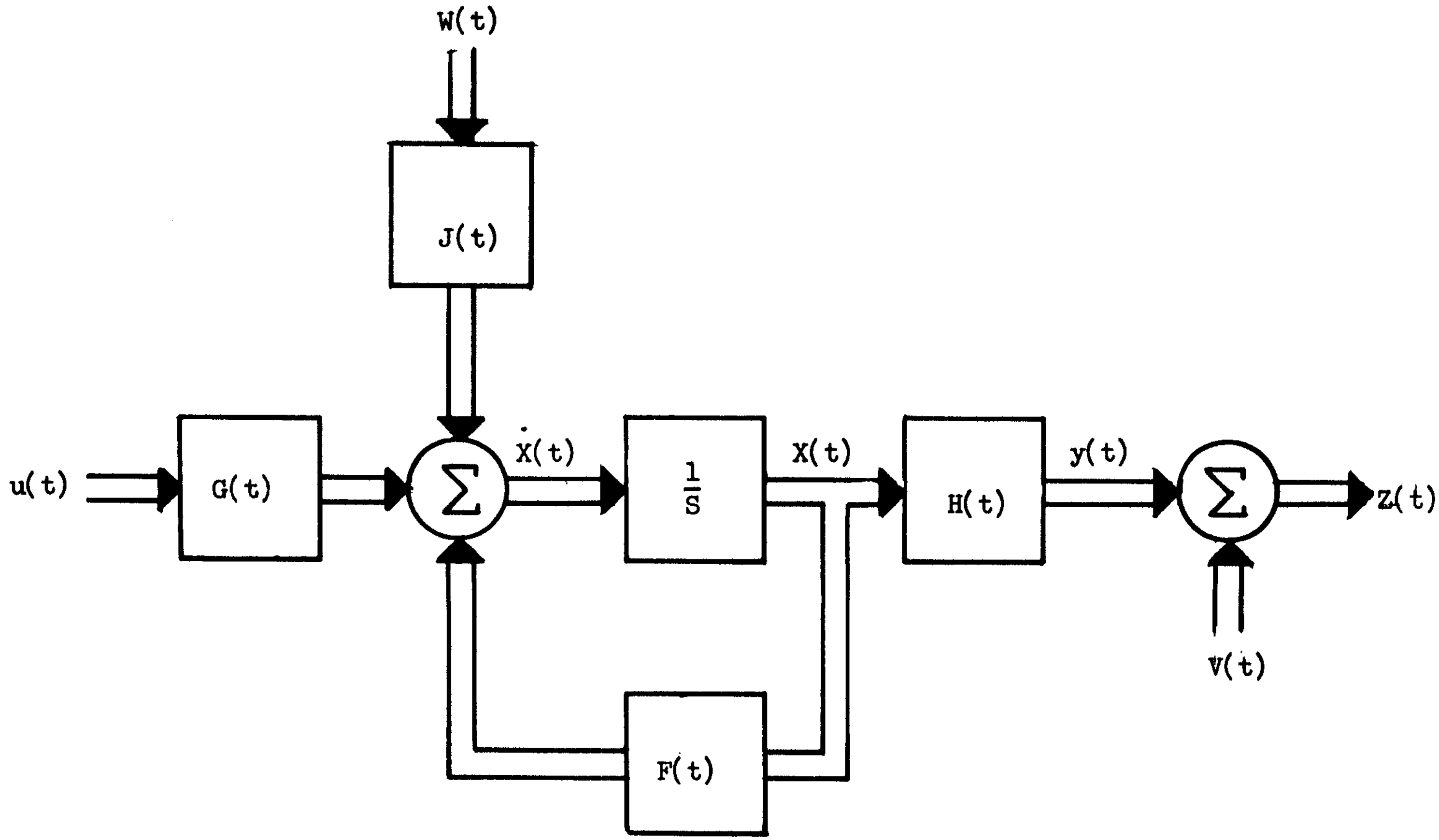


Figure 3. Block diagram of a linear dynamical system[10]

processes may be taken as ideal, then all information in regard to state variables, control inputs, disturbances, transfer characteristics, etc., is known, and the problem is reduced to finding a solution to the general time-varying control problem. That is, if all information is available obviating further measurement and all time-dependent characteristics are completely known for all time, the optimal solution can be determined exactly. Kalman[10] calls this an ideal adaptive control system. While the concept of an ideal adaptive system may have importance, most practical problems requiring the adaptive approach will not permit perfect measurement.

Varieties of Adaptive Control. The primary differences between adaptive control systems is in the way they acquire information about the control object and its environment. Given the required information the remaining design decisions are more or less determined by conventional servo theory, modern control theory, and good control system design practice. Aseltine[1], in a survey of adaptive control systems, identifies five types based on the kinds of measurements taken:

1. Passive adaptation: adaptation without system parameter changes, but through design for wide variations in environment.

2. Input signal adaptation: adjustment of system parameters in accordance with input signal characteristics.

3. Extremum adaptation: self-adjustment for maximum or minimum values of some systems variable(s).

4. System-characteristic adaptation: self-adjustments based on measurement of transfer characteristics.

5. System-variable adaptation: self-adjustment based on measurements of system variables.

Assuming Kalman's[10] representation of an adaptive control system, as shown in Figure 4, passive adaptation should not be classed as an adaptive process since there is no learning state and a fixed control law, $u(t) = K\,x(t)$, is used which is designed to give good average performance. Input adaptation is keyed to input signal characteristics such as in automatic gain control systems. Extremum and system-variable adaptation are very similar, but were separated due to the heavy emphasis given in practice to extremum adaptive systems. In extremum systems, system variables are varied to hunt system performance peaks. With system-variable adaptation, the same goal is sought but performed indirectly by controlling system variables on the basis of some secondary criteria. In system-characteristic adaptation, the transfer characteristics are measured by means of a test signal and the control law varied accordingly.

