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Expansion Theorems for Solutions of Two Generalizations of the Heat Equation

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I. INTRODUCTION

In this report we develop expansion theorems for classical solutions of the equations

$$\text{(I)} \qquad \frac{\partial u(x,t)}{\partial t} = (-1)^{j+1} \frac{\partial^q u(x,t)}{\partial x^q}, \quad q = 2j, \quad j = 1,2, \ldots ,$$

and

$$\text{(II)} \qquad \frac{\partial u(r,t)}{\partial t} = (-1)^{j+1} \Delta_n^j \cdot u(r,t), \quad j = 1,2, \ldots, n = 2,3, \ldots .$$

Here, Δ_n denotes the radially symmetric Laplacian operator. The coefficient $(-1)^{j+1}$ is needed to ensure the convergence of certain integrals which arise through the methods we employ. Theorems are developed which relate expansions of classical solutions, in terms of two basic solution sets, to i) a Huygens' principle and ii) the entireness properties of initial data and solution functions. By a Huygens' principle for a solution, we essentially mean that the solution at time $t + t'$ can be obtained from the solution at time t. We will make our meaning more precise later.

We obtain expansion theorems for solutions of Equation I in terms of the two basic solution sets: (i) the set $\{v_q^k(x,t)\}_{k=0}^{\infty}$ of <u>generalized heat polynomials</u>, and (ii) the set $\{w_q^k(x,t)\}_{k=0}^{\infty}$ of <u>associated functions.</u> The polynomials are defined to be the solutions of Equation I which satisfy the initial conditions $v_q^k(x,0) = \frac{x^k}{k!}$, $k = 0,1,2, \ldots$. They are a subset of a special set of polynomials first considered by E. T. Bell [2] and are related to a set investigated more recently by H. W. Gould and A. T. Hopper [10]. The set of associated functions is defined in terms of iterated derivatives of the fundamental solution $K_q(x,t)$ of Equation I. The function $K_q(x,t)$ here is defined by

$$K_q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixs-ts^q}\, ds, \quad t > 0 .$$

The results of J. Steinberg [17] are instrumental in developing the properties of the associated functions. His method of analysis is used to show that the sets $\{v_q^k(x,-t)\}$ and $\{w_q^k(x,t)\}$ are biorthogonal on $|x| < \infty$ for $t > 0$.

We also examine solutions of Equation II which have convergent expansions in terms of either the set $\{R_{j,n}^k(r,t)\}_{k=0}^{\infty}$ of <u>generalized radial heat polynomials</u> or the set $\{S_{j,n}^k(r,t)\}_{k=0}^{\infty}$ of <u>associated radial functions</u>. These sets are analogous to the sets $\{v_q^k(x,t)\}$ and $\{w_q^k(x,t)\}$ defined above. Furthermore, the sets $\{R_{j,n}^k(r,-t)\}$ and $\{S_{j,n}^k(r,t)\}$ form a biorthogonal system on $0 < r < \infty$ for $t > 0$.

For both problems I and II, one of the more important results is the decomposition of the kernel in the Poisson integral representation of solutions in terms of the basic solution sets. For example, it is shown for problem I, that

$$K_q(x-y,t+t') = \sum_{k=0}^{\infty} v_q^k(x,t) w_q^k(y,t') \quad ,$$

in an appropriate time strip. It is this result which allows the passage from the integral representation of a solution to a series representation of the desired type.

Series of polynomials, in general, are shown to be convergent in finite time strips, while series of associated functions converge in half planes. A typical theorem concerning expansions of solutions of Equation I in terms of the generalized heat polynomials is: Let the initial data, $\varphi(z)$, be an entire function which satisfies $|\varphi(z)| \leq B \exp(M|z|^{\frac{q}{q-1}})$. Then the solution, $u(x,t)$, has the series representation

$$u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t)$$

which converges for $|x| < \infty$, $|t| < \frac{1}{qm} \left(\frac{Mq}{q-1}\right)^{1-q}$.

The results are extended in a number of ways to include a larger class of related problems. With suitable restrictions, we find that the admission of strictly increasing time dependent coefficients leads to no significant changes in the theorems. Those results which do not depend on the Huygens' principle remain valid even if we admit continuous time dependent coefficients. Extensions to Euclidean n-space are also considered.

P. C. Rosenbloom and D. V. Widder [16] have made a detailed study of necessary and sufficient conditions for the validity of expanding solutions of the heat equation

$$(1) \qquad \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

in terms of two sets of special solutions. The first of these is the set $\{v_k(x,t)\}_{k=0}^{\infty}$ of heat polynomials. They are given explicitly by $v_k(x,t) = (-t)^{\frac{k}{2}} H_k(\frac{x}{(-4t)^{\frac{1}{2}}})$, $k = 0,1,2, \ldots$, where $H_k(x)$ is the usual Hermite polynomial of degree k. The second is the set $\{w_k(x,t)\}_{k=0}^{\infty}$ of associated functions where $w_k(x,t)$ is the Appell transform [1] of $v_k(x,t)$. The $w_k(x,t)$ are given explicitly by

$$(2) \qquad w_k(x,t) = K(x,t)v_k(\frac{x}{t}, -\frac{1}{t}), \quad k = 0,1,2, \ldots ,$$

where $K(x,t)$ denotes the fundamental solution of (1).

The biorthogonality of the sets $\{v_k(x,-t)\}$ and $\{w_k(x,t)\}$ on $|x| < \infty$ for $t > 0$ follows from the orthogonality relation for the Hermite polynomials. A Huygens' principle and entireness properties of the initial data and solution functions take on an important role in the development of their expansion theorems. A number of procedures are given for determining the coefficients in these expansions.

In a more recent paper, L. R. Bragg [6] has obtained similar results for solutions of the generalized heat equation

$$(3) \qquad \frac{\partial u(r,t)}{\partial t} = \Delta_{\mu} u(r,t)$$

where $\Delta_\mu = \partial^2/\partial r^2 + \frac{\mu-1}{r}\,\partial/\partial r, \quad \mu \geq 2.$

The solutions of Equation (3) are expanded in terms of two basic sets of solutions which are related to the generalized Laguerre polynomials. The entireness properties demanded of the solutions being represented give rise to certain differences in the theorems developed by Bragg and those developed by Rosenbloom and Widder.

A general theory for the expansion of solutions of certain linear homogeneous initial value problems on finite intervals has been developed by W. J. Davis [7]. His approach makes use of basic solution sets obtained through the application of formal solution operators to Boas-Buck type generating functions [4]. The convergence of expansions in terms of these sets is examined by means of the radical test. Both generalized and classical solutions can be represented by series of the indicated type.

There are important differences in the theorems we develop and the theorems of Rosenbloom and Widder and those of Bragg. First, refined asymptotic bounds, as are known for the Hermite and Laguerre polynomials, are not known for the special solution sets we use. We therefore find it necessary to use somewhat imprecise bounds which are obtained from the results of O. A. Ladyzhenskaya [13]. Moreover, we are unable to give an explicit relationship between the sets of special solutions as was done in [6] and [16]. This is because, as far as the author knows, there is no Appell-type transformation associated with Equations I and II when $j = 2, 3, \ldots$.

Finally, we should point out that the expansion theorems we obtain for solutions of Equations I and II are structurally the same. The methods used to prove analogous results preliminary to the expansion theorems are quite different. The corresponding expansion theorems, however, are proved in a similar manner. For these reasons, Sections 1 through 4 treat expansion theorem preliminaries for problem I, Sections 5 through 8 treat analogous results for problem II, and the expansion theorems for both problems are developed jointly in succeeding sections.

II. EXPANSION THEOREM PRELIMINARIES FOR EQUATION I

1. The Fundamental Solution of Equation I.

Consider the initial value problem related to the parabolic Equation I:

$$(1.1) \qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = (-1)^{j+1} \dfrac{\partial^q u(x,t)}{\partial x^q} \\ u(x,0) = \varphi(x), \quad q = 2j, \quad j = 1,2, \ldots . \end{cases}$$

Define the Fourier transform of a function $\varphi(x)$ by

$$\widetilde{\varphi(x)} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \varphi(s) e^{-ixs} ds .$$

Using the method of Fourier transforms, developed by Ladyzhenskaya [13] and I. Gelfand and G. Silov [9], it can be shown that if the initial data satisfies suitable growth conditions, then a solution of (1.1) has the integral representation

$$(1.2) \qquad u(x,t) = \int_{-\infty}^{\infty} K_q(x-y,t)\varphi(y)dy .$$

Moreover, this solution is unique in a class of functions which can be determined by the growth conditions on $\varphi(x)$. The kernel, or fundamental solution, $K_q(x,t)$, is defined by

$$(1.3) \qquad K_q(x,t) = (2\pi)^{-\frac{1}{2}} \widetilde{e^{-tx^q}}, \quad t > 0, \quad |x| < \infty .$$

An alternate, but formal, derivation of the fundamental solution which is more suited to our purposes will now be given. The Heaviside or Mikusinski operational calculus [14] associates with problem (1.1) the formal solution operator $e^{(-1)^{j+1}tD_x^q}$ which we interpret by

$$(1.4) \qquad e^{(-1)^{j+1}tD_x^q} \circ \varphi(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k(j+1)}t^k}{k!} D_x^{kq}\, \varphi(x)$$

whenever a meaning can be attached to the series. In particular, if $\varphi(x)$ is a polynomial, the expression (1.4) defines a classical solution of (1.1).

Assume now that $\varphi(x)$ has a Fourier transform $\widetilde{\varphi(x)}$. Then if $\widetilde{\varphi(x)}$ is absolutely integrable on $|x| < \infty$, we have, by the Fourier integral theorem,

$$\varphi(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(y) e^{-is(x-y)} ds\, dy.$$

A formal application of the solution operator (1.4) yields

$$u(x,t) = e^{(-1)^{j+1}tD_x^q} \circ \varphi(x)$$

$$= \int_{-\infty}^{\infty} \varphi(y) \left\{ (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ts^q - is(x-y)} ds \right\} dy$$

$$= \int_{-\infty}^{\infty} \varphi(y) K_q(x-y,t) dy$$

where, as before, $K_q(x,t) = (2\pi)^{-\frac{1}{2}} \widetilde{e^{-tx^q}}$.

It was this type of formal development which led I. I. Hirschman and D. V. Widder [12] to define the formal differential operator $e^{tD_x^2}$ by means of the Weierstrass transform. That is,

$$e^{tD_x^2} \circ \varphi(x) = \lim_{R \to \infty} \int_{-R}^{R} K(x-y,t)\varphi(y)dy,$$

whenever the limit on the right exists. The function $K(x,t)$ is the fundamental or source solution of the heat equation (1). If $f(x)*g(x)$ denotes the convolution integral

$$\int_{-\infty}^{\infty} f(x-y)g(y)dy,$$

then convolution integrals of the form $K_q(x,t)*\varphi(x)$ are generalized Weierstrass transforms. We will find that many of the important properties of the usual Weierstrass transform, discussed in [12], carry over to our more general situation.

If we denote the inverse Fourier transform of $\varphi(x)$ by $\overbrace{\varphi(x)}$, and observe that $K_q(x,t) = K_q(-x,t)$, we see that (1.3) may also be written

$$K_q(x,t) = (2\pi)^{-\frac{1}{2}} \overbrace{e^{-tx^q}} . \tag{1.5}$$

Furthermore, it is a well known fact that integrals of the form (1.3) represent entire functions of x for $t > 0$.

2. The Generalized Heat Polynomials and Associated Functions

E. T. Bell [2] has defined the polynomials Ψ_n, by

$$\Psi_n \equiv \Psi_n(t,\beta;j), \quad \Psi_{-n-1} \equiv 0, \qquad n = 0,1, \ldots ;$$

$$e^{h^j t} e^{h\beta} \equiv e^{h\Psi},$$

where $j > 0$ is an integer, β denotes an infinite sequence of independent variables, and t is an ordinary variable. Here

$e^{h\Psi}$ is used as a symbolic notation for $\sum_{n=0}^{\infty} \frac{h^n}{n!} \Psi_n(t,\beta;j)$.

Bell was mainly interested in (i) the $\Psi_n(t,\beta;j)$ as functions of t and (ii) certain ordinary differential equations (with respect to t) which these polynomials satisfy.

Gould and Hopper [10], by using the differential operator (1.4), have defined the set of polynomials

$$(2.1) \qquad g_k^q(x,h) = e^{hD_x^q} \cdot x^k, \quad q = 2j, \quad j = 1,2, \ldots, \quad k = 0,1,2, \ldots .$$

It is easily shown that the $g_k^q(x,h)$ are a special case of the $\Psi_n(t,\beta;j)$. The authors were mainly interested in obtaining operational identities connected with the polynomials $g_k^q(x,h)$.

In terms of the $g_k^q(x,h)$, we now define the set $\{v_q^k(x,t)\}_{k=0}^{\infty}$ of <u>generalized heat polynomials</u>, by

$$(2.2) \qquad v_q^k(x,t) = \frac{1}{k!} g_k^q(x,(-1)^{j+1}t), \; q = 2j, \; j = 1,2, \ldots .$$

Since the generalized heat polynomials play an important role in our development, we now summarize a number of their basic properties which follow from Bell's results. The polynomials are given explicitly by

$$(2.3) \qquad v_q^k(x,t) = \sum_{\ell=0}^{[\frac{k}{q}]} \frac{(-1)^{\ell(j+1)} t^{\ell} x^{k-q\ell}}{\ell! \, (k-q\ell)!}, \; q = 2j, \; j = 1,2, \ldots ,$$

and form an Appell set in the sense that

$$\text{(2.4)} \qquad \frac{\partial v_q^k(x,t)}{\partial x} = v_q^{k-1}(x,t), \quad k = 1,2, \ldots .$$

Finally, we have the generating relation

$$\text{(2.5)} \qquad e^{zx + (-1)^{j+1} t z^q} = \sum_{k=0}^{\infty} z^k v_q^k(x,t) \quad .$$

In order to see that the generalized heat polynomials satisfy Equation (1.1), differentiate both sides of (2.5) with respect to t to obtain

$$\sum_{k=0}^{\infty} z^k D_t v_q^k(x,t) = \sum_{\ell=0}^{\infty} (-1)^{j+1} z^{\ell+q} v_q^{\ell}(x,t) \ .$$

A comparison of coefficients of z^k gives

$$\text{(2.6)} \qquad D_t v_q^k(x,t) = (-1)^{j+1} v_q^{k-q}(x,t) = (-1)^{j+1} D_x^q v_q^k(x,t) \quad .$$

The final equality follows by q applications of (2.4).

Of necessity, the associated functions must be obtained in a manner quite different from that in [6] or [16], where the Appell transform was available. For the purpose of developing these functions, let S denote the space of infinitely differentiable functions $f(x)$ on $|x| < \infty$ which, together with all of their derivatives, tend to zero more rapidly than any power of $\frac{1}{|x|}$ as $|x| \longrightarrow \infty$. Let $A(z)$ generate the Appell set of polynomials $\{p_n(x)\}$ in the sense that $A(z)e^{zx} = \sum_{n=0}^{\infty} z^n p_n(x)$ and $D_x p_n(x) = p_{n-1}(x)$. Further, let $A(z)$ be regular in a neighborhood of the origin $(A(0) = 1)$ and let $1/A(iz) \in S$. Steinberg [17] has shown that the function

$q(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{-iyx}}{A(iy)} dy$ is in the space S. Moreover, the functions $q_n(x) = (-1)^n D_x^n q(x)$, $n = 0,1,2, \ldots$, satisfy the bi-orthonormality relation $\int_{-\infty}^{\infty} p_m(x) q_n(x) dx = \delta_{m,n}$. Here, $\delta_{m,n}$ is the usual Kronecker delta.

In view of the above discussion, (2.4), and (2.5), we observe that the set $\{v_q^k(x,-t)\}$ is an Appell set of polynomials generated by $e^{(-1)^j tz^q}$. Moreover, since $q = 2j$, $\left[e^{(-1)^j t(iz)^q}\right]^{-1} = e^{-tz^q}$ is an element of the space S for $t > 0$. Since

$$K_q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ty^q - iyx} dy ,$$

we are led to define the element $w_q^k(x,t)$ of the set $\{w_q^k(x,t)\}_{k=0}^{\infty}$ of associated functions by

(2.7) $\quad w_q^k(x,t) = (-1)^k D_x^k K_q(x,t), \quad t > 0,\ q = 2j, \quad j = 1,2, \ldots .$

It is not, however, immediately clear that the sets $\{v_q^k(x,-t)\}$ and $\{w_q^k(x,t)\}$ form a biorthogonal system on $|x| < \infty$ for $t > 0$. This is because it is not obvious that the integral

$$\int_{-\infty}^{\infty} v_q^k(x,-t) w_q^{\ell}(x,t) dx, \quad t > 0 ,$$

is independent of t for all values of ℓ and k. For this reason, we will give a direct proof of the biorthogonality relation. The proof is essentially Steinberg's [17].

<u>Lemma 2.1.</u> For $-\infty < x < \infty$ and $t > 0$, the sets $\{v_q^k(x,-t)\}$ and $\{w_q^k(x,t)\}$ satisfy the biorthogonality relation

$$(2.8)\qquad \int_{-\infty}^{\infty} v_q^k(x,-t)w_q^\ell(x,t)dx = \delta_{\ell,k}\,,\quad \ell,k = 0,1,2,\ \dots\ .$$

<u>Proof</u>. By the Fourier integral theorem and the fact that $K_q(x,t) = K_q(-x,t)$, we have

$$(2.9)\qquad e^{-ty^q} = \int_{-\infty}^{\infty} K_q(x,t)e^{iyx}dx,\quad t > 0\ .$$

Upon differentiating both sides of (2.9) n-times with respect to y, we obtain the relation

$$(2.10)\qquad \sum_{k=[\frac{n+q-1}{q}]}^{\infty} \frac{(-t)^k(qk)!y^{qk-n}}{k!(qk-n)!} = \int_{-\infty}^{\infty}(ix)^n e^{iyx}K_q(x,t)dx\ .$$

At $y = 0$, (2.10) reduces to

$$(2.11)\qquad \int_{-\infty}^{\infty}(ix)^n K_q(x,t)dx = \begin{cases} 0, & \text{if } n \neq qm \text{ for some integer } m; \\ \dfrac{(-t)^m(qm)!}{m!}, & \text{if } n = qm \text{ for some integer } m. \end{cases}$$

Since $w_q^k(x,t) \in S$, we have, after successive integration by parts,

$$(2.12)\qquad \int_{-\infty}^{\infty} v_q^k(x,-t)w_q^\ell(x,t)dx = \int_{-\infty}^{\infty} D_x^\ell v_q^k(x,-t)K_q(x,t)dx.$$

Clearly, if $\ell > k$, $D_x^\ell v_q^k(x,-t) \equiv 0$ and our result follows. If

$\ell = k$, $v_q^{\ell - k}(x,-t) = 1$, and $\int_{-\infty}^{\infty} K_q(x,t)dx = 1$ by (2.11) with $n = 0$. Finally, if $\ell < k$, we need only show that $K_q(x,t)$ is orthogonal to all $v_q^k(x,-t)$, $k = 1,2, \ldots$. Upon substituting the explicit form (2.3) of the polynomials into the integral on the right in (2.12), we have

$$\int_{-\infty}^{\infty} v_q^k(x,-t)K_q(x,t)dx = \sum_{\ell=0}^{[\frac{k}{q}]} \frac{(-1)^{j\ell} t^{\ell}}{\ell!(k-q\ell)!} \int_{-\infty}^{\infty} x^{k-q\ell} K_q(x,t)dx$$

$$= (-1)^{m(j+1)} \frac{t^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} = 0 \ .$$

We have made use of the fact that the integrals inside the sign of summation vanish unless $k = qm$ for some integer m, by (2.11).