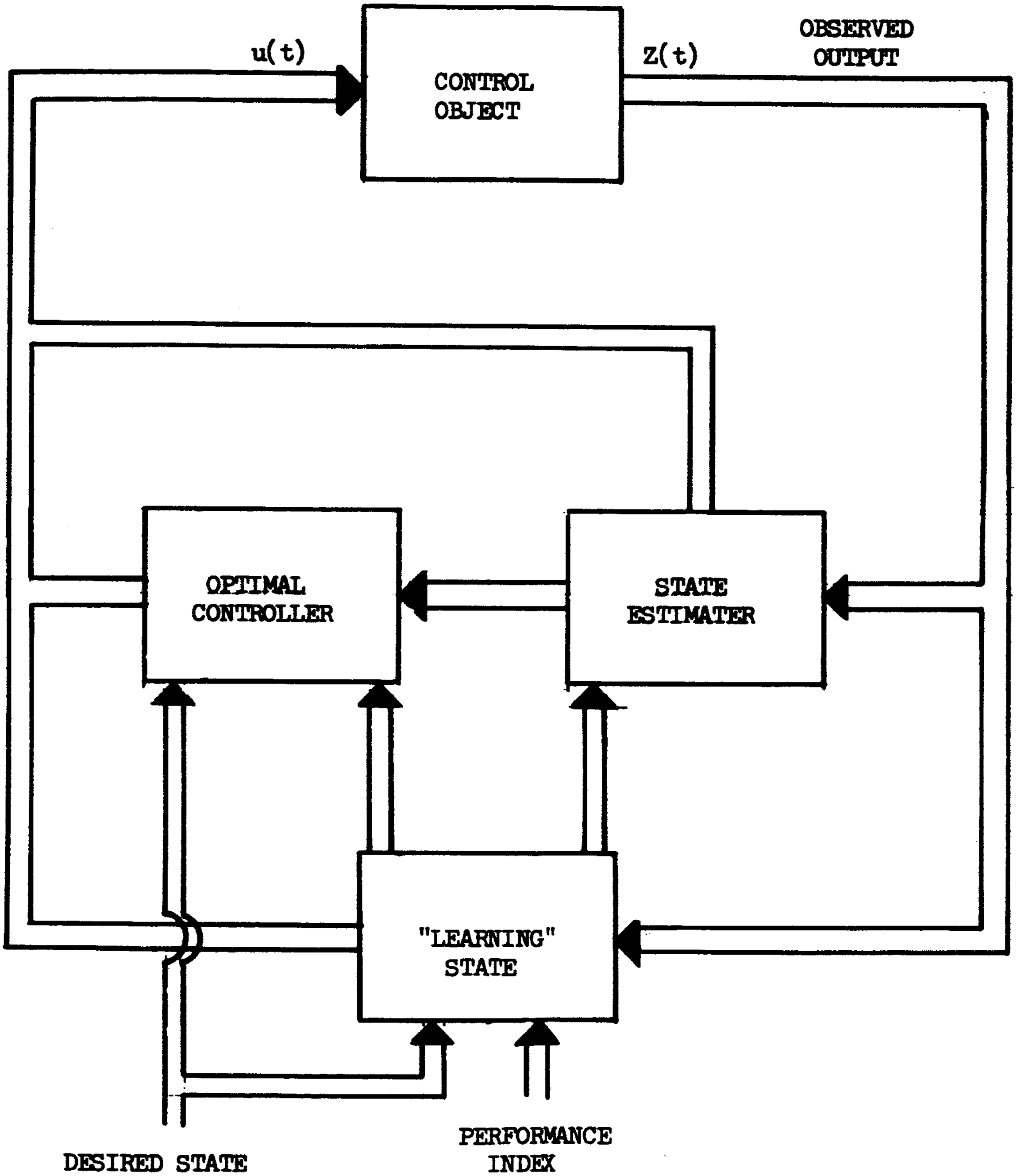


Figure 4. Kalman's representation of an adaptive control system[10]

Examining Figures 3 and 4, it may be seen that the following measurements may be taken: the input, the disturbance $w(t)$, errors in measurement $v(t)$, the state variables $x(t)$, and the matrices $G(t)$, $J(t)$, $F(t)$, and $H(t)$. All of these must be known or estimated to achieve optimal control, $u(t)$. On the other hand, each of the matrices or influencing signals may also be the object of control in order to effect a system with better performance. With this kind of complexity, a great variety of approaches to the adaptive control system problem is possible.

Summary. The adaptive control problem may be considered as a step or two up the hierarchy of control system problems as compared with these problems cited under optimal control theory. Given a full mathematical description of a controlled object and the associated environment, one may theoretically hope to find an optimal control solution. Given full and perfect measurement, but with an initially unknown system, a completely general technique must be available to fit any specific unknown system. Further, one must be able to find the solution quickly in order that the derived control is appropriate to a particular instant of time. If measurements are imperfect, then the situation is much more complex; how successful one will be depends entirely on how well the true situation can be estimated. This is a familiar, but still difficult, statistical problem.

Current Problems in Application

As may be apparent at this point, modern control theory is not a refined and mature theory, but is rather an embryo set of principles, theorems, and rules for special cases. The practicing system engineer cannot, of course, wait for a refined set of techniques, but must attempt to work with what is available. The problems of application serve, in fact, to define the current status of modern control theory.

Chang and Alexandro[5] point out that there are two principal difficulties in applying optimal control theories:

1. Physical difficulties: e.g., the existence of random noise and disturbances, and lack of knowledge and/or change in the plant.

2. Computational difficulties: e.g., the two-point boundary value problem with the Maximum Principle, and the requirement of large storage capacity with the dynamic programming approach.

Further, assuming the problems of demonstrating a satisfactory mathematical solution are solved, the literature indicates that there remains:

3. Problems of selecting a basis for optimization: e.g., in precisely defining the performance index.

4. Problems in finding practical and economical ways to mechanize the theory.

Difficulties in Defining the Physical Problem

Chang and Alexandro include a host of complications under the rubric of physical difficulties, but one critical aspect, which probably affects the applicability of optimal control theory more than any other, is the problem of delimiting the physical control. The theories of optimal control assume that the control problem can be completely defined in mathematical terms; that is, at least the dynamical equations describing the controlled system or plant, all system constraints, and all disturbances acting on the system must be completely known.

While this is a serious situation, it is one that the control engineer faces most of the time in designing complex physical systems. Physical descriptions of real systems are seldom simple, and the engineer is trained to avoid unnecessary complications in a sufficient understanding of the physical process. Therefore, he linearizes and approximates. Further, the control engineer seldom has full knowledge of the disturbances and inputs to which his system must react. At the start, then, in using optimal control theory the control engineer faces considerable equivocation in defining what it is he wishes to optimize. Rather than one precise physical description, he usually must work with several imprecise part-descriptions.

Unfortunately, there is a danger that the specific choices the engineer makes in defining his system may make major differences in the "optimal" solution. To take a case where the difficulty is clear, consider a given linear plant where the system equations involve the control linearly. If, as in virtually all hardware problems, the amount of control is limited ($u \leq 1$, the "hard" constraint), the Maximum Principle is applied, and as has been shown bang-bang control is a candidate for the optimum control. If, however, one re-defines the problem, eliminating the constraint on the control, but weighting the use of control heavily in the performance index (the "soft" constraint), then it can be reasonably expected that the solution will require a relatively small amount of control such that existing limited control will in fact suffice. In the latter case, applying Kalman's result, the optimal control is a linear feedback of the state variables. Clearly, a significant difference occurs depending upon whether the "soft" or the "hard" constraint is assumed. Depending on the constraints chosen, the optimal control law may be linear or nonlinear. The "hard" constraint may result in performing the control task in less time, and, therefore, be preferable. However, the "soft" constraint does not commit the design to maximum use of control, permitting a reserve for unexpected disturbances.

In short, optimal control theory does not aid the designer in defining the control problem, and the best design is probably not derived as an explicit mathematical solution given by optimal control theory. More likely the best design still results from an iterative, comparative, and judgmental process.

Computational Difficulties

The Maximum Principle. As noted, the solution for an optimal control in terms of the Maximum Principle gives the optimal control as a function of the Language multipliers, Ψ. At times, the form of the optimal control is evident, as in the case of bang-bang control, and the remaining details can be easily completed. However, in the general case, the auxiliary equations in Ψ must be solved before an explicit form for the optimal control is available. This is by no means an easy task.

Dynamic Programming. In dynamic programming, a solution is given in the form of a mapping from state space into a control space, so that each possible physical state has associated with it an optimal control action. Mechanization invariably involves a digital computer storing the appropriate control action for any feasible state the controlled object might assume as a result of control action and/or disturbances. For high dimensional systems, the total possible states for which one might wish to know the appropriate optimal control may be extremely large. Even for relatively simple physical systems, the storage capacity required may be more than that available in any digital computer. Bellman calls this "the curse of dimensionality"; however, it is conceivable that advances in digital computer technology may alleviate this problem somewhat.

Since the computational problems of dynamic programming and the Maximum Principle are quite different, it is possible the two techniques could be combined in a complementary fashion. In just such an attempt, Chang and Alexandro[5] joined the two techniques. The least-cost trajectories in a small neighborhood of the terminal state can be computed by dynamic programming without large computer storage capacity requirements. By making a guess and iterating, a few trajectories can be computed from the auxiliary equations in the Maximum Principle to go from the initial state to some point within the neighborhood of the terminal state. With a fast computer, a number of trajectories can be computed before a decision needs be made, and then the least-cost trajectory from these can be selected. Because the final point of the Pontryagin trajectory is a finite neighborhood rather than a single point, the trial process may not be very difficult. However, this method of combining techniques is not general, and usually will yield approximate, suboptimal, performance. When the order of the dynamical systems is high, when the dynamics are badly nonlinear, and when the performance index is other than minimum time or a quadratic, the computational difficulties involved in optimal control theory are severe.

Requirement for Feedback Control

The simplest mechanization implied by the Maximum Principle is to build a system which follows an optimum or nominal trajectory. However, this is an open loop solution in the sense that the trajectory is followed in a pre-programmed manner - even if closed-loop control devices are included to drive the system back to the nominal trajectory whenever disturbances cause a trajectory deviation. A problem with this approach is that once the system has strayed from the nominal trajectory, the future optimal path is no longer the pre-calculated one. Further, attempts to return to the original trajectory may lead to stability problems. A more desirable system results when the control system is mechanized with the optimal control as a function of the state variables. Then, if the system is disturbed from the nominal path, a new optimal trajectory is pursued rather than returning to the old trajectory.

As Bellman and Dreyfus[3] point out, "It is precisely the information needed to accomplish this - the optimal decision as a function of all possible reasonable states - that is produced in a dynamic programming calculation." This is distinctly an advantage for the technique of dynamic programming, although the solution as given by the Maximum Principle may frequently be transformed into the desired form. For linear systems and a quadratic performance index, Kalman's result gives the control law as a function of the state variables.

Even if the computational difficulties associated with the Maximum Principle and dynamic programming are solved, a problem still exists in implementing the closed-loop control law with existing hardware. As Truxal and Dorato[28] state: "In this connection it should be noted that the control law is generally a nonlinear function of the state of the plant. Here two separate problem arise: One is the problem of measuring the state of the plant, which often requires the measurement of high order derivatives, and the other is the problem of generating the required nonlinear function to the plant state." For a launch vehicle, for example, the state vector may consist of 14 or more terms, which results in a requirement for sensing and transmitting 14 channels of information. It is possible that this complexity is more trouble than it is worth, as Reynolds and Rynaski[25] note in regard to re-entry vehicle design, "Adding a differentiating network for the sideslip signal brings the total number of channels to eight. This is a lot of complexity. Also, the feedback gains must be programmed as a function of time or changed in some other manner during the re-entry. This also adds complexity. The advantages of designing a control system of this complexity are not apparent in the results." Since the measurement task may be severe, the system engineer may seek a suboptimal method that does not require feedback of the entire state vector of the system (e.g., Kalman's ASP may be used to find the optimal linear feedback based on incomplete. measurement of the state vector).

Selecting the Performance Index

All present approaches to optimal control theory assume as a beginning that some performance index is specified. That is, there is some overall "goodness" scale specified, and the task is to find a control input which will give a minimum or maximum on this scale. Implied in this assumption is an ordering of all systems which is complete and specifiable on a single dimension scale.

The technique of Kalman assumes a quadratic performance index given in terms of the matrices R, S, and Q. Given these matrices, and the linear system equations, the Automatic Synthesis Program (ASP) will yield a digital computer solution for the optimal control law. However, the engineer does not usually know how to specify the performance weighting matrices Q, R, and S. He knows that these weightings determine the relative emphasis in the definition of the optimum for terminal conditions, trajectory factors, and control usage, but he does not know how to translate qualitative definitions of "good control" into a quantitative definition in terms of Q, R, and S. Given the means for finding fast digital computer solutions, he is allowed the possibility of examining the control system response for a range of values for each matrix. As Kovatch[13] states: "There is some arbitrariness in the procedure since one has to specify the performance weighting matrices S, Q, and R, but this arbitrariness is expected on physical grounds since what may be considered as good performance for one application may not be good performance for another. However, with several runs of the ASP program one soon can see what gives the best performance for a desired mission."