In addition to the explicit form (2.3) of the generalized heat polynomials, we are able to prove the following Poisson integral representations:

Theorem 2.2. For $t \geq 0$, $-\infty < x < \infty$, the generalized heat polynomials have the representation

$$(2.13) \qquad v_q^k(x,t) = K_q(x,t) * \frac{x^k}{k!} = \int_{-\infty}^{\infty} K_q(x-y,t) \frac{y^k}{k!} dy \ .$$

Proof. By the definition of the convolution integral,

$$K_q(x,t) * \frac{x^k}{k!} = \frac{1}{k!}\int_{-\infty}^{\infty} K_q(y,t)(x-y)^k dy$$

$$= \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^\ell \binom{k}{\ell} x^{k-\ell}\int_{-\infty}^{\infty} K_q(y,t)y^\ell dy .$$

The integral under the sign of summation vanishes unless $\ell = qm$, $m = 0,1,2, \ldots,$ by (2.11). Thus, this last sum is equal to

$$\frac{1}{k!}\sum_{m=0}^{[\frac{k}{q}]}\binom{k}{qm} x^{k-qm}\,\frac{(-1)^{m(j+1)}t^m(qm)!}{m!}$$

$$= \sum_{m=0}^{[\frac{k}{q}]}(-1)^{m(j+1)}\,\frac{t^m x^{k-qm}}{m!(k-qm)!} = v_q^k(x,t), \quad \text{by} \quad (2.3).$$

Since the kernel is not defined when $t < 0$, (2.13) does not hold for all t. Nevertheless, we can obtain the following complex representation.

Theorem 2.3. For $0 < t < \infty$, $-\infty < x < \infty$, the generalized heat polynomials have the representation

(2.14) $$v_q^k(x,-t) = \frac{1}{k!}\int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t)(\alpha_2 y)^k dy ,$$

where $\alpha_1 = e^{\frac{i\pi}{q}}$ and $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$.

Proof. The integrand of the integral (2.14) is an entire function of $z = y + \alpha_1 x$. We may, therefore, apply Cauchy's integral theorem to show that

$$\frac{1}{k!}\int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t)(\alpha_2\ y)^k dy = \frac{1}{k!}\int_{-\infty}^{\infty} K_q(y,t)(x+\alpha_2 y)^k dy\ ,$$

for any fixed x. When $(x+\alpha_2 y)^k$ is expanded by the binomial theorem and the resultant series is integrated term by term, the last integral is found to be equal to

$$\frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}x^{k-\ell}\int_{-\infty}^{\infty} K_q(y,t)(\alpha_2 y)^{\ell}\, dy$$

$$= \frac{1}{k!}\sum_{m=0}^{[\frac{k}{q}]}\frac{(-i\alpha_2)^{qm}}{m!}\binom{k}{qm}(qm)!(-t)^m x^{k-qm}$$

$$= \sum_{m=0}^{[\frac{k}{q}]}\frac{(-1)^{m(j+1)}(-t)^m x^{k-qm}}{m!(k-qm)!} = v_q^k(x,-t)\ .$$

We have applied (2.11) in the above reduction.

3. Growth Bounds for Solutions of Equation I

In order to develop the expansion theorems, we need information concerning the behavior of the functions $v_q^k(x,t)$ and $w_q^k(x,t)$ for large k. To this end we make use of some basic results of O. A. Ladyzhenskaya [13]. Her results are discussed below but are restated in a form which fit our immediate needs.

Let $\bar{x}$ denote the point $(x_1,x_2, \ldots, x_n)$ of Euclidean n-space E_n, and let $\bar{x}\circ\bar{y}$ denote the usual scalar product $\sum_{i=1}^{n} x_i y_i$. Let $r = (\bar{x}\circ\bar{x})^{\frac{1}{2}} \equiv ||\bar{x}||$ and let $d\bar{x} = dx_1 dx_2 \ldots dx_n$. Consider the equation

$$(3.1) \qquad \frac{\partial u(\bar{x},t)}{\partial t} = (-1)^{j+1} A(D)\cdot u(\bar{x},t), \qquad j = 1,2, \ldots ,$$

where $A(\bar{y})$ is a polynomial of degree $q = 2j$, $j = 1,2, \ldots$.
Further, let $A(\bar{y})$ satisfy the conditions i) $A(\alpha\bar{y}) = \alpha^q A(\bar{y})$ and

ii) $\inf_{||\bar{y}||=1} A(\bar{y}) > 0$. Let $\mu = q(q-1)^{-1}$ and $\nu = (q-1)^{-1}$.

Define the numbers λ and m (both positive) by

$$(3.2) \qquad \begin{cases} \text{a)} \quad 2\lambda = \inf_{||\bar{y}||=1} A(\bar{y}) \\ \text{b)} \quad -m = \min \operatorname{Re} [A(\bar{\beta} + i\bar{W}) - \lambda||\beta||^q] , \end{cases}$$

the minimum taken over all $\bar{\beta}$ and all $\bar{W}$ satisfying $||\bar{W}|| = 1$.

Let $K(\bar{x},t) = (2\pi)^{-n} \widehat{e^{-tA(\bar{x})}}$ denote the fundamental solution of the equation under consideration.

For $t > 0$ and $\bar{x} \in E_n$,

$$(3.3) \qquad |K(\bar{x},t)| \leq C_1 t^{-\frac{n}{q}} e^{-Cr^{\mu} t^{1-\mu}}$$

where

$$(3.4) \qquad \begin{cases} \text{a)} \quad C = \frac{1}{\mu} (mq)^{-\nu} \\ \text{b)} \quad C_1 = \lambda^{-\frac{n}{q}} (2\pi)^{-n} \int_{E_n} e^{-A(||\bar{y}||)} d\bar{y} . \end{cases}$$

With the same restrictions on t and $\bar{x}$,

$$(3.5) \qquad |D_{\bar{x}}^k K(\bar{x},t)| \leq C_3 t^{-(\frac{k+n}{q})} e^{-Cr^{\mu} t^{1-\mu}} .$$

Here C is given by (3.4a), while

$$(3.6) \qquad C_3 = (2\pi)^{-n} \lambda^{-(\frac{k+n}{q})} \int_{E_n} ||\bar{y}||^k e^{-A(||\bar{y}||)} d\bar{y} .$$

Next, let $|\varphi(\bar{x})| \leq Be^{M||\bar{x}||^{\mu}}$ where B and M are constants. Then for $\bar{x} \in E_n$ and $0 < \alpha < 1$, it results that

$$(3.7) \qquad |K(\bar{x},t) * \varphi(\bar{x})| \leq C_4 \exp\left\{ \frac{Mr^{\mu}}{[1-(\frac{M}{C})^{q-1} \frac{t}{\alpha}]^{\mu-1}} \right\}$$

in the interval $0 \leq t < \alpha (\frac{C}{M})^{q-1}$. The constant C_4 is given by

$$(3.8) \qquad C_4 = BC_1 [C(1-\alpha^{\mu-1})]^{-\frac{n}{\mu}} \int_{E_n} e^{-A(||\bar{y}||)} d\bar{y}$$

while the constants C and C_1 are given by (3.4). Finally, we have

<u>Lemma 3.1</u>. Let $\lambda > 1$, $\mu = q(q-1)^{-1}$, $a \geq 0$, $b \geq 0$, and $q = 2j$, $j = 1,2, \ldots$. Then

$$(3.9) \qquad (a+b)^{\mu} \leq \lambda a^{\mu} + (1-\lambda^{1-q})^{1-\mu} b^{\mu} .$$

The above results apply directly to problem (1.1), where $n = 1$, $A(y) = y^q$, and $\lambda = \frac{1}{2}$. The desired growth bounds for the $v_q^k(x,t)$ and the $w_q^k(x,t)$ will become clear by the following elementary lemmas.

<u>Lemma 3.2</u>. For $t > 0$, $-\infty < x < \infty$,

$$(3.10) \qquad |w_q^k(x,t)| \leq \frac{1}{q\pi} \Gamma(\frac{k+1}{q}) (\frac{1}{\lambda t})^{\frac{k+1}{q}} e^{-C|x|^{\mu} t^{1-\mu}} ,$$

where C is given by (3.4a) and $\mu = q(q-1)^{-1}$.

Proof. Since $\int_{-\infty}^{\infty} |y|^k e^{-|y|^q} dy = \frac{2}{q}\Gamma(\frac{k+1}{q})$ and

$w_q^k(x,t) = (-1)^k D_x^k K_q(x,t)$ by (2.7), the result follows by (3.5).

Lemma 3.3. For $-\infty < x < \infty$, $0 \leq t < \infty$,

$$(3.11) \qquad |v_q^k(x,t)| \leq A \frac{1}{k!} \left[(\frac{t+\delta}{\alpha})(\frac{k}{Ce\mu})^{q-1} \right]^{\frac{k}{q}} e^{C(\frac{\alpha}{\delta})^{\nu}|x|^{\mu}}$$

for any α, $0 < \alpha < 1$, and any $\delta > 0$. Here C is given by (3.4a), $\mu = q(q-1)^{-1}$, $\nu = (q-1)^{-1}$, and A is a constant independent of k.

Proof. Recall that for $t \geq 0$, $v_q^k(x,t) = K_q(x,t) * \frac{x^k}{k!}$,

by (2.13). Further, for any $M > 0$, $|x|^k \leq (\frac{k}{Me\mu})^{\frac{k}{\mu}} e^{M|x|^{\mu}}$ as can be seen by maximizing the function $y = x^k e^{-Mx^{\mu}}$ for $x > 0$. By

(3.7), with $B = \frac{1}{k!}(\frac{k}{Me\mu})^{\frac{k}{\mu}}$, we see that

$$|v_q^k(x,t)| \leq A \frac{1}{k!} (\frac{k}{Me\mu})^{\frac{k}{\mu}} \exp \left\{ \frac{M|x|^{\mu}}{[1-(\frac{M}{C})^{q-1} \frac{t}{\alpha}]^{\mu-1}} \right\}$$

in $0 \leq t < \alpha (\frac{C}{M})^{q-1}$, $|x| < \infty$, for any α, $0 < \alpha < 1$. Here

$A = C_1 [C(1-\alpha^{\mu-1})]^{-\frac{1}{\mu}} \int_{\infty}^{\infty} e^{-y^q} dy$. Upon choosing $M = C(\frac{\alpha}{t+\delta})^{\nu}$,

we find that

$$\left(\frac{k}{Me\mu}\right)^{\frac{k}{\mu}} = \left[\left(\frac{t+\delta}{\alpha}\right)\left(\frac{k}{Ce\mu}\right)^{q-1}\right]^{\frac{k}{q}} \quad \text{and} \quad \frac{M}{[1-(\frac{M}{C})^{q-1}\frac{t}{\alpha}]^{\mu-1}} = C\left(\frac{\alpha}{\delta}\right)^{\frac{1}{q-1}} .$$

The resulting bound (3.11) is then valid for $0 \leq t < \infty$, $0 < \delta < \infty$, and any α, $0 < \alpha < 1$.

In order to obtain a bound on the generalized heat polynomial $v_q^k(x,t)$ when $t < 0$, we first prove the following extension of (3.7).

Lemma 3.4. Let $z = z_1 + iz_2$, z_1 and z_2 real. Let $\varphi(z)$ satisfy $|\varphi(z)| \leq Be^{M|z|^\mu}$ where $|z| = (z_1^2 + z_2^2)^{\frac{1}{2}}$, B and M are constants, and $M > 0$. Then, with $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$,

$$(3.12) \quad \left|\int_{-\infty}^{\infty} K_q(y,t)\varphi(x+\alpha_2 y)dy\right| \leq C_4 \exp\left\{\frac{M|x|^\mu}{[1-(\frac{M}{C})^{q-1}\frac{t}{\alpha}]^{\mu-1}}\right\}$$

for $0 < t < \alpha\left(\frac{C}{M}\right)^{q-1}$ and $0 < \alpha < 1$. The constant C_4 is given by (3.8).

Proof. By hypothesis and (3.3) we have

$$|I| \equiv \left|\int_{-\infty}^{\infty} K_q(y,t)\varphi(x+\alpha_2 y)dy\right|$$

$$\leq \int_{-\infty}^{\infty} BC_1 t^{-\frac{1}{q}} e^{-C|y|^\mu t^{1-\mu} + M|x+\alpha_2 y|^\mu} dy$$

$$\leq BC_1 t^{-\frac{1}{q}} \int_{-\infty}^{\infty} e^{M(|x|+|y|)^\mu - C|y|^\mu t^{1-\mu}} dy.$$

An application of Lemma (3.1) to $(|x|+|y|)^\mu$ in the last member of this inequality gives

$$|I| \le BC_1 t^{-\frac{1}{q}} e^{M(1-\lambda^{1-q})^{1-\mu}|x|^{\mu}} \int_{-\infty}^{\infty} e^{(M\lambda - Ct^{1-\mu})|y|^{\mu}} dy ,$$

for $\lambda > 1$. Let $\lambda = \frac{C}{M}(\frac{t}{\alpha})^{1-\mu}$ where $0 < \alpha < 1$. Then $\lambda > 1$ whenever $0 < t < \alpha (\frac{C}{M})^{q-1}$. For t in this range we find that

$$|I| \le BC_1 t^{-\frac{1}{q}} \left[\exp\left\{ \frac{M|x|^{\mu}}{[1-(\frac{C}{M})^{1-q} \frac{t}{\alpha}]^{\mu-1}} \right\}\right] \int_{-\infty}^{\infty} e^{-Ct^{1-\mu}(1-\alpha^{\mu-1})|y|^{\mu}} dy.$$

The desired result follows by introducing the change of variables $\xi = C^{\frac{1}{\mu}} t^{-\frac{1}{q}} (1-\alpha^{\mu-1})^{\frac{1}{\mu}} y$ in the integral on the right.

<u>Lemma 3.5.</u> For $0 < t < \infty$, $|x| < \infty$, $0 < \alpha < 1$, and $0 < \delta < \infty$,

$$(3.13) \qquad |v_q^k(x,-t)| \le A \frac{1}{k!} \left[(\frac{t+\delta}{\alpha})(\frac{k}{Ce\mu})^{q-1} \right]^{\frac{k}{q}} e^{C(\frac{\alpha}{\delta})^{\nu} |x|^{\mu}} .$$

Here C is given by (3.4a).

<u>Proof</u>. Let $\alpha_1 = e^{\frac{i\pi}{q}}$ and $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$. For $t > 0$,

$$(3.14) \qquad v_q^k(x,-t) = \frac{1}{k!} \int_{-\infty}^{\infty} K_q(y+\alpha_1 x, t)(\alpha_2 y)^k dy$$

$$= \frac{1}{k!} \int_{-\infty}^{\infty} K_q(y,t)(x+\alpha_2 y)^k dy ,$$

by (2.14) and an application of Cauchy's theorem. For any $M > 0$,

$|x+\alpha_2 y|^k \leq (|x|+|y|)^k \leq (\frac{k}{Ce\mu})^{\frac{k}{\mu}} e^{M(|x|+|y|)^\mu}$. Therefore, the hypotheses of Lemma (3.4) are satisfied. The result follows by choosing $M = C(\frac{\alpha}{t+\delta})^{\frac{1}{q-1}}$.

4. A Huygens' Principle and a Kernel Decomposition Theorem for Equation I.

Let us now recall the convolution theorem for Fourier transforms. That is, if $f(x)$ and $g(x)$ belong to $L(-\infty,\infty)$ and are the Fourier transforms of $F(x)$ and $G(x)$, respectively, then $\sqrt{2\pi}\, F(x)\, G(x) \in L(-\infty,\infty)$ and $\sqrt{2\pi}\, \widehat{F(x)\, G(x)} = \widehat{F(x)} * \widehat{G(x)}$. Since $K_q(x,t) = (2\pi)^{-\frac{1}{2}} \widehat{e^{-tx^q}}$, we have,

$$(4.1) \qquad K_q(x,t) * K_q(x,t') = K_q(x,t+t')$$

for $|x| < \infty$, $t > 0$, and $t' > 0$.

In the case of the heat equation, $(q = 2)$, it is known that the source solution $K(x,t)$ satisfies the identity

$$(4.2) \qquad \int_{-\infty}^{\infty} K(y+ix,t_1)\, K(iy-v,t_2)dy = K(x-v,t_2-t_1)$$

whenever $0 < t_1 < t_2$, [12, p. 177]. It is this result which allows Rosenbloom and Widder to obtain the decomposition relation for $K(x,t)$ when t is negative. Our next result extends this identity to the more general kernel $K_q(x,t)$.

Theorem 4.1. Let $\alpha_1 = e^{\frac{i\pi}{q}}$ and $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$. For $0 < t' < t$, $|x| < \infty$, and $|v| < \infty$,

$$(4.3) \qquad \int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t')K_q(\alpha_2 y-v,t)dy = K_q(x-v,t-t') \quad .$$

<u>Proof</u>. By Cauchy's integral theorem and the definition of $K_q(x,t)$,

$$(4.4) \qquad \int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t')K_q(\alpha_2 y-v,t)dy$$

$$= \int_{-\infty}^{\infty} K_q(y,t')K_q(x-v+\alpha_2 y,t)dy$$

$$= \int_{-\infty}^{\infty} K_q(y,t')(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ts^q-i(x-v)s-i\alpha_2 ys} ds\, dy$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ts^q-i(x-v)s} \left\{ \int_{-\infty}^{\infty} K_q(y,t')e^{-i\alpha_2 sy} dy \right\} ds.$$

The interchange of the order of integration is valid since all of the integrals involved are absolutely convergent in view of the bound (3.3) as applied to $K_q(x,t)$. In fact, the integral

$$(4.5) \qquad \int_{-\infty}^{\infty} K_q(y,t')e^{-i\alpha_2 sy}\, dy$$

converges uniformly in $|\mathrm{Im}\,(\alpha_2 s)| \leq p < \infty$ for any p which satisfies the inequality. Therefore, (4.5) represents an analytic function of $\alpha_2 s$ in $|\mathrm{Im}\,(\alpha_2 s)| < \infty$. The theory of analytic continuation shows that this is the only function analytic in the strip which assumes the values $e^{-t'z^q}$ when $z = \alpha_2 s$ is real. The last member of (4.4) then reduces to

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-(t-t')s^q-i(x-v)s} ds \;=\; K_q(x-v,t-t')$$

for $|x| < \infty$, $|v| < \infty$, $0 < t' < t$.

We now define the Huygens' property in agreement with Rosenbloom and Widder.

Definition. A function $u(x,t) \in H_q^*$ in $a < t < b$ if (i) $u(x,t) \in C^q$, (ii) $u(x,t)$ satisfies the equation (2.1), and (iii)

$$u(x,t) = K_q(x,t-t') * u(x,t') , \tag{4.6}$$

the integral converging for all t and t' in $a < t' < t < b$.

By (4.1), $K_q(x,t) \in H_q^*$ for $0 < t < \infty$. We now show that the elements of the sets $\{v_q^k(x,t)\}$ and $\{w_q^k(x,t)\}$ also belong to the class H_q^* in appropriate time strips.

Theorem 4.2. For $|x| < \infty$, $k = 0,1,2, \ldots$,

$$v_q^k(x,t) = K_q(x,t-t') * v_q^k(x,t') \quad \text{for} \quad |t| < \infty \tag{4.7}$$

and

$$w_q^k(x,t) = K_q(x,t-t') * w_q^k(x,t') \quad \text{for} \quad 0 < t < \infty . \tag{4.8}$$

Proof. We prove (4.8) first. By the definition (2.7) of $w_q^k(x,t')$ we have

$$w_q^k(x,t') = (-1)^k D_x^k K_q(x,t') = (2\pi)^{-\frac{1}{2}} \widehat{(-ix)^k e^{-t'x^q}} . \tag{4.9}$$

The last equality follows from the definition (1.3) of $K_q(x,t')$. Since the last integral is absolutely convergent for any $t' > 0$, we may apply the convolution theorem for Fourier transforms to obtain

$$K_q(x,t-t') * w_q^k(x,t') = (2\pi)^{-1} \widehat{e^{-(t-t')x^q}} * \widehat{(-ix)^k e^{-t'x^q}}$$

$$= (2\pi)^{-\frac{1}{2}} \widehat{(-ix)^k\ e^{-tx^q}} = w_q^k(x,t) \quad \text{for} \quad 0 < t' < t < \infty .$$

The proof of (4.7) proceeds in a different manner. In the right side of (4.7), replace $v_q^k(x,t')$ by its explicit form (2.3). Then,

(4.10) $$K_q(x,t-t') * v_q^k(x,t')$$

$$= \int_{-\infty}^{\infty} K_q(x-y,t-t') \left\{ \sum_{\ell=0}^{[\frac{k}{q}]} \frac{(-1)^{\ell(j+1)}(t')^{\ell}\ y^{k-q\ell}}{\ell!\ (k-q\ell)!} \right\} dy$$

$$= \sum_{\ell=0}^{[\frac{k}{q}]} \frac{(-1)^{\ell(j+1)}(t')^{\ell}}{\ell!} v_q^{k-q\ell}(x,t-t'), \qquad q = 2j,\ j = 1,2, \ldots .$$

We have used Theorem (2.13) to obtain the final sum. By (2.6),

(4.11) $$v_q^{k-q\ell}(x,t-t') = (-1)^{\ell(j+1)} D_t^{\ell} v_q^k(x,t-t') .$$

The series (4.10) is, therefore, equal to

$$\sum_{\ell=0}^{[\frac{k}{q}]} \left(\frac{t'}{\ell!}\right)^{\ell} D_t^{\ell} v_q^k(x,t-t')$$

when $t-t' > 0$. This last sum is just the Taylor's series representation of $v_q^k(x,t)$ about t'.