When a performance index is used as a device for picking a system which is desirable on other, subjective, grounds, the meaning of the term "optimal control" becomes unclear. However, as a device for finding system alternates to be subjected to discriminating control engineering judgment, the technique of using a variable performance index may be very valuable. Reynolds and Rynaski[25] used this approach successfully, but were also somewhat concerned with the nature of the real or ultimate performance index. They say, "Some of the responses are 'more optimum' than others on the basis of what we know to be a desirable transient response, although all the closed-loop responses optimize their specified performance index. The choice of relative weighting between output error and input gives a wide range of closed-loop dynamics and there is not a priori basis at present for choosing the weighting factor." Further: "Thus the performance index is used as a performance index - that is, we choose elements of the H and Q matrices to minimize what we would like to minimize from physical considerations - and it is used as a 'cut-and-try' parameter. The real criterion of performance is judgment applied during the 'cut-and-try' procedure."

Advantages of Optimal Control Theory

It is evident that there remains much to be accomplished in the field of optimal control theory. However, it should also be apparent that much has already been accomplished. The comment of Truxal and Dorato[28] is appropriate: "In spite of the difficulties cited...optimal control theory does provide an organized approach to the design of complex feedback systems. The computational difficulties encountered should, in fairness, be weighted against the rather sophisticated nature of the problem considered. Indeed, in many applications, especially nonlinear stochastic systems, the alternative to optimal control theory is no theory at all."

A number of specific advantages are pointed out by Tou and Joseph:[27]

1. It is not necessary to assume a configuration for the overall control system, but an optimal configuration can be derived. Modern control theory addresses itself to the synthesis problem directly, whereas past techniques have been analytical approaches based on an assumed initial form.

2. Extreme difficulties are encountered with conventional techniques in multi-variable control problems, but the modern approach allows relatively easy solution.

3. Design of time-varying control systems is facilitated.

4. Contrary to conventional procedures, no assumption is made that the system operates in a steady state.

5. The necessary calculations are appropriate to a digital computer. The present status of optimal control theory is such that one must be dubious about the value of application for the design of high order non-linear practical control systems. On the other hand, one must be encouraged by the promise of optimal control theory for future theoretical developments.

APPLICATIONS TO MANUAL CONTROL SYSTEMS

Allocation of Function to Man and Machine

Allocation of Human Control Function. In the synthesis of past hardware guidance and control systems, man's control function has been largely unspecified in a formal sense. For that matter, there has been no systematic way of assigning control function either automatic, semi-automatic, or manual. Part of the basic problem lies in the limits of control system synthesis techniques. Despite ambitious titles to the contrary, most of control theory lies in the domain of analysis and not synthesis. That is, given a fixed and known system configuration, a host of methods exist for the formal analysis of system response. However, in the area of synthesis (that is, starting with system goals and then rationally ordering hardware configurations to fit those goals), the state of the art is far more fluid and ambiguous. Synthesis in practice is accomplished by intuition, guess, and past experience without formal guidelines. Perhaps one of the problems is that there has been no way of specifying theoretically system goals.

Accordingly, man's function in guidance and control has been generated largely in haphazard and nonformal ways. At least five informal allocation techniques can be distinguished in past and present practice based on (1) what the automatic functions do not do, (2) traditional roles and preferences, (3) assumptions on the nature of human capabilities and limitations, (4) assumed formal descriptions of man's response characteristics, and (5) direct empirical assaults on the particular system.

1. In many cases, if a subsystem function description was possible, the control function was routinely automated. That the human controller might (or might not) perform the same function more efficiently was seldom considered. The more esoteric functions are given, usually without thorough analysis, to the human operator. The human operator was (and still is) relegated to the task of being adaptive, optimalizing, and nonlinear; that is, in translation, he is expected to do what the system designer does not know how to specify. Frequently, this is a post hoc decision where the human performs tasks discovered during test and operation that had not been anticipated during design. Also, he is expected to perform any control functions that might be necessary to save the system.

2. The major technique of assigning man's function is that of tradition and preference. In vehicle control, man traditionally has served as a prime control element directly controlling vehicle attitude and power. In subsonic aircraft, he also supplied, implicitly in design and explicitly in flight, the primary guidance function. Excessive attention was placed on vehicle stabilization and control with far less concern given to the guidance task; displays for the guidance task, such as position information and command data, have been accepted only with great difficulty. In the past fifteen years, flight vehicles and their missions have undergone such radical change that it is surprising to see the traditional assignment of human control function still pressed. In some cases, there is serious doubt as to whether the traditional manual control functions are either desireable or even feasible.[20] Much useless argument has been expended on this point in the design of space vehicle guidance and control systems; this effort might have been much better spent considering man for other more promising and more important guidance and control subsystem tasks.

3. A third technique is to assume some basic properties of the human controller and design with respect to optimization of the human function alone. This technique is best characterized by the approach of Birmingham and Taylor[4], a design philosophy that has had a greater impact on manual control system synthesis than any other set of concepts. In brief, it dictates: "Design the man-machine system so that (1) the bandpass required of the man never exceeds three radians per second, and (2) the transfer function required of the man is, mathematically, always as simple as possible, and, whenever practicable, no more complex than that of a simple amplifier."[4](page 8) The problem is to find a design with suitable performance in which the human operator, acting as an amplifier, can be imbedded.

Birmingham and Taylor offer a demonstration by choosing a design of a tracking system designed to follow a constant velocity input. Since an amplifier simply transmits a signal, possibly modified in amplitude, there are a number of places in the system where the human controller, acting as an amplifier, could be placed without changing overall system function. Further, by block diagram manipulations, a number of equivalent systems can be created that suggest other system roles for the human.[6] If the initial design is a good one, it is then possible for the human to perform well at a simple task with resultant excellent system performance. If circumstances dictate that the operator must assume more extensive participation in system control, it is also clear what other functions (e.g., differentiation, integration) he must assume.

A debate that ensued at the time of the introduction of the Birmingham-Taylor concept arose on the question: If the operator's function is that of a simple amplifier, or if the function is precisely known, why not replace him with an amplifier or appropriate required mechanisms? The answer frequently given is correct: If the human operator can be replaced by a simple amplifier, one should do just that. This leads to a curious line of argument. Man's best role is as an amplifier. However, if the system requires an amplifier, man should not be used.

In practice, the situation is somewhat more complex. In most hardware systems, there is often a need to satisfy obvious system functions for which man appears to be the only possible candidate. If, for example, the human operator is, in addition to simple tracking, serving as a sensor in gathering information, or serving as a primary source of control power, or serving as a backup controller in case of automatic system failure, or is needed for many other virtues accompanying the human controller, then replacing him requires much more than placing an amplifier in the system block diagram.