Our next two results will be stated without proof since their proofs are identical to those given in [16] for the case $q = 2$.

<u>Lemma 4.3</u>. If $u(x,t) = \int_{-\infty}^{\infty} K_q(x-y,t)d\alpha(y)$, the integral converging absolutely in $0 < t < c$, then $u(x,t) \in H_q^*$ there.

Lemma 4.4. If $u(x,t) \in H_q^*$ in $a < t < b$ and $v(x,t) \in H_q^*$ in $a < -t < b$, and if

$$\int_{-\infty}^{\infty} |u(x,t)| \int_{-\infty}^{\infty} K_q(x-y,t'-t)|v(y,-t')| dy\, dx < \infty$$

for $a < t < t' < b$, then

$$\int_{-\infty}^{\infty} u(x,t)v(x,-t)dx \tag{4.12}$$

is independent of t for $a < t < b$.

We now have the following kernel decomposition theorem.

Theorem 4.5. For $|x| < \infty$, $|y| < \infty$, and $-\min(t',\lambda t'/m) < t < \lambda t'/m$,

$$K_q(x-y,t+t') = \sum_{k=0}^{\infty} v_q^k(x,t)w_q^k(y,t') \quad . \tag{4.13}$$

The constants λ and m are defined by (3.2a) and (3.2b), respectively.

Proof. We first prove the result in the range $0 \leq t < \frac{\lambda t'}{m}$. By (4.1),

$$K_q(y-x,t'+t) = K_q(y-x,t')* K_q(x,t) \tag{4.14}$$

whenever $t' > 0$ and $t > 0$. Since $K_q(y-x,t')$ is an entire function of x for any $t' > 0$,

$$K_q(y-x,t') = \sum_{k=0}^{\infty} \frac{x^k}{k!} (-1)^k D_y^k K_q(y,t') \tag{4.15}$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} w_q^k(y,t') \quad .$$

Upon substituting this series representation of $K_q(y-x,t')$ into (4.14) we have, formally,

$$K_q(y-x,t'+t) = K_q(x,t) * \sum_{k=0}^{\infty} \frac{x^k}{k!} w_q^k(y,t')$$

$$= \sum_{k=0}^{\infty} \left(K_q(x,t) * \frac{x^k}{k!}\right) w_q^k(y,t') = \sum_{k=0}^{\infty} v_q^k(x,t) w_q^k(y,t') .$$

The reduction to the last series follows from Theorem (2.13), which is valid since $t \geq 0$. A sufficient condition for the validity of the formal term by term integration is

$$(4.16) \qquad \sum_{k=0}^{\infty} |w_q^k(y,t')| \int_{-\infty}^{\infty} K_q(x-s,t) \frac{|s|^k}{k!} ds < \infty .$$

Using the estimates

$$(4.17) \qquad \left(\frac{p}{e}\right)^p \left(\frac{2\pi}{p}\right)^{\frac{1}{2}} < \Gamma(p) < \left(\frac{p}{e}\right)^p \left(\frac{2\pi}{p}\right)^{\frac{1}{2}} e^{\frac{\pi}{24p} + \frac{1}{12p^2}}$$

(see for example, Hille [11, p. 235]) and the bounds (3.10) and (3.11), it follows that series (4.16) is dominated by

$$(4.18) \qquad F(x,y;t,t') \sum_{k=0}^{\infty} \frac{1}{k!} \left[\left(\frac{t+\delta}{\alpha\lambda t'}\right)\left(\frac{k}{Ce\mu}\right)^{q-1}\right]^{\frac{k}{q}} \left(\frac{k+1}{q}\right)^{\frac{k+1}{q} - \frac{1}{2}} \cdot e^{R(k) - \left(\frac{k+1}{q}\right)}$$

for $0 < \alpha < 1$, $0 < \delta < \infty$. In (4.18) ,

$$R(k) = \frac{q\pi}{24(k+1)} + \frac{q^2}{12(k+1)^2} \quad \text{and} \quad F(x,y;t,t')$$

denotes all factors which are independent of k. By applying the ratio test to series (4.18) and using the fact that

$(Ce\mu)^{-\frac{1}{\mu}} = e^{\frac{1-q}{q}} (mq)^{\frac{1}{q}}$, we have

$$\lim_{k \to \infty} \left[\frac{(t+\delta)m}{\alpha\lambda t'}\right]^{\frac{1}{q}} \left(\frac{k+1}{k}\right)^{\frac{k}{\mu}} \left(\frac{k+2}{k+1}\right)^{\frac{k+2}{q} - \frac{1}{2}} e^{R(k+1)-R(k)-1}$$

$$= \left[\frac{(t+\delta)m}{\alpha\lambda t'}\right]^{\frac{1}{q}} .$$

This proves that series (4.16) converges provided $t + \delta < \frac{\alpha\lambda t'}{m}$. Since α is arbitrary in $(0,1)$ and δ is arbitrary in $(0,\infty)$ we may choose α suitably close to 1 and δ sufficiently small to obtain the desired result.

The addition formula (4.1) fails if $t < 0$ and we must proceed differently in order to prove (4.13) in the range $-\min(t', \frac{\lambda t'}{m}) < t < 0$. By Theorem (4.1),

$$(4.19) \qquad K_q(x-y, t'-t) = \int_{-\infty}^{\infty} K_q(s+\alpha_1 x, t) K_q(\alpha_2 s-y, t') ds,$$

for $0 < t < t'$, with $\alpha_1 = e^{\frac{i\pi}{q}}$ and $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$. Upon expanding $K_q(\alpha_2 s-y, t')$ about the point y and substituting the resultant series into the integral (4.19), we find that, formally,

$$(4.20) \qquad K_q(x-y, t'-t)$$

$$= \int_{-\infty}^{\infty} K_q(s+\alpha_1 x, t) \left\{ \sum_{k=0}^{\infty} (-1)^k (D_y^k K_q(y, t')) \frac{(\alpha_2 s)^k}{k!} \right\} ds$$

$$= \sum_{k=0}^{\infty} (-1)^k D_y^k K_q(y,t') \int_{-\infty}^{\infty} K_q(s+\alpha_1 x,t) \frac{(\alpha_2 s)^k}{k!} ds$$

$$= \sum_{k=0}^{\infty} v_q^k(x,-t) w_q^k(y,t') \quad .$$

We have used the definition of $w_q^k(y,t')$ and Theorem (2.3) to obtain the final series. The interchange of the order of summation and integration is valid if

$$(4.21) \qquad \sum_{k=0}^{\infty} |w_q^k(y,t')| \int_{-\infty}^{\infty} |K_q(s+\alpha_1 x,t)| \frac{|\alpha_2 s|^k}{k!} ds < \infty \; .$$

In view of the bounds (3.10) and (3.13) this series is bounded by the series (4.18), which converges in $0 < t < \frac{\lambda t'}{m}$. However, if $\frac{\lambda t'}{m} > t'$, the complex addition formula (4.3) is not valid in this larger interval. We have therefore proved (4.20) only in $0 < t < \min(t', \frac{\lambda t'}{m})$. But this is just (4.13) with $-t$ substituted for t and the proof of the theorem is now complete.

III. EXPANSION THEOREM PRELIMINARIES FOR EQUATION II

5. The Poisson Integral Representation

Let $\bar{x}$ denote the n-vector $(x_1, x_2, \ldots, x_n)$ and let $\bar{x} \cdot \bar{y}$ denote the usual scalar product $\sum_{i=1}^{n} x_i y_i$. Let $r = (\bar{x} \cdot \bar{x})^{\frac{1}{2}}$ and T denote the transformation from Cartesian coordinates $(x_1, x_2, \ldots, x_n)$ to spherical coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{n-1})$ where $0 \le r < \infty$, $0 \le \theta_i \le \pi$, $i = 1,2,\ldots,n-2$, and $0 \le \theta_{n-1} \le 2\pi$. The required transformation is given explicitly by $x_1 = r\text{Cos}\,\theta_1$, $x_i = r\text{Sin}\,\theta_1 \text{Sin}\,\theta_2 \cdots \text{Sin}\,\theta_{i-1}$ $\text{Cos}\,\theta_i$, $i = 2, \ldots, n-1$, and $x_n = r\text{Sin}\,\theta_1 \text{Sin}\,\theta_2 \cdots \text{Sin}\,\theta_{n-1}$. When we say that a function $\psi_1(\bar{x})$ is radially symmetric we mean that under the transformation T: $\psi_1(\bar{x}) \rightarrow \psi(r(\bar{x})) \equiv \psi(r)$.

We now prove a lemma which we will use in this and subsequent sections.

Lemma 5.1. Let $\varphi_1(\bar{x})$ be radially symmetric and, further, assume that $\widetilde{\varphi_1(\bar{x})}$ exists for all $\bar{x} \in E_n$. Then for $n \ge 2$ (an integer)

$$(5.1) \qquad \widetilde{\varphi_1(\bar{x})} = r^{1-\frac{n}{2}} \int_0^\infty \varphi(\xi)\xi^{\frac{n}{2}} J_{\frac{n}{2}-1}(r\xi)d\xi \quad .$$

Here $J_\nu(x)$ denotes the Bessel function of order ν of the first kind.

Proof. Let $\xi = (\bar{s} \cdot \bar{s})^{\frac{1}{2}}$ and introduce the coordinates $(\xi, \theta_1, \theta_2, \ldots, \theta_{n-1})$ into the integral defining $\widetilde{\varphi_1(\bar{x})}$. Then

$$(5.2) \quad \widetilde{\varphi_1(\bar{x})} = (2\pi)^{-\frac{n}{2}} \int_0^\infty \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \varphi(\xi) e^{-i(\bar{x}\cdot\bar{y})} |I| d\theta_1 \cdots d\theta_{n-1} d\xi,$$

where $y = (\xi \operatorname{Cos} \theta_1, \xi \operatorname{Sin} \theta_1 \operatorname{Cos} \theta_2, \ldots, \xi \operatorname{Sin} \theta_1 \operatorname{Sin} \theta_2 \ldots \operatorname{Sin} \theta_{n-1})$ and $|I| = \xi^{n-1} \operatorname{Sin}^{n-2} \theta_1 \operatorname{Sin}^{n-3} \theta_2 \ldots \operatorname{Sin} \theta_{n-2}$ denotes the Jacobian of the transformation T. Integrating first with respect to θ_1, we must evaluate the integral

$$I_1 = \int_0^\pi \operatorname{Sin}^{n-2} \theta_1 \, e^{-i\xi(x_1 \operatorname{Cos} \theta_1 + y_2 \operatorname{Sin} \theta_1)} \, d\theta_1 \quad \text{where}$$

$y_2 = x_2 \operatorname{Cos} \theta_2 + x_3 \operatorname{Sin} \theta_2 \operatorname{Cos} \theta_3 + \ldots + x_n \operatorname{Sin} \theta_2 \ldots \operatorname{Sin} \theta_{n-1}$.

Upon expanding the exponential and integrating term by term, we obtain

$$I_1 = \sum_{\ell=0}^{\infty} (-\xi)^\ell \sum_{m_1=0}^{[\frac{\ell}{2}]} \frac{(ix_1)^{2m_1} (iy_2)^{\ell-2m_1}}{(2m_1)!(\ell-2m_1)!} \beta\left(\frac{2m_1+1}{2}, \frac{\ell-2m_1+n-1}{2}\right).$$

Next, integrate with respect to θ_2. Then we must evaluate the integral

$$I_2 = \int_0^\pi (x_2 \operatorname{Cos} \theta_2 + y_3 \operatorname{Sin} \theta_2)^{\ell-2m_1} \, d\theta_2$$

where $y_3 = x_3 \operatorname{Cos} \theta_3 + \ldots + x_n \operatorname{Sin} \theta_3 \ldots \operatorname{Sin} \theta_{n-1}$. By an application of the binomial theorem and term by term integration, we obtain

$$I_2 = \sum_{m_2=0}^{[\frac{\ell-2m_1}{2}]} \binom{\ell-2m_1}{2m_2} (ix_2)^{2m_2} (iy_3)^{\ell-2(m_1+m_2)} \beta\left(\frac{2m_2+1}{2}, \frac{\ell-2m_1-2m_2+n-1}{2}\right).$$

Through a repeated application of the above procedure we find that (5.2) reduces to

$$(5.3) \quad \widetilde{\varphi_1(\bar{x})} = \left(\tfrac{1}{2}\right)^{\frac{n}{2}-1} \int_0^\infty \varphi(\xi) \xi^{n-1} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \left(\frac{r\xi}{2}\right)^{2\ell}}{\ell! \Gamma(\ell + \frac{n}{2})} \, d\xi .$$

The series in (5.3) is just $(\frac{r\xi}{2})^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r\xi)$ and we obtain

$$\widetilde{\varphi_1(\bar{x})} = r^{1-\frac{n}{2}} \int_0^\infty \varphi(\xi)\xi^{\frac{n}{2}} J_{\frac{n}{2}-1}(r\xi)d\xi \ .$$

Let us now consider the equation

$$(5.4) \qquad \frac{\partial u_1(\bar{x},t)}{\partial t} = (-1)^{j+1} \left(\sum_{i=1}^{n} D_i^2\right)^j \cdot u_1(\bar{x},t), \ j = 1,2,\dots, n=2,3,\dots,$$

where $D_i = \partial/\partial x_i$, $i = 1,2, \dots, n$. Expansion theorems for solutions of (5.4) when $j = 1$ (the n-dimensional heat equation) have been given by Widder [18]. If we assume that $u_1(\bar{x},t)$ is radially symmetric, then under the transformation T, Equation (5.4) transforms into

$$(5.5) \qquad \frac{\partial u(r,t)}{\partial t} = (-1)^{j+1} \Delta_n^j \circ u(r,t), \ j = 1,2, \ \dots, \ n = 2,3, \ \dots,$$

where $\Delta_n = \partial^2/\partial r^2 + \frac{n-1}{r}\,\partial/\partial r$ is the radially symmetric Laplacian operator.

Analogous to our treatment of Equation (1.1), it can be shown that the fundamental solution of Equation (5.4) is given by

$$(5.6) \qquad F_j(\bar{x},t) = (2\pi)^{-\frac{n}{2}} \widetilde{e^{-t(\bar{x}\cdot\bar{x})^j}} \ .$$

The Poisson integral representation for a solution of (5.4), subject to the initial condition $u_1(\bar{x},0) = \varphi_1(\bar{x})$, is given by

$$(5.7) \qquad u_1(\bar{x},t) = \int_{E_n} F_j(\bar{y},t)\varphi_1(\bar{x}-\bar{y})d\bar{y}, \ j = 1,2, \ \dots \ .$$

$$(5.11) \quad \int_0^\infty \varphi(y) y^{\frac{n}{2}} \int_{E_n} e^{-t(\bar{s}\cdot\bar{s})^j - i\bar{x}\cdot\bar{s}} (\bar{s}\cdot\bar{s})^{\frac{1}{2} - \frac{n}{4}} J_{\frac{n}{2}-1}(y(\bar{s}\cdot\bar{s})^{\frac{1}{2}}) d\bar{s}\, dy .$$

The inner integral (5.11) is the n-dimensional Fourier transform of a radially symmetric function. We can now apply Lemma (5.1) to obtain

$$(5.12) \qquad u(r,t) = \int_0^\infty \varphi(y) K_{j,n}(r,y;t) dy .$$

This is the desired Poisson integral representation with kernel

$$(5.13) \qquad K_{j,n}(r,y;t) = y^{\frac{n}{2}} r^{1-\frac{n}{2}} \int_0^\infty s e^{-ts^{2j}} J_{\frac{n}{2}-1}(rs) J_{\frac{n}{2}-1}(ys) ds.$$

The initial data $\varphi(y)$ must be restricted to ensure the convergence of (5.12).

We define the Hankel transform of order ν of a function $\varphi(r)$ by

$$(5.14) \qquad \Phi(r) = \mathcal{H}_\nu(r, \varphi(r)) \equiv \int_0^\infty s\varphi(s) J_\nu(rs) ds .$$

Observe that Lemma (5.1) then gives a direct connection between the n-dimensional Fourier transform of a radially symmetric function $\varphi_1(\bar{x})$ and the Hankel transform of order $\frac{n}{2}-1$ of $\varphi(r)$, $r = (\bar{x}\cdot\bar{x})^{\frac{1}{2}}$. In particular, we should note that by (5.8),

$$(5.15) \qquad F_{j,n}(r,t) = (2\pi)^{-\frac{n}{2}} r^{1-\frac{n}{2}} \mathcal{H}_{\frac{n}{2}-1}(r, r^{\frac{n}{2}-1} e^{-tr^{2j}})$$

for $t > 0$. Further, (5.13) can be written as either

$$(5.16) \qquad K_{j,n}(r,y;t) = y^{\frac{n}{2}} r^{1-\frac{n}{2}} \mathcal{H}_{\frac{n}{2}-1}(y, e^{-ty^{2j}} J_{\frac{n}{2}-1}(ry))$$

or

$$(5.17) \qquad K_{j,n}(r,y;t) = y^{\frac{n}{2}} r^{1-\frac{n}{2}} \mathcal{H}_{\frac{n}{2}-1}(r, e^{-tr^{2j}} J_{\frac{n}{2}-1}(ry)) .$$

Finally, let us note the interchange property

$$(5.18) \qquad K_{j,n}(r,y;t) = (y/r)^{n-1} K_{j,n}(y,r;t) .$$

6. The Generalized Radial Heat Polynomials and Associated Functions

Associated with problem (5.5) is the formal solution operator $e^{(-1)^{j+1} t \Delta_n^j}$ which we will interpret by

$$(6.1) \qquad e^{(-1)^{j+1} t \Delta_n^j} \circ \varphi(r) = \sum_{\ell=0}^{\infty} (-1)^{\ell(j+1)} \frac{t^\ell}{\ell!} (\Delta_n^{\ell j} \circ \varphi(r))$$

whenever we can attach a meaning to the series on the right. We define the generalized radial heat polynomial $R_{j,n}^k(r,t)$ to be the classical solution of (5.5) which satisfies $R_{j,n}^k(r,0) = \frac{r^{2k}}{k!}$. The explicit form of the polynomials is readily obtained by applying the formal operator $e^{(-1)^{j+1} t \Delta_n^j}$ to $r^{2k}/k!$ and using the definition (6.1). Thus,

$$(6.2) \qquad R_{j,n}^k(r,t) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell(j+1)}}{\ell!} t^\ell \left[\Delta_n^{\ell j} \cdot \frac{r^{2k}}{k!} \right]$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell(j+1)}}{\ell!\, k!} t^\ell (j\ell)! \binom{k}{k-\ell j} 2^{2\ell} \left\{ \prod_{i=1}^{\ell j} [2(k-i)+n] \right\} r^{2(k-\ell j)}$$

$$= \Gamma(\tfrac{n}{2}+k) \sum_{\ell=0}^{[\frac{k}{j}]} (-1)^{\ell(j+1)} \frac{(4^j t)^\ell \, r^{2(k-\ell j)}}{\ell!(k-\ell j)!\,\Gamma(\frac{n}{2}+k-\ell j)} .$$

When $j = 1$, (6.2) is, except for a constant factor, the radial heat polynomial $R_k^n(r,t)$ studied by Bragg [6].

Although the polynomials $R_{j,n}^k(r,t)$, $k = 0,1, \ldots$, do not form an Appell set in the usual sense it can be shown by induction that they do satisfy the relation

$$(6.3) \qquad \Delta_n^m R_{j,n}^k(r,t) = \frac{4^m \Gamma(\frac{n}{2} + k)}{\Gamma(\frac{n}{2} + k-m)} R_{j,n}^{k-m}(r,t), \quad m \leq k.$$

In order to see that the $R_{j,n}^k(r,t)$, $k = 0,1, \ldots$, are solutions of Equation (5.5), we differentiate the explicit form (6.2) with respect to t to obtain

$$(6.4) \qquad D_t R_{j,n}^k(r,t)$$

$$= \Gamma(\tfrac{n}{2} + k) \sum_{\ell=0}^{[\frac{k-j}{j}]} \frac{(-1)^{(\ell+1)(j+1)} 4^{j(\ell+1)} t^{\ell} r^{2(k-j-\ell j)}}{\ell!(k-j-\ell j)!\Gamma(\frac{n}{2} + k-j-\ell j)}$$

$$= \frac{(-1)^{j+1} 4^j \Gamma(\frac{n}{2} + k)}{\Gamma(\frac{n}{2} + k-j)} R_{j,n}^{k-j}(r,t) = (-1)^{j+1} \Delta_n^j R_{j,n}^k(r,t) .$$

The final equality here follows from (6.3) by taking $j = m$.

We define the <u>associated function</u> $S_{j,n}^k(r,t)$ by the relation

$$(6.5) \qquad S_{j,n}^k(r,t) = \Delta_n^k \cdot F_{j,n}(r,t), \; r > 0, \; t > 0, \; k = 0,1,2, \ldots .$$

By making use of the well known recurrence relation

$$2\nu \, J_\nu(z) = zJ_{\nu+1}(z) + zJ_{\nu-1}(z) ,$$

it can be shown by induction that

$$(6.6)\qquad \Delta_n^k \cdot [r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r\xi)] = (-1)^k \xi^{2k} r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r\xi) .$$

From definition (6.5), relation (5.8), and (6.6), we then have the more useful determination,

$$(6.7)\quad S_{j,n}^k(r,t) = \Delta_n^k \cdot \left\{ (2\pi)^{-\frac{n}{2}} r^{1-\frac{n}{2}} \int_0^\infty \xi^{\frac{n}{2}} e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(r\xi)d\xi \right\}$$

$$= (-1)^k (2\pi)^{-\frac{n}{2}} r^{1-\frac{n}{2}} \int_0^\infty \xi^{\frac{n}{2}+2k} e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(r\xi)d\xi .$$

The indicated operation can be carried out under the sign of integration since the final integral is absolutely convergent.

Theorem 6.1. For $t > 0$, the sets $\{R_{j,n}^k(r,-t)\}_{k=0}^\infty$ and $\{S_{j,n}^k(r,t)\}_{k=0}^\infty$ satisfy the biorthogonality relation

$$(6.8)\qquad \int_0^\infty W_n(r) R_{j,n}^p(r,-t) S_{j,n}^q(r,t)dr$$

$$= \begin{cases} 0 & \text{if } p \neq q , \\ 2^{2q+\frac{n}{2}-1}\Gamma(\frac{n}{2}+q) & \text{if } q = p . \end{cases}$$

The weight factor $W_n(r)$ is given by

$$(6.9)\qquad W_n(r) = (2\pi)^{\frac{n}{2}} r^{n-1} .$$

Proof. We will prove this result in essentially the same way we proved (2.8). By our definition of the Hankel transform, (6.7) can also be written

$$(6.10)\qquad (-1)^q(2\pi)^{\frac{n}{2}}\, r^{\frac{n}{2}-1} S^q_{j,n}(r,t) = \mathcal{H}_{\frac{n}{2}-1}\left(r, r^{\frac{n}{2}+2q-1} e^{-tr^{2j}}\right) .$$

This implies, by Hankel's inversion formula, that

$$(6.11)\qquad \xi^{2q} e^{-t\xi^{2j}} = \int_0^\infty (-1)^q (2\pi r)^{\frac{n}{2}}\, \xi^{1-\frac{n}{2}} S^q_{j,n}(r,t) J_{\frac{n}{2}-1}(r\xi)dr .$$

By an application of the differential operator

$(\partial^2/\partial\xi^2 + \frac{n-1}{\xi}\,\partial/\partial\xi)^s$ to both sides of (6.11) and an evaluation of the result at $\xi = 0$ we obtain the relation

$$(6.12)\qquad \int_0^\infty r^{2s+n-1} S^q_{j,n}(r,t)dr$$

$$= \begin{cases} 0 & \text{if either } q > s \text{ or } q \le s \text{ and} \\ & s-q \ne jm \text{ for some integer } m, \\ (-1)^{j(m+1)} \dfrac{t^m 2^{2s+\frac{n}{2}-1}}{m!\,(2\pi)^{n/2}}\, s!\,\Gamma(\frac{n}{2}+s) & \text{if} \\ & q \le s \text{ and } s-q = jm \text{ for some integer } m. \end{cases}$$

Now substitute the explicit representation (6.2) of the polynomial $R^p_{j,n}(r,-t)$ into the integral (6.8) to obtain

$$(6.13)\qquad \int_0^\infty W_n(r) R^p_{j,n}(r,-t) S^q_{j,n}(r,t)dr$$

$$= \Gamma(\tfrac{n}{2}+p) \sum_{k=0}^{[\frac{p}{j}]} \frac{(-1)^{kj}(4^j t)^k}{k!(p-kj)!\Gamma(\frac{n}{2}+p-kj)} \int_0^\infty r^{2(p-kj)+n-1} S^q_{j,n}(rt)dr .$$

The result follows by (6.12) and an analysis of the possibilities $p > q$, $p < q$, or $p = q$.

<u>Theorem 6.2</u>. For $0 < r < \infty$, $t \geq 0$, the generalized radial heat polynomials have the representation

$$(6.14)\qquad R^k_{j,n}(r,t) = \int_0^\infty K_{j,n}(r,y;t)\,\frac{y^{2k}}{k!}\,dy, \quad k = 0, 1, 2, \ldots .$$

<u>Proof.</u> By Hankel's inversion formula, (5.16) implies that

$$(6.15)\qquad e^{-t\xi^{2j}}(r\xi)^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(r\xi) = \int_0^\infty K_{j,n}(r,y;t)(y\xi)^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(y\xi)dy.$$

Now, by (6.6),

$$(6.16)\qquad (\partial^2/\partial\xi^2 + \tfrac{n-1}{\xi}\,\partial/\partial\xi)^m \int_0^\infty K_{j,n}(r,y;t)(y\xi)^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(y\xi)dy$$

$$= \int_0^\infty K_{j,n}(r,y;t)(-1)^m y^{1-\frac{n}{2}+2m}\xi^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(y\xi)dy .$$

The last member of (6.16), when evaluated at $\xi = 0$, reduces to

$$(6.17)\qquad (-1)^m(\tfrac{1}{2})^{\frac{n}{2}-1}\frac{1}{\Gamma(\frac{n}{2})}\int_0^\infty K_{j,n}(r,y;t)y^{2m}dy, \quad m = 0,1,2, \ldots .$$

Upon applying the operator $(\partial^2/\partial\xi^2 + \frac{n-1}{\xi}\,\partial/\partial\xi)^m$ to the left member of (6.15) we have

$$(6.18)\qquad (\partial^2/\partial\xi^2 + \tfrac{n-1}{\xi}\,\partial/\partial\xi)^m \cdot e^{-t\xi^{2j}}(r\xi)^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(r\xi)$$

$$= (\tfrac{1}{2})^{\frac{n}{2}-1}\sum_{\ell=0}^{[\frac{m}{j}]}\sum_{k=m-\ell j}^{\infty}\frac{(-1)^{\ell+k}m!\,\Gamma(\frac{n}{2}+\ell j+k)4^{m-2k}}{\ell!\,k!\,\Gamma(\frac{n}{2}+k)\Gamma(\frac{n}{2}+\ell j+k-m)}\binom{\ell j+k}{\ell j+k-m}t^{\ell}r^{2k}\xi^{2(\ell j+k-m)}.$$

At $\xi = 0$, the last member of (6.18) reduces to

$$(6.19)\qquad (\tfrac{1}{2})^{\frac{n}{2}-1}(-1)^m\, m!\,\frac{1}{\Gamma(\frac{n}{2})}R^k_{j,n}(r,t), \quad m = 0,1,2, \ldots .$$

The result follows by equating (6.19) and (6.17).

Using a similar type of reasoning, we can also prove

Theorem 6.3. For $0 < r < \infty$ and $t > 0$, the generalized radial heat polynomials have the representation

$$(6.20) \qquad R^k_{j,n}(r,-t) = \frac{1}{k!}\int_0^\infty K_{j,n}(\alpha_1 r, y; t)(\alpha_2 y)^{2k} dy$$

where $\alpha_1 = e^{\frac{i\pi}{2j}}$ and $\alpha_2 = e^{\frac{(2j-1)i\pi}{2j}}$.

7. Growth Bounds for Solutions of Equation II

The growth bounds for the generalized radial heat polynomials and the associated set $\{S^k_{j,n}(r,t)\}$ can be obtained by using the results of Ladyzhenskaya discussed in Section 3. Recall, by Lemma (5.1), that $F_j(\bar{x},t) = F_{j,n}(r,t)$. We can, therefore, apply Ladyzhenskaya's result (3.3) to obtain

$$(7.1) \qquad |F_{j,n}(r,t)| \leq C_1 t^{-\frac{n}{2j}} e^{-Cr^{\mu}t^{1-\mu}}$$

where $\mu = 2j(2j-1)^{-1}$. The constants C_1 and C are given by (3.4).

By applying Lemma (5.1) to relation (6.7) we see that in Cartesian coordinates $S^k_{j,n}(r,t)$ has the representation

$$(7.2) \qquad S^k_j(\bar{x},t) = (-1)^k(2\pi)^{-\frac{n}{2}} \widehat{(\bar{x}\cdot\bar{x})^k e^{-t(\bar{x}\cdot\bar{x})^j}} .$$

In the same way that Ladyzhenskaya proved (3.5), we can show that

$$(7.3) \qquad |S^k_{j,n}(r,t)| \leq C_5\Gamma(\tfrac{n+2k}{2j})\, (\lambda t)^{-(\frac{n+2k}{2j})} e^{-Cr^{\mu}t^{1-\mu}} .$$

The constant C_5 depends only on n and j while λ is defined by (3.2a).

The needed bound on the polynomial $R^k_{j,n}(r,t)$ can also be obtained from our previous work. For $t \geq 0$ we have

$$R^k_j(\bar{x},t) = \int_{E_n} F_j(\bar{x}-\bar{y},t)\,\frac{(\bar{y}\cdot\bar{y})^k}{k!}\,d\bar{y}\ .$$

Since $|\bar{y}\cdot\bar{y}|^k \leq \left(\frac{2k}{Me\mu}\right)^{\frac{2k}{\mu}} e^{My^\mu}$ for any $M > 0$, $y = (\bar{y}\cdot\bar{y})^{\frac{1}{2}}$, we can reason as in the proof of (3.7) that

$$(7.4)\qquad |R^k_{j,n}(r,t)| \leq A\,\frac{1}{k!}\left[\left(\frac{t+\delta}{\alpha}\right)\left(\frac{2k}{Ce\mu}\right)^{2j-1}\right]^{\frac{k}{j}} e^{C\left(\frac{\alpha}{\delta}\right)^{\frac{1}{2j-1}} r^\mu}$$

for $t \geq 0$, $0 < \delta < \infty$, $0 < \alpha < 1$, and $\mu = 2j(2j-1)^{-1}$. Here C is given by (3.4a) and A is a constant independent of k.

Similarly, we have

$$(7.5)\qquad |R^k_{j,n}(r,-t)| \leq A\,\frac{1}{k!}\left[\left(\frac{t+\delta}{\alpha}\right)\left(\frac{2k}{Ce\mu}\right)^{2j-1}\right]^{\frac{k}{j}} e^{C\left(\frac{\alpha}{\delta}\right)^{\frac{1}{2j-1}} r^\mu}$$

for $t > 0$, $0 < \delta < \infty$, $0 < \alpha < 1$, and $\mu = 2j(2j-1)^{-1}$. Again A is some constant independent of k and C is given by (3.4a).

8. A Huygens' Principle and a Kernel Decomposition Theorem for Equation II

Let us first show that the time translation property holds for the kernel $K_{j,n}(r,y;t)$.

Theorem 8.1. For $r \geq 0$, $y \geq 0$, $t > 0$, and $t' > 0$,

$$(8.1)\qquad K_{j,n}(r,y;t+t') = \int_0^\infty K_{j,n}(r,\xi;t)K_{j,n}(\xi,y;t')d\xi \ .$$

Proof. From (5.13), the definition of $K_{j,n}(\xi,y;t)$, the integral (8.1) is equal to

$$(8.2)\quad \int_0^\infty K_{j,n}(r,\xi;t)\xi^{1-\frac{n}{2}}y^{\frac{n}{2}}\int_0^\infty \eta e^{-t'\eta^{2j}}J_{\frac{n}{2}-1}(y\eta)J_{\frac{n}{2}-1}(\xi\eta)d\eta\, d\xi$$

$$= \int_0^\infty \eta e^{-t'\eta^{2j}}J_{\frac{n}{2}-1}(y\eta)y^{\frac{n}{2}}\left\{\int_0^\infty \xi^{1-\frac{n}{2}}K_{j,n}(r,\xi;t)J_{\frac{n}{2}-1}(\xi\eta)d\xi\right\} d\eta \ .$$

The interchange of order of integration is justified by the absolute convergence of the integrals involved. By Hankel's inversion theorem, as applied to (5.16), we have

$$(8.3)\quad \int_0^\infty \xi^{1-\frac{n}{2}}K_{j,n}(r,\xi;t)J_{\frac{n}{2}-1}(\xi\eta)d\xi = r^{1-\frac{n}{2}}e^{-t\eta^{2j}}J_{\frac{n}{2}-1}(r\eta) \ .$$

The value of the integral (8.3), when substituted into (8.2), yields

$$(8.4)\qquad y^{\frac{n}{2}}r^{1-\frac{n}{2}}\int_0^\infty \eta e^{-(t+t')\eta^{2j}}J_{\frac{n}{2}-1}(y\eta)J_{\frac{n}{2}-1}(r\eta)d\eta$$

$$= K_{j,n}(r,y;t+t') \ .$$

Theorem 8.2. Let $\alpha_1 = e^{\frac{i\pi}{2j}}$ and $\alpha_2 = e^{\frac{(2j-1)i\pi}{2j}}$. Then

$$(8.5)\qquad K_{j,n}(r,y;t'-t) = \int_0^\infty K_{j,n}(\alpha_1 r,\xi;t)K_{j,n}(\alpha_2\xi,y;t')d\xi$$

for $r > 0$, $y > 0$, $t' > 0$, and $0 < t < t'$.

Proof. Upon substituting the integral representation (5.13) of $K_{j,n}(\alpha_2\xi,y;t')$ into the integral (8.5), we obtain,

$$(8.6)\quad \int_0^\infty K_{j,n}(\alpha_1 r,\xi;t)(\alpha_2\xi)^{1-\frac{n}{2}}y^{\frac{n}{2}}\int_0^\infty \eta e^{-t'\eta^{2j}}J_{\frac{n}{2}-1}(y\eta)J_{\frac{n}{2}-1}(\alpha_2\xi r)d\eta\, d\xi$$

$$=\int_0^\infty \eta e^{-t'\eta^{2j}}y^{\frac{n}{2}}J_{\frac{n}{2}-1}(y\eta)\int_0^\infty(\alpha_2\xi)^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(\alpha_2\xi\eta)K_{j,n}(\alpha_1 r,\xi;t)d\xi\, d\eta \quad .$$

The interchange of the order of integration is valid since all integrals are absolutely convergent. The inner integral of the second member of (8.6) can be evaluated as follows. Substitute the series expansion for the Bessel function into the integrand and formally integrate term by term. By (6.3), we obtain,

$$(8.7)\quad \int_0^\infty(\alpha_2\xi)^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(\alpha_2\xi\eta)K_{j,n}(\alpha_1 r,\xi;t)d\xi$$

$$=\left(\frac{\eta}{2}\right)^{\frac{n}{2}-1}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\eta}{2}\right)^{2k}}{\Gamma\left(\frac{n}{2}+k\right)}R^k_{j,n}(r,-t) \quad .$$

The term by term integration will be valid if

$$(8.8)\quad \sum_{k=0}^{\infty}\frac{1}{\Gamma\left(\frac{n}{2}+k\right)}\left(\frac{\eta}{2}\right)^{2k}\frac{1}{k!}\left|\int_0^\infty K_{j,n}(\alpha_1 r,\xi;t)\xi^{2k}d\xi\right|<\infty \quad .$$

By (7.5) and the ratio test, (8.8) converges absolutely for all $\eta > 0$ and $t > 0$. If we now substitute the explicit form (6.2) of the $R^k_{j,n}(r,-t)$ into the second member of (8.7), we have

$$(8.9)\quad (\tfrac{\eta}{2})^{\frac{n}{2}-1}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{\eta}{2})^{2k}}{\Gamma(\frac{n}{2}+k)}\Gamma(\tfrac{n}{2}+k)\sum_{\ell=0}^{[\frac{k}{j}]}\frac{(-1)^{\ell(j+1)}(-4^j t)^{\ell}\, r^{2(k-\ell j)}}{\ell!\,\Gamma(\frac{n}{2}+k-\ell j)(k-\ell j)!}$$

$$= (\tfrac{\eta}{2})^{\frac{n}{2}-1}\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+\ell j}(\eta/2)^{2(k+\ell j)}(-1)^{\ell(j+1)}(-4^j t)^{\ell} r^{2k}}{\ell!\,k!\,\Gamma(\frac{n}{2}+k)}$$

$$= r^{1-\frac{n}{2}} e^{t\eta^{2j}} J_{\frac{n}{2}-1}(r\eta) .$$

The second member of (8.7) is, therefore, equal to

$$(8.10)\quad y^{\frac{n}{2}} r^{1-\frac{n}{2}} \int_0^{\infty} \eta e^{-(t'-t)\eta^{2j}} J_{\frac{n}{2}-1}(y\eta) J_{\frac{n}{2}-1}(r\eta)\, d\eta$$

$$= K_{j,n}(r,y;t'-t) , \qquad 0 < t < t' .$$

For problem (5.5) we will define the Huygens' principle as follows: A function $u(r,t)$ belongs to the class H_j^* in $a < t < b$ if (i) $u(r,t) \in C^q$, (ii) $u(r,t)$ satisfies Equation (6.5), and (iii) if

$$(8.11)\qquad u(r,t) = \int_0^{\infty} K_{j,n}(r,y;t-t')\, u(y,t')dy,$$

the integral converging for $a < t' < t < b$.

<u>Theorem 8.3</u>. For $r > 0$, $y > 0$, and $k = 0,1,2, \ldots,$

$$(8.12)\qquad R_{j,n}^k(r,t) = \int_0^{\infty} K_{j,n}(r,y;t-t')R_{j,n}^k(y,t')dy, \; |t| < \infty ,$$

and

$$(8.13)\qquad S_{j,n}^k(r,t) = \int_0^{\infty} K_{j,n}(r,y;t-t')R_{j,n}^k(y,t')\, dy, 0 < t < \infty .$$

Proof. From the definition (6.5) of $S^k_{j,n}(y,t')$, the integral (8.13) is equal to

$$(8.14)\quad \int_0^\infty K_{j,n}(r,y;t-t')\frac{(-1)^k}{(2\pi)^{\frac{n}{2}}}y^{1-\frac{n}{2}}\int_0^\infty \xi^{\frac{n}{2}+2k}e^{-t'\xi^{2j}}J_{\frac{n}{2}-1}(y\xi)d\xi\, dy$$

$$= \int_0^\infty \xi^{\frac{n}{2}+2k}e^{-t'\xi^{2j}}\frac{(-1)^k}{(2\pi)^{\frac{n}{2}}}\int_0^\infty y^{1-\frac{n}{2}}K_{j,n}(r,y;t-t')J_{\frac{n}{2}-1}(y\xi)dy\, d\xi .$$

The interchange of the order of integration is justified by absolute convergence. The inner integral of the second member of (8.14) can be evaluated by Hankel's integral theorem, as applied to (5.16), to give the desired result.