In design, the qualitative objective in the Birmingham-Taylor approach is reasonable: Wherever possible, simplify the human operator's task. This approach has many potential advantages; increased accuracy can be expected, variability of human performance is reduced, his task load over several simultaneous tasks is simplified, and so forth. But the quantitative objective seems curiously impractical. If a function can be clearly specified to require only amplification, it is doubtful that man should perform that function even if he does it well. Further, in the full complexity of most present day systems, the most common problem is the question of possibly adapting man to perform essential and difficult tasks that cannot be automated (if, indeed, they can be precisely described). The human operator offers many potential advantages for complex system operation; to stress as his exclusive role simple amplification does not face the design problem. And that problem is the rational allocation of control function - automatic, semi-automatic, or manual - to perform the host of subsystem functions that are characteristic of every practical system. If the human factors specialist has only a simple amplifier to offer for the solution of this allocation problem, he does little service to the design process. In fact, he fails to offer what he really has - a flexible and effective system element capable of performing well beyond amplification. Because we cannot, at this time, formally specify his adaptive, nonlinear, and at least partially optimalizing characteristics, does not mean we can ignore them. Had we done so prior to 1906 and since there would still be no manned flight at all.

4. From the theoretical standpoint, it would be ideal if there were available a formal representation of the input-output characteristics of the human operator in a mathematical language consistent with the description of the machine elements. In man-machine servo systems, total closed-loop system <u>analysis</u> would then be possible with good predictions of man-machine system performance responses. The objective of the work on the human transfer function[2, 18] has been to approach that ideal. Putting aside the problem of the adequacy of any known linear or quasi-linear transfer function to describe human behavior, it should be noted that any reasonable representation can serve a useful design purpose on the allocation problem.

For example, a common design necessity is to make preliminary feasibility judgments as to possible guidance and control concepts early in the design of a vehicle. Configurations are assumed, and close-loop analyses performed. If a configuration has manual or semi-automatic modes, a transfer function of the human controller is essential for analysis. In one exceedingly complex hardware case[19], it was found that the McRuer-Krendel[18] generalized human transfer function could be used to predict feasible manual and semi-automatic control modes (with verification, based on subsequent empirical simulation). While the level of prediction was weak and essentially qualitative in that only acceptable, marginal, and unacceptable modes could be classified, this information is itself of considerable value during initial study and design where the guidance and control problem is very broad and vague.

5. When all else fails, empirical simulation remains as a major design tool. For some, full mission simulation is, indeed, the method of choice. Insofar as prediction from ground-based simulation data to inflight operations is valid, there is warm comfort in the ability to check human control functions in a mission-simulated environment. In practice, however, certain complications arise. First, simulation is extremely expensive and time-consuming. Unless extensive simulation facilities are immediately available, construction and programming delays can result in data produced after the design fact. Second, in full mission simulation, there is seldom the flexibility one desires in trying, and accepting or rejecting, a variety of control modes. As a preliminary design tool for the allocation of control function, the flight simulator is an unwieldly and costly device. For verification and modification of design and operator training on a prototype or operational design, it is an essential part of the development process for manned flight vehicles.

These methods, then, comprise the bag of tools - mixed with guess, intuition, and experience - that the control system engineer and the human factors specialist has available in the design of the total guidance and control system for a manned flight vehicle. It seems not unreasonable that we might desire a somewhat more structured approach to control system synthesis.

Modern Control Theory. And it is precisely a structured approach that modern optimal control theory attempts to provide. As should be apparent, modern control theory concentrates on the description of system objectives and the formal statement of how these objectives can be optimally achieved. If the current difficulties described in a previous section cannot be overcome, modern control theory may become an unsuccessful approach to the design of practical control systems. There is, however, some hope and some optimism that optimal control theory will be a valid and useful approach to the synthesis of satisfactory control systems.

1. At best, modern control theory provides a formal statement of the optimal control process. In practice, it may be that the final synthesis is suboptimal, or even a combination of techniques with heuristic, iterative, methods may result. In either case, a precise standard is provided against which designs can be evaluated.

2. It has been pointed out that optimal control theory yields a system description in functional terms without any specification as to the nature of the system elements that will be required to perform the function. In this sense, the theory does not infer an allocation of function. However, the functions to be performed are made quite explicit, and the mechanization decisions can at least be performed against a precise standard.

3. System design will, in practice, go through a number of iterations in utilizing and compensating for the characteristics of specific system elements. In mechanizing a bang-bang control, delays in switching may cause a limit-cycle behavior so that a saturating amplifier might be used instead of a pure on-off device, providing bang-bang control in the large but linear and stable control near the phase-plane origin. If this were the case, the initial allocation of functions may have dictated a relay for switching that, under later inspection, would have to be replaced by a saturating amplifier.

4. In every case, the basic allocation of function problem will arise: Given an optimal control process with specified control functions, what control mechanization - automatic, semi-automatic, or manual - will be required to perform these control functions? Again, the control task is specified in terms of the input (the state vector) and the relation of the controller output to the input (the control law), but the exact mechanization may be purely manual, purely automatic, or some intermediate combination. Through optimal control theory, a functional system synthesis is achieved which allows allocation of function to man or machine based on capability to perform that function.

There are, at the present time, no empirical studies available establishing optimal control processes and comparing mechanization alternatives. Such studies, set in a variety of vehicle and mission contexts, would be of considerable value not only for the mechanization problem but also for further practice in the use of optimal control theory. For manual control, such studies would provide an interesting context for the investigation of manual contributions in a variety of task forms.

Manual Control and Crew Station Design

Manual Control Allocation. There is no universal optimal control, and, therefore, unique manual control task implied by modern optimal control theory. A wide range of potential manual control tasks are possible. For some systems, the optimal control will be proportional to some weighting (possibly nonlinear and time-variable) of the state variables. Depending on the nature of control provided, the constraints imposed, and the performance index, bang-bang control is another very likely candidate for the optimal control. In short, the specific control function may range from discrete to continuous or some combination (e.g., the singular control problem).