The proof of (8.12) differs but slightly from the proof of (4.7). The next two results can be proved in an obvious manner.

Lemma 8.4. If $u(r,t) \in H^*_j$ in $a < t < b$ and $v(r,t) \in H^*_j$ in $a < -t < b$, and if

$$(8.15)\quad \int_0^\infty |u(r,t)| \int_0^\infty y^{n-1}K_{j,n}(y,z;t'-t)|v(z,-t')|dz\, dr < \infty$$

in $a < t < t' < b$, then

$$(8.16)\quad \int_0^\infty r^{n-1}u(r,t)v(r,-t)dr$$

is constant in $a < t < b$.

Lemma 8.5. If $u(r,t) = \int_0^\infty K_{j,n}(r,y;t)d\alpha(y)$, the integral converging absolutely in $0 < t < c$, then $u(r,t) \in H^*_j$ there.

We are now in a position to examine the decomposition relation for $K_{j,n}(r,y;t)$. We will have need of the following lemma which is, in fact, a special case of the theorem we wish to prove.

Lemma 8.6. For $t > 0$, $y > 0$,

$$(8.17)\quad K_{j,n}(r,y;t) = (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{\ell=0}^{\infty} (\tfrac{r}{2})^{2\ell} \frac{S_{j,n}^{\ell}(y,t)}{\ell!\,\Gamma(\frac{n}{2}+\ell)} .$$

The function $W_n(y)$ is given by (6.9).

Proof. In the integral (5.13) for $K_{j,n}(r,y;t)$, substitute the series representation of $J_{\frac{n}{2}-1}(r\xi)$. Then, integrating term by term,

$$(8.16)\quad K_{j,n}(r,y;t) = r^{1-\frac{n}{2}} y^{\frac{n}{2}} \int_0^{\infty} \xi e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(y\xi) J_{\frac{n}{2}-1}(r\xi)\, d\xi$$

$$= r^{1-\frac{n}{2}} y^{\frac{n}{2}} \int_0^{\infty} \xi e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(y\xi) (\tfrac{r\xi}{2})^{\frac{n}{2}-1} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} (\frac{r\xi}{2})^{2\ell}}{\ell!\,\Gamma(\frac{n}{2}+\ell)}\, d\xi$$

$$= (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{\ell=0}^{\infty} (\tfrac{r}{2})^{2\ell} \frac{S_{j,n}^{\ell}(y,t)}{\ell!\,\Gamma(\frac{n}{2}+\ell)} .$$

The formal term by term integration is valid if

$$(8.17)\quad \sum_{\ell=0}^{\infty} (\tfrac{r}{2})^{2\ell} \frac{1}{\ell!\,\Gamma(\frac{n}{2}+\ell)} \left| S_{j,n}^{\ell}(y,t) \right| < \infty .$$

The series (8.17) is observed to be dominated by

$$(8.18)\quad C_5 t^{-\frac{n}{2j}} e^{-Cy^{\mu} t^{1-\mu}} \sum_{\ell=0}^{\infty} \left(\frac{r^2}{\lambda t^{1/j}}\right)^{\ell} \frac{\Gamma(\frac{n+2\ell}{2j})}{\ell!\,4^{\ell}\,\Gamma(\frac{n}{2}+\ell)},$$

by (7.3). By the ratio test, the series (8.18) converges for all $r > 0$ and $t > 0$.

We should observe that r may be complex in the above proof. The series (8.16) then has an infinite radius of convergence.

Theorem 8.7. For $r > 0$, $y > 0$, and

$-\min(\frac{\lambda t'}{m}, t') < t < \frac{\lambda t'}{m}$,

$$K_{j,n}(r,y;t+t') = (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{\ell=0}^{\infty} \frac{R_{j,n}^{\ell}(r,t) S_{j,n}^{\ell}(y,t')}{4^{\ell}\,\Gamma(\frac{n}{2}+\ell)} . \tag{8.19}$$

The constants λ and m are defined by (3.2a) and (3.2b), respectively.

Proof. For $0 \le t < t'$, we have, by (8.1) and Lemma (8.6),

$$\begin{aligned} K_{j,n}(r,y;t+t') &= \int_0^{\infty} K_{j,n}(r,\xi;t) K_{j,n}(\xi,y;t')\,d\xi \\ &= \int_0^{\infty} K_{j,n}(r,\xi;t) (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{\ell=0}^{\infty} (\tfrac{\xi}{2})^{2\ell} \frac{S_{j,n}^{\ell}(y,t')}{\ell!\,\Gamma(\frac{n}{2}+\ell)}\,d\xi \\ &= (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{\ell=0}^{\infty} \frac{R_{j,n}^{\ell}(r,t) S_{j,n}^{\ell}(y,t')}{4^{\ell}\,\Gamma(\frac{n}{2}+\ell)} . \end{aligned} \tag{8.20}$$

For $t < 0$ we will use the complex addition formula (8.5) and Lemma (8.6) to obtain

$$\begin{aligned} K_{j,n}(r,y;t'-t) &= \int_0^{\infty} K_{j,n}(\alpha_1 r,\xi;t) K_{j,n}(\alpha_2\xi,y;t')\,d\xi \\ &= (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \int_0^{\infty} K_{j,n}(\alpha_1 r,\xi;t) \sum_{\ell=0}^{\infty} (\tfrac{\alpha_2 r}{2})^{2\ell} \frac{S_{j,n}^{\ell}(y,t')}{\ell!\,\Gamma(\frac{n}{2}+\ell)}\,d\xi \\ &= (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{\ell=0}^{\infty} \frac{R_{j,n}^{\ell}(r,t) S_{j,n}^{\ell}(y,t')}{4^{\ell}\,\Gamma(\frac{n}{2}+\ell)}, \text{ in } 0 < t < t' . \end{aligned} \tag{8.21}$$

The proof can be completed in the usual way by using the bounds (7.3), (7.4), (7.5), and an application of the ratio test.

IV. EXPANSIONS OF SOLUTIONS IN TERMS OF THE BASIC SETS

9. Representations in Terms of the Polynomials

In this section we will examine series of the form:

$$(9.1)\quad \begin{cases} \text{I.} \quad u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t) \,, \\ \\ \text{II.} \quad u(r,t) = \sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t) \,. \end{cases}$$

The designations I and II refer here, and throughout the remainder of the report, to the problems (I) and (II), respectively. Analogous results for the two problems will be stated in a single theorem by using this notation. We will prove only one of the results contained in each theorem. The proof of the second result is entirely similar.

We now prove a lemma that will be of frequent use.

Lemma 9.1. I. If $u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t)$ converges at (x_0, t_0') where $x_0 > 0$, $t_0' = (-1)^{j+1}\, t_0$, $t_0 > 0$, then

$$a_k = O[k!\left(\frac{e^{q-1}}{qt_0 k^{q-1}}\right)^{\frac{k}{q}}] \quad \text{as} \quad k \longrightarrow \infty, \quad q = 2j \,.$$

II. If $u(r,t) = \sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t)$ converges at (r_0, t'_0) where $r_0 > 0$, $t_0' = (-1)^{j+1} t_0$, $t_0 > 0$, then

$$b_k = O[\left(\frac{e^{j-1}}{jt_0 4^j k^{j-1}}\right)^{\frac{k}{j}}] \quad \text{as} \quad k \longrightarrow \infty, \quad j = 1,2, \ldots \,.$$

Proof. We prove I. Define the polynomial $\bar{v}_q^k(x,t)$ by the relation

$$(9.2) \qquad \bar{v}_q^k(x,t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} v_q^k(x,t) \quad .$$

Then $\sum_{k=0}^{\infty} a_k v_q^k(x,t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \bar{v}_q^k(x,t)$. By (2.3) and (9.2), we see that at the point (x_0,t_0')

$$(9.3) \qquad \bar{v}_q^k(x_0,t_0') = \sum_{\ell=0}^{[\frac{k}{q}]} \frac{k!t_0^\ell x_0^{k-q\ell}}{\ell!(k-q\ell)!}, \quad q = 2j,\ j = 1,2,\dots \ .$$

Since this is a sum of positive terms,

$$(9.4) \qquad \bar{v}_q^{mq+s} \geq \frac{t_0^m x_0^q (mq+s)!}{m!\ s!}$$

for $m = 0,1,2,\ \dots,$ and $s = 0,1,2,\ \dots,\ q-1$. By hypothesis, for all k sufficiently large and a suitable constant M, $|\frac{a_k}{k!}| \leq \frac{M}{|\bar{v}_q^k(x,t)|}$.

We may now use Stirling's formula in (9.4) to obtain

$$(9.5) \qquad |\frac{a_{mq+s}}{(mq+s)!}| \leq \frac{Ms!}{x_0^q} (\frac{m!}{t_0^m(mq+s)!})$$

$$\leq M^* (\frac{m}{t_0 e})^{m + \frac{s}{q}} (\frac{e^q}{(mq+s)^q})^{m + \frac{s}{q}}$$

$$\leq M^* (\frac{e^{q-1}}{qt_0(mq+s)^{q-1}})^{\frac{mq+s}{q}}, \quad M^* = M^*(x_0,q) \ .$$

By setting $k = qm+s$, (9.5) reduces to

$$(9.6) \qquad |a_k| \leq M^* k! (\frac{e^{q-1}}{qt_0 k^{q-1}})^{\frac{k}{q}},$$

for all k sufficiently large.

M. Bocher [5] and Rosenbloom and Widder [16] discuss the fact that, in general, a series of heat polynomials converges in a strip $|t| < \sigma$. This behavior was also noticed by Bragg [6] for series of radial heat polynomials. Our next result shows that this is also true for series of the form (9.1I) and (9.1II).

Lemma 9.2. I. If $\limsup_{k \to \infty} |a_k|^{q/k} (\frac{qe}{k}) = \frac{1}{\sigma m} < \infty$,

then $\sum_{k=0}^{\infty} a_k v_q^k(x,t)$ converges absolutely for $|t| < \sigma$.

II. If $\limsup_{k \to \infty} |b_k|^{\frac{j}{k}} 4j(\frac{4k}{e})^{j-1} = \frac{1}{\sigma m} < \infty$,

then $\sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t)$ converges absolutely for $|t| < \sigma$. In both I and II above, m is the constant given by (3.2b).

Proof. Let us prove the second part of this Lemma. By hypothesis, for any θ, $0 < \theta < 1$, there exists an integer $N = N(\theta)$ such that for all $k \geq N$, $|b_k| \leq (\frac{e^{j-1}}{j 4^j \sigma m \theta k^{j-1}})^{\frac{k}{j}}$. For $t \geq 0$ we can use the bound (7.4) and the left half of inequality (4.17) to show that

$$\sum_{k=N}^{\infty} |b_k| \; |R_{j,n}^k(r,t)| \leq A_1 \sum_{k=N}^{\infty} (\frac{t+\delta}{\alpha\sigma\theta})^{\frac{k}{j}} (\frac{1}{2\pi k})^{\frac{1}{2}} \quad \text{for } 0 < \alpha < 1,$$

$0 < \delta < \infty$, and $0 < \theta < 1$. By the ratio test, the last series converges for $0 \leq t < \sigma$.

The proof of Lemma (9.2II) for $t < 0$ follows in precisely the same way. This is due to the similarity in the bounds on the polynomials when $t < 0$.

<u>Theorem 9.3</u>. I. If $u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t)$ converges for $|x| < \infty$, $|t| < \sigma$, then $u(x,t)$ satisfies Equation (1.1) for $|x| < \infty$ and $|t| < \min(\sigma,\frac{\sigma}{m})$.

II. If $u(r,t) = \sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t)$ converges for $0 < r < \infty$, $|t| < \sigma$, then $u(r,t)$ satisfies Equation (5.5) for $0 < r < \infty$ and $|t| < \min(\sigma,\sigma/m)$. The constant m is given by (3.2b).

<u>Proof</u>. We will prove Theorem (9.3I). If we formally differentiate $u(x,t)$ term by term, we obtain (by (2.4))

(9.6) $$D_x^q u(x,t) = \sum_{k=0}^{\infty} a_{k+q} v_q^k(x,t) \ ,$$

while (by (2.6))

(9.7) $$D_t u(x,t) = (-1)^{j+1} \sum_{k=0}^{\infty} a_{k+q} v_q^k(x,t), \ q = 2j, \ j = 1,2,\dots \ .$$

The proof will be complete when we show that, for any $\ell = 0,1, \dots, q$, the series

(9.8) $$\sum_{k=0}^{\infty} a_{k+\ell} v_q^k(x,t)$$

converges uniformly in every closed rectangle $(|t| \leq T, |x| \leq R)$ contained in the strip $|t| < \min(\sigma,\sigma/m)$.

Choose t_0 to satisfy $0 < t_0 < \sigma$. Then, by Lemma (9.1I) $a_k = O[k!(\frac{e^{q-1}}{qt_0k^{q-1}})^{\frac{k}{q}}]$ as $k \longrightarrow \infty$. Now choose T so that $0 < T < \min(t_0, \frac{t_0}{m})$ and let R be any finite positive number. Then, by (3.11) and (3.13),

$|v_q^k(x,t)| \leq A \frac{1}{k!} e^{C(\frac{\alpha}{\delta})^{\nu} R^{\mu}} \left[(\frac{T+\delta}{\alpha})(\frac{k}{Ce\mu})^{q-1} \right]^{\frac{k}{q}}$ in the closed rectangle $|t| \leq T$, $|x| \leq R$. Here, A is a constant independent of k while α and δ are arbitrary in $(0,1)$ and $(0,\infty)$, respectively. By the M-test, series (9.8) will, therefore, converge uniformly wherever

$$(9.9) \qquad \sum_{k=N}^{\infty} \left(\frac{e^{q-1}}{qt_0(k+\ell)^{q-1}\theta}\right)^{\frac{k+\ell}{q}} \left(\frac{(k+\ell)!}{k!}\right)\left(\frac{T+\delta}{\alpha}\right)^{\frac{k}{q}} \left(\frac{k}{Ce\mu}\right)^{\frac{k}{q}(q-1)}$$

converges. We have used the fact that for any θ, $0 < \theta < 1$, there is an integer $N = N(\theta)$ such that $|a_k| \leq k!\left(\frac{e^{q-1}}{qt_0k^{q-1}\theta}\right)^{\frac{k}{q}}$ for all $k \geq N$. The ratio of the $(k+1)^{st}$ to the k^{th} term of (9.9) is observed to be

$\left(\frac{(T+\delta)m}{\alpha\theta\, t_0}\right)^{\frac{1}{q}} \left(\frac{k+1}{k}\right)^{\frac{k}{\mu}} \left(\frac{k+\ell}{k+\ell+1}\right)^{\frac{k+\ell}{\mu}} \left(\frac{k+\ell+1}{k+1}\right)^{\frac{1}{q}}$, $\mu = q(q-1)^{-1}$. The limit of this ratio as $k \longrightarrow \infty$ is $\left(\frac{m(T+\delta)}{\alpha t_0\theta}\right)^{\frac{1}{q}}$, which is less than one provided $(T+\delta) < (\alpha t_0\theta)m^{-1}$. Upon choosing δ arbitrarily close to zero and α and θ sufficiently close to one, we see that series (9.9) converges if $T < \frac{t_0}{m}$. Since $T < \min(t_0, \frac{t_0}{m}) \leq \frac{t_0}{m}$, we have shown that (9.8) converges uniformly in $|t| \leq T < \min(t_0, \frac{t_0}{m})$, $|x| \leq R$. The desired result now follows by allowing t_0 to approach σ.

The next theorem relates the Huygens' principle to expansions of the form (9.1I) and (9.1II).

Theorem 9.4. I. If $u(x,t) \in H_q^*$ in $|t| < \sigma$ and if

$$\int_{-\infty}^{\infty} |u(y,t')| \, e^{-C|y|^{\mu}|t'|^{1-\mu}} \, dy \tag{9.10}$$

converges for $-\min(\sigma, \frac{\lambda\sigma}{m}) < t' < \frac{\lambda\sigma}{m}$, then

$u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t)$ for $-\min(\sigma, \frac{\lambda\sigma}{m}) < t < \frac{\lambda\sigma}{m}$.

The coefficient a_k has the determination,

$$a_k = \int_{-\infty}^{\infty} u(y,-t) w_q^k(y,t) dy. \tag{9.11}$$

Moreover, if $u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t)$ converges for $|t| < \sigma$,

then $u(x,t) \in H_q^*$ in $|t| < \min(\sigma,\sigma/m)$.

II. If $u(r,t) \in H_j^*$ for $|t| < \sigma$ and if

$$\int_{0}^{\infty} |u(y,t')| W_n(y) e^{-C|y|^{\mu}|t'|^{1-\mu}} \, dy \tag{9.12}$$

converges for $-\min(\sigma,\frac{\lambda\sigma}{m}) < t' < \frac{\lambda\sigma}{m}$, then

$u(r,t) = \sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t)$ for $-\min(\sigma,\frac{\lambda\sigma}{m}) < t < \frac{\lambda\sigma}{m}$.

The coefficient b_k has the determination

$$\int_{0}^{\infty} \left(\tfrac{1}{2}\right)^{\frac{n}{2}-1} W_n(y) \, \frac{S_{j,n}^k(y,t)}{4^k \Gamma(\frac{n}{2}+k)} \, u(y,-t) dy. \tag{9.13}$$

Here, $W_n(y)$ is given by (6.9). Moreover, if $u(r,t) = \sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t)$

converges for $|t| < \sigma$, then $u(r,t) \in H_j^*$ for $|t| < \min(\sigma,\frac{\sigma}{m})$.

Proof. We will prove Theorem (9.4II). Let us suppose that in $|t| < \sigma$,

$$(9.14) \qquad u(r,t) = \int_0^\infty K_{j,n}(r,y;t-t')u(y,t')dy.$$

Choose t' so that $-\sigma < t' < 0$. Then, by Theorem (8.7),

$$K_{j,n}(r,y;t-t') = (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{k=0}^{\infty} \frac{R^k_{j,n}(r,t)S^k_{j,n}(y,-t')}{4^k\,\Gamma(\frac{n}{2}+k)}$$

for $-\min(|t'|, \frac{\lambda|t'|}{m}) < t < \frac{\lambda|t'|}{m}$. In this interval,

$$(9.15) \qquad u(r,t) = \int_0^\infty u(y,t')(\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \sum_{k=0}^{\infty} \frac{R^k_{j,n}(r,t)S^k_{j,n}(y,-t')}{4^k\Gamma(\frac{n}{2}+k)} dy$$

$$= \sum_{k=0}^{\infty} \{ \int_0^\infty (\tfrac{1}{2})^{\frac{n}{2}-1} W_n(y) \frac{S^k_{j,n}(y,-t')u(y,t')}{4^k\Gamma(\frac{n}{2}+k)} dy \} R^k_{j,n}(r,t)$$

$$= \sum_{k=0}^{\infty} b_k R^k_{j,n}(r,t) \; .$$

The formal term by term integration is valid if

$$(9.16) \qquad C_5'(\tfrac{1}{2})^{\frac{n}{2}-1} \{ \int_0^\infty W_n(y)|u(y,t')| e^{-C|y|^\mu |t'|^{1-\mu}} dy \} \, S < \infty$$

where

$$(9.17) \qquad S = \sum_{k=0}^{\infty} |R^k_{j,n}(r,t)| \Gamma(\tfrac{n+2k}{2j}) \frac{(\lambda|t'|)^{-(\frac{n+2k}{2j})}}{4^k\Gamma(\frac{n}{2}+k)} \; .$$

The integral (9.16) converges by hypothesis. The series S is dominated by

$$(9.18) \qquad A_1 \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n+2k}{2j})(\lambda|t'|)^{-(\frac{n+2k}{2j})}}{4^k\Gamma(\frac{n}{2}+k)k!} (\tfrac{|t|+\delta}{\alpha})^{\frac{k}{j}} (\tfrac{2k}{Ce\mu})^{\frac{k(2j-1)}{j}} \quad ,$$

by (7.4). We have also used (7.4) to obtain (9.16) and (9.17). By the ratio test, series (9.18) is observed to converge for

$|t| < \frac{\lambda|t'|}{m}$. If $\frac{\lambda}{m} > 1$, however, the kernel decomposition theorem (8.7) is not valid in so large an interval. Thus, we have obtained the desired result in $-\min(|t'|, \frac{\lambda|t'|}{m}) < t < \frac{\lambda|t'|}{m}$. The coefficient b_k was obtained in the course of the proof. That b_k is independent of t follows from the fact that both $u(y,-t)$ and $S^k_{j,n}(y,t)$ are in the class H^*_j and therefore satisfy Lemma (8.4).