At the present time, no thorough classification of the potential range of manual control tasks has been compiled. The number of application studies published to date has been small. Some of the control functions studied would appear to be within the range of human control. For example, one might infer strongly that bang-bang control functions such as suggested by the Maximum Principle can be performed by the human provided the display of switching curves is possible (Figure 1). Proportional control is a task which the human has performed in a variety of vehicle settings. Combinations of control functions (e.g., the singular control problem) are also well within the realm of consideration. In some cases, it is difficult to decide just what the manual control task would be. The computer algorithm for Bellman's Principle of Optimality diagrammed in Figure 2 pertains only to the computational requirements, and does not directly imply the specific manual control function.

Across this wide potential range of control tasks, it is interesting to examine the direct implications for crew station design and display-control techniques that might be required. In some cases, displays are required that do not differ from previous display technology, but in other cases rather different display forms are suggested. Three cases may be noted: (1) display of the state vector, (2) "quickened" display, and (3) display of switching curves.

Display of the State Vector. As noted several times, optimal control theory provides control as some function of the state variables. It follows directly that information with respect to the state vector must be displayed to the human controller in order that his control function be optimal. The form of display may vary considerably: (1) A one-dimensional display per control dimension indicating weighted state variable information, (2) a multi-dimensional display of state variables with switching surfaces, or (3) simply individual display of each state variable.

The latter display configuration would be necessary for full status display to the human operator. It would be consistent with the custom originated in aircraft display systems in presenting position, velocity, and acceleration information in the form of altitude, rate of climb, and acceleration, and so forth. Perhaps the main advantage of optimal control theory in this case would be the precise specification of the state vector and the complete set of information categories that must be displayed. It would not, however, alleviate the problem of the proper techniques of displaying this information either in individual displays or in terms of integrated display systems.

In discussing the problem of the requirement for feedback control, it was noted that practical control problems may require the feedback of large numbers of state variables.[25] It may not be necessary in practice to include all elements of the state vector, and it was specifically noted that Kalman's ASP could be used to find optimal linear feedback even with incomplete measurement of the state vector. In the allocation of tasks, it is feasible to consider distributing the specific state variables between man and machine. Further, man could provide a multi-mode combination with monitoring of automatic feedback and possible direct intervention in case of feedback failure.

"Quickened" Display. If the optimal control is a linear combination of, or even a nonlinear function of, the state vector, the designer has the option of displaying a signal proportional to the desired control. This type of display would be directly analogous to that term "quickened" by Birmingham and Taylor[4]. If the signal did not exceed the bandpass capability of the human operator, one could be relatively confident that the resultant manual control system would be optimal. With current theoretical methods, such a display would demand relatively fixed system response, although conceivably some adaptation might be provided with, for example, changing performance criteria. There is some danger in this particular display technique in that the human controller participates in a rather blind fashion. Performance may be appropriate for a particular system mode, but, to provide flexibility and to allow emergency backup modes, display of the individual state variables may also be necessary.

Display of Switching Curves. A third type of display - switching curves - is specifically applicable to bang-bang control. Where bang-bang control is indicated as optimal or even as a good sub-optimal control, switching curve display is indicated, and represents a new type of display configuration. Figure 1 illustrates this type of display.

For second order systems, the display of switching curves to a human operator is quite feasible since only a plane is required for display. Happily, Platzer[22, 23] has provided direct experimental evidence indicating that the human operator performs excellently with a phase-plane display. Since the primary information with a phase-plane display is whether the system is on the switching curves (or, if not, on which side), it is reasonable to attempt the control task with a one-dimensional display showing on which side of, and how far from, the switching curve the system is. Platzer[23] directly investigated this display - called the ϕ-display -, and reported quite satisfactory human performance.

Unfortunately, higher order dynamical systems require representation in a multidimensional state space and complex switching characteristics. A third order system, for example, would require switching in a three-dimensional volume. A three-dimensional display is feasible, and dimension reducing techniques should be investigated. However, the situation is complex, and higher order systems appear quite intractable. With respect to display mechanization, Doll and Stout[7] note: "Practical difficulties arise from the fact that a programmed controller for an nth order system requires at least an (n-1)-variable function generator. Lack of suitable two-variable function generators has hampered even laboratory investigations of third-order systems."

Controller Design. In addition to influencing information display requirements, a specific optimal control function is likely to restrict the class of desirable human operator outputs. Even then, a wide range of control types might be indicated: (1) pure bang-bang control, (2) linear control that saturates at some specified control level, or (3) bang-bang control with a provision for an emergency, reserve, control level. In any case, given a specific optimal control task, it is reasonable to expect that the controller characteristics will be carefully tailored to the specific task. This suggests the possibility both of a variety of controller devices and precise care in the nature of the allowable control output generated by the human controller. Traditional practice in the design of control devices will be of limited aid.

Performance Measurement

The Performance Index. As has been noted, optimal control theory requires that the designer can specify a single performance index with which it is possible to order all system designs from best to worst. The performance index yields a single rating number, but it is possible (and probably necessary) that it represent a combined weighting of a number of different performance factors. An example would be generating a single performance index from a combined weighting of deviation from a nominal trajectory, control efficiency, and terminal error.

If a fully satisfactory performance index is available, critical information would be available which is normally unknown in detailed system design. In the ideal case, the designer allocating control functions has a precise definition of system goals. If a necessary and sufficient collection of performance factors have been weighted into the performance index, the designer has in fact generated formal criteria for the particular tradeoffs to be achieved. This information would be invaluable for objectively comparing alternative mechanizations.

Current Problems. Unfortunately, the choice of a specific performance index would appear to be somewhat artificial and arbitrary. Whereas the technique demands that a precise performance index be specified, the theory does not aid the designer in selecting one, and he is no more capable of defining system goals than before the advent of optimal control theory.

It is true, however, that optimal control theory does provide rather precise specifications of the functions of particular system elements. The delimitation of individual system element goals aids in the selection of performance measures of these particular functions. Given a mathematical description of the function of a given element - that is, what it should do - it is usually quite easy to invent measures of whether an element performs the assigned task and how well. Combining these measures into a single valid and reliable performance index is a matter that could stand a great deal more methodological study.

There are some aspects of system performance measurement where modern optimal control theory does not appear to be of help. As has been suggested elsewhere[21], the system designer is normally interested in general evaluation criteria for system stability, performance, adaptability, reliability, and acceptability. Conventional servo analytic techniques have stressed system stability. Modern control theory obviously centers on performance although it should be of assistance in defining the requirements for stability, performance, and adaptability. In the discussion of adaptive control, attention was called to measurement of adaptive control system response (Figures 3 and 4).

However, there are no immediately apparent implications in modern control theory for the measurement of reliability and acceptance. It is possible that these criteria may not enter directly into the performance index. Given a set of alternative mechanisms evaluated by the performance index, a second evaluation can be made with respect to the relative reliabilities of the alternates. The cost of the alternates also represents a significant scale for evaluation. Rather than attempting to incorporate cost and reliability into the performance index, it is perhaps more reasonable to add these two as separate and subsequent criteria.

The measurement of acceptability is pertinent only with respect to manual or semi-automatic control modes, and essentially asks the question as to whether or not these modes have acceptable handling qualities.[2] This is, in effect, a judgment of performance, and it seems reasonable to explore the possibility of incorporating the handling qualities performance scale in some way. Past practice in generating handling qualities judgments has been to find experimentally the direct relation between pilot judgments and specific aerodynamic parameters. Since these same parameters may be directly related to the elements of the state vectors in optimal control theory, it is at least theoretically possible to correlate handling qualities measures to the state vector. Insofar as applicable literature exists, handling qualities data may suggest constraints on both the elements of the state vector and the control assigned to each state. One advantage of optimal control theory is that it must deal with the entire policy space. If formal correlations are possible between either the state vector and/or the control space, handling qualities measurement could be applied for the first time to the performance of the total system rather than to an isolated collection of opinions on individual aerodynamic parameters.

Somewhat simpler conceptually is to see if the performance index dictated by modern control theory could be formally related to the existing scales of handling qualities. If so, the performance index could be directly evaluated on the basis of handling qualities for every configuration where manual or semi-automatic control modes were involved. Under this approach, acceptability, reliability, and cost would be additional evaluation criteria after the performance index had been generated.

Conceptual Frameworks for Manual Control Theory

Transfer Function and State Vector Representation. Apart from the system context, a matter of no little technical importance is the quantitative theory applied directly to the description of human controller behavior. To date, the major theoretical advances have been made within the framework of conventional servo analysis techniques and particularly in the quasi-linear describing function approach of McRuer and Krendel[18]. They have provided perhaps the most common and certainly the most tested form of transfer function for the human operator in a tracking task:

$$\frac{C(S)}{G(S)} = \frac{K_p\, e^{-S\tau}\,(1 + T_L S)}{(1 + T_N S)\,(1 + T_I S)}$$

with the five critical parameters usually labelled: K_p = static gain; $e^{-S\tau}$ = reaction time delay; T_L = anticipation lead time constant; T_N = neuromuscular lag time constant; T_I = error smoothing lag time constant. This transfer function has met with success in closed-loop tracking tasks. It has been most extensively investigated with only error display and with a single input and output dimension. The transfer function is expressed in terms of deviation on the compensatory display.

It has been pointed out that it is possible to represent a dynamical system as either a high order differential or as a simultaneous system of first order differential equations. The two forms are equivalent in terms of predicting the resultant system motion. However, the advantage of using the simultaneous system of equations is that they clearly indicate the transition of system states. In the transfer function just given, if the controlled element is of high order, the state variables are suppressed. In the state vector representation, the optimal control for a multi-dimensional system would be expressed as:

$$\begin{bmatrix} u_1 (t) \\ u_2 (t) \\ \cdot \\ \cdot \\ \cdot \\ u_r (t) \end{bmatrix} = \begin{bmatrix} k^1 (x^1, x^2, \ldots, x^n) \\ k^1 (x^1, x^2, \ldots, x^n) \\ \cdot \\ \cdot \\ \cdot \\ k^r (x^1, x^2, \ldots, x^n) \end{bmatrix}$$

and if the optimal control were a linear feedback, the control would be of the following form:

$$\begin{bmatrix} u_1 (t) \\ u_2 (t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_r (t) \end{bmatrix} = \begin{bmatrix} k_{11}(t), k_{12}(t), \ldots., k_{1n}(t) \\ k_{21}(t), k_{22}(t), \ldots., k_{2n}(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ k_{r1}(t), k_{r2}(t), \ldots., k_{rn}(t) \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{bmatrix}$$

Such a form gives a direct transfer function representation of multi-dimensional systems.

It is characteristic of the human controller that he is most commonly used in multi-dimensional high order systems. At the outset it would seem potentially desirable to measure the transfer function of the human operator in the state vector form. The McRuer-Krendel transfer function was developed for a simple display, a simple tracking situation where information with respect to the state vector could be obtained only by differentiation of the input error signal by the human. However, in the complex vehicle case, the human operator is usually given separate display of the state vector (e.g., altitude, rate of climb, acceleration, etc.). Optimal control theory would indicate that the control should depend upon this information, and it is certainly reasonable to assume that the human operator uses this information in the multi-dimensional control case.

It can therefore be contended that an expression of human operator control would most naturally be given as a function of the state vector. This in effect would be a direct mapping of a manual policy space, and would be desirable for the measurement of human operator control behavior. It would be particularly useful in dealing with multi-dimensional systems in which the operator is imbedded. If modern control theory is successful, it would further establish a form for the human operator, as a control mechanization mode, that would be analytically consistent with that theoretical approach. To date, there has been no attempt to re-define human control tasks in this manner. It is probable that one difficulty will be that detailed available knowledge is not sufficient to specify the behavior involved, let alone map this control behavior into the appropriate state vector.

The Inverse Optimal Control Problem. While not directly related to the mainstream of modern control theory, the following question might be asked: Given a particular control for a dynamical system, under what performance criteria is this control optimal? In the case of the human controller, the question becomes: Under what performance criteria is his performance optimal or nearly so? What criteria is he trying to optimize?

If an answer exists for this inverse optimal control problem, it is probably not unique. In a particular case, some performance index may be optimized, but very likely other performance indices will be optimized as well. In fact, it may be possible to generate performance indices that are optimized by any control chosen. What makes the problem of direct interest to manual control is the often observed fact that the human controller, instructions not withstanding, brings with him a set of strategies which he imposes on the manual control problem. He can be expected to assume some performance criteria, which may or may not be the one requested by instructions.

One particular type of performance criteria is that of the handling qualities requirements already mentioned. It is usually necessary that a manual control system design meet with pilot acceptance. For flight vehicles, the requirements specified on the control system to elicit pilot acceptance are referred to as handling qualities requirements. These dictate a match between the static and dynamic characteristics of the pilot and vehicle. Reynolds and Rynaski[25] comment: "An optimal control system, in order to provide good handling qualities, must ... satisfy two criteria:

1. The minimum of the quadratic performance index must be obtained.

2. The dynamic characteristics of the resulting system must be within the area defined by the pilot as desirable."

As noted in the preceding section, the problem then becomes one of achieving a trade-off, in some fashion, between the two criteria. Reynolds and Rynaski[25] offer three techniques for approaching the problem, but all rest on knowing the relationship between control and the performance index. In their case, using Kalman's linear technique, it seemed feasible to select a quadratic performance index such that the optimal control was one yielding pilot acceptance.