In order to prove the second part of Theorem (9.4II), suppose that $u(r,t) = \sum_{k=0}^{\infty} b_k R^k_{j,n}(r,t)$ converges in $|t| < \sigma$. Choose t' to satisfy $0 < t' < \min(\sigma, \frac{\sigma}{m})$ and form the integral

(9.19) $$\int_0^\infty K_{j,n}(r,y;t+t')u(y,-t')dy \ .$$

Now substitute the series for $u(y,-t')$ into (9.19) to obtain

(9.20) $$\begin{aligned}\int_0^\infty K_{j,n}(r,y;t+t') \sum_{k=0}^{\infty} b_k R^k_{j,n}(y,-t')dy \\ = \sum_{k=0}^{\infty} b_k \int_0^\infty K_{j,n}(r,y;t+t')R^k_{j,n}(y,-t')dy \\ = \sum_{k=0}^{\infty} b_k R^k_{j,n}(r,t) = u(r,t) \ .\end{aligned}$$

We have used the fact that $R^k_{j,n}(r,t) \in H^*_j$ in the reduction to the final series. The formal term by term integration is valid provided

(9.21) $$\sum_{k=0}^{\infty} |b_k| \int_0^\infty K_{j,n}(r,y;t+t')|R^k_{j,n}(y,-t')|dy < \infty$$

for t' in $(0, \min(\sigma,\frac{\sigma}{m}))$. The series (9.21) is bounded by

We must, of course, suitably restrict $\varphi_1(\bar{x})$ to ensure that (5.7) converges.

The fundamental solution of (5.5) can be obtained from (5.6) by an application of Lemma (5.1). That is, since $e^{-t(\bar{x}\cdot\bar{x})^j}$ is radially symmetric, we have, by Lemma (5.1),

$$(5.8)\quad F_{j,n}(r,t) = (2\pi)^{-\frac{n}{2}} r^{1-\frac{n}{2}} \int_0^\infty \xi^{\frac{n}{2}} e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(r\xi)d\xi \ .$$

The Poisson integral representation for solutions of (5.5) is not as readily obtained. In the integral (5.7), introduce the coordinates $(y,\theta_1,\theta_2, \ldots, \theta_{n-1})$, $y = (\bar{y}\cdot\bar{y})^{\frac{1}{2}}$, to obtain

$$(5.9)\qquad \int_{E_n} \varphi_1(\bar{y})F_j(\bar{x}-\bar{y},t)d\bar{y}$$

$$= (2\pi)^{-\frac{n}{2}} \int_0^\infty \varphi(y)y^{n-1} \int_{E_n} e^{-t(\bar{s}\cdot\bar{s})^j - i\bar{x}\cdot\bar{s}} I_1 d\bar{s}\ dy \ .$$

Here $I_1 = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} e^{i\bar{s}\cdot\bar{y}_1} \text{Sin}^{n-2}\theta_1\ \text{Sin}^{n-3}\theta_2 \ldots \text{Sin}\ \theta_{n-2} d\theta_1 \ldots d\theta_{n-1}$

and $\bar{y}_1 = (y\text{Cos}\ \theta_1,\ y\text{Sin}\ \theta_1\ \text{Cos}\ \theta_2,\ \ldots,\ y\text{Sin}\ \theta_1\ \text{Sin}\ \theta_2 \ldots \text{Sin}\ \theta_{n-1})$.

We can evaluate I_1 in the same way that we evaluated the integral (5.2). After carrying out the indicated reduction we find that

$$(5.10)\qquad I_1 = (2\pi)^{\frac{n}{2}}(sy)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(sy), \quad s = (\bar{s}\cdot\bar{s})^{\frac{1}{2}} \ .$$

The second member of (5.9) is then equal to

$$(9.22) \qquad \sum_{k=0}^{\infty} |b_k| \frac{A}{k!}\left(\frac{t'+\delta}{\alpha}\right)^{\frac{k}{j}} \left(\frac{2k}{Ce\mu}\right)^{\frac{k}{j}(2j-1)} \cdot |I|,$$

where $I = \int_0^\infty K_{j,n}(r,y;t+t')e^{C(\frac{\alpha}{\delta})^\nu |y|^\mu} dy$. We have made use of (7.5) to obtain the dominating series.

If we consider the function $e^{C(\frac{\alpha}{\delta})^\nu |y|^\mu}$ as initial data, it is clear that the integral I results from transforming the integral

$$(9.23) \qquad \int_{E_n} e^{C(\frac{\alpha}{\delta})^\nu (\bar{y}\cdot\bar{y})^{\mu/2}} F_j(\bar{x}-\bar{y},t+t')d\bar{y}$$

by means of the spherical transformation T. We can apply (3.7) to the integral (9.23) to show that the integral I converges for $-t' < t < t' + \beta(\frac{\delta}{\alpha})$. Here $0 < \alpha < 1$, $0 < \beta < 1$, and $0 < \delta < \infty$.

Choose t_0 so that $0 < t' < \min(t_0, \frac{t_0}{m}) < \min(\sigma, \frac{\sigma}{m})$. Then, $0 < t_0 < \sigma$ and by Lemma (9.1II), $b_k = O[(\frac{e^{j-1}}{jt_0 4^j k^{j-1}})^{\frac{k}{j}}]$ as $k \longrightarrow \infty$. The series (9.22), therefore, is dominated by the series

$$A \sum_{k=N}^{\infty} \frac{1}{k!} \left[\left(\frac{e^{j-1}}{j\theta t_0 4^j k^{j-1}}\right) \left(\frac{t'+\delta}{\alpha}\right) \left(\frac{2k}{Ce\mu}\right)^{2j-1} \right]^{\frac{k}{j}}$$

where $N = N(\theta)$, $0 < \theta < 1$. This series converges provided $t' + \delta < \frac{\alpha t_0 \theta}{m}$, as can be seen by applying the ratio test. Letting $\delta \longrightarrow 0$, $\alpha \longrightarrow 1$, $\theta \longrightarrow 1$, and choosing t_0 sufficiently close to σ we find that

$$(9.24) \qquad u(r,t) = \int_0^\infty K_{j,n}(r,y;t+t')u(y,-t')dy$$

for $|t| < t' < \min(\sigma,\frac{\sigma}{m})$. Since (9.24) is an absolutely convergent Poisson-Stieltjes integral, we can apply Lemma (8.5) to obtain the proof of the theorem.

Let us now examine the growth properties of initial data. The following definition will make our meaning precise. For example, see [3,p.8].

<u>Definition</u>. An entire function $f(z)$ is of growth (ρ,τ) if and only if $\limsup_{k\to\infty} \frac{k}{e\rho} |a_k|^{\rho/k} \leq \tau$. The numbers $a_k(k=0,1,\ldots)$ are the coefficients in the Taylor's series representation of $f(z)$ about the origin.

<u>Theorem 9.5</u>.I. If $u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t)$ converges in $|t| < \sigma$, then

a) $u(x,t) = K_q(x,t) * \varphi(x)$ for $0 \leq t < \min(\sigma,\frac{\sigma}{m})$

and

b) $u(x,-t) = \int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t)\varphi(\alpha_2 y)dy$ for $0 < t < \min(\sigma,\frac{\sigma}{m})$

for some entire function $\varphi(z)$ of growth $(\mu,\mu^{-1}(q\sigma)^{-\nu})$.

II. If $u(r,t)=\sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t)$ converges in $|t| < \sigma$, then

a) $u(r,t) = \int_0^\infty K_{j,n}(r,y;t)\psi(y)dy$ for $0 \leq t < \min(\sigma,\frac{\sigma}{m})$

and

b) $u(r,-t) = \int_0^\infty K_{j,n}(\alpha_1 r, y; t)\psi(\alpha_2 y)dy$ for $0 < t < \min(\sigma, \frac{\sigma}{m})$

for some entire function $\psi(z)$ of growth $(\frac{\mu}{2}, \mu^{-1}(2j\sigma)^{-\nu})$ in z^2.

In both I and II, $\alpha_1 = e^{\frac{i\pi}{q}}$, $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$, and $q = 2j$, $j = 1,2, \ldots$.

Proof. We will prove Theorem (9.5I). For $t \geq 0$, $v_q^k(x,t) = K_q(x,t) * \frac{x^k}{k!}$ by (2.13). Formally then,

$$\sum_{k=0}^{\infty} a_k v_q^k(x,t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} K_q(x,t) * x^k \tag{9.25}$$

$$= K_q(x,t) * \varphi(x) ,$$

where $\varphi(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$. The interchange of the order of summation and integration will be permissible if the final integral (9.25) is absolutely convergent. By (3.7), if $\varphi(x)$ has growth (μ,τ), then $K_q(x,t) * \varphi(x)$ is absolutely convergent in $0 \leq t < \beta(\frac{c}{\tau})^{q-1}$ for any β, $0 < \beta < 1$.

Choose t_0 so that $0 < \min(t_0, \frac{t_0}{m}) < \min(\sigma, \frac{\sigma}{m})$. By Lemma (9.1I), $a_k = 0\left[k!\left(\frac{e^{q-1}}{qt_0 k^{q-1}}\right)^{\frac{k}{q}}\right]$. Thus,

$$\limsup_{k \to \infty} \frac{k}{e\mu} \left|\frac{a_k}{k!}\right|^{\mu/k} \leq \limsup_{k \to \infty} \left(\frac{k}{\mu e}\right) M^{\mu/k} \left(\frac{e}{k}\right)\left(\frac{1}{qt_0}\right)^{\frac{1}{q-1}}$$

$$\leq \mu^{-1}(qt_0)^{-\nu}, \quad \nu = (q-1)^{-1} .$$

That is, $\varphi(x)$ has growth $(\mu, \mu^{-1}(qt_0)^{-\nu})$. By our earlier remarks, the final integral (9.25) converges absolutely for $0 \leq t < \beta \frac{t_0}{m}$,

$0 < \beta < 1$. By choosing β sufficiently close to 1 and allowing $t_0 > \sigma$, we obtain the proof of the theorem.

We must proceed differently when $t < 0$. By Theorem (2.14),

$v_q^k(x,-t) = \frac{1}{k!}\int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t)(\alpha_2 y)^k dy$ for $t > 0$ with

$\alpha_1 = e^{\frac{i\pi}{q}}$ and $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$. Then, formally,

$$(9.26) \qquad \sum_{k=0}^{\infty} a_k v_q^k(x,-t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t)(\alpha_2 y)^k dy$$

$$= \int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t)\varphi(\alpha_2 y)dy$$

where $\varphi(\alpha_2 y) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(\alpha_2 y)^k$. It is clear that $\varphi(z)$ is of growth $(\mu,\mu^{-1}(\sigma m)^{-\nu})$. We can apply Lemma (3.4) to show that $\int_{-\infty}^{\infty} K_q(y+\alpha_1 x,t)\varphi(\alpha_2 y)dy$ converges absolutely for $0 < t < \beta(\frac{t_0}{m})$, $0 < \beta < 1$. Here, as before, t_0 has been selected to satisfy $0 < \min(t_0, \frac{t_0}{m}) < \min(\sigma,\frac{\sigma}{m})$ in order to apply Lemma (9.1I). The desired result now follows by choosing β sufficiently close to 1 and t_0 sufficiently close to σ.

To conclude this section, we will show to what extent the growth properties of the initial data determine the time strip in which a series of the form (9.1I) or (9.1II) converges.

Theorem 9.6. I. Let the initial data, $\varphi(z) = \sum_{k=0}^{\infty} C_k z^k$ be an entire function of growth (μ,M), $M > 0$. Then the solution corresponding to $\varphi(x)$ is $u(x,t) = \sum_{k=0}^{\infty} a_k v_q^k(x,t)$ which converges absolutely for $|x| < \infty$, $|t| < \frac{1}{mq}(M\mu)^{1-q}$. Here, $a_k = k!c_k$.

II. Let the initial data, $\psi(z) = \sum_{k=0}^{\infty} d_k z^{2k}$ be an entire function of growth $(\mu/2, M)$ in z^2. Then the solution corresponding to $\psi(r)$ is $u(r,t) = \sum_{k=0}^{\infty} b_k R_{j,n}^k(r,t)$ which converges absolutely for $0 < r < \infty$, $|t| < \frac{1}{2mj}(M\mu)^{1-2j}$. Here, $b_k = k!d_k$ and $\mu = 2j(2j-1)^{-1}$.

<u>Proof</u>. We prove Theorem (9.6I). By hypothesis, $|\varphi(x)| \leq Be^{M|x|^{\mu}}$ for a suitable constant B. For initial data with these growth properties, Ladyzhenskaya [13] has shown that the unique classical solution of problem (1.1) has the representation

$u(x,t) = K_q(x,t) * \varphi(x)$ for $|x| < \infty$, $0 \leq t < \frac{\beta}{mq}(\frac{1}{M\mu})^{q-1}$, $0 < \beta < 1$.
In this time interval,

$$u(x,t) = K_q(x,t) * \varphi(x) = \sum_{k=0}^{\infty} k!c_k v_q^k(x,t). \tag{9.27}$$

The last member of (9.27) follows from (2.13). The formal term by term integration is valid wherever

$$\sum_{k=0}^{\infty} k!|c_k| \int_{-\infty}^{\infty} K_q(x-y,t)| \frac{y^k}{k!}|dy < \infty . \tag{9.28}$$

By hypothesis, for any θ, $0 < \theta < 1$, there is an integer $N = N(\theta)$ such that for all $k \geq N$, $|c_k| \leq (\frac{Me\mu}{k\theta})^{\frac{k}{\mu}}$. The series (9.28) is, therefore, dominated by

$$Ae^{C(\frac{\alpha}{\delta})^{\nu}|x|^{\mu}} \sum_{k=0}^{\infty} \left[(\frac{t+\delta}{\alpha}) \frac{M^{q-1} mq\, \mu^{\mu-1}}{\theta^{q-1}} \right]^{\frac{k}{q}} \tag{9.29}$$

for $0 < \delta < \infty$, $0 < \alpha < 1$, $0 < \theta < 1$. We have used (3.11) to obtain (9.29). The series (9.29) converges by the root-test provided $t + \delta < \frac{\alpha}{mq} (\frac{\theta}{M\mu})^{q-1}$. Since α, δ, and θ are arbitrary in their respective intervals, we have proved the desired result for $0 \leq t < \frac{1}{mq} (M\mu)^{1-q}$.

In order to prove Theorem (9.6I) when $t < 0$, form the integral

(9.30) $$u(x,-t) = \int_{-\infty}^{\infty} K_q(y,t)\varphi(x+\alpha_2 y)dy, \quad t > 0.$$

As usual, $\alpha_2 = e^{\frac{(q-1)i\pi}{q}}$. By Lemma (3.4), the integral converges absolutely for $|x| < \infty$, $0 < t < \frac{\beta}{mq}(\frac{1}{M\mu})^{q-1}$ when $\varphi(z)$ has growth (μ,M). Upon substituting the series for $\varphi(x+\alpha_2 y)$ into the integral (9.30) and formally integrating term by term we obtain

(9.31) $$\int_{-\infty}^{\infty} K_q(y,t) \sum_{k=0}^{\infty} C_k(x+\alpha_2 y)^k dy = \sum_{k=0}^{\infty} k! C_k v_q^k(x,-t) .$$

The last series is obtained by applying (2.14). With the bound (3.13) on $v_q^k(x,-t)$ and the fact that $|C_k| \leq (\frac{Me\mu}{k\theta})^{\frac{k}{\mu}}$, $0 < \theta < 1$, we can show in the usual way that the formal term by term integration is valid for $0 < t < \frac{1}{mq}(M\mu)^{1-q}$.

10. Representations in Terms of the Associated Functions

We now examine series of the form:

$$(10.1)\quad \begin{cases} \text{I.} & u(x,t) = \sum_{k=0}^{\infty} a_k w_q^k(x,t) \;, \\ \text{II.} & u(r,t) = \sum_{k=0}^{\infty} b_k S_{j,n}^k(r,t) \;. \end{cases}$$

Here, as in Section 9, the designations I and II refer to Equations (I) and (II), respectively. We begin with a series of elementary lemmas.

Lemma 10.1. I. For $|x| < \infty$ and $t > 0$,

$$(10.2)\qquad w_q^k(x,t) = \frac{(-1)^k}{q\pi} \sum_{\ell=[\frac{k+1}{2}]}^{\infty} (-1)^\ell \frac{x^{2\ell-k}}{(2\ell-k)!} \left(\frac{1}{t}\right)^{\frac{2\ell+1}{q}} \Gamma\left(\frac{2\ell+1}{q}\right) \;.$$

II. For $r > 0$ and $t > 0$,

$$(10.3)\qquad S_{j,n}^k(r,t) = \frac{(-1)^k}{j}\left(\frac{1}{4\pi}\right)^{\frac{n}{2}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(\frac{n+2k+2\ell}{2j}\right)}{\ell!\,\Gamma\left(\ell+\frac{n}{2}\right)} \left(\frac{r}{2}\right)^{2\ell} t^{-\left(\frac{n+2k+2\ell}{2j}\right)} \;.$$

Proof. We will prove Lemma (10.1I). Recall that

$$K_q(x,t) = (2\pi)^{-\frac{1}{2}} \widehat{e^{-tx^q}} \;, \; t > 0, \; |x| < \infty,$$

by (1.3). Formally,

$$(10.4)\qquad \begin{aligned} K_q(x,t) &= \frac{1}{2\pi}\int_0^\infty e^{-ts^q + ixs}\, ds \\ &= \frac{1}{\pi}\sum_{\ell=0}^{\infty} \frac{(-1)^\ell x^{2\ell}}{(2\ell)!}\int_0^\infty s^{2\ell} e^{-ts^q} ds \\ &= \frac{1}{q\pi}\sum_{\ell=0}^{\infty} (-1)^\ell \frac{x^{2\ell}}{(2\ell)!} \left(\frac{1}{t}\right)^{\frac{2\ell+1}{q}} \Gamma\left(\frac{2\ell+1}{q}\right) \;. \end{aligned}$$

The term by term integration can be justified by the ratio test, the final series converging absolutely for all $t > 0$ and $|x| < \infty$. From the definition (2.7) of $w_q^k(x,t)$, we have,

$$(10.5)\qquad w_q^k(x,t) = (-1)^k D_x^k K_q(x,t)$$

$$= \frac{(-1)^k}{qk} \sum_{\ell=[\frac{k+1}{2}]}^{\infty} (-1)^{\ell} \frac{x^{2\ell-k}}{(2\ell-k)!} \left(\frac{1}{t}\right)^{\frac{2\ell+1}{q}} \Gamma\left(\frac{2\ell+1}{q}\right) \quad .$$

The formal term by term differentiation can be justified by any of the standard tests.

<u>Lemma 10.2.</u> I. If $\sum_{k=0}^{\infty} |a_k||w_q^k(ix,t)|$ converges at (x_0,t_0), $x_0 > 0$, $t_0 > 0$, then

$$|a_k| = O\left(\frac{t_0 qe}{k}\right)^{\frac{k}{q}} \quad \text{as} \quad k \longrightarrow \infty \; .$$

II. If $\sum_{k=0}^{\infty} |b_k||S_{j,n}^k(ir,t)|$ converges at (r_0,t_0), $r_0 > 0$, $t_0 > 0$, then

$$|b_k| = O\left(\frac{t_0 je}{k}\right)^{\frac{k}{j}} \quad \text{as} \quad k \longrightarrow \infty \; .$$

<u>Proof</u>. Let us prove Lemma (10.2II). By (10.3), we observe that

$$(10.6)\qquad |S_{j,n}^k(ir_0,t_0)| = \frac{1}{j}\left(\frac{1}{4\pi}\right)^{\frac{n}{2}} \sum_{\ell=0}^{\infty} \frac{r_0^{2\ell} t_0^{-\left(\frac{n+2k+2\ell}{2j}\right)}}{\ell! 4^{\ell}\, \Gamma(\ell + \frac{n}{2})} \Gamma\left(\frac{n+2k+2\ell}{2j}\right) \; .$$

Upon choosing the term corresponding to $\ell = j$, we have,

$$(10.7)\qquad |S_{j,n}^k(ir_0,t_0)| \geq \left(\frac{1}{4\pi}\right)^{\frac{n}{2}} \frac{r_0^{2j} t_0^{-\left(\frac{n+2k+2j}{2j}\right)}}{j 4^j j!\, \Gamma(j + \frac{n}{2})} \Gamma\left(\frac{n+2k}{2j} + 1\right)$$

$$\geq M^* \left(\frac{2k+n}{2jt_0 e}\right)^{\frac{k}{j}} \geq M^* \left(\frac{k}{jt_0 e}\right)^{\frac{k}{j}} \quad .$$

By hypothesis, for a suitable constant M and all k sufficiently large,

$$(10.8)\qquad |b_k| \le M|S^k_{j,n}(ir_0,t_0)|^{-1} \le \frac{M}{M^*}\left(\frac{jt_0e}{k}\right)^{\frac{k}{j}} .$$

While series of the form (9.1I) or (9.1II), in general, converge in strips ($|t| < \sigma$), our next result will show that series of associated functions, in general, converge in half planes ($0 \le \sigma < t$). This behavior was noticed in [6] and [16] for problems I and II with $j = 1$.

Lemma 10.3. I. If $\limsup_{k\to\infty} \frac{k}{qe}|a_k|^{\frac{q}{k}} = \sigma \ge 0$, then $u(x,t) = \sum_{k=0}^{\infty} a_k w^k_q(x,t)$ converges absolutely for $0 \le \frac{\sigma}{\lambda} < t$.

II. If $\limsup_{k\to\infty} \frac{k}{je}|b_k|^{\frac{j}{k}} = \sigma \ge 0$, then $u(r,t) = \sum_{k=0}^{\infty} b_k S^k_{j,n}(r,t)$ converges absolutely for $0 \le \frac{\sigma}{\lambda} < t$.

Here and in I, λ is the constant defined by (3.2a).

Proof. We will prove Lemma (10.3II). By hypothesis, for any θ, $0 < \theta < 1$, there exists an integer $N = N(\theta)$ such that for all $k \ge N$, $|b_k| \le \left(\frac{\sigma je}{\theta k}\right)^{\frac{k}{j}}$. Thus, by (7.3),

$$(10.9)\qquad \sum_{k=N}^{\infty}|b_k||S^k_{j,n}(r,t)| \le C_5 e^{-Cr^{\mu}t^{(1-\mu)}} \sum_{k=N}^{\infty}\left(\frac{\sigma je}{\theta k}\right)^{\frac{k}{j}}(\lambda t)^{-\left(\frac{n+2k}{2j}\right)}\Gamma\left(\frac{n+2k}{2j}\right).$$

If we use the right half of inequality (4.17) to estimate $\Gamma(\frac{n+2k}{2j})$ in the last series, the resultant series converges absolutely, by an application of the ratio test, provided $t > \frac{\sigma}{\theta\lambda}$. The proof of the lemma now follows by choosing θ sufficiently close to one.

Theorem 10.4. I. Let $\varphi(x) = \sum_{k=0}^{\infty} a_k' x^k$ be an entire function of growth (q,σ), $\sigma \geq 0$. Then

$$u(x,t) = (2\pi)^{-\frac{1}{2}} \widehat{\varphi(x)e^{-tx^q}} \tag{10.10}$$

is a solution of problem (1.1) for $|x| < \infty$, $0 \leq \sigma < t$. Moreover, $u(x,t)$ has the series representation

$$u(x,t) = \sum_{k=0}^{\infty} a_k w_q^k(x,t), \quad |x| < \infty, \quad 0 \leq \sigma < t. \tag{10.11}$$

The coefficients a_k are given by $a_k = (-i)^{-k} a_k'$, $k = 0,1,2, \ldots$.

II. Let $\psi(r) = \sum_{k=0}^{\infty} b_k' r^{2k}$ be an entire function of growth (j,σ) in r^2, $\sigma \geq 0$. Then

$$u(r,t) = r^{1-\frac{n}{2}} \mathcal{H}_{\frac{n}{2}-1}(r, r^{\frac{n}{2}-1} \psi(r) e^{-tr^{2j}}) \tag{10.12}$$

satisfies Equation (5.5) for $r > 0$, $0 \leq \sigma < t$. Moreover, $u(r,t)$ has the series representation

$$u(r,t) = \sum_{k=0}^{\infty} b_k S_{j,n}^k(r,t) \tag{10.13}$$

for $0 \leq \sigma < t$, $r > 0$. The coefficients are given by

$b_k = (-1)^k (2\pi)^{\frac{n}{2}} b_k'$, $k = 0,1,2, \ldots$.

Proof. Let us prove Theorem (10.4I). For a suitable constant B, $|\varphi(x)| \leq Be^{\sigma|x|^q}$, by hypothesis. Then,

$$|u(x,t)| = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \varphi(s) e^{-ts^r - ixs} ds \right|$$

$$\leq \frac{B}{2\pi} \int_{-\infty}^{\infty} e^{-(t-\sigma)s^q} ds.$$

Thus, $u(x,t)$ is defined by a uniformly convergent integral for $0 \le \sigma < t$, $|x| < \infty$. Similarly, the integrals

$$(10.14) \qquad \int_{-\infty}^{\infty} D_t\{\varphi(s)e^{-ixs-ts^q}\}ds$$

and

$$(10.15) \qquad \int_{-\infty}^{\infty} D_x^k\{\varphi(s)e^{-ixs-ts^q}\}ds, \; k = 0,1,2, \ldots,q,$$

converge uniformly for $0 \le \sigma < t$, $|x| < \infty$. As a result,

$$D_t u(x,t) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} s^q\varphi(s)e^{-ixs-ts^q}ds$$

$$= (-1)^{j+1}\frac{1}{2\pi}\int_{-\infty}^{\infty} (-is)^q\varphi(s)e^{-ixs-ts^q}ds$$

$$= (-1)^{j+1} D_x^q u(x,t), \quad |x| < \infty, \; 0 \le \sigma < t.$$

In order to prove the second part of the theorem, we observe that, formally,

$$(10.16) \qquad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixs-ts^q}\left(\sum_{k=0}^{\infty} a_k' s^k\right)ds$$

$$= \sum_{k=0}^{\infty} a_k' \frac{1}{2\pi}\int_{-\infty}^{\infty} s^k e^{-ixs-ts^q}ds$$

$$= \sum_{k=0}^{\infty} (-i)^{-k} a_k' \, w_q^k(x,t) .$$

The reduction to the final series follows from (2.7). The term by term integration will be valid wherever

$$(10.17) \qquad \int_{-\infty}^{\infty} e^{-ts^q} \sum_{k=0}^{\infty} |a_k'||s|^k ds < \infty .$$

But, by hypothesis, $\sum_{k=0}^{\infty} |a_k'| |s|^k < Be^{\sigma' s^q}$ for every $\sigma' > \sigma$ and a suitable constant B. The integral (10.17) is, therefore, bounded by $B \int_{-\infty}^{\infty} e^{-(t-\sigma')s^q} ds$ for every $\sigma' > \sigma$ and converges whenever $t > \sigma' > \sigma \geq 0$.

<u>Theorem 10.5</u>.I. Let $\sum_{k=0}^{\infty} |a_k| |w_q^k(ix,t)|$ converge for $t > \sigma \geq 0$. Then

(10.18) $$u(x,t) = \sum_{k=0}^{\infty} a_k w_q^k(x,t) = \frac{1}{\sqrt{2\pi}} \widehat{\varphi(x)e^{-tx^q}}$$

for $0 \leq \sigma - t$, where $\varphi(x)$ is an entire function of growth (q,σ).

II. Let $\sum_{k=0}^{\infty} |b_k| |S_{j,n}^k(ir,t)|$ converge for $0 \leq \sigma < t$. Then

(10.19) $$u(r,t) = \sum_{k=0}^{\infty} b_k S_{j,n}^k(r,t)$$
$$= (2\pi)^{-\frac{n}{2}} r^{1-\frac{n}{2}} \mathcal{H}_{\frac{n}{2}-1}(r, r^{\frac{n}{2}-1} \psi(r) e^{-tr^{2j}})$$

in $0 \leq \sigma < t$, where $\psi(r)$ is an entire function of growth (j,σ) in r^2.

<u>Proof</u>. We prove Theorem (10.5II). It is clear that $\sum_{k=0}^{\infty} b_k S_{j,n}^k(r,t)$ converges at each point (r,t) where $\sum_{k=0}^{\infty} |b_k| |S_{j,n}^k(ir,t)|$ converges. Choose $\sigma' > \sigma \geq 0$. Then by Lemma (10.2II),

$$|b_k| = O\left(\frac{\sigma' je}{k}\right)^{\frac{k}{j}} \quad \text{as } k \to \infty .$$

Now, formally,

$$u(r,t) = \sum_{k=0}^{\infty} b_k S_{j,n}^{k}(r,t) \tag{10.20}$$

$$= \sum_{k=0}^{\infty} b_k(-1)^k(2\pi)^{-\frac{n}{2}} r^{1-\frac{n}{2}} \int_0^\infty \xi^{\frac{n}{2}+2k} e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(r\xi)d\xi$$

$$= (2\pi)^{-\frac{n}{2}} r^{1-\frac{n}{2}} \int_0^\infty \xi^{\frac{n}{2}} e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(r\xi)\psi(\xi)d\xi .$$

Here, $\psi(\xi) = \sum_{k=0}^{\infty} (-1)^k b_k \xi^{2k}$. Since $|b_k| \leq M(\frac{\sigma' je}{k})^{\frac{k}{j}}$ for a suitable constant M and all k sufficiently large, $\psi(\xi)$ is of growth (j,σ) in ξ^2. That is,

$$\limsup_{k\to\infty} \frac{k}{je}\left[(\frac{\sigma' je}{k})^{\frac{k}{j}}\right]^{\frac{j}{k}} \leq \sigma' \quad \text{for all } \sigma' > \sigma \geq 0.$$

For a suitable constant B, $|\psi(\xi)| \leq Be^{\sigma|\xi|^{2j}}$ and $r^{1-\frac{n}{2}} \int_0^\infty \xi^{\frac{n}{2}} e^{-t\xi^{2j}} |J_{\frac{n}{2}-1}(r\xi)||\psi(\xi)|d\xi < \infty$ for $t > \sigma \geq 0$. This justifies the formal interchange of the order of summation and integration performed in (10.20).

<u>Corollary</u>. I. If $\sum_{k=0}^{\infty} |a_k||w_q^k(ix,t)|$ converges for $t > \sigma \geq 0$, then $u(x,t) = \sum_{k=0}^{\infty} a_k w_q^k(x,t)$ belongs to the class H_q^* there.

II. If $\sum_{k=0}^{\infty} |b_k||S_{j,n}^k(ir,t)|$ converges for $t > \sigma \geq 0$, then $u(r,t) = \sum_{k=0}^{\infty} b_k S_{j,n}^k(r,t)$ belongs to the class H_j^* there.

Proof. By Theorem (10.5II),

$$(10.21)\qquad \int_0^\infty K_{j,n}(r,y;t-t')u(y,t')dy$$

$$= \int_0^\infty K_{j,n}(r,y;t-t')(2\pi)^{-\frac{n}{2}} y^{1-\frac{n}{2}} \int_0^\infty \xi^{\frac{n}{2}} e^{-t'\xi^{2j}} \psi(\xi) J_{\frac{n}{2}-1}(y\xi)d\xi\, dy,$$

where $\psi(\xi)$ satisfies $|\psi(\xi)| \leq Be^{\sigma|\xi|^{2j}}$. In view of this bound, we can interchange the order of integration in the final integral (10.21). The last member of (10.21) is then equal to

$$\int_0^\infty \xi^{\frac{n}{2}} e^{-t'\xi^{2j}} \psi(\xi) \int_0^\infty (2\pi)^{-\frac{n}{2}} y^{1-\frac{n}{2}} K_{j,n}(r,y;t-t') J_{\frac{n}{2}-1}(y\xi) dy\, d\xi$$

$$= \int_0^\infty (2\pi)^{-\frac{n}{2}} \xi^{\frac{n}{2}} r^{1-\frac{n}{2}} \psi(\xi) e^{-t\xi^{2j}} J_{\frac{n}{2}-1}(r\xi)d\xi = u(r,t)$$

for $0 \leq \sigma < t' < t$. We have used the inverse Hankel transform relation as applied to (5.16) in order to obtain the last integral.

Theorem 10.6. I. If $u(x,t) \in H_q^*$ for $0 \leq \sigma < \min(t,\frac{\lambda t}{m})$ and if

$$(10.22)\qquad \int_{-\infty}^{\infty} |u(y,t')| e^{C(\frac{\alpha}{\delta})^{\nu} |y|^{\mu}} dy$$

converges for $0 \leq \sigma < t'$, $0 < \alpha < 1$, $0 < \delta < \infty$, then

$$(10.23)\qquad u(x,t) = \sum_{k=0}^{\infty} a_k w_q^k(x,t) \quad \text{for} \quad 0 \leq \sigma < \min(t,\frac{\lambda t}{m}).$$

The coefficients a_k have the determination

$$a_k = \int_{-\infty}^{\infty} v_q^k(y,-t')u(y,t')dy, \quad k = 0,1,2, \ldots .$$

II. If $u(r,t) \in H_j^*$ for $0 \leq \sigma < \min(t, \frac{\lambda t}{m})$ and if

$$(10.24) \qquad \int_0^\infty |u(y,t')| e^{c(\frac{\alpha}{\delta})^\nu |y|^\mu} W_n(y)dy$$

converges for $0 \leq \sigma < t'$, $0 < \alpha < 1$, $0 < \delta < \infty$, then

$$(10.25) \qquad u(r,t) = \sum_{k=0}^\infty b_k S_{j,n}^k(r,t) \quad \text{for} \quad 0 \leq \sigma < \min(t, \frac{\lambda t}{m}) .$$

The coefficients b_k have the determination

$$b_k = (\tfrac{1}{2})^{\frac{n}{2} + 2k-1} \frac{1}{\Gamma(\frac{n}{2} + k)} \int_0^\infty W_n(y) R_{j,n}^k(y,-t')u(y,t')dy.$$

Here, $W_n(y)$ is given by (6.9).

<u>Proof</u>. We prove Theorem (10.6I). By Theorem (4.5),

$$K_q(x-y,t-t') = \sum_{k=0}^\infty v_q^k(y,-t')w_q^k(x,t) \quad \text{for } 0 < t' < \min(t, \frac{\lambda t}{m}).$$

Since $u(x,t) \in H_q^*$ for $0 \leq \sigma < \min(t, \frac{\lambda t}{m})$,

$$(10.26) \qquad u(x,t) = \int_{-\infty}^\infty K_q(x-y,t-t')u(y,t')dy$$

$$= \int_{-\infty}^\infty u(y,t') \sum_{k=0}^\infty v_q^k(y,-t')w_q^k(x,t)dy$$

$$= \sum_{k=0}^\infty w_q^k(x,t) \int_{-\infty}^\infty v_q^k(y,-t')u(y,t')dy.$$

The formal interchange of the order of summation and integration is valid if

$$(10.27) \qquad \sum_{k=0}^\infty |w_q^k(x,t)| \left| \int_{-\infty}^\infty |u(y,t')| |v_q^k(y,-t')| dy < \infty \right. .$$

The series (10.27) is dominated by

$$(10.28) \qquad C' \sum_{k=0}^{\infty} \left[\left(\frac{t'+\delta}{\alpha\lambda t}\right) \left(\frac{k}{Ce\mu}\right)^{q-1} \right]^{\frac{k}{q}} \Gamma\left(\frac{k+1}{q}\right) \cdot I$$

with $I = \int_{-\infty}^{\infty} |u(y,t')| \, e^{C(\frac{\alpha}{\delta})^{\nu}|y|^{\mu}} dy$. We have used (3.10) and (3.13) to obtain (10.28). Here, $0 < \alpha < 1$, $0 < \delta < 1$, and C' consists of all factors which are independent of k. The integral I converges by hypothesis while the series (10.28) converges by the ratio test for $t' < \min(t, \frac{\lambda t}{m})$. The coefficients a_k, $k = 0,1,2, \ldots,$ were obtained in the course of the proof.

V. COMPARISONS AND GENERALIZATIONS

11. Comparisons with Known Results

We now indicate to what extent certain results of the preceding sections compare with the results of Rosenbloom and Widder [16] and Bragg [6] for the case $j = 1$. The principle differences are in the intervals in which certain results are valid. Thus, for example, in [16, p.227] it is shown that

$$K_2(x-y, t+t') = \sum_{k=0}^{\infty} v_2^k(x,t) w_2^k(y,t') \tag{11.1}$$

for $-t' < t < t'$. If we evaluate the constants λ and m, by (3.2a) and (3.2b), respectively, we find, when $j = 1$, that $\lambda = \frac{1}{2}$ and $m = 1$. Then, when $j = 1$, Theorem (4.5) reduces to (11.1) but only for $-\frac{t'}{2} < t < \frac{t'}{2}$. It would, therefore, appear that our results are somewhat more restrictive than necessary. This difference arises from the bounds (3.10), (3.11), and (3.13).

We now indicate how these bounds can be strengthened in the case $j = 1$. The proof is that given by Ladyzhenskaya in [13] except for the change indicated.

Recall that

$$K_q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ts^q + ixs} ds, \quad t > 0. \tag{11.2}$$

Introduce the change of variables $s = \theta\beta$, $\theta = \eta\left(\frac{|x|}{t}\right)^{\nu}$, where η is a constant to be fixed later and $\nu = (q-1)^{-1}$, $q = 2j$, $j = 1,2,\ldots$. Then (11.2) is just

(11.3) $$K_q(x,t) = \frac{\theta}{2\pi}\int_{-\infty}^{\infty} e^{-t\theta^q\beta^q+ix\theta\beta}\,d\beta .$$

Let $\sigma = r^{\mu} t^{1-\mu}$ with $r = |x|$. Then,

(11.4) $$-t\theta^q\beta^q + ix\theta\beta = -\sigma(\eta^q\beta^q - \frac{i\eta x\beta}{r})$$

and the integrand in (11.3) is observed to be an entire function of β. By an application of Cauchy's integral theorem, the integral (11.3) can be evaluated by integrating along any path parallel to the real β axis. Thus we may replace β by $\beta + iw$ where $w = \frac{x}{r}$. The exponent is then equal to

(11.5) $$-\sigma\left\{\eta^q[(\beta + iw)^q - \lambda|\beta|^q] - i\eta\frac{x\beta}{r} + \eta\right\} - \lambda t|s|^q,$$

where Ladyzhenskaya defines λ to be the constant (3.2a).

Let us modify her proof at this point by letting λ be any positive number to be fixed later. Now let

$Q = \eta^q[(\beta+iw)^q - \lambda|\beta|^q] - i\eta\frac{x\beta}{r} + \eta$ and let $Q_1 = (\beta+iw)^q - \lambda|\beta|^q$. Then

(11.6) $$\text{Re } Q = \eta^q \text{Re } Q_1 + \eta \geq -m\eta^q + \eta$$

where $-m = \min \text{Re } Q_1$. The minimum is taken over all β and all w such that $|w| = 1$. Observe, at this point, that $\text{Re } Q_1 = (1-\lambda)\beta^q + P(\beta)$, where $P(\beta)$ is a polynomial of degree q-2. Hence, $-m$ will be finite provided $1 - \lambda > 0$. However, when $q = 2$, $P(\beta) \equiv -1$ and $-m \equiv -1$ for all λ, $0 < \lambda \leq 1$. In the case $q = 2$, therefore, we will choose $\lambda = 1$. For $q = 2j$, $j = 2,3,\ldots$, we will choose λ, as Ladyzhenskaya did, to be the constant (3.2a).

When $q = 2$, we have,

(11.7) $$\text{Re } Q \geq -\eta^2 + \eta \equiv g(\eta) .$$

If we choose η to maximize $g(\eta)$ $(\eta = \frac{1}{2})$ and let C denote the $\max_{\eta > 0} (g(\eta)) = \frac{1}{4}$, then

(11.8) $$-\sigma \text{ Re } Q - \lambda t|s|^q \leq -\frac{1}{4}\sigma - t|s|^q .$$

Finally,

(11.9) $$|K_2(x,t)| \leq \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{-ts^2} ds = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} .$$

It is interesting to note that the last member of (11.9) is the fundamental solution of the heat equation, in addition to being the bound (3.3) when $q = 2$ and $\lambda = m = 1$. In this sense, the bound (3.3) is the best possible.