Standards for Evaluating Tracking Performance. In the conventional tracking task, there exists frequently the problem of defining goodness of performance. Inconsistencies between conventional performance measures and questions concerning the validity of these measures have not clarified this problem.[21] From the point of view of optimal control theory, a number of interesting questions can be asked. How well can the tracking task be performed? In what ways are the operator's control functions optimal? Can the human operator learn to be optimal with different performance criteria? Objective answers to these questions would be considerably helped if objective criteria for optimal control performance were available coupled with a performance index.

The work of Thomas[26] was directly addressed to this problem based on the Maximum Principle and Dynamic Programming. He concluded that these techniques provided absolute standards "...against which the performance of a human controller can be quantitatively compared." He created a scale of manual control efficiency by scaling operator performance resulting in no control as 0% and performance resulting in optimal control as 100%; thereby the operator's score could be given on this scale as percent effectiveness. Obviously, this requires a quantitative performance index defining optimal control. Measurement techniques of this kind would be necessary to evaluate quantitatively alternate mechanizations of automatic, semi-automatic, and manual control functions.

An Evaluation

The development of modern optimal control theory in the past decade represents a radically new theoretical approach to the description of complex, multi-dimensional, control systems. It provides a framework for the direct attack on nonlinear, higher order, and adaptive control processes, and most important it has as its prime objective the precise statement of optimal control processes.

Within the context of modern optimal control theory, manual control is subsumed as a problem in the mechanization of the control. That is, given the control defined as the optimal, a decision is required as to whether that control will be mechanized by automatic, semi-automatic, or manual control modes. No formal rules have been established on which that decision can be made. Further, only broad hints of possible applications of modern control theory to manual control can be given at this time. It is apparent that much theoretical and empirical work remains to be performed. A very large investment of effort will be required to evaluate whether or not optimal control theory is in detail applicable to the problems of manual control. At this time, it will be a matter for the individual investigator to decide whether or not such an investment is warranted.

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APPENDIX

Pontryagin Maximum Principle: Matrix Form Expanded

The mathematical expressions used in this paper are predominantly in matrix notation. It is important that this fact be recognized for a satisfactory understanding of the optimization techniques discussed here. To illustrate, the major computational steps for the Pontryagin Maximum Principle are shown in generalized matrix notation form.

The state vector (x) and the control vector (u) are shown as arbitrary n and r element column matrices. In practice, each matrix will contain a finite number of elements corresponding to the significant states and control dimensions of the system problem. As an example, analysis of a re-entry vehicle might be noted.

State vector (x). The state vector will consist of important plant characteristics. For a re-entry vehicle these might be:

x^1: angle of attack

x^2: side-slip angle

x^3: pitch

x^4: pitch rate

x^5: temperature

x^6: velocity

and so forth as the problem requires.

Control vector (u). For the same re-entry vehicle, the control vector might consist of:

u_1: elevator deflection

u_2: thrust

u_3: speed brake deflection

and so forth.

Performance index (J). The performance index (J) must be a monotonic-increasing function. Therefore, f^o always must be positive. There are, of course, many performance criteria from which we may select. If performing the control task in minimal time is selected, this corresponds to $f^o = 1$; i.e.,

$$J = \int_{t_o}^{t_f} dt = t_f - t_o$$

As described in the text, the auxiliary equations must then be formed. Step (2), below, shows this matrix equation with general elements. When the indicated multiplication is performed, each row of the matrix takes the form shown.

The function H is next formed (Step 3) by combining the auxiliary equations with the original set of equations in Step 1. Upon multiplication, H becomes a scalar, the summation of the product of corresponding Lagrange multipliers, Ψ, and functions f^{α}.

The Maximum Principle (Step 4) states that an optimal control, u*, from all admissible controls, u, maximizes H.

1. Given (x), (u), (J):

$$x = \begin{bmatrix} x^0 \\ x^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{bmatrix} \qquad f(x,u,t) = \begin{bmatrix} f^0(x,u,t) \\ f^1(x,u,t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f^r(x,u,t) \end{bmatrix}$$

$$\dot{x}^1 = (f^1(x,u,t)$$

$$\dot{x}^2 = f^2(x,u,t)$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\dot{x}^n = f^n(x,u,t)$$

$$J = \int_{t_o}^{t_f} f^0(x,u,t)\ dt \text{ yields } \dot{x}^0 = f^0(x,u,t)$$

2. Form the auxiliary equations:

$$\psi = \begin{bmatrix} \psi_0 \\ \cdot \\ \cdot \\ \cdot\cdot \\ \cdot \\ \psi_n \end{bmatrix} \qquad \begin{bmatrix} \psi_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \psi_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f^0}{\partial x^1} & \frac{\partial f^1}{\partial x^1} & \cdots\cdot & \frac{\partial f^n}{\partial x^1} \\ \frac{\partial f^0}{\partial x^2} & \frac{\partial f^1}{\partial x^2} & \cdot & \cdot \\ & & & \vdots \\ \frac{\partial f^0}{\partial x^n} & \frac{\partial f^1}{\partial x^n} & & \frac{\partial f^n}{\partial x^n} \end{bmatrix} X \begin{bmatrix} \psi_0 \\ \cdot \\ \cdot \\ \cdot\cdot \\ \psi_n \end{bmatrix}$$

$$\psi_i = - \sum_{\alpha=0}^{n} \frac{\partial f}{\partial x^i} \psi_\alpha \qquad i = 0, 1, \ldots n$$

3. Combine 1 and 2 to form function H:

$$H(\psi, x, u) = \psi^T f = \begin{bmatrix} \psi_0, & \ldots\ldots, & \psi_n \end{bmatrix} \begin{bmatrix} f^0(x,u,t) \\ f^1(x,u,t) \\ \cdot \\ \cdot \\ f^n(x,u,t) \end{bmatrix} = \sum_{\alpha=0}^{n} \psi_\alpha \quad f^\alpha(x,u,t)$$

4. If u^* is an optimal control, then

$$H(\psi, x, u^*) = M$$

where M is the maximum of H evaluated over all controls, u .