When $q = 2$, the same modification, choosing $\lambda = 1$, can also be made in the bounds (3.10), (3.11), and (3.13). In this way, we are able to extend the interval in which Theorem (4.5) is valid to $-t' < t < t'$. Similar modifications allow many of our results pertaining to the radial problem (5.5) to be reduced to the analogous results in [6] when $\mu = \eta \geq 2$ is an integer.

12. Generalizations and Extensions

Let us briefly indicate several ways in which the preceding theory can be extended to more general equations. Since the proofs will, for the main part, be similar to those already given we will omit them and emphasize points of difference.

We first alter Equation (1.1) by admitting a time dependent coefficient. Let $a(t)$ be a continuous function of t and consider the problem

$$(12.1)\qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = (-1)^{j+1}\, a(t)\, \dfrac{\partial^q u(x,t)}{\partial x^q} \\ u(x,0) = \varphi(x), \quad q = 2j,\ j = 1,2, \ldots . \end{cases}$$

Let $b(t) = \int_0^t a(s)\, ds$ and define the set P by

$$(12.2)\qquad P = \{t \mid b(t) > 0\} .$$

For simplicity, let us further assume that P is a connected subset of $|t| < \infty$.

The formal solution operator associated with problem (12.1) is

$$(12.3)\qquad e^{(-1)^{j+1} b(t) D_x^q}, \quad q = 2j,\ j = 1,2, \ldots .$$

We will interpret this operator in the usual way. Using the work of Gelfand and Silov [9], it can be shown that the fundamental solution of (12.1) has the representation

$$(12.4)\qquad K_q(x,b(t)) = (2\pi)^{-\frac{1}{2}}\, \widehat{e^{-b(t)x^q}}, \quad t \in P .$$

The solution polynomials are defined by

$$(12.5)\qquad v_q^k(x,b(t)) = e^{(-1)^{j+1} b(t) D_x^q} \circ \frac{x^k}{k!}, \quad k = 0,1,2, \ldots,$$

and the associated functions are defined by

$$(12.6)\qquad w_q^k(x,b(t)) = (-1)^k D_x^k K_q(x,b(t)), \quad t \in P .$$

The sets $\{v_q^k(x,-b(t))\}$ and $\{w_q^k(x,b(t))\}$ can be shown to be biorthogonal on $|x| < \infty$ for $t \in P$ in the same manner that we proved (2.8) and (6.8).

We may use reasoning similar to that given in Section 11 to develop the needed growth bounds. A semi-group property for the fundamental solution assumes the form

$$(12.7) \qquad K_q(x,b(t)+b(t')) = K_q(x,b(t)) * K_q(x,b(t')).$$

for t and t' in the set P. This relation can be proved by applying the convolution formula for Fourier transforms.

It is clear by now that the main changes which will occur will simply be the replacement of t by $b(t)$ and the intervals in which the various results are valid. A difference of more significance occurs in the Huygens' principle.

Let us further restrict $b(t)$ to be a strictly increasing function on some interval $a < t < b$ contained in P. We will then say that a function $u(x,t)$ is a member of the class $\overline{H}_q^*$ on $a < t < b$ if (i) $u(x,t) \epsilon C^q$, (ii) $u(x,t)$ satisfies Equation (12.1), and (iii) if

$$(12.8) \qquad u(x,t) = K_q(x,b(t)-b(t')) * u(x,t')$$

for all t and t' satisfying $a < t' < t < b$. The restriction that $b(t)$ be strictly increasing is needed to ensure that $b(t)-b(t') > 0$ when $t' < t$. Otherwise, $K_q(x,b(t)-b(t'))$ is not necessarily defined for all t and t' in the interval.

It is easily verified that the elements of the sets $\{v_q^k(x,b(t))\}$ and $\{w_q^k(x,b(t))\}$ belong to $\overline{H}_q^*$ in appropriate intervals. Results similar to those in Sections 9 and 10 can be derived in the obvious manner.

Let us now consider extending our results to E_n, n-dimensional Euclidean space. There are several ways to consider extensions of Equation (1.1) to E_n and we will restrict our attention to two such ways. Our notation for the calculus of n-variables will be consistent with that given in Section 5.

First, consider the initial value problem

$$(12.9) \qquad \begin{cases} \dfrac{\partial u(\bar{x},t)}{\partial t} = (-1)^{j+1}\left(\sum_{k=1}^{n} a_k(t) \dfrac{\partial^q}{\partial x_k^q}\right) \cdot u(\bar{x},t) \\ u(\bar{x},0) = \varphi(\bar{x}), \quad q = 2j,\ j = 1,2, \ldots . \end{cases}$$

We require each $a_k(t)$, $k = 1,2,\ldots, n$, to be a continuous function of t. Widder [18] has considered an expansion theory for solutions of Equation (12.9) when $q = 2$ and each $a_k(t) \equiv 1$.

Let $b_k(t) = \int_0^t a_k(s)ds$ and let $P_k = \{t | b_k(t) > 0\}$.

Let $P' = \bigcap_{k=1}^{n} P_k$ and assume that P' is a connected subset of $|t| < \infty$. Let $\bar{b}(t) = (b_1(t), b_2(t), \ldots, b_n(t))$ and $D_i = \dfrac{\partial}{\partial x_i}$.

The formal solution operator associated with Equation (12.9) is

$$(12.10) \qquad e^{(-1)^{j+1}\left(\sum_{k=1}^{n} b_k(t)D_k^q\right)} \circ \varphi(\bar{x})$$

$$= \prod_{k=1}^{n}\left(e^{(-1)^{j+1} b_k(t)D_x^q} \circ \varphi(\bar{x})\right) .$$

We will interpret the operator by the second member whenever $e^{(-1)^{j+1}b_k(t)D_k^q} \circ \varphi(\bar{x})$ is meaningful for each $k = 1,2, \ldots, n$.

The kernel or fundamental solution can be obtained in the usual way and is given by

$$(12.11) \qquad K_q(\bar{x},\ \bar{b}(t)) = (2\pi)^{-\frac{n}{2}} e^{-\overbrace{\sum_{k=1}^{n} b_k(t)x_k^q}}, \quad t \in P' .$$

Here we have the n-dimensional Fourier transform with respect to $\bar{x}$. Observe that $K_q(\bar{x},\bar{b}(t))$ is related to the kernel (12.4) by

$$(12.12) \qquad K_q(\bar{x},\bar{b}(t)) = \prod_{k=1}^{n} K_q(x_k,b_k(t)) .$$

Let $\bar{k}$ denote the multi index $(k_1,k_2, \ldots, k_n)$. Let $\bar{x}^{\bar{k}}$ denote the product $x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}$ and let $\bar{k}!$ denote the product $k_1!\ k_2! \cdots k_n!$. Similarly, $D_{\bar{x}}^{\bar{k}}$ shall denote the differential operator $D_1^{k_1}D_2^{k_2} \cdots D_n^{k_n}$ and $|\bar{k}|$ shall denote $k_1 + k_2 + k_3 + \cdots + k_n$.

With this notation, we define the set of solution polynomials by

$$(12.13) \qquad v_{q,n}^{\bar{k}}(\bar{x},\bar{b}(t)) = e^{(-1)^{j+1} \sum_{k=1}^{n} b_k(t)D_k^q} \circ \frac{\bar{x}^{\bar{k}}}{\bar{k}!} .$$

The polynomial $v_{q,n}^{\bar{k}}(\bar{x},\bar{b}(t))$ is related to the polynomials (12.5) by

$$(12.14) \qquad v_{q,n}^{\bar{k}}(\bar{x},\bar{b}(t)) = \prod_{\ell=1}^{n} v_q^{k_\ell}(x_\ell,b_\ell(t)) .$$

We define the associated set of functions by

$$(12.15) \qquad w_{q,n}^{\bar{k}}(\bar{x},\bar{b}(t)) = (-1)^{|\bar{k}|} D_{\bar{x}}^{\bar{k}} K_{q,n}(\bar{x},\bar{b}(t)), \quad t \in P' .$$

Here,

$$(12.16) \qquad w_{q,n}^{\bar{k}}(\bar{x},\bar{b}(t)) = \prod_{\ell=1}^{n} w_q^{k_\ell}(x_\ell,\ b_\ell(t))$$

where $w_{q,n}^{k_\ell}(x_\ell,\ b_\ell(t))$ is given by (12.6). From (12.14) and

(12.16) we can easily deduce the biorthogonality of the sets $\{v^{\bar{k}}_{q,n}(\bar{x},-\bar{b}(t))\}$ and $\{w^{\bar{k}}_{q,n}(\bar{x},\bar{b}(t))\}$ for $\bar{x} \in E_n$ and $t \in P'$.

The growth bounds can be obtained either directly from Ladyzhenskaya's results or by using relations (12.14) and (12.16) in conjunction with the growth bounds on the functions $v_q^{k_\ell}(x_\ell, b_\ell(t))$ and $w_q^{k_\ell}(x_\ell, b_\ell(t))$.

A semi-group property for $K_{q,n}(\bar{x},\bar{b}(t))$ is given by

$$K_{q,n}(\bar{x},\bar{b}(t)) * K_{q,n}(\bar{x},\bar{b}(t')) = K_{q,n}(\bar{x},\bar{b}(t) + \bar{b}(t')) \tag{12.17}$$

for any t and t' in P'. We will say that $u(\bar{x},t)$ satisfies the Huygens' principle on $a < t < b$ if (i) $u(\bar{x},t) \in C^q$, (ii) $u(\bar{x},t)$ satisfies Equation (12.9) there, and (iii) if

$$u(\bar{x},t) = K_{q,n}(\bar{x},\bar{b}(t) - \bar{b}(t')) * u(\bar{x},t') \tag{12.18}$$

for $a < t' < t < b$. Each $b_k(t)$ must be a strictly increasing function of t on the interval $a < t < b$ contained in P'. On appropriate intervals, we can show that the kernel, the polynomials, and the associated functions satisfy relation (12.18) for suitably restricted $b_k(t)$, $k = 1,2, \ldots, n$.

The kernel decomposition theorem and expansion theorems similar to those previously discussed can be developed by arguments analogous to those we have used before. The time intervals in which these results are valid will depend strongly on properties of the functions $b_k(t)$.

Finally, let us consider the problem

$$(12.19) \quad \begin{cases} \dfrac{\partial u(\bar{x},t)}{\partial t} = (-1)^{j+1} \Delta_n^j \cdot u(\bar{x},t) \\ u(\bar{x},0) = \varphi(\bar{x}), \quad j = 1,2, \ldots, \quad n = 2,3, \ldots \quad , \end{cases}$$

where Δ_n is the Laplacian operator in n-dimensional Cartesian coordinates. We have discussed this problem in detail under the assumption of radial symmetry. However, in the absence of radial symmetry, it is not entirely clear how one should proceed.

The fundamental solution of the equation is

$$(12.20) \qquad K_{j,n}(\bar{x},t) = (2\pi)^{-\frac{n}{2}} \widehat{e^{-t(\bar{x}\cdot\bar{x})^j}}, \quad t > 0,$$

and the formal solution operator is

$$(12.21) \qquad e^{(-1)^{j+1}t\Delta_n^j} \circ \varphi(\bar{x}) = \sum_{k=0}^{\infty} \frac{(-1)^{k(j+1)}t^k}{k!} \Delta_n^{jk} \circ \varphi(\bar{x}).$$

We must be careful to restrict our interpretation of $\Delta_n^{\ell} \circ \varphi(\bar{x})$ according to whether or not the order of differentiation of $\varphi(\bar{x})$ with respect to the variables x_i, $i = 1,2, \ldots, n$, is commutative.

Define the solution polynomial $v_{j,n}^{\bar{k}}(\bar{x},t)$ by

$$(12.22) \qquad v_{j,n}^{\bar{k}}(\bar{x},t) = e^{(-1)^{j+1}t\Delta_n^j} \circ \frac{\bar{x}^{\bar{k}}}{\bar{k}!}$$

and the element $w_{j,n}^{\bar{k}}(\bar{x},t)$ of the associated set by

$$(12.23) \qquad w_{j,n}^{\bar{k}}(\bar{x},t) = (-1)^{|\bar{k}|} D_{\bar{x}}^{\bar{k}} K_{j,n}(\bar{x},t) .$$

The explicit form of the polynomials is

$$(12.24) \qquad v_{j,n}^{\bar{k}}(\bar{x},t) = \sum_{\substack{\alpha_i \le \ell_i \le \beta_i \\ i=1,2,\ldots,n}} \frac{(-1)^{\ell_1(j+1)} t^{\ell_1}}{\ell_1 !} \binom{j\,\ell_1}{\ell_1, \ell_2, \ldots, \ell_n} \frac{x^{\overline{k-2\ell}}}{(\overline{k-2\ell})!}$$

where $(\overline{k-2\ell}) = (k_1-2j\ell_1-2\ell_2, k_2-2\ell_2-2\ell_3, \ldots, k_n-2\ell_n)$. In the notation for the calculus of n-variables, the results for the present problem have the same appearance as the results developed for problem (1.1). The proofs are also similar in the reasoning involved but can be described as cumbersome at best.

As an example, let us consider the proof of the biorthogonality of the sets $\{v_{j,n}^{\bar{k}}(\bar{x},-t)\}$ and $\{w_{j,n}^{\bar{k}}(\bar{x},t)\}$ for $\bar{x} \in E_n$ and $t > 0$.

Since $w_{j,n}^{\bar{k}}(\bar{x},t) \longrightarrow 0$ as $\bar{x} \longrightarrow \infty$, we have, after successive integration by parts,

$$(12.25) \qquad \int_{E_n} v_{j,n}^{\bar{k}}(\bar{x},-t) w_{j,n}^{\bar{\ell}}(\bar{x},t) d\bar{x} = \int_{E_n} v_{j,n}^{\bar{k}-\bar{\ell}}(\bar{x},-t) K_{j,n}(\bar{x},t) d\bar{x} .$$

If $\ell_i > k_i$ for any $i = 1,2, \ldots, n$, then $v_{j,n}^{\bar{k}-\bar{\ell}}(\bar{x},-t) \equiv 0$ and we have the desired result. If $\ell_i = k_i$ for all $i = 1,2,\ldots,n$, then

$$(12.26) \qquad \int_{E_n} v_{j,n}^{\bar{k}}(\bar{x},-t) w_{j,n}^{\bar{\ell}}(\bar{x},t) d\bar{x} = \int_{E_n} K_{j,n}(\bar{x},t) d\bar{x} .$$

By Fourier's integral theorem,

$$\int_{E_n} K_{j,n}(\bar{x},t)d\bar{x} = \int_{E_n} e^{i\bar{y}\cdot\bar{x}} K_{j,n}(\bar{x},t)d\bar{x} \Big|_{\bar{y}=0}$$

$$= e^{-t(\bar{y}\cdot\bar{y})^j} \Big|_{\bar{y}=0} = 1.$$

To complete the proof, we need to show that all $v^{\bar{k}}_{j,n}(\bar{x},-t)$ are orthogonal to $K_{j,n}(\bar{x},t)$. Upon substituting the explicit form of the polynomials into the second member of (12.25) we observe that we must evaluate integrals of the form

(12.27) $$\int_{E_n} \bar{x}^{\bar{r}} K_{j,n}(\bar{x},t)d\bar{x} \quad .$$

We do this in the usual way. Thus

(12.28) $$\int_{E_n} i^{|\bar{r}|} \bar{x}^{\bar{r}} e^{i\bar{y}\cdot\bar{x}} K_{j,n}(\bar{x},t)d\bar{x} = D^{\bar{r}}_{\bar{y}} e^{-t(\bar{y}\cdot\bar{y})^j}$$

$$= \sum_{k=\tau_0}^{\infty} \frac{(-1)^k t^k}{k!} \sum_{k_1=\tau_1}^{jk} \sum_{k_2=\tau_2}^{k_1} \cdots \sum_{k_{n-1}}^{k_{n-2}} \begin{pmatrix} jk \\ k_1,\ldots,k_{n-1} \end{pmatrix} .$$

$$\cdot \frac{(2jk-2k_1)!(2k_1-2k_2)!\ldots(2k_{n-2}-2k_{n-1})!(2k_{n-1})!}{(2jk-2k_1-r_1)!(2k_1-2k_2-r_2)!\ldots(2k_{n-1}-r_n)\ !} \quad .$$

$$y_1^{2jk-2k_1-r_1}\, y_2^{2k_1-2k_2-r_2} \cdots y_n^{2k_{n-1}-r_n} \quad .$$

The lower limits of summation, τ_i, are computed by ensuring that the factorials which appear in the denominator are well defined. Thus,

$$\tau_{n-1} = [\frac{r_n+1}{2}] \quad \text{and} \quad \tau_i = [\frac{r_{i+1}+r_{i+2}+\ldots+r_n+1}{2}],\ i = 0,1,\ \ldots,\ n-1.$$

The series (12.28) vanishes identically at $\bar{y} = 0$ unless

$2jk-2k_1-r_1 = 0$, $2k_i-2k_{i+1}-r_{i+1}=0, i=1,2, \ldots, n-2$, and $2k_{n-1}-r_n = 0$. These conditions imply that each r_i is even and also that $\sum_{i=1}^{n} r_i = 2jk$. With this information it is easily shown that we obtain a non-zero contribution from the series (12.28) only when $k = \tau_0$, $k_i = \tau_i$, $i = 1,2, \ldots, n-1$. Thus,

$$(12.29) \qquad \int_{E_n} i^{|\bar{r}|} \bar{x}^{\bar{r}} K_{j,n}(\bar{x},t)d\bar{x}$$

$$= \begin{cases} 0 \quad \text{if a) } r_i \neq 2s_i,\ i = 1,\ldots,n, \\ \quad \text{or b) } \sum_{i=1}^{n} r_i \neq 2j\ell \quad \text{for some integer } \ell, \\ \\ \dfrac{(-t)^{\ell}(j\ell)!(\bar{r})!}{\ell!\ \bar{s}!} \quad \text{if a) } r_i = 2s_i,\ i=1,\ldots,n, \\ \text{and b) } \sum_{i=1}^{n} r_i = 2j\ell \quad \text{for some integer } \ell. \end{cases}$$

By (12.29) and after some simplification, we observe that

$$(12.30) \qquad \int_{E_n} v_{j,n}^{\bar{r}}(\bar{x},-\bar{t})K_{j,n}(\bar{x},t)d\bar{x}$$

$$= \sum_{k=0}^{\ell} \frac{(-1)^{\ell(j+1)+k}\ t^{\ell}}{k!\ \bar{s}!(\ell-k)!}(jk)!(j\ell-jk)! \sum_{k_1=0}^{jk} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} \binom{s_1}{jk-k_1}\binom{s_2}{k_1-k_2}\cdots\binom{s_n}{k_{n-1}}.$$

We may reduce the inner sums by using the well known Vandermond convolution [15]. Thus,

$$\sum_{k_{n-1}=0}^{k_{n-2}} \binom{s_{n-1}}{k_{n-2}-k_{n-1}}\binom{s_n}{k_{n-1}} = \binom{s_{n-1}+s_n}{k_{n-2}},$$

$$\sum_{k_{n-2}=0}^{k_{n-3}} \binom{s_{n-2}}{k_{n-3}-k_{n-2}} \binom{s_{n-1}+s_n}{k_{n-2}} = \binom{s_{n-2}+s_{n-1}+s_n}{k_{n-3}} ,$$

$$\vdots$$

$$\sum_{k_1=0}^{jk} \binom{s_1}{k_j - k_1} \binom{s_2+s_3+\ldots+s_n}{k_1} = \binom{s_1+s_2+\ldots+s_n}{jk} = \binom{\ell j}{jk} .$$

The series (12.30) is, therefore, equal to

$$\frac{t^{\ell}}{\bar{r}!} \sum_{k=0}^{\ell} \frac{(-1)^{\ell(j+1)+k} (jk)!(j\ell-jk)!}{k!\ (\ell-k)!} \binom{\ell j}{jk}$$

$$= \frac{(-1)^{\ell(j+1)} t^{\ell} (\ell j)!}{\bar{r}!\ \ell!} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} = 0,$$

which concludes the proof of the biorthogonality.

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